

Random Walks on Sierpiński Graphs: Hyperbolicity and Stochastic Homogenization

Vadim A. Kaimanovich

Vienna, Preprint ESI 1169 (2002)

June 12, 2002

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via <http://www.esi.ac.at>

Random Walks on Sierpiński Graphs: Hyperbolicity and Stochastic Homogenization

Vadim A. Kaimanovich

Abstract. We introduce two new techniques to the analysis on fractals. One is based on the presentation of the fractal as the boundary of a countable Gromov hyperbolic graph, whereas the other one consists in taking all possible “backward” extensions of the above hyperbolic graph and considering them as the classes of a discrete equivalence relation on an appropriate compact space. Illustrating these techniques on the example of the Sierpiński gasket (the associated hyperbolic graph is called the Sierpiński graph), we show that the Sierpiński gasket can be identified with the Martin and the Poisson boundaries for fairly general classes of Markov chains on the Sierpiński graph.

1. Introduction

The aim of this paper is to introduce two new techniques to the analysis on fractals (which, as testified by the present volume, is a very active and quickly developing area).

Our approach works for any fractal generated by an IFS (iterated function system) consisting of similarities with the same scaling factor. However, leaving out idle generalities, we introduce these techniques just on the famous example of the (d -dimensional) Sierpiński gasket \mathbf{G} determined by a simplex Δ in the Euclidean space \mathbb{R}^d , and apply them to the problem of the realization of the Sierpiński gasket as the boundary of an appropriate countable Markov chain. The latter problem, first considered by Denker and his collaborators Sato and Koch in a recent series of papers [15], [16], [17] (see Section 4.3 for more details), was the starting point of the present work.

One technique relates fractals to the *hyperbolic geometry*, and it is based on the presentation of the fractal as the *boundary* of an appropriate countable *Gromov hyperbolic graph*. For defining the graph associated with the Sierpiński gasket we begin with the “vertical” Cayley tree of the free semigroup generated by the IFS from the definition of the gasket. We identify the tree vertices with the images of the base simplex Δ under the corresponding maps. In order to take into account the spatial configuration on each horizontal level, this tree is further *augmented* by “horizontal” edges joining simplices with non-empty intersections.

We call the resulting graph \mathcal{G} the (*d-dimensional*) *Sierpiński graph*. [This term had already been used in a somewhat different context, see [23], [28], [53]. The object which is called “Sierpiński graph” in these papers is also known under the name of the “graphical Sierpiński gasket” [4] and is isomorphic to the horizontal layers of certain extended Sierpiński graphs in the sense of our Definition 2.9, see Section 2.7 below for more details.]

In the above construction one could actually take an arbitrary subset $K \subset \mathbb{R}^d$ instead of the simplex Δ . If the diameter of K is small, then it does not allow one to see the spatial interaction between different branches of the IFS, and the corresponding graph is just a tree. However, for any compact set K containing the simplex Δ the arising graph is quasi-isometric to the Sierpiński graph.

Yet another hyperbolic metric on the Sierpiński graph (quasi-isometric to the graph distance) can be obtained by using the fact that the group $\text{Sim}(\mathbb{R}^d)$ of similarities of the Euclidean space \mathbb{R}^d acts simply transitively both on the set of all simplices similar to Δ and on the hyperbolic space \mathbf{H}^{d+1} , which gives rise to an embedding of the Sierpiński graph into the hyperbolic space \mathbf{H}^{d+1} .

The other technique is brought forward in order to resolve the putative contradiction between the highly symmetric appearance of the Sierpiński gasket and the absence of a sufficiently big symmetry group (there is just a semigroup action responsible for the self-similar structure of the Sierpiński gasket). We replace the usual “homogeneity” synonymous to the presence of a symmetry group with the *stochastic homogeneity* characterized by the presence of a *discrete Borel equivalence relation* with a finite stationary measure.

We make the definitions of the Sierpiński gasket and of the Sierpiński graph “bilateral” by extending them from the “microscopic” to the “macroscopic” scale. Namely, instead of just taking smaller and smaller subsimplices of the given simplex Δ one may also go “backwards” by embedding Δ into bigger and bigger simplices. In terms of the theory of dynamical systems this procedure corresponds to passing to the *natural extension* of the associated unilateral full shift. The alphabet \mathcal{A} of this shift is the set of contractions from the IFS used in the definition of the Sierpiński gasket, so that in our setup it can also be identified with the vertex set of the original simplex Δ .

In this way, any string \mathbf{a} from the compact space of left-infinite strings $\mathcal{A}_{-\infty}^0 = \{(\dots, a_{-2}, a_{-1}, a_0) : a_i \in \mathcal{A}\}$ determines a (non-compact) *extended Sierpiński gasket* $\mathbf{G}(\mathbf{a}) \supset \mathbf{G}$ and the associated *extended Sierpiński graph* $\mathcal{G}(\mathbf{a}) \supset \mathcal{G}$. Further, there is a Borel graph structure on the *weak tail equivalence relation* \sim on the space $\mathcal{A}_{-\infty}^0$ (\equiv the orbit equivalence relation of the natural right action of the free semigroup generated by \mathcal{A} on $\mathcal{A}_{-\infty}^0$) such that the class $[\mathbf{a}]_{\sim}$ endowed with this structure is isomorphic to $\mathcal{G}(\mathbf{a})$ for all $\mathbf{a} \in \mathcal{A}_{-\infty}^0$ except for a countable number of virtually periodic strings.

By endowing the space $\mathcal{A}_{-\infty}^0$ with an appropriate probability measure m we may now apply ergodic methods in order to obtain statements valid for almost all (with respect to the measure m) extended graphs $\mathcal{G}(\mathbf{a})$ and gaskets $\mathbf{G}(\mathbf{a})$, which,

in turn, lead to statements about the original Sierpiński graph \mathcal{G} and Sierpiński gasket \mathbf{G} (provided the added parts of $\mathcal{G}(\mathbf{a})$ and $\mathbf{G}(\mathbf{a})$ do not interfere “too much” with what happens on \mathcal{G} and \mathbf{G}). Note that this construction is quite different from what is usually meant by “random fractals” (e.g., see [26], [27] and the references therein).

Due to the lack of space (and time) we chose to concentrate on the detailed explanation of the two aforementioned techniques and on the reduction of the problems concerning the boundary behaviour of Markov chains on the Sierpiński graph to the frameworks of the hyperbolic geometry and of the theory of discrete equivalence relations. On the other hand, the final implementation of this reduction is often just sketched as it follows the same lines as already known results from these disciplines.

The paper has the following structure.

Section 2 is auxiliary: we introduce the Sierpiński gasket (Section 2.1), define the Sierpiński graph (Section 2.2), discuss various actions of the group of similarities (Section 2.4) and a symbolic coding of the Sierpiński gasket and graph (Section 2.5). Further we define the extended Sierpiński gaskets and graphs (Section 2.7) and realize them in terms of strong and weak tail equivalence relations on the symbolic space $\mathcal{A}_{-\infty}^0$ (Sections 2.8 and 2.9).

Section 3 is devoted to geometric properties of Sierpiński graphs. We begin with a general discussion of trees and certain related graphs. Both for *rooted trees* (i.e., ones with a fixed vertex) and *remotely rooted trees* (i.e., ones with a fixed boundary point) there is a well-defined notion of belonging to the same “generation” (level) with respect to the root. In order to obtain an *augmented tree* one adds to a rooted tree new “horizontal” edges satisfying a certain natural condition. In Section 3.4 we formulate a simple necessary and sufficient condition for Gromov hyperbolicity of augmented rooted trees (Theorems 3.13 and 3.15) and give an explicit description of their hyperbolic boundary (Theorem 3.16). In Section 3.5 we show that the Sierpiński graph (which is an augmented tree) satisfies the above hyperbolicity condition and identify its hyperbolic boundary with the Sierpiński gasket (Theorem 3.21). In the same way, the hyperbolic boundary of an extended Sierpiński graph is the one-point compactification of the associated extended Sierpiński gasket (Theorem 3.23). We also show that the Euclidean metric on the Sierpiński gasket is uniformly Hölder equivalent to a family of natural metrics of “hyperbolic origin” (Theorem 3.25). In Section 3.6 we discuss the quasi-isometric embedding of the Sierpiński graph into the hyperbolic space \mathbf{H}^{d+1} determined by the group of similarities of \mathbb{R}^d and related issues from the combinatorial group theory and conformal dynamics. Finally, in Section 3.7 we establish *non-amenability* of the Sierpiński graph \mathcal{G} and of the extended Sierpiński graphs $\mathcal{G}(\mathbf{a})$.

In Section 4 we apply general results from the theory of Markov chains on hyperbolic spaces and on equivalence relations to the objects associated with the

Sierpiński gasket which were constructed in Sections 2 and 3. By using the Ancona theory we identify in Theorem 4.6 the Martin boundary for a class of bounded range Markov operators on the Sierpiński graph (resp., on an extended Sierpiński graph) with the Sierpiński gasket (resp., with the one-point compactification of the associated extended Sierpiński gasket). In particular, this class contains the simple random walks (Theorem 4.7). On the other hand, by using the entropy theory of random walks on equivalence relations we also obtain a description of the Poisson boundary for a family of Markov chains on “typical” extended Sierpiński graphs $\mathcal{G}(\mathbf{a})$ under the finite first moment condition (which is much weaker than the bounded range assumption used for the identification of the Martin boundary). The situation here is similar to what happens with random walks on non-unimodular groups [11], [40]. Namely, everything is determined by the sign of the drift with respect to the remote root. If the drift is zero or directed towards the root, then the Poisson boundary is trivial, whereas if the drift is directed from the root, then the Poisson boundary can be identified with the extended Sierpiński gasket $\mathbf{G}(\mathbf{a})$ (Theorem 4.10). In Theorem 4.11 we prove that in the latter case the Hausdorff dimension of the harmonic measure is expressed as the familiar ratio of the entropy and the exponent (\equiv drift). Finally, in Section 4.6 we ask the intriguing question about the singularity of the harmonic measure of the simple random walk on the Sierpiński graph with respect to the Hausdorff (\equiv uniform) measure on the Sierpiński gasket (Problem 4.14) and discuss several related topics.

I would like to thank Professors Peter Grabner and Wolfgang Woess for inviting me to the excellent multi-faceted conference “Fractals in Graz 2001” and for their collaboration and invariable patience during the editorial process. Significant parts of the paper were written during the author’s visits to Technische Universität Graz, Università di Roma “La Sapienza” (organized by the Gruppo Nazionale di Analisi Matematica, Probabilità e Applicazioni – GNAMPA) and the Ben-Gurion University of the Negev at Beer-Sheva (organized by the Dozor Fellowship Visiting Fund) whose support is gratefully acknowledged.

2. The Sierpiński gasket and the Sierpiński graph

2.1. The Sierpiński gasket

Throughout the paper we shall fix a simplex

$$\Delta = \Delta(\{p_\alpha\}) = \left\{ \sum_{\alpha} t_{\alpha} p_{\alpha} : t_{\alpha} \geq 0, \sum_{\alpha} t_{\alpha} = 1 \right\} \subset \mathbb{R}^d$$

spanned by its vertex set $\{p_\alpha\}$, where α runs through the alphabet

$$\mathcal{A} = \{1, 2, \dots, d+1\}.$$

Denote by

$$\text{Sim}(\mathbb{R}^d) = \{g : x \mapsto ax + b, a \in \mathbb{R}_+, b \in \mathbb{R}^d\}$$

the group of *similarities* of the Euclidean space \mathbb{R}^d , which acts simply transitively on the set

$$\mathfrak{S} = \{\Sigma = g\Delta : g \in \text{Sim}(\mathbb{R}^d)\} \quad (1)$$

of all simplices similar to Δ . If the *scaling factor* a of a similarity $g : x \mapsto ax + b$ is not equal to 1, then g has a unique fixed point x_0 and satisfies the formula

$$gx - x_0 = a(x - x_0) .$$

By

$$g_\alpha \in \text{Sim}(\mathbb{R}^d) : \quad g_\alpha x - p_\alpha = \frac{1}{2}(x - p_\alpha) , \quad x \in \mathbb{R}^d, \alpha \in \mathcal{A} , \quad (2)$$

denote the similarities with the scaling factor $\frac{1}{2}$ and the fixed points p_α , i.e., g_α are uniquely determined by the conditions

$$g_\alpha p_\beta = \begin{cases} p_\alpha , & \alpha = \beta , \\ \frac{1}{2}(p_\alpha + p_\beta) , & \alpha \neq \beta . \end{cases}$$

Denote by

$$S = \text{sgr}(\{g_\alpha\}) \subset G = \text{gr}(\{g_\alpha\}) \subset \text{Sim}(\mathbb{R}^d) \quad (3)$$

the semigroup (resp., the group) generated by the similarities g_α , $g \in \mathcal{A}$.

Let us now define inductively a sequence of subsets of the set of simplices \mathfrak{S} :

$$\vec{\Delta}^0 = \{\Delta\} , \quad \vec{\Delta}^{n+1} = \left\{ g_\alpha \Sigma : \alpha \in \mathcal{A}, \Sigma \in \vec{\Delta}^n \right\} , \quad n \geq 0 ,$$

(the simplices from $\vec{\Delta}^n$ are called *level n simplices*), and put

$$\vec{\Delta} = \bigcup_{n \geq 0} \vec{\Delta}^n \subset \mathfrak{S} . \quad (4)$$

Definition 2.1. *The compact set*

$$\mathbf{G} = \bigcap_n \mathbf{G}^n , \quad \text{where} \quad \mathbf{G}^n = \bigcup_{\Sigma \in \vec{\Delta}^n} \Sigma , \quad n \geq 0 ,$$

is called the Sierpiński gasket determined by the simplex Δ .

The first 3 iterations of the construction from Definition 2.1 are presented in Fig. 1; see Notices of Amer. Math. Soc., **46** (1999), No. 10 for a full colour “front page coverage” of the Sierpiński gasket.

2.2. The Sierpiński graph

Recall that a *graph* X is determined by a *vertex set* $\mathcal{V}(X)$ and an *edge set* $\mathcal{E}(X) \subset \mathcal{V}(X) \times \mathcal{V}(X)$. Slightly abusing the notation we shall often identify the graph with its vertex set. A graph X is *non-oriented* if the set $\mathcal{E}(X)$ is symmetric, and *contains no loops* if $\mathcal{E}(X)$ does not intersect the diagonal in $\mathcal{V}(X) \times \mathcal{V}(X)$. Two vertices $x, y \in \mathcal{V}(X)$ are called *neighbours* if $(x, y) \in \mathcal{E}(X)$. The *degree* $\deg(x)$ of a vertex x is the number of its neighbours. If any two vertices $x, y \in \mathcal{V}(X)$ can be joined with a chain of edges from $\mathcal{E}(X)$, then the graph X is called *connected*. The minimal

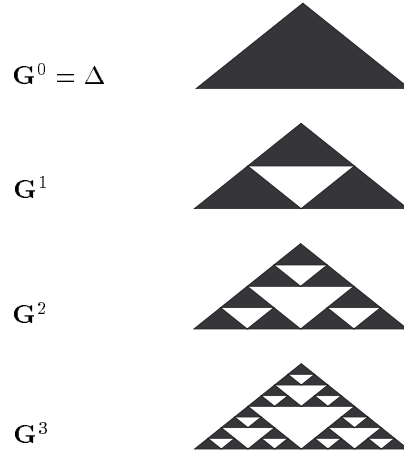


FIGURE 1

length of such a chain is called the *graph distance* on $\mathcal{V}(X)$ and it is denoted by $\text{dist}(x, y)$.

Convention 2.2. *Throughout the paper all graphs (unless otherwise specified) are assumed to be countably infinite, non-oriented, with no loops, to have uniformly bounded vertex degrees, and to be connected.*

Definition 2.3. *The Sierpiński graph \mathcal{G} (of dimension d) is the graph whose vertex set*

$$\mathcal{V}(\mathcal{G}) = \vec{\Delta}$$

is the set (4) of all simplices used in the construction of the Sierpiński gasket, and the edge set

$$\mathcal{E}(\mathcal{G}) = \mathcal{E}^v(\mathcal{G}) \cup \mathcal{E}^h(\mathcal{G})$$

is a union of the sets of vertical and horizontal edges, respectively, where

$$\begin{aligned} \mathcal{E}^v(\mathcal{G}) &= \left\{ (\Sigma, \Sigma'), (\Sigma', \Sigma) : \exists n \geq 0 \text{ with } \Sigma \in \vec{\Delta}^n, \Sigma' \in \vec{\Delta}^{n+1}, \text{ and } \Sigma' \subset \Sigma \right\}, \\ \mathcal{E}^h(\mathcal{G}) &= \left\{ (\Sigma, \Sigma') : \exists n \geq 0 \text{ with } \Sigma, \Sigma' \in \vec{\Delta}^n, \text{ and } \Sigma \cap \Sigma' \neq \emptyset \right\}. \end{aligned}$$

In other words, the vertical edges of \mathcal{G} are those of the natural partition tree structure on $\vec{\Delta}$, whereas the horizontal edges take into account the spatial configuration of simplices from each level $\vec{\Delta}^n$ by joining those simplices whose intersection is non-empty.

2.3. Symbolic spaces

Denote by

$$\mathcal{A}_m^n = \prod_{k=m}^n \mathcal{A}, \quad \mathcal{A}_m^* = \bigcup_{n: m \leq n < \infty} \mathcal{A}_m^n, \quad \mathcal{A}_*^n = \bigcup_{m: -\infty < m \leq n} \mathcal{A}_m^n$$

various spaces of *strings* $\mathbf{a} = (a_k)$ of symbols from the alphabet \mathcal{A} (the numbers m and n in these notations are allowed to take the values $\pm\infty$ as well). We shall also use the following notations:

- $|\mathbf{a}| = n - m + 1$ — the *length* of a string $\mathbf{a} \in \mathcal{A}_m^n$;
- $[\mathbf{a}] = n$ — the *stretch* of a string $\mathbf{a} \in \mathcal{A}_{-\infty}^n \subset \mathcal{A}_{-\infty}^*$;
- $\mathbf{a}_{m'}^{n'} \in \mathcal{A}_{m'}^{n'}$ — the *truncation* of a string $\mathbf{a} \in \mathcal{A}_m^n$ determined by the integers m', n' with $m \leq m' \leq n' \leq n$;
- $\alpha_m^n \in \mathcal{A}_m^n$ — the string whose entries are all equal to a symbol $\alpha \in \mathcal{A}$;
- $\overline{\mathbf{a}\mathbf{b}}$ — the *concatenation* of two strings $\mathbf{a} \in \mathcal{A}_k^m$ and $\mathbf{b} \in \mathcal{A}_{m+1}^n$;
- $T : \mathcal{A}_m^n \rightarrow \mathcal{A}_{m-1}^{n-1}$ — the *shift* defined by the formula $[T\mathbf{a}]_k = [\mathbf{a}]_{k+1}$;
- $U : \mathcal{A}_1^\infty \rightarrow \mathcal{A}_1^\infty$ — the (non-invertible) *unilateral shift* defined by the formula $U\mathbf{a} = (T\mathbf{a})_1^\infty$.

We shall refer to the strings from the space \mathcal{A}_1^* as *words* and denote by \emptyset the empty word (of length $|\emptyset| = 0$). Then the set

$$\mathcal{A}_\emptyset^* = \{\emptyset\} \cup \mathcal{A}_1^*$$

endowed with the multiplication

$$\mathbf{a}\mathbf{b} = \overline{\mathbf{a}(T^{-|\mathbf{a}|}\mathbf{b})}, \quad \mathbf{a}, \mathbf{b} \in \mathcal{A}_\emptyset^*,$$

becomes the *free semigroup* on the alphabet \mathcal{A} . This multiplication naturally extends to the right action of the semigroup \mathcal{A}_\emptyset^* on the space $\mathcal{A}_{-\infty}^*$ of left infinite strings by the formula

$$(\mathbf{a}, \mathbf{w}) \mapsto \mathbf{a}\mathbf{w} = \overline{\mathbf{a}(T^{-[\mathbf{a}]}\mathbf{w})}, \quad \mathbf{a} \in \mathcal{A}_{-\infty}^*, \mathbf{w} \in \mathcal{A}_\emptyset^*. \quad (5)$$

Below we shall also need the right action of \mathcal{A}_\emptyset^* on $\mathcal{A}_{-\infty}^0$ by the formula

$$(\mathbf{a}, \mathbf{w}) \mapsto \mathbf{a} \cdot \mathbf{w} = T^{|\mathbf{w}|}(\mathbf{a}\mathbf{w}), \quad \mathbf{a} \in \mathcal{A}_{-\infty}^*, \mathbf{w} \in \mathcal{A}_\emptyset^*. \quad (6)$$

2.4. Actions on the space of simplices

It is clear (see Fig. 1) that the map

$$\mathbf{w} \mapsto \begin{cases} g_{w_1} g_{w_2} \cdots g_{w_n}, & \mathbf{w} = (w_1 w_2 \cdots w_n), n \geq 1, \\ \text{Id}, & \mathbf{w} = \emptyset. \end{cases}$$

is an isomorphism between the free semigroup \mathcal{A}_\emptyset^* and the semigroup $S \subset \text{Sim}(\mathbb{R}^d)$ (here Id denotes the identity of $\text{Sim}(\mathbb{R}^d)$). Denote by

$$\mathbf{w}x = g_{w_1} g_{w_2} \cdots g_{w_n} x = 2^{-|\mathbf{w}|} x + \sum_{k=1}^{|\mathbf{w}|} 2^{-k} p_{w_k}, \quad x \in \mathbb{R}^d,$$

the resulting (left) action of $\mathcal{A}_\emptyset^* \cong S$ on \mathbb{R}^d , and by

$$\mathbf{w}\Sigma = \{\mathbf{w}x : x \in \Sigma\}$$

the associated (left) action of $\mathcal{A}_\emptyset^* \cong S$ on the space of simplices \mathfrak{S} (1). Since S is a subsemigroup of the group $\text{Sim}(\mathbb{R}^d)$, we shall also use (in the obvious sense) the notations $(\mathbf{w})^{-1}x$, etc., for $\mathbf{w} \in \mathcal{A}_\emptyset^*$.

Since the group $\text{Sim}(\mathbb{R}^d)$ acts simply transitively on \mathfrak{S} , any simplex $\Sigma \in \mathfrak{S}$ can be uniquely presented as

$$\Sigma = g^\Sigma \Delta, \quad g^\Sigma \in \text{Sim}(\mathbb{R}^d).$$

Therefore, the right action of the group $\text{Sim}(\mathbb{R}^d)$ on itself determines by the formula

$$\Sigma g = g^\Sigma g \Delta, \quad \Sigma \in \mathfrak{S}, \quad (7)$$

a *right* action of the group $\text{Sim}(\mathbb{R}^d)$ on \mathfrak{S} . Obviously,

$$\Delta g = g \Delta \quad \forall g \in \text{Sim}(\mathbb{R}^d).$$

Remark 2.4. *Unlike the left action $\Sigma \mapsto g\Sigma$, the right action (7) is defined on the space of simplices \mathfrak{S} only and does not correspond to any action on the space \mathbb{R}^d .*

The resulting right action

$$\Sigma \mathbf{w} = \Sigma g_{w_1} g_{w_2} \cdots g_{w_n}$$

of the free semigroup $\mathcal{A}_\emptyset^* \cong S$ on \mathfrak{S} admits the following natural interpretation. Let us denote by $p_\alpha^\Sigma = g^\Sigma p_\alpha$ the vertex of a simplex $\Sigma = g^\Sigma \Delta \in \mathfrak{S}$ corresponding to the vertex p_α of the reference simplex Δ . By $g_\alpha^\Sigma \in \text{Sim}(\mathbb{R}^d)$ denote the similarity with the scaling factor $\frac{1}{2}$ and the fixed point p_α^Σ , so that $g_\alpha^\Sigma = g^\Sigma g_\alpha (g^\Sigma)^{-1}$, where g_α are the similarities (2) associated with the vertices of the reference simplex Δ , and let

$$\varphi_\alpha(\Sigma) = g_\alpha^\Sigma \Sigma. \quad (8)$$

Then

$$\varphi_\alpha(\Sigma) = g_\alpha^\Sigma \Sigma = g_\alpha^\Sigma g^\Sigma \Delta = g^\Sigma g_\alpha (g^\Sigma)^{-1} g^\Sigma \Delta = g^\Sigma g_\alpha \Delta = \Sigma g_\alpha,$$

so that

$$\Sigma \mathbf{w} = \Sigma g_{w_1} g_{w_2} \cdots g_{w_n} = \varphi_{w_n} \circ \cdots \circ \varphi_{w_2} \circ \varphi_{w_1}(\Sigma)$$

By extending the notation (4), let us denote by

$$\vec{\Sigma} = \{\Sigma \mathbf{w} : \mathbf{w} \in \mathcal{A}_\emptyset^*\} \subset \mathfrak{S}$$

the set consisting of the simplex Σ and all smaller simplices obtained from Σ by an iterative application of the transformations φ_α (8).

2.5. Symbolic coding

We shall identify the vertex set $\mathcal{V}(\mathcal{G}) = \vec{\Delta}$ of the Sierpiński graph with the set of words \mathcal{A}_\emptyset^* by the map

$$\mathcal{A}_\emptyset^* \rightarrow \vec{\Delta}, \quad \mathbf{a} \mapsto \mathbf{a}\Delta = \Delta\mathbf{a}.$$

In these symbolic terms the “vertical” part $(\mathcal{V}(\mathcal{G}), \mathcal{E}^v(\mathcal{G}))$ of the Sierpiński graph \mathcal{G} is isomorphic to the (right) Cayley tree of the free semigroup $\mathcal{A}_\emptyset^* \cong S$.

For giving a symbolic description of the “horizontal” part of \mathcal{G} notice that two distinct simplices $\mathbf{a}\Delta, \mathbf{b}\Delta$ from the same level $\vec{\Delta}^n$ have a non-empty intersection (consisting of a single point) if and only if the words \mathbf{a} and \mathbf{b} have the form

$$\begin{cases} \mathbf{a} = \mathbf{c}\alpha\beta^k \\ \mathbf{b} = \mathbf{c}\beta\alpha^k \end{cases} \quad \text{with } \mathbf{c} \in \mathcal{A}_\emptyset^*, \alpha \neq \beta \in \mathcal{A}, k \geq 0, \quad (9)$$

in which case $\mathbf{c}\Delta$ is the minimal simplex from $\vec{\Delta}$ containing both $\mathbf{a}\Delta$ and $\mathbf{b}\Delta$, and

$$\mathbf{a}\Delta \cap \mathbf{b}\Delta = \left\{ \sum_{k=1}^{|c|} 2^{-k} p_{c_k} + 2^{-|c|-1} (p_\alpha + p_\beta) \right\} = \left\{ \mathbf{c} \left(\frac{1}{2} (p_\alpha + p_\beta) \right) \right\}.$$

Let

$$Q = \left\{ \mathbf{c} \left(\frac{1}{2} (p_\alpha + p_\beta) \right) : \mathbf{c} \in \mathcal{A}_\emptyset^*, \alpha, \beta \in \mathcal{A} \right\} \subset \Delta. \quad (10)$$

Following [16], let us define the *conjugate* of a word $\mathbf{a} \in \mathcal{A}_\emptyset^*$ as

$$\mathbf{a}^* = \begin{cases} \mathbf{c}\beta\alpha^k, & \mathbf{a} = \mathbf{c}\alpha\beta^k, \mathbf{c} \in \mathcal{A}_\emptyset^*, \alpha, \beta \in \mathcal{A}, k \geq 1, \\ \mathbf{a}, & \mathbf{a} = \alpha^k, \alpha \in \mathcal{A}, k \geq 0. \end{cases} \quad (11)$$

Then, by (9), two words $\mathbf{a} \neq \mathbf{b} \in \mathcal{A}_1^n$ are joined with a horizontal edge in the Sierpiński graph if and only if either $\mathbf{a}_1^{n-1} = \mathbf{b}_1^{n-1}$ or $\mathbf{a}^* = \mathbf{b}$. Therefore, the vertices of the Sierpiński graph \mathcal{G} are classified by their degrees in the following way:

- (i) The “root” \emptyset , for which $\deg(\emptyset) = d + 1$. The neighbours of \emptyset are the 1-letter words $\alpha, \alpha \in \mathcal{A}$.
- (ii) The “corner vertices” $\mathbf{a} = \alpha^n, \alpha \in \mathcal{A}, n > 0$, for which $\deg(\mathbf{a}) = 2d + 2$. Each of these vertices has 1 neighbour α^{n-1} from the preceding $(n - 1)$ -th level, d neighbours $\alpha^{n-1}\beta, \beta \in \mathcal{A} \setminus \{\alpha\}$, from the same n -th level, and $d + 1$ neighbours $\alpha^n\beta, \beta \in \mathcal{A}$, from the next $(n + 1)$ -th level.
- (iii) For all other (“ordinary”) vertices $\deg(\mathbf{a}) = 2d + 3$. Each of them has 1 neighbour \mathbf{a}_1^{n-1} from the preceding $(n - 1)$ -th level and $d + 1$ neighbours $\mathbf{a}\beta, \beta \in \mathcal{A}$, from the next $(n + 1)$ -th level, whereas among $d + 1$ neighbours of \mathbf{a} from the same n -th level d ones $\mathbf{a}_1^{n-1}\beta, \beta \in \mathcal{A} \setminus \{\alpha_n\}$ are the “siblings” of \mathbf{a} (i.e., they have the same first $n - 1$ letters), and the remaining neighbour \mathbf{a}^* is a “distant relative”.

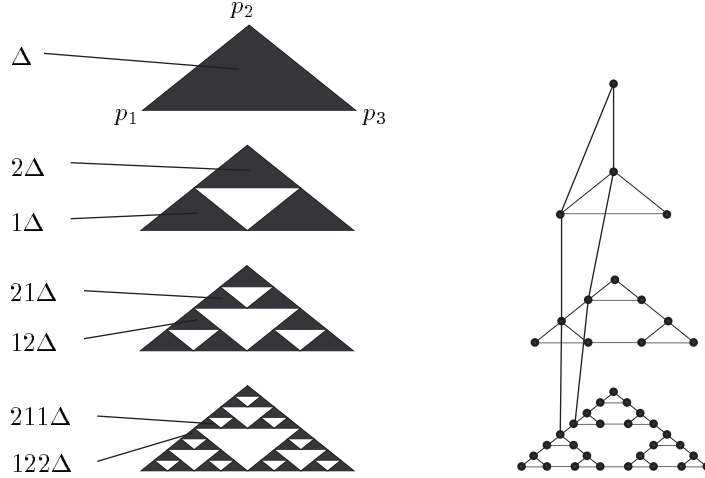


FIGURE 2

For any $\mathbf{a} \in \mathcal{A}_1^\infty$ the simplices $\Delta_n(\mathbf{a}) = \mathbf{a}_1^n \Delta$ decrease on n , their intersection consists of the single point

$$\pi(\mathbf{a}) = \sum_{k=1}^{\infty} 2^{-k} p_{a_k} \in \mathbf{G}, \quad (12)$$

and

$$\mathbf{a}_1^n x \xrightarrow{n \rightarrow \infty} \pi(\mathbf{a}) \quad \forall x \in \mathbb{R}^d.$$

The coding map $\pi : \mathcal{A}_1^\infty \rightarrow \mathbf{G}$ is one-to-one on $\mathbf{G} \setminus Q$, whereas for any “rational” point from the set Q (10) its π -preimage consists of two coding strings $c\alpha\beta^\infty$ and $c\beta\alpha^\infty$ (cf. with analogous boundary expansions for Fuchsian groups, see [52]).

Proposition 2.5 ([16, Proposition 4.2]). *The coding map π (12) establishes a homeomorphism between the Sierpiński gasket \mathbf{G} and the quotient of the space \mathcal{A}_1^∞ (endowed with the product topology) by the equivalence relation consisting of the preimages of π .*

Remark 2.6. *In the degenerate case $d = 1$ if $p_1 = 0, p_2 = 1$, then \mathbf{G} coincides with the unit interval $\Delta = [0, 1]$, the coding π corresponds to the dyadic expansion, and the subset Q consists of dyadic-rational numbers.*

2.6. Natural extension

The constructions of the Sierpiński gasket (Definition 2.1) and the Sierpiński graph (Definition 2.3) being “unilateral”, it is natural (and useful for applications) to make them “bilateral” by extending from the “microscopic” to the “macroscopic”

scale. Namely, instead of just taking smaller and smaller subsimplices of the given simplex Δ one may also go “backwards” by embedding Δ into bigger and bigger simplices. In terms of the theory of dynamical systems this procedure corresponds to passing from the space of unilateral strings \mathcal{A}_1^∞ (endowed with the unilateral shift U) to its *natural extension* $\mathcal{A}_{-\infty}^\infty$ (endowed with the bilateral shift T); see [14] for a general definition of the natural extension of a non-invertible dynamical system.

Let

$$\Delta_n = \Delta_n(\mathbf{a}) = \mathbf{a}_1^n \Delta = \Delta \mathbf{a}_1^n, \quad n \geq 0, \quad (13)$$

be the simplices associated with a string $\mathbf{a} \in \mathcal{A}_1^\infty$ (see Section 2.5). Note that the simplices associated with \mathbf{a} and its shift $U\mathbf{a}$ are connected with the formula

$$\Delta_n(U\mathbf{a}) = g_{a_1}^{-1} \Delta_{n+1}(\mathbf{a}) \quad \forall \mathbf{a} \in \mathcal{A}_1^\infty, n \geq 0. \quad (14)$$

Then

$$\Delta_{n+1} = \Delta \mathbf{a}_1^{n+1} = (\Delta \mathbf{a}_1^n)_{a_{n+1}} = \Delta_n a_{n+1} = \varphi_{a_{n+1}}(\Delta_n), \quad (15)$$

or, in plain words, Δ_{n+1} is obtained by contracting Δ_n towards the vertex $p_{a_{n+1}}^{\Delta_n}$ (cf. Fig. 2). By using formula (15), we may now extend the definition (13) from positive indices n to all $n \in \mathbb{Z}$ by putting for any $\mathbf{a} \in \mathcal{A}_{-\infty}^\infty$ and $n < 0$

$$\Delta_n(\mathbf{a}) = \varphi_{a_{n+1}}^{-1} \circ \cdots \circ \varphi_{a_0}^{-1}(\Delta) = \Delta (\mathbf{a}_{n+1}^0)^{-1},$$

or, equivalently,

$$\Delta_n(\mathbf{a}) = g_{a_0}^{-1} \cdots g_{a_{n+1}}^{-1}(\Delta) = (\mathbf{a}_{n+1}^0)^{-1} \Delta \quad (16)$$

(see Fig. 3 and Fig. 4). With this extended definition of the sequence $\Delta_n(\mathbf{a})$ formula (14) (where the unilateral shift U is replaced with the bilateral shift T) now holds for all $\mathbf{a} \in \mathcal{A}_{-\infty}^\infty$ and $n \in \mathbb{Z}$. If $n < 0$, we shall use the notation (16) for unilateral strings $\mathbf{a} \in \mathcal{A}_{-\infty}^0$ as well.

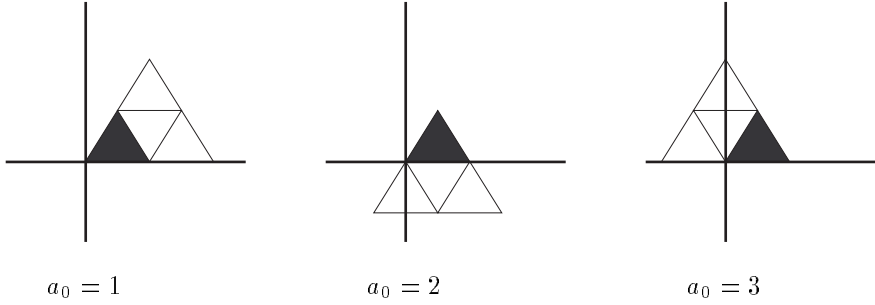


FIGURE 3

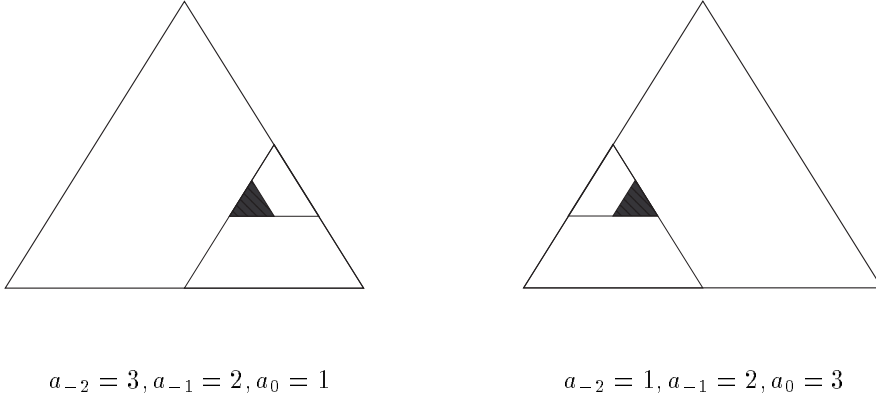


FIGURE 4

Remark 2.7. Formula (16) is precisely analogous to the one arising in the theory of random walks on groups when passing from unilateral to bilateral walks, see the discussion in [33].

Remark 2.8. If each of the symbols from \mathcal{A} occurs in the string $\mathbf{a}_{-\infty}^0$ infinitely often, then the simplices $\Delta_n(\mathbf{a})$ exhaust the space \mathbb{R}^d .

2.7. Extended Sierpiński gaskets and graphs

Given a string $\mathbf{a} \in \mathcal{A}_{-\infty}^0$ put

$$\mathbf{G}_n(\mathbf{a}) = (\mathbf{a}_{n+1}^0)^{-1} \mathbf{G} ,$$

so that

$$\Delta_n(\mathbf{a}) \supset \mathbf{G}_n(\mathbf{a}) \supset \mathbf{G}_{n+1}(\mathbf{a}) \quad \forall n < 0 ,$$

and

$$\mathbf{G}_m(\mathbf{a}) \cap \Delta_n(\mathbf{a}) = \mathbf{G}_n(\mathbf{a}) \quad \forall m \leq n .$$

In the same way we define the graphs

$$\mathcal{G}_n(\mathbf{a}) = (\mathbf{a}_{n+1}^0)^{-1} \mathcal{G} ,$$

so that the vertex set of $\mathcal{G}_n(\mathbf{a})$ is $\overrightarrow{\Delta_n(\mathbf{a})}$. Then

$$\mathcal{G}_n(\mathbf{a}) \supset \mathcal{G}_{n+1}(\mathbf{a}) \quad \forall n < 0 .$$

Definition 2.9. The set

$$\mathbf{G}_{-\infty}(\mathbf{a}) = \bigcup_n \mathbf{G}_n(\mathbf{a}) \subset \mathbb{R}^d .$$

and the graph

$$\mathcal{G}_{-\infty}(\mathbf{a}) = \bigcup_n \mathcal{G}_n(\mathbf{a}) .$$

are called the extended Sierpiński gasket and the extended Sierpiński graph determined by the string $\mathbf{a} \in \mathcal{A}_{-\infty}^{\infty}$, respectively.

The extended Sierpiński graphs $\mathcal{G}(\mathbf{a})$ (resp., the associated gaskets $\mathbf{G}(\mathbf{a})$) are “more homogeneous” than the original Sierpiński graph \mathcal{G} (resp., the gasket \mathbf{G}), as here the root \emptyset is “moved to infinity” and replaced with a “mythical ancestor” in the terminology of Cartier [10]; cf. [11]. For this reason the graphs $\mathcal{G}(\alpha_{-\infty}^0)$ and the gaskets $\mathbf{G}(\alpha_{-\infty}^0)$ had already been considered, for example, in [5], [23]. However, the graphs $\mathcal{G}(\alpha_{-\infty}^0)$ still have “corner vertices” whose degree is smaller than the degree of all other “ordinary vertices” (cf. Section 2.5). In order to deal with this nuisance and to have a graph with a constant degree of vertices, in the above papers a mirror copy of $\mathcal{G}(\alpha_{-\infty}^0)$ was then attached to $\mathcal{G}(\alpha_{-\infty}^0)$. However, it is much more natural to apply a “stochastic homogenization” instead and to consider a “random” graph (resp., gasket) picked from the family of graphs $\mathcal{G}(\mathbf{a})$ (resp., of gaskets $\mathbf{G}(\mathbf{a})$) according to an appropriate probability measure on $\mathcal{A}_{-\infty}^0$, see below Section 4.5.

2.8. Strong tail equivalence relation

Definition 2.10. Let us denote the strong tail equivalence relation on $\mathcal{A}_{-\infty}^*$ by

$$\mathbf{a} \approx \mathbf{b} \iff \exists N \in \mathbb{Z} : a_n = b_n \ \forall n \leq N, \quad \mathbf{a}, \mathbf{b} \in \mathcal{A}_{-\infty}^*,$$

and let

$$R_{\approx} = \{(\mathbf{a}, \mathbf{b}) : \mathbf{a} \approx \mathbf{b}\} \subset \mathcal{A}_{-\infty}^* \times \mathcal{A}_{-\infty}^*$$

be the Borel set consisting of all pairs of \approx -equivalent strings. By

$$[\mathbf{a}]_{\approx} = \{\mathbf{b} \in \mathcal{A}_{-\infty}^* : \mathbf{a} \approx \mathbf{b}\}$$

we denote the strong equivalence class of a string $\mathbf{a} \in \mathcal{A}_{-\infty}^*$.

Remark 2.11. The strong equivalence relation coincides with the orbit equivalence relation of the action (5) of the free semigroup $\mathcal{A}_{\emptyset}^*$ on $\mathcal{A}_{-\infty}^*$. Namely, for any two strings $\mathbf{a}, \mathbf{b} \in \mathcal{A}_{-\infty}^*$

$$\mathbf{a} \approx \mathbf{b} \iff \exists \mathbf{c} \in \mathcal{A}_{-\infty}^*, \mathbf{w}, \mathbf{w}' \in \mathcal{A}_{\emptyset}^* : \mathbf{a} = \mathbf{c}\mathbf{w}, \mathbf{b} = \mathbf{c}\mathbf{w}'.$$

Let

$$\gamma(\mathbf{a}, \mathbf{b}) = \lim_{n \rightarrow -\infty} \left(\mathbf{a}_n^{[a]} \right)^{-1} \mathbf{b}_n^{[b]} \in G \subset \text{Sim}(\mathbb{R}^d)$$

be the G -valued Gibbs cocycle on the strong tail equivalence relation \approx [38] (the expression in the right-hand side of the above formula stabilizes on n by the definition of the equivalence relation \approx). As in Section 2.5, we shall identify the vertex set $\overrightarrow{\Delta_n(\mathbf{a})}$ of the graph $\mathcal{G}_n(\mathbf{a})$ with the set of strings

$$\{\mathbf{b} \in [\mathbf{a}]_{\approx} : \mathbf{b}_k = \mathbf{a}_k \text{ for } k \leq n\}$$

by the map

$$\mathbf{b} \mapsto \gamma(\mathbf{a}, \mathbf{b})\Delta \in \mathfrak{S}. \quad (17)$$

The identification (17) respects the embeddings $\mathcal{G}_n(\mathbf{a}) \supset \mathcal{G}_{n+1}(\mathbf{a})$, so that we have

Proposition 2.12. *For any string $\mathbf{a} \in \mathcal{A}_{-\infty}^0$ the map (17) establishes a one-to-one correspondence between the equivalence class $[\mathbf{a}]_{\approx} \subset \mathcal{A}_{-\infty}^*$ and the vertex set of the extended Sierpiński graph $\mathcal{G}(\mathbf{a})$.*

The extended Sierpiński graphs $\mathcal{G}(\mathbf{a})$ can be described in symbolic terms in the same way as it was done with the Sierpiński graph in Section 2.5. Namely, let us extend the definition of the conjugacy $\mathbf{a} \mapsto \mathbf{a}^*$ from $\mathcal{A}_{\emptyset}^*$ to $\mathcal{A}_{-\infty}^*$ by putting

$$\begin{cases} (b\alpha\beta^n)^* = b\beta\alpha^n & , \quad \mathbf{b} \in \mathcal{A}_{-\infty}^*, \alpha, \beta \in \mathcal{A}, n > 0, \\ (\alpha_{-\infty}^n)^* = (\alpha_{-\infty}^n) & , \quad \alpha \in \mathcal{A}. \end{cases}$$

Then the set of edges of the Sierpiński graph $\mathcal{G}(\mathbf{a})$ (realized under the identification (17) on the equivalence class $[\mathbf{a}]_{\approx}$) is a union

$$\mathcal{E}_{\approx}(\mathbf{a}) = \mathcal{E}_{\approx}^v(\mathbf{a}) \cup \mathcal{E}_{\approx}^h(\mathbf{a}) \subset [\mathbf{a}]_{\approx} \times [\mathbf{a}]_{\approx}$$

of the set of vertical edges

$$\mathcal{E}_{\approx}^v(\mathbf{a}) = \{(b, b\alpha), (b\alpha, b) : b \in [\mathbf{a}]_{\approx}, \alpha \in \mathcal{A}\}, \quad (18)$$

and of the set of horizontal edges

$$\mathcal{E}_{\approx}^h(\mathbf{a}) = \{(b\alpha, b\beta) : b \in [\mathbf{a}]_{\approx}, \alpha, \beta \in \mathcal{A}\} \cup \{(b, b^*) : b \in [\mathbf{a}]_{\approx}, b \neq b^*\}. \quad (19)$$

Remark 2.13. *In the same way as in Section 2.5, one may also obtain a coding of the Sierpiński gasket $\mathbf{G}(\mathbf{a})$ by bilateral strings $\mathbf{b} \in \mathcal{A}_{-\infty}^\infty$ with $\mathbf{b}_{-\infty}^0 \approx \mathbf{a}$ which is one-to-one on the complement of a countable subset of $\mathbf{G}(\mathbf{a})$ (where it is two-to-one).*

It is clear that $\mathcal{E}_{\approx}(\mathbf{a}) = \mathcal{E}_{\approx}(\mathbf{b})$ whenever $\mathbf{a} \approx \mathbf{b}$, so that actually we have a well-defined graph structure on any equivalence class $[\mathbf{a}]_{\approx} \subset \mathcal{A}_{-\infty}^*$. Moreover, the union

$$\mathcal{E}_{\approx} = \bigcup_{\mathbf{a} \in \mathcal{A}_{-\infty}^*} \mathcal{E}_{\approx}(\mathbf{a}) \quad (20)$$

is a Borel subset of R_{\approx} . Thus, the triple $(\mathcal{A}_{-\infty}^*, \approx, \mathcal{E}_{\approx})$ is a *graphed equivalence relation* (see [1], [34] for a discussion of this notion), which allows one to consider the graphs $\mathcal{G}(\mathbf{a})$ not only individually, but also as members of a “collection” consisting of all \approx equivalence classes in $\mathcal{A}_{-\infty}^*$.

2.9. Weak tail equivalence relation

The disadvantage of the space $\mathcal{A}_{-\infty}^*$ is in its non-compactness. Therefore, it is convenient to modify the definitions from Section 2.8 by replacing $\mathcal{A}_{-\infty}^*$ with the compact space $\mathcal{A}_{-\infty}^0$ and allowing in return for shifted strings to remain equivalent.

Definition 2.14. *Let us denote the weak tail equivalence relation on $\mathcal{A}_{-\infty}^*$ by*

$$\mathbf{a} \sim \mathbf{b} \iff \exists t \in \mathbb{Z} : \mathbf{a} \sim T^t \mathbf{b}, \quad \mathbf{a}, \mathbf{b} \in \mathcal{A}_{-\infty}^*,$$

and let

$$R_{\sim} = \{(\mathbf{a}, \mathbf{b}) : \mathbf{a} \sim \mathbf{b}\} \subset \mathcal{A}_{-\infty}^* \times \mathcal{A}_{-\infty}^*$$

be the Borel set consisting of all pairs of \sim -equivalent strings. By

$$[\mathbf{a}]_{\sim} = \{\mathbf{b} \in \mathcal{A}_{-\infty}^* : \mathbf{a} \sim \mathbf{b}\}$$

we denote the weak equivalence class of a string $\mathbf{a} \in \mathcal{A}_{-\infty}^*$. We shall also use the notations

$$R_{\sim}^0 = R_{\sim} \cap \mathcal{A}_{-\infty}^0 \times \mathcal{A}_{-\infty}^0$$

and

$$[\mathbf{a}]_{\sim}^0 = [\mathbf{a}]_{\sim} \cap \mathcal{A}_{-\infty}^0$$

for the restriction of the equivalence relation \sim to $\mathcal{A}_{-\infty}^0$.

Remark 2.15. The restriction of the weak equivalence relation onto $\mathcal{A}_{-\infty}^0$ coincides with the orbit equivalence relation of the action (6) of the free semigroup $\mathcal{A}_{\mathcal{S}}^*$ on $\mathcal{A}_{-\infty}^0$, cf. Remark 2.11.

Clearly, a string $\mathbf{a} \in \mathcal{A}_{\infty}^*$ may be \sim -equivalent to its own shift $T^t \mathbf{a}$, $t \in \mathbb{Z}$ if and only if it is \sim -equivalent to a periodic string. In this case we say that the string \mathbf{a} is *residually periodic*. Denote by $\Pi \subset \mathcal{A}_{\infty}^*$ the countable set of all residually periodic strings. Then the formula

$$T^{\sigma(\mathbf{a}, \mathbf{b})} \mathbf{a} \approx \mathbf{b} \quad (21)$$

uniquely defines the \mathbb{Z} -valued *synchronization cocycle* on the restriction of the equivalence relation \sim onto $\mathcal{A}_{-\infty}^* \setminus \Pi$. In particular,

$$\sigma(\mathbf{a}, T^t \mathbf{a}) = t \quad \forall \mathbf{a} \in \mathcal{A}_{-\infty}^* \setminus \Pi, t \in \mathbb{Z}.$$

Proposition 2.16. For any $\mathbf{a} \in \mathcal{A}_{-\infty}^0 \setminus \Pi$ the map

$$[\mathbf{a}]_{\sim} \rightarrow [\mathbf{a}]_{\sim}^0, \quad \mathbf{b} \mapsto \mathbf{c} = T^{[\mathbf{b}]} \mathbf{b}, \quad (22)$$

and the inverse map

$$[\mathbf{a}]_{\sim}^0 \rightarrow [\mathbf{a}]_{\sim}, \quad \mathbf{c} \mapsto \mathbf{b} = T^{-[\mathbf{b}]} \mathbf{c} = T^{-\sigma(\mathbf{b}, \mathbf{c})} \mathbf{c} = T^{-\sigma(\mathbf{a}, \mathbf{c})} \mathbf{c} \quad (23)$$

establish a one-to-one correspondence between the equivalence classes $[\mathbf{a}]_{\sim} \subset \mathcal{A}_{-\infty}^*$ and $[\mathbf{a}]_{\sim}^0 \subset \mathcal{A}_{-\infty}^0$.

Given $\mathbf{a} \in \mathcal{A}_{-\infty}^0$, let us now put

$$\mathcal{E}_{\sim}(\mathbf{a}) = \mathcal{E}_{\sim}^v(\mathbf{a}) \cup \mathcal{E}_{\sim}^h(\mathbf{a}) \subset [\mathbf{a}]_{\sim}^0 \times [\mathbf{a}]_{\sim}^0,$$

where

$$\mathcal{E}_{\sim}^v(\mathbf{a}) = \{(\mathbf{b}, \mathbf{b} \cdot \alpha), (\mathbf{b} \cdot \alpha, \mathbf{b}) : \mathbf{b} \in [\mathbf{a}]_{\sim}^0, \alpha \in \mathcal{A}\},$$

$$\mathcal{E}_{\sim}^h(\mathbf{a}) = \{(\mathbf{b} \cdot \alpha, \mathbf{b} \cdot \beta) : \mathbf{b} \in [\mathbf{a}]_{\sim}^0, \alpha, \beta \in \mathcal{A}\} \cup \{(\mathbf{b}, \mathbf{b}^*) : \mathbf{b} \in [\mathbf{a}]_{\sim}^0, \mathbf{b} \neq \mathbf{b}^*\}.$$

The only difference between these definitions and formulas (18), (19) is that here the action (5) is replaced with the action (6). Then the Borel set

$$\mathcal{E}_{\sim} = \bigcup_{\mathbf{a} \in \mathcal{A}_{-\infty}^0} \mathcal{E}_{\sim}(\mathbf{a}) \subset R_{\sim}^0 \quad (24)$$

determines a structure of a graphed equivalence relation on $(\mathcal{A}_{-\infty}^0, \sim)$.

Proposition 2.17. *For any $\mathbf{a} \in \mathcal{A}_{-\infty}^0 \setminus \Pi$ the maps (22) and (23) establish an isomorphism of the graphs $([\mathbf{a}]_{\approx}, \mathcal{E}_{\approx}(\mathbf{a}))$ and $([\mathbf{a}]_{\sim}^0, \mathcal{E}_{\sim}(\mathbf{a}))$.*

Remark 2.18. *Let X be a Borel space endowed with a right Borel action of the free semigroup $\mathcal{A}_{\mathcal{O}}^*$. The union of all (non-oriented) edges of the form $(x\alpha, x)$, $(x\alpha, x\beta)$ and $(x\alpha, x\alpha^*)$ (with $x \in X$, $\alpha, \beta \in \mathcal{A}$, $\mathbf{a} \in \mathcal{A}_{\mathcal{O}}^*$) determines then a graph structure on the orbit equivalence relation of this action. The “vertical” part of this structure (consisting of the edges $(x\alpha, x)$) coincides with the Schreier graph structure of the action with respect to the generating set \mathcal{A} (cf. [6]). The graph structures (20) and (24) are the specializations of this general construction to the actions (5) and (6), respectively.*

Remark 2.19. *In geometric terms the synchronization cocycle $\sigma(\mathbf{a}, \mathbf{b})$ (21) is equal to the difference between the levels of \mathbf{b} and \mathbf{a} in the graph $\mathcal{G}(\mathbf{a}) \cong \mathcal{G}(\mathbf{b})$, i.e., coincides with the Busemann cocycle on $\mathcal{G}(\mathbf{a})$ (see below Section 3.3 for more details).*

Remark 2.20. *The synchronization cocycle σ (21) on the weak tail equivalence relation $(\mathcal{A}_{-\infty}^0 \setminus \Pi, \sim)$ (i.e., in view of the previous Remark, the Busemann cocycle with respect to the graph structure \mathcal{E}_{\sim} (24)) is cohomologically non-trivial in the Borel category in contrast to the trivial Busemann cocycle $(\mathbf{a}, \mathbf{b}) \mapsto [\mathbf{b}] - [\mathbf{a}]$ on the strong tail equivalence relation $(\mathcal{A}_{-\infty}^*, \approx)$ with respect to the graph structure \mathcal{E}_{\approx} (20).*

3. Geometric properties of Sierpiński graphs

3.1. Rooted trees

Recall that a connected graph is called a *tree* if it contains no cycles. By ∂T we denote the *space of ends* of a tree T , i.e., the totally disconnected space which is the projective limit of the (finite discrete) spaces of infinite connected components of $T \setminus K_n$, where connected finite sets K_n exhaust T . The space of ends ∂T serves as the boundary of the *end compactification* of T .

Definition 3.1. *Given a tree T and a vertex $o \in T$, we shall call the couple (T, o) a rooted tree.*

For a rooted tree (T, o) put

$$|x| = |x|_o = \text{dist}(o, x) \quad (25)$$

and denote by

$$T_n = \{x \in T : |x| = n\}, \quad n \geq 0 \quad (26)$$

the sphere in T of radius n centered at the root o . We shall refer to the set T_n as the *n -th level* of the rooted tree (T, o) . For any point $x \in T$ and a number $0 \leq k \leq |x|$ put

$$x^{[-k]} = [o, x] \cap T_{|x|-k}, \quad (27)$$

in other words, $x^{[-k]}$ (the k -th *predecessor* of x) is the point on the geodesic segment $[o, x]$ at distance k from x (see Fig. 5, where $|x| = 3$).

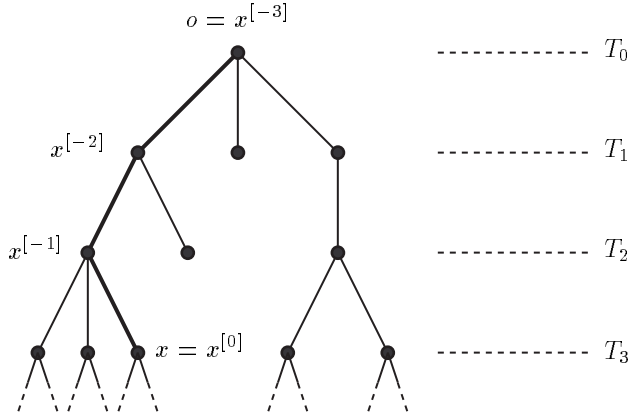


FIGURE 5

For a rooted tree (T, o) the space of ends ∂T can be identified with the space of all infinite *geodesic rays* (i.e., isometric embeddings of \mathbb{Z}_+ into T) $\bar{x} = (x_n)$ issued from o whose (totally disconnected) topology is determined by the pointwise convergence of geodesic rays. The end compactification of T is homeomorphic to its *visual compactification*, in which a sequence of points $x_n \in T$ converges if and only if the geodesic segments $[o, x_n]$ converge pointwise.

3.2. Augmented rooted trees

Definition 3.2. Let (T, o) be a rooted tree, and let

$$\mathcal{E}^h \subset \mathcal{V}(T) \times \mathcal{V}(T)$$

be a symmetric set such that

$$(x, y) \in \mathcal{E}^h \implies |x| = |y|, (x^{[-k]}, y^{[-k]}) \in \mathcal{E}^h \forall k > 0.$$

Then the graph X with the vertex set

$$\mathcal{V}(X) = \mathcal{V}(T)$$

and the edge set

$$\mathcal{E}(X) = \mathcal{E}(T) \cup (\mathcal{E}^h \setminus \text{diag})$$

is called an *augmented rooted tree*. We shall call the edges from $\mathcal{E}(T)$ *vertical* and the edges from $\mathcal{E}^h \setminus \text{diag}$ *horizontal* (see Fig. 6).

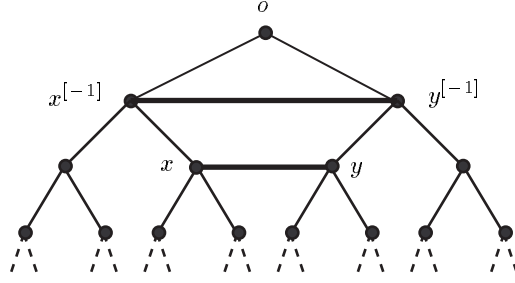


FIGURE 6

We shall extend the notations (25) – (27) from rooted trees to augmented rooted trees.

Denote by dist the graph distance in an augmented rooted tree X , and by dist_n the graph distance on its n -th level X_n (where, as usually, we put $\text{dist}_n(x, y) = \infty$ when x and y belong to different connected components of X_n). Obviously,

$$\text{dist}(x^{[-1]}, y^{[-1]}) \leq \text{dist}(x, y) \quad \forall x, y \in T. \quad (28)$$

Definition 3.3. We shall say that a geodesic segment

$$x = z_0, z_1, \dots, z_d = y, \quad d = \text{dist}(x, y)$$

joining two points x, y in an augmented rooted tree is canonical if it consists of two vertical segments (one or both of which may possibly be empty) with an intermediate horizontal segment, i.e.,

$$z_i = \begin{cases} x^{[-i]}, & i \leq m, \\ y^{[i-d]}, & i \geq n \end{cases} \quad \text{and} \quad |z_m| = |z_{m+1}| = \dots = |z_{n-1}| = |z_n| \quad (29)$$

for certain integers m, n with $0 \leq m \leq n \leq d$ (see Fig. 7).

Proposition 3.4. Any two points x, y in an augmented rooted tree can be joined with a canonical geodesic.

Proof. By (28) the “moves”

$$\begin{cases} (u, v, v^{[-1]}) \mapsto (u, u^{[-1]}, v^{[-1]}) \\ (u^{[-1]}, u, v) \mapsto (u^{[-1]}, v^{[-1]}, v) \end{cases}, \quad |u| = |v|,$$

consisting in “lifting” horizontal edges do not increase the distance. Applying them to an arbitrary geodesic segment joining x and y eventually gives a required geodesic segment of the form (29). \square

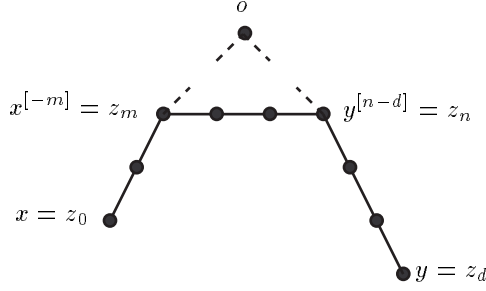


FIGURE 7

Remark 3.5. *Definition 3.3 and Proposition 3.4 are applicable to geodesic rays and to infinite bilateral geodesics as well.*

Definition 3.6. *A geodesic square R in an augmented rooted tree X is a quadruple of points $x, y, x', y' \in X$ such that*

- (i) $|x| = |y|$;
- (ii) $x' = x^{[-k]}, y' = y^{[-k]}$ for a certain $k > 0$;
- (iii) $\text{dist}(x, y) = \text{dist}(x', y') = \text{dist}(x, x') = \text{dist}(y, y') = k$.

The number k is called the size of the rectangle R .

Inequality (28) and Proposition 3.4 imply

Proposition 3.7. *If $(x, y, x^{[-k]}, y^{[-k]})$ is a geodesic square of size k in an augmented rooted tree X , then*

$$\text{dist}_{|x|-i}(x^{[-i]}, y^{[-i]}) = k \quad \forall 0 \leq i \leq k,$$

and any geodesic segment in X joining the points $x^{[-i]}$ and $y^{[-i]}$ lies in the level $X_{|x|-i}$.

Remark 3.8. *An isometric embedding π of the square $\{0, 1, \dots, k\}^2 \subset \mathbb{Z}^2$ into an augmented rooted tree X such that*

$$|\pi(0, 0)| = |\pi(k, 0)|, \quad |\pi(0, k)| = |\pi(k, k)|, \quad |\pi(0, 0)| - |\pi(0, k)| = k \quad (30)$$

clearly determines a geodesic square of size k in X in the sense of Definition 3.6. Proposition 3.7 shows that, conversely, any geodesic square of size k gives rise to an isometric embedding $\pi : \{0, 1, \dots, k\}^2 \rightarrow X$ satisfying conditions (30).

3.3. Remotely rooted trees

Definition 3.9. *Given a tree T and a point $\omega \in \partial T$, we shall call the couple (T, ω) a remotely rooted tree.*

Remark 3.10. *In a less “botanical” terminology “rooted trees” are called pointed, and “remotely rooted trees” are called pointed at infinity.*

Denote by $\beta = \beta_\omega$ the *Busemann cocycle* on T defined as

$$\beta(x, y) = \lim_{z \rightarrow \omega} [\text{dist}(y, z) - \text{dist}(x, z)] . \quad (31)$$

One may consider the above formula as a regularization of the formal expression

$$\beta(x, y) = \text{“dist}(y, \omega) - \text{dist}(x, \omega)\text{”} ,$$

i.e., the Busemann cocycle is a “difference” between the “distances” from the points y and x to the point at infinity ω . We shall refer to the level sets

$$T_x = \{y \in T : \beta(x, y) = 0\} \quad (32)$$

of the Busemann cocycle as *horizontal levels* in a remotely rooted tree T , so that $\beta(x, y)$ is the signed distance between the heights of the levels of the points x and y (see Fig. 8, where $\beta(x, y) = -1$).

By extending the notation (27), we shall denote by $x^{[-k]}$, $k \geq 0$ the k -th *predecessor* of a vertex x uniquely determined by the relation

$$\beta(x, x^{[-k]}) = -k , \quad x^{[-k]} \in [x, \omega] , \quad (33)$$

where $[x, \omega)$ is the geodesic ray joining x and the remote root ω (see Fig. 8). Clearly,

$$\beta(x, y) = \beta(x^{[-k]}, y^{[-k]}) \quad \forall x, y \in T, k \geq 0 .$$

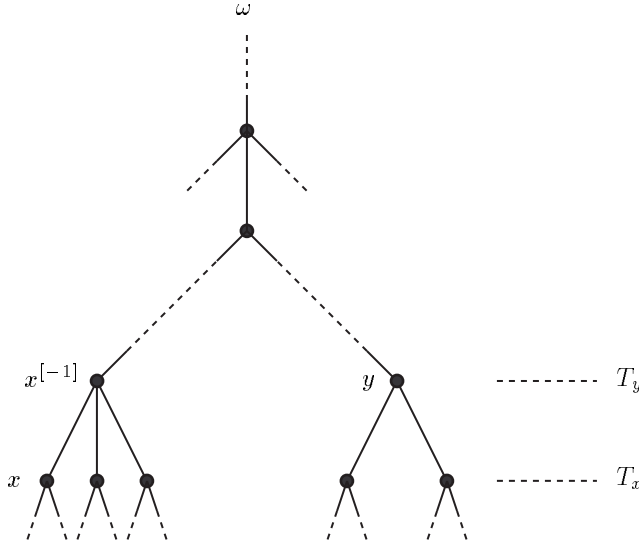


FIGURE 8

As it follows from the definition (31), remotely rooted trees can be considered as limits of rooted trees in the sense that the partitions of T into the spherical levels

(26) with respect to a sequence of roots o_n converge pointwise to the partition of T into the level sets (32) of the Busemann cocycle β_ω if and only if the sequence o_n converges to ω (cf. the discussion at the end of Section 2.7).

Definition 3.11. Let (T, ω) be a remotely rooted tree, and let a symmetric set

$$\mathcal{E}^h \subset \mathcal{V}(T) \times \mathcal{V}(T)$$

be such that

$$(x, y) \in \mathcal{E}^h \implies \beta(x, y) = 0, (x^{[-k]}, y^{[-k]}) \in \mathcal{E}^h \forall k > 0.$$

Then the graph X with the vertex set

$$\mathcal{V}(X) = \mathcal{V}(T)$$

and the edge set

$$\mathcal{E}(X) = \mathcal{E}(T) \cup (\mathcal{E}^h \setminus \text{diag})$$

is called an augmented remotely rooted tree. Following Definition 3.2, we shall call the edges from $\mathcal{E}(T)$ vertical and the edges from $\mathcal{E}^h \setminus \text{diag}$ horizontal.

As in Section 3.2, we shall extend the notations (31) – (33) from remotely rooted trees to augmented remotely rooted trees.

3.4. Hyperbolicity

The Gromov product on a graph X (with respect to a reference point o) is defined as

$$(x|y)_o = \frac{1}{2} [\text{dist}(o, x) + \text{dist}(o, y) - \text{dist}(x, y)]. \quad (34)$$

A graph X is called Gromov hyperbolic if there exists a constant $\delta > 0$ such that the δ -ultrametric inequality

$$(x|y)_o \geq \min\{(x|z)_o, (y|z)_o\} - \delta$$

is satisfied for all $o, x, y, z \in X$. Equivalently, X is Gromov hyperbolic if all geodesic triangles in X are uniformly *thin*, i.e., one can always choose a point on each of the sides of a geodesic triangle in such a way that the pairwise distances between these points are uniformly bounded.

The *hyperbolic boundary* ∂X of a hyperbolic graph X is defined as the space of equivalence classes of asymptotic geodesic rays in X (i.e., those which lie within a finite distance one from another). For any two points $x \in X, \xi \in \partial X$ there exists a geodesic ray (not necessarily unique!) issued from x and belonging to the class ξ (i.e., *joining* x and ξ). In the same way, any two distinct points $\xi_- \neq \xi_+ \in \partial X$ can be joined by a bilateral geodesic (once again, not necessarily unique) whose positive (resp., negative) geodesic ray belongs to the class ξ_+ (resp., ξ_-). The definition of the Gromov product (34) can be extended to the case when one of the arguments belong to ∂X by putting

$$(x|\xi)_o = \sup\{(x|y_n)_o : (y_n) \text{ is a geodesic ray joining } o \text{ and } \xi\}.$$

Analogously (by taking the supremum over all geodesic rays joining o with the points $\eta, \xi \in \partial X$) one also defines the Gromov product when both arguments

are boundary points. There exists an absolute constant $C > 0$ (depending on the hyperbolicity constant δ only) such that for any bilateral geodesic γ joining any two points $\eta \neq \xi \in \partial X$ and any reference point $o \in X$

$$|(\eta|\xi)_o - \text{dist}(o, \gamma)| \leq C . \quad (35)$$

Below we shall usually leave out the reference point o by assuming that it is fixed once and for all (in particular, for augmented rooted trees we shall always take for o the root of the underlying tree).

The hyperbolic boundary ∂X is the boundary of the *hyperbolic compactification* of X : a sequence of points $x_n \in X$ converges in this compactification if $(x_n|x_m) \rightarrow \infty$, and the limit is a point $\xi \in \partial X$ if $(x_n|\xi) \rightarrow \infty$ (in particular, any geodesic ray converges to the point of ∂X determined by its asymptotic equivalence class). For any sufficiently small $\varepsilon > 0$ the topology of ∂X is metrizable by a metric ρ_ε uniformly equivalent to $\exp[-\varepsilon(\cdot|\cdot)]$, i.e., such that for a certain constant $C > 0$

$$\frac{1}{C} \leq e^{\varepsilon(\eta|\xi)} \cdot \rho_\varepsilon(\eta, \xi) \leq C \quad \forall \eta \neq \xi \in \partial X . \quad (36)$$

The hyperbolic compactification of a tree is homeomorphic to its end (\equiv visual) compactification. See [24], [9], [22] for more details concerning Gromov hyperbolic spaces.

Definition 3.12. *An augmented rooted tree X satisfies “no big squares” condition if the size of geodesic squares in X is bounded (cf. Definition 3.6)*

Theorem 3.13 ([35]; cf. [9, Theorems 11.11, 11.13]). *An augmented rooted tree X is Gromov hyperbolic if and only if it satisfies the “no big squares” condition.*

Remark 3.14. *The hyperbolicity of an augmented rooted tree (X, o) implies that the lengths of horizontal segments in canonical geodesics on X (see Definition 3.3) are uniformly bounded (the geodesic triangle whose base is such a horizontal segment and the lateral sides are the geodesics joining its endpoints with the root is thin). This can be also directly deduced from the “no big squares” condition.*

Since the definition of the Gromov hyperbolicity is local in the sense that it only involves geodesic triangles in the space, Theorem 3.13 immediately carries over to augmented remotely rooted trees.

Theorem 3.15. *An augmented remotely rooted tree X is Gromov hyperbolic if and only if it satisfies the “no big squares” condition.*

One can explicitly describe the hyperbolic boundary ∂X of an augmented rooted tree (X, o) . Indeed, geodesic rays issued from the root o are the same on T and on X . The boundary ∂T of the tree T is the space of all such rays, and it is projected onto ∂X by the map which assigns to any ray its asymptotic equivalence class with respect to the graph distance on X (this map is onto because any asymptotic equivalence class contains a ray issued from the root). More precisely, denote by $\tilde{\cdot}$ the *asymptotic equivalence relation* on ∂T :

$$\tilde{x} \tilde{\cdot} \tilde{y} \iff \exists C > 0 \text{ such that } \text{dist}(x_n, y_n) \leq C \quad \forall n \geq 0 ,$$

which is the envelope of the set

$$\{(\bar{x}, \bar{y}) \in \partial T \times \partial T : (x_n, y_n) \in \mathcal{E}^h \quad \forall n \geq 0\}.$$

Then the definition of the hyperbolic boundary implies

Theorem 3.16. *The hyperbolic boundary ∂X of a Gromov hyperbolic augmented rooted tree (X, o) is homeomorphic to the quotient of the boundary ∂T of the underlying rooted tree (T, o) by the equivalence relation $\tilde{\sim}$.*

One can also give a more explicit description of the metrics ρ_ε (36) on ∂X , or, equivalently, of the Gromov product on ∂X . Let

$$\mathfrak{h}(\eta, \xi) = \min\{\text{dist}(o, \gamma)\}, \quad \eta \neq \xi \in \partial X, \quad (37)$$

where the minimum is taken over all bilateral geodesics joining η and ξ . By Proposition 3.4 and Remark 3.5, such a geodesic can be chosen to be canonical, so that $\mathfrak{h}(\eta, \xi)$ is the minimal distance from the root to the horizontal segments of these canonical geodesics. Then inequality (35) implies

Proposition 3.17. *There is a constant $C > 0$ such that*

$$|\mathfrak{h}(\eta, \xi) - (\eta|\xi)| \leq C \quad \forall \eta \neq \xi \in \partial X,$$

so that any metric ρ_ε is uniformly equivalent to $\exp[-\varepsilon \mathfrak{h}(\cdot, \cdot)]$.

3.5. Hyperbolicity of the Sierpiński graphs

We shall now apply the above arguments to the Sierpiński graph \mathcal{G} and the extended graphs $\mathcal{G}(\mathbf{a})$, $\mathbf{a} \in \mathcal{A}_{-\infty}^0$.

Proposition 3.18. *The Sierpiński graph \mathcal{G} is an augmented rooted tree whose underlying tree is the Cayley graph of the free semigroup $\mathcal{A}_\mathcal{O}^*$. Any extended Sierpiński graph $\mathcal{G}(\mathbf{a})$, $\mathbf{a} \in \mathcal{A}_{-\infty}^0$ is an augmented remotely rooted tree.*

Any level $\vec{\Delta}^n \cong \mathcal{A}_1^n$ of the Sierpiński graph \mathcal{G} can be embedded into \mathbb{R}^d by the map which assigns to a simplex $\Sigma \in \vec{\Delta}^n$ its baricenter $\dot{\Sigma}$. Denote by d_n the metric on $\vec{\Delta}^n$ induced by the Euclidean metric on \mathbb{R}^d under this embedding.

Proposition 3.19. *The metrics $2^n d_n$ are uniformly quasi-isometric to the graph metrics dist_n on $\vec{\Delta}^n$, i.e., there exists a constant $C > 1$ such that*

$$\frac{1}{C} \leq \frac{2^n d_n(\Sigma_1, \Sigma_2)}{\text{dist}_n(\Sigma_1, \Sigma_2)} \leq C \quad \forall \Sigma_1 \neq \Sigma_2 \in \vec{\Delta}^n, \quad n > 0. \quad (38)$$

Proof. If two simplices $\Sigma, \Sigma' \in \vec{\Delta}^n$ are neighbours, then obviously

$$\frac{1}{C} \leq 2^n d_n(\Sigma, \Sigma') \leq C$$

for a constant C which depends on the original simplex Δ only, which proves the right-hand inequality in (38).

For proving the left-hand side inequality let us take a Euclidean geodesic ℓ joining the barycenters of two simplices $\Sigma, \Sigma' \in \vec{\Delta}^n$ and endow ℓ with the length

parameterization. By slightly moving the endpoints of ℓ we may assume without loss of generality that ℓ does not intersect the set Q_n of vertices of level n simplices. Therefore, the sequence of simplices

$$\Sigma = \Sigma_0, \Sigma_1, \dots, \Sigma_N = \Sigma' \in \vec{\Delta}^n$$

consecutively intersected by ℓ is well defined, and each of the intersections

$$[\tau_{2k}, \tau_{2k+1}] = \ell \cap \Sigma_k$$

is not a single point. We shall call the points τ_i *marks* on ℓ . Then the distance between two consecutive marks $\tau_{i+1} - \tau_i$ may be “small” (compared to the size of simplices from $\vec{\Delta}^n$, i.e., to 2^{-n}) only near a point from Q_n , the length of any consecutive series of “small” differences is uniformly bounded, and each such series is preceded and followed by a difference at least comparable with 2^{-n} .

We shall now build a path in the graph $\vec{\Delta}^n$ by joining the simplices Σ and Σ' in the following way.

- The segments $[\tau_{2k}, \tau_{2k+1}]$ are assigned the simplices Σ_k .
- The segments $[\tau_{2k-1}, \tau_{2k}]$ correspond to intersections of ℓ with the connected components Ω_α of the complement $\Delta \setminus \mathbf{G}^n$ (see Definition 2.1). We shall assign to each such segment the shortest possible chain of simplices from $\vec{\Delta}^n$ going “around” the associated component Ω_α , see Fig. 9. The length of such a chain is uniformly comparable with $2^n(\tau_{2k} - \tau_{2k-1})$ unless the difference $\tau_{2k} - \tau_{2k-1}$ is “small”.

By using the above properties of the differences $\tau_{i+1} - \tau_i$ one can now see that the length of the constructed path is uniformly dominated by $2^n|\ell|$. \square

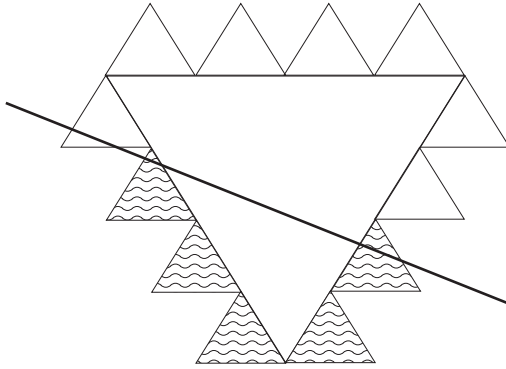


FIGURE 9

Corollary 3.20. *The Sierpiński graph \mathcal{G} satisfies “no big squares” condition.*

Theorem 3.13, Theorem 3.16 and Proposition 2.5 now immediately imply

Theorem 3.21. *The Sierpiński graph \mathcal{G} is Gromov hyperbolic, and its hyperbolic boundary $\partial\mathcal{G}$ is homeomorphic to the Sierpiński gasket \mathbf{G} .*

Remark 3.22. *Another example of Gromov hyperbolic augmented trees is provided by the graphs C_K associated by Elek [19] to any compact K contained in the Euclidean cube $[0, 1]^d$. Their vertices are binary subcubes of $[0, 1]^d$, and the hyperbolic boundary ∂K is homeomorphic to K . See the recent survey [7] for a general discussion of boundaries of Gromov hyperbolic spaces.*

In the same way, Theorem 3.15 implies

Theorem 3.23. *Any extended Sierpiński graph $\mathcal{G}(\mathbf{a})$, $\mathbf{a} \in \mathcal{A}_{-\infty}^0$, is Gromov hyperbolic, and its hyperbolic boundary $\partial\mathcal{G}(\mathbf{a})$ is homeomorphic to the one-point compactification $\mathbf{G}(\mathbf{a}) \cup \{\omega\}$ of the extended Sierpiński gasket $\mathbf{G}(\mathbf{a})$.*

Moreover, it turns out that the Euclidean metric on the Sierpiński gasket \mathbf{G} is uniformly Hölder equivalent to the metrics ρ_ε (36).

Theorem 3.24. *There exists a constant $C > 0$ such that*

$$\frac{1}{C} \leq \|\eta - \xi\| \cdot 2^{(\eta|\xi)} \leq C \quad \forall \eta \neq \xi \in \mathbf{G}.$$

Proof. Let us take a canonical bilateral geodesic γ joining the points η and ξ and realizing $\mathfrak{h}(\eta, \xi) = n$ (37), see Definition 3.3, Remark 3.5 and Proposition 3.17. Denote by

$$\Sigma_n \supset \Sigma_{n+1} \supset \dots \{\eta\}, \quad \Sigma'_n \supset \Sigma'_{n+1} \supset \dots \{\xi\},$$

the vertical rays of this geodesic going to the points η and ξ , respectively. Since $\eta \in \Sigma_n$, $\xi \in \Sigma'_n$, Remark 3.14 implies that

$$\|\eta - \xi\| \leq C \cdot 2^{-n}.$$

Conversely, the simplices Σ_{n+1} and Σ'_{n+1} are not neighbours in the Sierpiński graph \mathcal{G} (for otherwise one could have shortened γ by directly connecting Σ_{n+1} and Σ'_{n+1}), which by the definition of the Sierpiński graph means that $\Sigma_{n+1} \cap \Sigma'_{n+1} = \emptyset$. Therefore,

$$\frac{1}{C} \cdot 2^{-n} \leq \|\eta - \xi\|,$$

which by Proposition 3.17 ends the proof. \square

Theorem 3.25. *The Euclidean metric on the Sierpiński gasket \mathbf{G} and the metric ρ_ε on $\mathbf{G} \cong \partial\mathcal{G}$ are uniformly Hölder equivalent in the sense that there exists a constant $C > 0$ such that*

$$\frac{1}{C} \leq \frac{\|\eta - \xi\|^{(\log 2)/\varepsilon}}{\rho_\varepsilon(\eta, \xi)} \leq C \quad \forall \eta \neq \xi \in \mathbf{G}.$$

3.6. Embedding into the hyperbolic space

The group $\text{Sim}(\mathbb{R}^d)$ acts simply transitively by isometries on the hyperbolic space \mathbf{H}^{d+1} realized in the upper half-space model as $\mathbf{H}^{d+1} \cong \mathbb{R}^d \times \mathbb{R}_+$ (see [37] for a general discussion of the relationship between the hyperbolic and similarity structures). Since $\text{Sim}(\mathbb{R}^d)$ also acts simply transitively on the space \mathfrak{S} of all simplices similar to Δ , we may identify \mathfrak{S} with the hyperbolic space \mathbf{H}^{d+1} by the map

$$g\Delta \mapsto gz, \quad g \in \text{Sim}(\mathbb{R}^d), \quad (39)$$

where $z \in \mathbf{H}^{d+1}$ is a chosen reference point.

Theorem 3.26 ([35]; cf. [19, Theorem 4]). *The embedding of the Sierpiński graph \mathcal{G} into the hyperbolic space \mathbf{H}^{d+1} determined by formula (39) is a quasi-isometry with respect to the graph metric on \mathcal{G} and the hyperbolic metric on \mathbf{H}^{d+1} .*

Below are several comments on the relationship between the objects associated with the Sierpiński gasket and the hyperbolic geometry.

1. The Sierpiński gasket \mathbf{G} is the limit set (in \mathbf{H}^{d+1}) of the semigroup $S \cong \mathcal{A}_\emptyset^* \subset \text{Sim}(\mathbb{R}^d)$ (3).

2. The “Sierpiński group” $G \subset \text{Sim}(\mathbb{R}^d)$ (3) is isomorphic to the similarity group of the dyadic-rational space $\mathbb{Z}^d[\frac{1}{2}]$, in particular, for $d = 1$ it coincides with the group $\langle a, b | ab^2 = ba \rangle$, see Remark 2.6. Although G is not a discrete subgroup of $\text{Iso}(\mathbb{R}^d)$, it is a lattice in the product of the real and dyadic similarity groups. The group G has two natural boundaries which are the real and the dyadic d -spaces, see [39], [20] for the case $d = 1$. The Sierpiński gasket belongs to the “real” boundary which describes the behaviour of our dynamics (iteration of contractions g_α) at $+\infty$, whereas the strings $\mathbf{a} \in \mathcal{A}_{-\infty}^0$ which describe the dynamics at $-\infty$ can be interpreted as points of the “dyadic boundary”. It would be interesting to understand to what extent the rich geometry of the Sierpiński gaskets and graphs could be interpreted just in terms of the group G and its free semigroup S .

3. The situation characterized by the presence of two different structures responsible for the behaviour of the dynamics at $+\infty$ and $-\infty$, respectively, occurs also in the theory of iterations of rational maps. Lyubich and Minsky [45] constructed a Riemann surface lamination \mathcal{A}_f and a hyperbolic 3-lamination \mathcal{H}_f associated with an endomorphism f of the Riemann sphere $\overline{\mathbb{C}}$. The leaves of \mathcal{A}_f are planes endowed with a complex affine (\equiv real similarity) structure, whereas the leaves of \mathcal{H}_f are pointed at infinity hyperbolic 3-spaces whose boundary planes are the leaves of \mathcal{A}_f . Both laminations are endowed with an action of the natural extension \hat{f} of the rational map f which is minimal on \mathcal{A}_f and properly discontinuous on \mathcal{H}_f , so that the latter action gives rise to the quotient hyperbolic lamination $\mathcal{M}_f = \mathcal{H}_f / \hat{f}$. The “forward” dynamics of f is described by the leafwise Julia sets, whereas the “backward” dynamics is described by the transversal structure of these laminations (the transversals of these laminations are, roughly speaking, backward trajectories of f on $\overline{\mathbb{C}}$). This picture also bears close resemblance to the

dynamics of the geodesic flow on hyperbolic manifolds and of associated Kleinian groups, see [37].

In our context the product of the boundaries of the group G at $+\infty$ and $-\infty$ (see Comment 2 above) can be considered as a \mathbb{R}^d -lamination \mathcal{L} with a leafwise similarity structure. The forward dynamics is described by the leafwise Sierpiński gaskets $\mathbf{G}(\mathbf{a})$, $\mathbf{a} \in \mathcal{A}_{-\infty}^0$, whereas the backward dynamics is described by the transverse structure of \mathcal{L} . The group G acts minimally on \mathcal{L} and properly discontinuously on the associated \mathbf{H}^{d+1} -lamination \mathcal{H} . We shall return to a more detailed description of this construction elsewhere.

3.7. Amenability

For a subset A of a graph X denote by

$$\partial A = \{x \in A : \exists y \in X \setminus A \text{ with } (x, y) \in \mathcal{E}(X)\}$$

its *boundary*. Recall that a connected graph X is called amenable if it contains finite sets $A \subset X$ with arbitrarily small *isoperimetric ratio*

$$(\text{card } \partial A) / (\text{card } A) \tag{40}$$

(we remind the reader Convention 2.2 made in Section 2.2). According to a criterion of Gromov (see [12]) a graph X is non-amenable if and only if there exists a map $\varphi : X \rightarrow X$ and a constant $C > 0$ such that

$$\text{dist}(x, \varphi(x)) \leq C, \quad \text{card } \{\varphi^{-1}(x)\} \geq 2 \quad \forall x \in X,$$

In particular, a tree is amenable if and only if it contains arbitrarily long geodesic segments without branching.

Since adding new edges to the same vertex set may only make the graph distance smaller, Gromov's criterion implies

Theorem 3.27. *Under conditions of Convention 2.2, if a rooted tree (T, o) is non-amenable, then any its augmentation (X, o) is also non-amenable.*

The converse is not true even under the assumption that the augmented tree (X, o) is hyperbolic. For an example let (T', o) be the rooted tree obtained by adding to a binary rooted tree (T, o) a new geodesic ray γ' issued from o . Then T' is amenable, whereas T is non-amenable. Choose a ray γ in T , and add to T' all horizontal edges joining γ and γ' . Then the resulting augmented rooted tree (X', o) is roughly isometric to the original binary tree (T, o) , so that X' is non-amenable.

Theorem 3.28. *The Sierpiński graph \mathcal{G} and all the extended Sierpiński graphs $\mathcal{G}(\mathbf{a})$, $\mathbf{a} \in \mathcal{A}_{-\infty}^0$, are non-amenable.*

Proof. The underlying binary trees of the Sierpiński graph \mathcal{G} and of the extended Sierpiński graphs $\mathcal{G}(\mathbf{a})$ are non-amenable, so that the claim follows from Theorem 3.27. \square

4. Random walks on Sierpiński graphs

In this Section we apply general results from the theory of Markov chains on hyperbolic spaces and on equivalence relations to the objects associated with the Sierpiński gasket which were constructed in Sections 2 and 3. Therefore, the exposition in this Section is sketchier, and detailed proofs are often replaced with references to analogous cases treated with the same general methods.

4.1. Markov chains and Markov operators

A *Markov chain* on a countable *state space* X is determined by the family of *transition probabilities*

$$\pi_x = p(x, \cdot), \quad x \in X,$$

or, equivalently, by the associated *Markov operator*

$$Pf(x) = \sum_{y \in X} p(x, y)f(y)$$

By $\pi_x^n = p_n(x, \cdot)$ we denote the n -step transition probabilities of the Markov chain.

There are several conditions connecting transition probabilities of a Markov chain with a graph structure on its state space X . A Markov chain is said to be of *nearest neighbour type* if $p(x, y) > 0$ only if x and y are neighbours, and is said to be of *bounded range* if there is a constant $D > 0$ such that $p(x, y) = 0$ whenever $\text{dist}(x, y) \geq D$ (where $\text{dist}(\cdot, \cdot)$ denotes the graph metric on X). For the *simple random walk* on a graph X the transition probabilities π_x are equidistributed among the neighbours of x , i.e.,

$$p(x, y) = \begin{cases} \frac{1}{\deg(x)}, & \text{if } (x, y) \in \mathcal{E}(X); \\ 0, & \text{otherwise.} \end{cases}$$

A Markov operator on a graph X is called *irreducible* if any vertex $y \in X$ can be attained from any other vertex $x \in X$ with positive probability, i.e., if there exists $n \geq 0$ with $p_n(x, y) > 0$, and it is called *uniformly irreducible* if there exist an integer $N > 0$ and a number $\varepsilon > 0$ such that whenever two points $x, y \in X$ are neighbours there exists $n \leq N$ with $p_n(x, y) \geq \varepsilon$. In particular, the simple random walk on X is always uniformly irreducible.

The *spectral radius* of an irreducible Markov operator is defined as

$$\rho(P) = \limsup_{n \rightarrow \infty} (p_n(x, y))^{1/n}.$$

By irreducibility the limit in the above formula does not depend on the choice of the points $x, y \in X$.

Theorem 4.1 ([18]). *A graph X is amenable if and only the spectral radius of the simple random walk on X is 1.*

Remark 4.2. *This theorem is actually valid for a much larger class of reversible random walks on X , see [30].*

4.2. Boundaries of Markov operators

There are two principal notions of a *boundary* of a Markov chain. The *Poisson boundary* is defined in the measure theoretical category, and the *Martin boundary* is defined in the topological category.

More precisely, the Poisson boundary \mathcal{G} of a Markov chain is defined as the space of ergodic components of the time shift in its path space and is endowed with a natural *harmonic measure class* $[\nu]$. For any starting point $x \in X$ the image ν_x of the measure \mathbf{P}_x in the path space (corresponding to starting the chain at time 0 from the point x) under the projection onto the Poisson boundary is called the *harmonic measure* of the point x . The harmonic measures ν_x are absolutely continuous with respect to the class $[\nu]$ and satisfy the *stationarity condition*

$$\nu_x = \sum_y p(x, y) \nu_y ,$$

so that any function $\hat{f} \in L^\infty(\mathcal{G}, [\nu])$ determines by the *Poisson formula*

$$f(x) = \langle \hat{f}, \nu_x \rangle$$

a bounded P -invariant function on X (such functions are called *harmonic*). In fact, the Poisson formula establishes an isometry between the space $L^\infty(\mathcal{G}, [\nu])$ and the space of bounded P -harmonic functions on X .

The Martin boundary is defined in terms of the *Green kernel*

$$G(x, y) = \sum_{n=0}^{\infty} p_n(x, y)$$

of the Markov operator P . Namely, one first embeds the space X into the space of positive functions on itself by the map $y \mapsto G(\cdot, y)$. The projectivization of the latter space by the multiplicative action of \mathbb{R}_+ (which amounts to replacing the Green kernel with the *Martin kernel* $K(x, y) = G(x, y)/G(o, y)$, where o is a fixed reference point), gives an embedding of X into a compact space, after which it only remains to take the closure of X in this compact space (in this cursory description we assume for simplicity that the operator P is irreducible). The resulting compactification is called the *Martin compactification* of the state space X determined by the operator P , and its boundary is called the *Martin boundary*.

By the construction, the points of the Martin boundary can be identified with the (projective classes) of positive *superharmonic* functions f on X (i.e., such that $Pf \leq f$). The Martin boundary contains (the projective classes of) all minimal positive harmonic functions (\equiv the extremal rays in the cone of positive harmonic functions). For any point $x \in X$ the condition $f(x) = 1$ allows one to choose a representative in each ray of the cone of positive harmonic functions (i.e., this condition determines a *base* B_x of the cone). Then any positive harmonic function φ has a unique representing measure ν_x^f concentrated on the extremal points of the convex set B_x . The Martin boundary endowed with the family of the representing measures $\nu_x^{\mathbf{1}}$ of the constant function $\mathbf{1}$ is isomorphic to the Poisson boundary.

Moreover, almost all sample paths of the Markov chain converge in the Martin compactification, and for any $x \in X$ the measure ν_x^1 is the hitting measure on the Martin boundary corresponding to the starting point x .

For a more detailed discussion of the theory of boundaries of Markov chains on graphs see the author's articles [31], [33], the book by Woess [56] and the references therein.

4.3. The Martin boundary of Sierpiński gasket

The fundamental results of Ancona give a description of the Martin boundary on hyperbolic graphs and general Gromov hyperbolic spaces [2] (see also the exposition in the book [56]).

Theorem 4.3. *Let P be a uniformly irreducible bounded range Markov operator on a hyperbolic graph X with $\rho(P) < 1$. Then the Martin compactification of P is homeomorphic to the hyperbolic compactification of X , in particular, the Martin boundary of P is homeomorphic to the hyperbolic boundary ∂X .*

Remark 4.4. *Under the conditions of Theorem 4.3 the harmonic measure class on the hyperbolic boundary ∂X is purely non-atomic, and the operator P satisfies the boundary Harnack principle, which implies that the Radon-Nikodym derivatives of the harmonic (\equiv hitting) measures*

$$\frac{d\nu_x}{d\nu_y}(\xi), \quad x, y \in X, \xi \in \partial X,$$

extend to Hölder continuous functions on ∂X with respect to the metrics ρ_ε (36), see [2], [3].

In view of Theorem 4.1 we have

Theorem 4.5. *If X is a non-amenable hyperbolic graph, then the Martin boundary of the simple random walk on X is homeomorphic to the hyperbolic boundary ∂X .*

Theorems 3.21 and 3.23 imply

Theorem 4.6. *Let P be a uniformly irreducible bounded range Markov operator on the Sierpiński graph \mathcal{G} (resp., on the augmented Sierpiński graph $\mathcal{G}(\mathbf{a})$, $\mathbf{a}\mathcal{A}_{-\infty}^0$). If $\rho(P) < 1$, then the Martin boundary of P is homeomorphic to the Sierpiński gasket \mathbf{G} (resp., to the one-point compactification $\mathbf{G}(\mathbf{a}) \cup \{\omega\}$ of the extended Sierpiński gasket $\mathbf{G}(\mathbf{a})$).*

In particular, in view of Theorems 3.28 and 4.5 we have

Theorem 4.7. *The Martin boundary of the simple random walk on the Sierpiński graph \mathcal{G} (resp., on the augmented Sierpiński graph $\mathcal{G}(\mathbf{a})$, $\mathbf{a}\mathcal{A}_{-\infty}^0$) is homeomorphic to the Sierpiński gasket \mathbf{G} (resp., to the one-point compactification $\mathbf{G}(\mathbf{a}) \cup \{\omega\}$ of the extended Sierpiński gasket $\mathbf{G}(\mathbf{a})$).*

Remark 4.8. *The Hölder continuity of the Radon–Nikodym derivatives of harmonic measures (see Remark 4.4) can be used to show that the harmonic measure class of the simple random walk on the Sierpiński graph determines a Gibbs measure on the symbolic space \mathcal{A}_1^∞ (which provides a coding of the Sierpiński gasket \mathbf{G} as explained in Section 2.5), cf. [48], [55], [51], [44].*

Denker and his collaborators Sato and Koch [16], [17], [15] considered the random walk on the Sierpiński graph \mathcal{G} for which the transition probabilities from a point $\mathbf{a} \in \mathcal{A}_\emptyset^*$ are equidistributed among the offsprings of \mathbf{a} and of its conjugate \mathbf{a}^* (11), i.e.,

$$p(\mathbf{a}, \mathbf{b}) = \begin{cases} \frac{1}{d+1}, & \mathbf{a} = \mathbf{a}^*, \mathbf{b} = \mathbf{a}\alpha, \alpha \in \mathcal{A}, \\ \frac{1}{2d+2}, & \mathbf{a} \neq \mathbf{a}^*, \mathbf{b} = \mathbf{a}\alpha, \mathbf{a}^*\alpha, \alpha \in \mathcal{A}, \\ 0, & \text{otherwise.} \end{cases} \quad (41)$$

In particular, they proved (by a direct computation of the Green and Martin kernels) that the Martin boundary of this chain is homeomorphic to the Sierpiński gasket. This random walk always moves from the n -th level in the Sierpiński graph to the next $(n+1)$ -th level, so that it is not irreducible in the sense of Section 4.1, and the results of Theorem 4.3 are not applicable in this situation. However, due to the absence of returns for this random walk, its Green kernel is given just by the n -step transition probabilities. Therefore, the Green kernel is obviously multiplicative along geodesics issued from the root of the Sierpiński graph. Since the almost multiplicativity of the Green kernel along geodesics in a hyperbolic space is the main ingredient of Ancona’s approach, his methods could be actually adapted to this situation as well.

4.4. Random walks on equivalence relations

Recall that a *discrete equivalence relation* R on a Borel set X is an equivalence relation which is Borel as a subset of $X \times X$ and whose classes $[x]$ are at most countable. The transition probabilities $\pi_x = p(x, \cdot)$, $x \in X$ of a *Markov chain on equivalence relation* R are required to be concentrated on the class $[x]$ for any $x \in X$ and to be Borel (as functions on R). These transition probabilities give rise to the *global* Markov chain with the state space X and to *local* Markov chains on each equivalence class $[x]$.

If the global state space X is compact, and the transition probabilities π_x depend on x continuously in the weak* topology, then by compactness considerations there exists a probability measure m on X which is stationary with respect to the global chain. If no local chain has a finite stationary measure, then the measure m is necessarily purely non-atomic. Standard results from the ergodic theory of stationary Markov chains imply that the measure m can be always chosen to be *ergodic*, i.e., not decomposable into a convex combination of two different stationary measures. This definition of ergodicity is equivalent to saying that the time shift in the path space of the global chain is ergodic with respect to the invariant measure \mathbf{P}_m (whose one-dimensional distributions are m), or, that the state space

X does not contain any non-trivial absorbing subsets with respect to the global chain, see [50], [29].

Suppose now that the equivalence relation R is in addition endowed with a graph structure (determined by a Borel subset $\mathcal{E} \subset R$, see the discussion at the end of Section 2.8), and let $\text{dist}(\cdot, \cdot)$ be the associated graph distance on the equivalence classes. We shall say that the global Markov chain on the graphed equivalence relation (R, \mathcal{E}) determined by a family of transition probabilities π_x has a *finite first moment* with respect to a stationary measure m if

$$\int \sum_y \text{dist}(x, y) p(x, y) dm(x) < \infty. \quad (42)$$

Clearly, if the first moments $\sum_y \text{dist}(x, y) p(x, y)$ of the transition probabilities π_x are uniformly bounded on x (in particular, if all the local chains on the equivalence classes have uniformly bounded range), then condition (42) is satisfied for any stationary measure m .

An additive *cocycle* of the equivalence relation R is a function $c : R \rightarrow \mathbb{R}$ which satisfies the chain rule

$$c(x, y) + c(y, z) = c(x, z)$$

for all triples of equivalent points $x, y, z \in X$. A cocycle is *Lipschitz* with respect to the graph structure \mathcal{E} if there exists a constant $C > 0$ such that

$$c(x, y) \leq C \cdot \text{dist}(x, y) \quad \forall (x, y) \in R.$$

If the transition probabilities π_x have a finite first moment with respect to a stationary measure m , then the *drift* of a Lipschitz cocycle is defined as

$$\delta = \delta(X, R, \{\pi_x\}, m, c) = \int \sum_y c(x, y) p(x, y) dm(x),$$

so that if the measure m is ergodic then

$$\frac{1}{n} c(x_0, x_n) \rightarrow \delta$$

for \mathbf{P}_m -a.e. sample path (x_n) of the global chain on X and in the space $L^1(\mathbf{P}_m)$.

The methods of the *entropy theory* of random walks on groups (see [39], [33] and the references therein) can be carried over to the Markov chains on equivalence relations and give criteria of triviality and of identification of the Poisson boundaries of local Markov chains on the classes of the equivalence relation analogous to those for random walks on groups, see [32], [36].

4.5. The Poisson boundary of extended Sierpiński gaskets

We shall now apply the considerations from the previous Section to the weak tail equivalence relation \sim on the compact set $\mathcal{A}_{-\infty}^0$ (see Definition 2.14) endowed with the graph structure \mathcal{E}_{\sim} (24). In particular, for any weak* continuous family of transition probabilities on \sim -classes there is a stationary measure on $\mathcal{A}_{-\infty}^0$.

Remark 4.9. Apparently, in our situation the stationary measure should be unique under reasonable conditions on the transition probabilities $\pi_{\mathbf{a}}$, $\mathbf{a} \in \mathcal{A}_{-\infty}^0$ (for example, for the simple random walk with respect to the graph structure \mathcal{E}_{\sim}). To take the simplest example, it is well-known to be the case if the transition probabilities $\pi_{\mathbf{a}}$ are determined by a random walk on the free semigroup $\mathcal{A}_{\emptyset}^*$ via the action (6), i.e., $p(\mathbf{a}, \mathbf{a} \cdot \mathbf{w}) = \mu(\mathbf{a})$ for a certain non-degenerate probability measure μ on $\mathcal{A}_{\emptyset}^*$, see [33] and the references therein (cf. also an analogous uniqueness result for the Brownian motion on foliations in [21]).

The synchronization cocycle σ (21) is obviously Lipschitz with respect to the graph structure \mathcal{E}_{\sim} with the constant $C = 1$.

Theorem 4.10 (cf. [11], [40]). *Let $\{\pi_{\mathbf{a}}\}$ be the family of transition probabilities of a Markov chain on the weak tail equivalence relation $(\mathcal{A}_{-\infty}^0, \sim)$ with a finite first moment with respect to the graph structure \mathcal{E}_{\sim} , and let m be a purely non-atomic ergodic stationary measure on $\mathcal{A}_{-\infty}^0$. Depending on the sign of the drift δ of the synchronization cocycle σ the following three cases occur:*

- (i) *If $\delta < 0$, then \mathbf{P}_m -a.e. sample path $(\mathbf{a}_0, \mathbf{a}_1, \dots)$ converges to the remote root of the equivalence class $[\mathbf{a}_0]$, and the Poisson boundary of m -a.e. local Markov chain is trivial.*
- (ii) *If $\delta = 0$, then the Poisson boundary of m -a.e. local Markov chain is trivial.*
- (iii) *If $\delta > 0$, then \mathbf{P}_m -a.e. sample path $(\mathbf{a}_0, \mathbf{a}_1, \dots)$ converges to a point of the Sierpiński gasket $\mathbf{G}(\mathbf{a}_0)$ (considered as a subset of the hyperbolic boundary of the Sierpiński graph $\mathcal{G}(\mathbf{a}_0)$). For m -a.e. string $\mathbf{a} \in \mathcal{A}_{-\infty}^0$ the Poisson boundary of the local Markov chain on the equivalence class $[\mathbf{a}]$ is isomorphic to the Sierpiński gasket $\mathbf{G}(\mathbf{a})$ endowed with the associated family of hitting probabilities.*

In the case (iii) the harmonic measure class $[\nu^{\mathbf{a}}]$ on a.e. Sierpiński gasket $\mathbf{G}(\mathbf{a})$ is purely non-atomic [36]. By removing a countable set of points (cf. Remark 2.13) we obtain an increasing sequence of partitions ζ_n of $\mathbf{G}(\mathbf{a})$ whose elements are the interiors of the n -th level simplices of the Sierpiński graph $\mathcal{G}(\mathbf{a})$. Then the approach from [32] in combination with Theorem 3.24 implies

Theorem 4.11. *Under conditions of Theorem 4.10, if $\delta > 0$ then for m -a.e. $\mathbf{a} \in \mathcal{A}_{-\infty}^0$ the Hausdorff dimension of the harmonic measure class $[\nu^{\mathbf{a}}]$ on the Sierpiński gasket $\mathbf{G}(\mathbf{a})$ is*

$$HD[\nu^{\mathbf{a}}] = \frac{1}{\log 2} \cdot \frac{h}{\delta}, \quad (43)$$

where the asymptotic entropy h is the number defined as

$$h = - \lim_{n \rightarrow \infty} \frac{1}{n} \log p_n(\mathbf{a}_0, \mathbf{a}_n)$$

(this limit exists \mathbf{P}_m -a.e. and the space $L^1(\mathbf{P}_m)$).

The Sierpiński graph \mathcal{G} (resp., the gasket \mathbf{G}) is contained in all the extended Sierpiński graphs $\mathcal{G}(\mathbf{a})$ (resp., the gaskets $\mathbf{G}(\mathbf{a})$). For a random walk on the classes

of the weak tail equivalence relation, *a priori*, the restrictions of transition probabilities from $\mathcal{G}(\mathbf{a})$ to \mathcal{G} are all different and the restrictions of the harmonic measure classes $[\nu^{\mathbf{a}}]$, $\mathbf{a} \in \mathcal{A}_{-\infty}^0$ to \mathbf{G} are pairwise singular for the strings \mathbf{a} from different weak tail equivalence classes. However, if the restrictions of the transition probabilities from the extended Sierpiński graphs $\mathcal{G}(\mathbf{a})$ to the Sierpiński graph \mathcal{G} are all the same, then under relatively mild conditions one can show that the behaviour on \mathcal{G} does not depend “too much” on what happens on the complement $\mathcal{G}(\mathbf{a}) \setminus \mathcal{G}$. This allows one to apply the results obtained for a.e. random graph $\mathcal{G}(\mathbf{a})$ to the concrete Sierpiński graph G . For example,

Proposition 4.12. *For a string $\mathbf{a} \in \mathcal{A}_{-\infty}^0$ consider the simple random walk on the extended Sierpiński graph $\mathcal{G}(\mathbf{a})$, and denote by $[\nu^{\mathbf{a}}]$ the arising harmonic measure class on the extended Sierpiński gasket $\mathbf{G}(\mathbf{a})$. Then the restriction $[\nu^{\mathbf{a}}]_{\mathbf{G}}$ of the class $[\nu^{\mathbf{a}}]$ to the Sierpiński gasket $\mathbf{G} \subset \mathbf{G}(\mathbf{a})$ is equivalent to the harmonic measure class $[\nu]$ on \mathbf{G} determined by the simple random walk on the Sierpiński graph \mathcal{G} .*

Sketch of the proof. The simple random walk on \mathcal{G} is obtained by reflecting the simple random walk on $\mathcal{G}(\mathbf{a})$ on the boundary of \mathcal{G} in $\mathcal{G}(\mathbf{a})$. Thus, $[\nu]$ is absolutely continuous with respect to $[\nu^{\mathbf{a}}]_{\mathbf{G}}$. Conversely, the boundary of \mathbf{G} in $\mathbf{G}(\mathbf{a})$ consists of at most 3 points, so that it is negligible with respect to $[\nu^{\mathbf{a}}]_{\mathbf{G}}$. Therefore, a.e. sample path of the simple random walk on $\mathcal{G}(\mathbf{a})$ which converges to a point in $\mathbf{G} \subset \mathbf{G}(\mathbf{a})$ eventually coincides with a certain sample path of the simple random walk on \mathcal{G} . \square

Corollary 4.13. *The Hausdorff dimension of the harmonic measure class on the Sierpiński gasket \mathbf{G} determined by the simple random walk on the Sierpiński graph is given by formula (43), where h and δ are the asymptotic entropy and the drift of the synchronization cocycle, respectively, determined by any stationary measure of the simple random along the classes of the weak tail equivalence relation \sim endowed with the graph structure \mathcal{E}_{\sim} (cf. Remark 4.9).*

4.6. The singularity problem

The problem of comparing the harmonic measure with other natural measures on the boundary arises in numerous situations: negatively curved Riemannian manifolds, random walks on groups, products of random matrices, conformal dynamics, see the references below. In all known cases coincidence of the harmonic measure type with other natural measure types inevitably implies that the considered system must belong to a certain very special subclass. However, the results of this type are notoriously difficult and heavily exploit the specifics of the considered class of systems (cf. the entirely different approaches used in [25], [13], [43], [49], [41], [46], [47]). The problem remains open in many interesting situations. Let us just mention the following problem. Let $G = \pi_1(M)$ be the fundamental group of a compact negatively curved manifold M . Is it true that the harmonic measure of any finitely supported random walk on G is singular with respect to the Hausdorff measure on the sphere at infinity of the universal covering manifold? Yet another

closely connected problem is that of describing finitely generated groups admitting a “maximal entropy” random walk, i.e., such that $h = lv$, where h is the entropy, l is the linear rate of escape, and v is the growth of the group (e.g., see the recent paper [54] and the references therein).

The Sierpiński gasket \mathbf{G} carries a natural uniformly distributed measure λ , which is the image of the uniform Bernoulli measure on \mathcal{A}_1^∞ under the map π (12) and coincides with the $\log(d+1)/\log 2$ -dimensional Hausdorff measure on \mathcal{G} .

Problem 4.14. *Is the harmonic measure class $[\nu]$ on the Sierpiński gasket \mathbf{G} determined by the simple random walk on the Sierpiński graph \mathcal{G} singular with respect to the Hausdorff measure λ ?*

Below are several comments to this problem.

1. For the random walk on the Sierpiński graph \mathcal{G} with the transition probabilities (41) considered by Denker and collaborators the harmonic measure coincides with the Hausdorff measure due to the very special choice of the transition probabilities (actually, the time n transition probability from the root \emptyset is precisely the uniform measure on the n -th level of the Sierpiński graph). However, for the simple random walk on \mathcal{G} the situation becomes non-trivial due to the presence of the horizontal transitions, so that there is no *a priori* reason for the equivalence (let alone coincidence) of the harmonic and the Hausdorff measures. For example, let us look at Fig. 10 where a fragment of a horizontal level of the Sierpiński graph is shown (on the left-hand side of the picture are the triangles represented as graph vertices on the right-hand side). This fragment is the 3-neighbourhood of a set Z consisting of 3 “siblings” (represented as black triangles on the left-hand side of the picture and as black dots on the right-hand side). If the initial distribution is equidistributed on the set Z , then after 5 steps of the simple random walk its restriction onto Z is no longer uniformly distributed (because of an additional cycle the two points to the right will have higher probabilities than the point on the left).



FIGURE 10

2. Since λ is the maximal entropy measure of the Bernoulli shift on \mathcal{A}_1^∞ , Remark 4.8 in combination with the uniqueness of the measure of the maximal entropy for the Bernoulli shift implies that the singularity of $[\nu]$ and λ is actually equivalent to the Hausdorff dimension of ν being strictly smaller than the Hausdorff dimension of \mathcal{G} (cf. Corollary 4.13).

3. An example of a “natural” measure on the Sierpiński gasket singular with respect to the Hausdorff measure is provided by Kusuoka’s *energy measure* [42], [8]. Actually, the arguments in these papers shows that its Hausdorff dimension is strictly less than the Hausdorff dimension of the Sierpiński gasket. It would be interesting to better understand the dynamical properties of the energy measure (for example, is it a Gibbs measure?).

References

- [1] S. R. Adams, *Trees and amenable equivalence relations*, Ergodic Theory Dynam. Systems, **10** (1990), 1–14.
- [2] A. Ancona, *Théorie du potentiel sur les graphes et les variétés*, in: Ecole d’été de Probabilités de Saint-Flour XVIII — 1988, Lecture Notes in Math., **1427**, Springer, Berlin, 1990, 1–112.
- [3] M. T. Anderson, R. Schoen, *Positive harmonic functions on complete manifolds of negative curvature*, Ann. of Math. (2), **121** (1985), 429–461.
- [4] M. T. Barlow, R. F. Bass, *Random walks on graphical Sierpinski carpets*, in: Random walks and discrete potential theory (Cortona, 1997), 26–55, Sympos. Math., XXXIX, Cambridge Univ. Press, Cambridge, 1999.
- [5] M. T. Barlow, E. A. Perkins, *Brownian motion on the Sierpiński gasket*, Probab. Theory Related Fields, **79** (1988), 543–623.
- [6] L. Bartholdi, R. Grigorchuk, V. Nekrashevich, *From fractal groups to fractal sets*, these Proceedings.
- [7] N. Benakli, I. Kapovich, *Boundaries of hyperbolic groups*, preprint (2002).
- [8] O. Ben-Bassat, R. S. Strichartz, A. Teplyaev, *What is not in the domain of the Laplacian on Sierpiński gasket type fractals*, J. Funct. Anal., **166** (1999), 197–217.
- [9] J. W. Cannon, *The theory of negatively curved spaces and groups*, in: Ergodic theory, symbolic dynamics, and hyperbolic spaces. Papers from the Workshop on Hyperbolic Geometry and Ergodic Theory held in Trieste, April 17–28, 1989. Edited by T. Bedford, M. Keane and C. Series, Oxford University Press, New York, 1991, 315–369.
- [10] P. Cartier, *Fonctions harmoniques sur un arbre*, in: Convegno di Calcolo delle Probabilità, INDAM, Rome, 1971, Symposia Mathematica, **9**, Academic Press, London, 1972, 203–270.
- [11] D. I. Cartwright, V. A. Kaimanovich, W. Woess, *Random walks on the affine group of local fields and of homogeneous trees*, Ann. Inst. Fourier (Grenoble), **44** (1994), 1243–1288.

- [12] T. Ceccherini-Silberstein, R. I. Grigorchuk, P. de la Harpe, *Amenability and paradoxical decompositions for pseudogroups and discrete metric spaces*, Proc. Steklov Inst. Math., **224** (1999), 57–97.
- [13] P. Chassaing, G. Letac, M. Mora, *Brocot sequences and random walks in $SL(2, \mathbb{R})$* , in: Probability measures on groups, VII (Oberwolfach, 1983), Lecture Notes in Math., **1064**, Springer, Berlin, 1984, 36–48.
- [14] I. P. Cornfeld, S. V. Fomin, Ya. G. Sinai, *Ergodic theory*, Grundlehren der Mathematischen Wissenschaften, **245**, Springer-Verlag, New York, 1982.
- [15] M. Denker, S. Koch, *A Poisson formula for harmonic functions on the Sierpiński gasket*, Forum Math., **12** (2000), 435–448.
- [16] M. Denker, H. Sato, *Sierpiński gasket as a Martin boundary. I. Martin kernels*, Potential Anal., **14** (2001), 211–232.
- [17] M. Denker, H. Sato, *Sierpiński gasket as a Martin boundary. II. The intrinsic metric*, Publ. Res. Inst. Math. Sci., **35** (1999), 769–794.
- [18] J. Dodziuk, *Difference equations, isoperimetric inequality and transience of certain random walks*, Trans. Amer. Math. Soc., **1984**, 787–794.
- [19] G. Elek, *The l^p -cohomology and the conformal dimension of hyperbolic cones*, Geom. Dedicata, **68** (1997), 263–279.
- [20] B. Farb, L. Mosher, *A rigidity theorem for the solvable Baumslag–Solitar groups. With an appendix by Daryl Cooper*, Invent. Math., **131** (1998), 419–451.
- [21] L. Garnett, *Foliations, the ergodic theorem and Brownian motion*, J. Funct. Anal., **51** (1983), 285–311.
- [22] *Sur les groupes hyperboliques d'après Mikhael Gromov*, Edited by E. Ghys and P. de la Harpe. Progress in Mathematics, **83**, Birkhäuser, Boston, MA, 1990.
- [23] P. J. Grabner, W. Woess, *Functional iterations and periodic oscillations for simple random walk on the Sierpiński graph*, Stochastic Processes. Appl., **69** (1997), 127–138.
- [24] M. Gromov, *Hyperbolic groups*, in: Essays in group theory, Math. Sci. Res. Inst. Publ., **8**, Springer, New York, 1987, 75–263.
- [25] Y. Guivarc'h, J. Le Jan, *Asymptotic winding of the geodesic flow on modular surfaces and continued fractions*, Ann. Sci. Ecole Norm. Sup. (4), **26** (1993), 23–50.
- [26] B. M. Hambly, *Brownian motion on a homogeneous random fractal*, Probab. Theory Rel. Fields, **94** (1992), 1–38.
- [27] B. M. Hambly, T. Kumagai, S. Kusuoka, X. Y. Zhou, *Transition density estimates for diffusion processes on homogeneous random Sierpinski carpets*, J. Math. Soc. Japan **52** (2000), 373–408.
- [28] O. D. Jones, *Transition probabilities for the simple random walk on the Sierpiński graph*, Stochastic Process. Appl., **61** (1996), 45–69.
- [29] V. A. Kaimanovich, *Measure-theoretical boundaries of Markov chains, 0–2 laws and entropy*, in: Harmonic analysis and discrete potential theory (Frascati, 1991), Plenum, New York, 1992, 145–180.
- [30] V. A. Kaimanovich, *Dirichlet norms, capacities and generalized isoperimetric inequalities for Markov operators*, Potential Anal., **1** (1992), 61–82.

- [31] V. A. Kaimanovich, *Boundaries of invariant Markov operators: the identification problem*, in: Ergodic theory of \mathbb{Z}^d actions, Warwick, 1993–1994, London Math. Soc. Lecture Note Ser., **228**, Cambridge University Press, Cambridge, 1996, 127–176.
- [32] V. A. Kaimanovich, *Hausdorff dimension of the harmonic measure on trees*, Ergodic Theory Dynam. Systems, **18** (1998), 631–660.
- [33] V. A. Kaimanovich, *The Poisson formula for groups with hyperbolic properties*, Ann. of Math. (2), **152** (2000), 659–692.
- [34] V. A. Kaimanovich, *Equivalence relations with amenable leaves need not be amenable*, in: Topology, ergodic theory, real algebraic geometry, Amer. Math. Soc. Transl. Ser. 2, **202**, Amer. Math. Soc., Providence, RI, 2001, 151–166.
- [35] V. A. Kaimanovich, *Fractals and hyperbolicity*, preprint (2002).
- [36] V. A. Kaimanovich, *Random walks on equivalence relations*, preprint (2002).
- [37] V. A. Kaimanovich, M. Lyubich, *Conformal and harmonic measures on laminations associated with rational maps*, preprint (2001).
- [38] V. A. Kaimanovich, K. Schmidt, *Ergodicity of cocycles. I. General theory*, preprint (2001).
- [39] V. A. Kaimanovich, A. M. Vershik, *Random walks on discrete groups: boundary and entropy*, Ann. Probab., **11** (1983), 457–490.
- [40] V. A. Kaimanovich, W. Woess, *Boundary end entropy of space homogeneous Markov chains*, Ann. Probab., **30** (2002).
- [41] A. Katok, *Four applications of conformal equivalence to geometry and dynamics*, Ergodic Theory Dynam. Systems, **8*** (1988), Charles Conley Memorial Issue, 139–152.
- [42] S. Kusuoka, *Dirichlet forms on fractals and products of random matrices*, Publ. Res. Inst. Math. Sci., **25** (1989), 659–680.
- [43] F. Ledrappier, *Applications of dynamics to compact manifolds of negative curvature*, in: Proceedings of the International Congress of Mathematicians (Zürich, 1994), Birkhäuser, Basel, 1995, 1195–1202.
- [44] F. Ledrappier, *Some asymptotic properties of random walks on free groups*, in: Topics in probability and Lie groups: boundary theory, CRM Proc. Lecture Notes, **28**, Amer. Math. Soc., Providence, RI, 2001, 117–152.
- [45] M. Lyubich, Y. Minsky, *Laminations in holomorphic dynamics*, J. Differential Geom., **47** (1997), 17–94.
- [46] M. Lyubich, A. Volberg, *A comparison of harmonic and maximal measures for rational functions*, in: Approximation by solutions of partial differential equations (Hanstholm, 1991), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **365**, Kluwer, Dordrecht, 1992, 127–139.
- [47] M. Lyubich, A. Volberg, *A comparison of harmonic and balanced measures on Cantor repellers*, in: Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993), J. Fourier Anal. Appl., Special Issue, 1995, 379–399.
- [48] N. G. Makarov, A. L. Volberg, *On the harmonic measure of discontinuous fractals*, LOMI preprint E-6-86, 1986.
- [49] F. Przytycki, M. Urbański, A. Zdunik, *Harmonic, Gibbs and Hausdorff measures on repellers for holomorphic maps. I*, Ann. of Math. (2), **130** (1989), 1–40.

- [50] M. Rosenblatt, *Markov processes. Structure and asymptotic behaviour*, Grundlehren der Mathematischen Wissenschaften, **184**, Springer-Verlag, New York - Heidelberg, 1971.
- [51] C. Series, *Martin boundaries of random walks on Fuchsian groups*, Israel J. Math., **44** (1983), 221–242.
- [52] C. Series, *Geometrical methods of symbolic coding*, in: Ergodic theory, symbolic dynamics, and hyperbolic spaces. Papers from the Workshop on Hyperbolic Geometry and Ergodic Theory held in Trieste, April 17–28, 1989. Edited by T. Bedford, M. Keane and C. Series, Oxford University Press, New York, 1991, 125–151.
- [53] E. Teufl, *The average displacement of the simple random walk on the Sierpiński graph*, preprint (2001).
- [54] A. M. Vershik, *Dynamic theory of growth in groups: entropy, boundaries, examples*, Russian Math. Surveys, **55** (2000), no. 4, 667–733.
- [55] A. Volberg, *On the dimension of harmonic measure of Cantor repellers*, Michigan Math. J., **40** (1993), 239–258.
- [56] W. Woess, *Random walks on infinite graphs and groups*, Cambridge Tracts in Mathematics, **138**, Cambridge University Press, Cambridge, 2000.

CNRS UMR 6625 (IRMAR)

Université Rennes-1

Campus Beaulieu

35042 Rennes, France

E-mail address: `kaimanov@univ-rennes1.fr`