ESI **The Erwin Schrödinger International Institute for Mathematical Physics**

Pasteurgasse 4/7
A-1090 Wien, Austria

Bogomolny–type Equations for Gravitating Gauged Sigma Models

P. Forgács
Z. Horváth
P.A. Horváthy

Vienna, Preprint ESI 117 (1994) August 10, 1994

Supported by Federal Ministry of Science and Research, Austria
Available via WWW.ESI.AC.AT

Bogomolny–type equations
for
gravitating gauged sigma models

P. FORGÁCS [1], Z. HORVÁTH [2] and P. A. HORVÁTHY [3]

Abstract. It is shown that the complete set of field equations of a gauged sigma model coupled to gravity is satisfied by a set of first-order equations when the metric is static.

[1] Research Institute for Particle and Nuclear Physics, H-1525 BUDAPEST 114, P.O.B. 49, Hungary; e-mail: forgacs@rmki.kfki.hu

[2] Institute for Theoretical Physics, Eötvös University, H–1088 BUDAPEST, Puskin u. 5-7 (Hungary); e-mail: zalanh@ludens.elte.hu

[3] Département de Mathématiques, Université de Tours, Parc de Grandmont, F–37200 TOURS (France); e-mail: horvathy@univ-tours.fr

1. Introduction

In the past few years, there has been a growing interest to study the effects of gravity on classical solutions of Yang–Mills (YM) [1] and of spontaneously broken gauge theories (YMH) [2,3]. It has been found that, in the presence of gravity, new nonperturbative phenomena appear even for weak coupling. However, due the complexity of the equations, most investigations have been numerical. In YMH theories without a Higgs potential (in flat spacetime) a number of striking (analytic) results based on the first order Bogomolny equations [4] have been obtained. It is therefore natural to look for a similar simplification also in the presence of gravity. Then the immediate problem one faces is the lack of a positive definite energy functional in the static case whose variation would yield the field equations and which could be bounded from below in the spirit of Ref. [4]

If the metric is static and it is fixed (background), the generalization of the flat space Bogomolny eqs. for the curved-spaced YMH theory without scalar potential have been found in Ref. [5]. Compatibility with the Einstein eqs., however, excluded any non-abelian solution (in the spherically symmetric case).

It has been found in string theories that, when the four-metric is *conformally flat*, solutions of the flat-space self-duality eqs. can be imbedded into the string background field eqs. (with graviton, YM, dilaton and the antisymmetric tensor) [6]. Similarly, when a YMH theory is coupled to N=2 supergravity, Bogomolny-type eqs. were found when the four-metric is of the special conformstatic form $\mathrm{e}^{2\phi}dt^2 - \mathrm{e}^{-2\phi}dx^i dx^i$ [7]. In Refs. [8] resp. [9] the Bogomolny-type eqs. of Ref. [5] have been rediscovered for this special conformstatic metric in an EYMH theory (with a coupling of the Higgs field to the Ricci scalar in [8]) resp. in an EYMH theory with a dilaton [9].

The aim of this paper is to show that in a *gauged sigma model* (the infinite mass limit of a YMH theory), the Bogomolny–type eqs. of Ref. [5] yield solutions to the full coupled system with a static metric. No additional constraints appear. This result appears to be somewhat surprising, since, in flat space, no corresponding first order system exists and that the metric of the three-space is completely general. These results are easily generalized when a dilaton is also included.

2. Gravitating sigma model

We consider the following action, corresponding to a gauged non–linear sigma model, coupled to gravity:

$$S = -\int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}\,R + \tfrac{1}{4}\mathrm{Tr}\ F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\mathrm{Tr}\ (D_\mu \mathbf{n}\, D^\mu \mathbf{n})\right]\ , \tag{1}$$

where G is Newton's constant, R is the Ricci scalar, and the scalar field $\mathbf{n}$ is in the adjoint representation of the gauge group G satisfying the constraint $\mathrm{Tr}\ \mathbf{n}^2 = 1$. This model can also be viewed as the $M_H \to \infty$ limit of a spontaneously broken gauge theory with the Higgs fields in the adjoint representation (M_H being the mass of the Higgs fields).

The field equations following from the action (1) are:

$$\frac{1}{\sqrt{-g}} D_\mu\big(\sqrt{-g}\, F^{\mu\nu}\big) = \big[\mathbf{n}, D^\nu \mathbf{n}\big], \tag{2.a}$$

$$\frac{1}{\sqrt{-g}} D_\mu\big(\sqrt{-g}\, D^\mu \mathbf{n}\big) = -\mathrm{Tr}\ (D^\mu \mathbf{n} D_\mu \mathbf{n})\mathbf{n}\,, \tag{2.b}$$

together with the Einstein equations

$$R_{\mu\nu} = 8\pi G\big(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T\big)\,. \tag{3}$$

The energy-momentum tensor is $T_{\mu\nu} = T^{YM}_{\mu\nu}[F] + T^{Sc}_{\mu\nu}[\mathbf{n}]$, where

$$\begin{aligned} T^{YM}_{\mu\nu}[F] &= \mathrm{Tr}\ \big\{F_{\mu\rho}F^{\rho}{}_{\nu} - \tfrac{1}{4} g_{\mu\nu} F_{\sigma\rho}F^{\sigma\rho}\big\}\ , \\ T^{Sc}_{\mu\nu}[\mathbf{n}] &= \mathrm{Tr}\ \big\{(D_\mu \mathbf{n} D_\nu \mathbf{n}) - \tfrac{1}{2} g_{\mu\nu}(D_\sigma \mathbf{n} D^\sigma \mathbf{n})\big\}\ . \end{aligned} \tag{4}$$

Note that T, the trace of $T_{\mu\nu}$, is that of the scalar part alone, $T = -\mathrm{Tr}\ (D_\sigma \mathbf{n} D^\sigma \mathbf{n})$.

We now restrict ourselves to static, purely-magnetic, ($A_0 = 0$), configurations. The metric is then of the form $g_{\mu\nu} = (g_{00}, -\widetilde{g}_{ij})$.

The Yang–Mills–Higgs theory in a static background based on the action (1), but without the constraint Tr $\mathbf{n}^2 = 1$ was considered some time ago. This corresponds to the $M_H = 0$ limit. The field equations of this system are (2.a), and (2.b) with zero on the right side, can be solved by a system of first order equations. This system, generalizing the flat space Bogomolny–equations, has been found to be [5]:

$$D_i \mathbf{n} + (\partial_i u)\mathbf{n} = \tfrac{1}{2}\sqrt{\widetilde{g}}\,\epsilon_{ijk}F^{jk} \quad \text{where} \quad u = \ln\sqrt{g_{00}} \tag{5}$$

solves the field equations, provided the metric satisfies the constraint

$$\widetilde{\triangle} \ln\sqrt{g_{00}} = \frac{1}{\sqrt{\widetilde{g}}}\,\partial_i\big(\sqrt{\widetilde{g}}\,\partial^i \ln\sqrt{g_{00}}\big) = 0\,. \tag{6}$$

When one considers the *coupled* Einstein–Yang–Mills equations the constraint (6) turns out to be very restrictive. In the spherically symmetric case the only solution is the extreme Reissner–Nordstrøm metric and an abelian gauge field.

In the gauged sigma model (1) we are considering, no first-order (Bogomolny–type) equations are known even in flat space. The main result of this paper is that the first order equations (5) yield solutions of the coupled system (2.a, 2.b, 3).

First, it is easy to see that Eqs. (5) imply the Yang-Mills Eq. (2.a): Just contract Eqs. (5) with ϵ^{ilm} and D_m and exploit Eqs. (5). In fact, this derivation remains true

for more general YM theories coupled to scalars in the adjoint representation (e.g. YMH theory with potential).
Contracting Eqs. (5) with D_i and using the Bianchi identities one finds that Eqs. (2.b) (the scalar eqs.) are also satisfied provided the constraint

$$\widetilde{\triangle} \ln \sqrt{g_{00}} = \mathrm{Tr}\ (D_i \mathbf{n} D^i \mathbf{n}) , \tag{7}$$

holds. Clearly, the r.h.s of (7) — the modification of the previous constraint (6) — arises due to the r.h.s. of (2.b). For a YMH theory (with potential) in a curved background a similar eq. could be derived with an appropriately modified r.h.s. What makes this gauged sigma model so special, is that the (00) component of the Einstein eqs. (3) turn out to be *identical* to Eq. (7), when using the first order system (5) and the sigma model constraint Tr $\mathbf{n}^2 = 1$. To see this we note that $R_{00} = \sqrt{g_{00}}\, \widetilde{\triangle} \sqrt{g_{00}}$, so the corresponding Einstein equation is

$$\frac{1}{\sqrt{g_{00}}}\, \widetilde{\triangle} \sqrt{g_{00}} = (4\pi G)\, \tfrac{1}{2} \mathrm{Tr}\ F_{ij} F^{ij} \, . \tag{8}$$

The r.h.s. of Eq. (8) can be rewritten with the use of the the first order system (5) and Tr $\mathbf{n}^2 = 1$. Then Eq. (8) becomes equivalent to (7), provided

$$4\pi G = 1, \tag{9}$$

which amounts to setting the mass of the gauge fields (M_W) equal to the Planck mass. We note that with use of the Einstein eqs. (7) can be rewritten in the following way:

$$\widetilde{\triangle} \ln \sqrt{g_{00}} - \frac{1}{2} R = 0 \, . \tag{10}$$

Eq. (10) involves only the metric and its derivatives. The remaining Einstein eqs. determine the $\widetilde{g}_{ij}$ components of the metric and no additional constraints arise. Therefore a class of solutions of the full system is obtained by solving the first order Eqs. (5) instead of the second order field eqs. (2.a, 2.b).

Interestingly, by coupling a dilaton field to our sigma model, the first order system (5) can easily be modified to take the dilaton into account. Then the action (1) is replaced by

$$- \int \left[\frac{1}{16\pi G} \left(R - \tfrac{1}{2} \partial_\mu \sigma \partial^\mu \sigma \right) + \tfrac{1}{4} \mathrm{Tr}\ \left(e^{\gamma\sigma} F_{\mu\nu} F^{\mu\nu} \right) - \tfrac{1}{2} \mathrm{Tr}\ \left(D_\mu \mathbf{n}\, D^\mu \mathbf{n} \right) \right] \sqrt{-g}, \tag{11}$$

where γ is a constant. The equations of motion are then given as

$$\frac{1}{\sqrt{-g}} D_\mu \left(\sqrt{-g}\, e^{\gamma\sigma} F^{\mu\nu} \right) = \left[\mathbf{n}, D_\nu \mathbf{n} \right] , \tag{12.a}$$

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, \partial^\mu \sigma \right) = (4\pi G \gamma) e^{\gamma\sigma} \mathrm{Tr}\ F_{\mu\nu} F^{\mu\nu} \, . \tag{12.b}$$

The field equations of $\mathbf{n}$ are again given by Eqs. (2.b). The Einstein equations are as in (3), where the stress energy tensor has to be replaced by

$$T_{\mu\nu} = e^{\gamma\sigma} T^{YM}_{\mu\nu}[F] + T^{Sc}_{\mu\nu}[\mathbf{n}] + \frac{1}{16\pi G} T^{Sc}_{\mu\nu}[\sigma]\,, \tag{13}$$

where $T^{Sc}_{\mu\nu}[\sigma]$ is $T^{Sc}_{\mu\nu}[\mathbf{n}]$ with σ replacing $\mathbf{n}$. Still restricting ourselves to static systems, one finds that by comparing the (00) component of the Einstein equations with the dilaton equation (12.b) yields the following relation between σ and g_{00}:

$$\partial_i \left(\sqrt{-g}\,\partial^i (\sigma - 2\gamma \ln \sqrt{g_{00}}) \right) = 0\,. \tag{14}$$

With the obvious modification of Eqs. (5)

$$D_i \mathbf{n} + (\partial_i u)\mathbf{n} = \tfrac{1}{2}\sqrt{\tilde{g}}\,\epsilon_{ijk} e^{\gamma\sigma/2} F^{jk}\,, \tag{15}$$

one easily finds that (15) implies the YM Eqs. (12.a), when

$$\sigma = 2\gamma \log \sqrt{g_{00}}\,, \quad \text{and} \quad u = (1+\gamma^2)\log\sqrt{g_{00}}\,. \tag{16}$$

An analogous computation as before shows that the scalar Eq. (2.b) is satisfied provided

$$\widetilde{\triangle} u - \tfrac{1}{2}\gamma\, \partial_i \sigma\, \partial^i u = \mathrm{Tr}\,(D_i \mathbf{n} D^i \mathbf{n})\,, \tag{17}$$

holds, which is the counterpart of the constraint equation (7). Finally, a very similar calculation to the one without the dilaton shows that Eq. (17) will be identical to the (00) component of the Einstein equations if

$$4\pi G(1+\gamma^2) = 1\,. \tag{18}$$

Interestingly, condition (17) then reduces again to (10).

For the special conformstat metric

$$V^2 dt^2 - V^{-2} dx^i dx^i\,, \tag{19}$$

it has been observed in [8] and [9] that the static reduction of what corresponds to the action (1) resp. (11), can be bounded modulo a surface term yielding the correspondig special form of Eqs. (5) resp. (15). Eq. (9) is identically satisfied by the special conformstat class of metrics (18).

3. Spherically symmetric equations

Let us now examine briefly the simplest nontrivial case, namely that of spherical symmetry and gauge group $SU(2)$. The first order eqs. for this case have been discovered in Ref. [10]. Parametrize the line element as

$$ds^2 = A^2 \mu^2 \, dt^2 - \frac{1}{\mu^2} dr^2 - r^2 d\Omega \tag{20}$$

and consider the minimal spherically symmetric ansatz:

$$A_0 = 0 \,, \qquad A_i^a = \epsilon_{iak} \, \frac{x^k}{r^2} \big(1 - W(r)\big) \,, \qquad n^a = \frac{x^a}{r} \,. \tag{21}$$

The gauge field equation and the two Einstein eqs. can be written as:

$$(A\mu^2 W')' = AW \left(\frac{W^2 - 1}{r^2} + 1 \right) , \tag{22.a}$$

$$2r\mu\mu' = 1 - \mu^2 - 2\alpha^2 \mu^2 W'^2 - 2\alpha^2 \left[\frac{(W^2-1)^2}{2r^2} + W^2 \right] , \tag{22.b}$$

$$\frac{A'}{A} = \frac{2\alpha^2}{r} W'^2 \,, \tag{22.c}$$

where $\alpha^2 = 4\pi G$. The Bogomolny-type equations (5) read

$$\mu \, W' = W, \qquad (A\mu)' = A \frac{W^2 - 1}{r^2} \,. \tag{23}$$

One can check directly that Eq. (22.a) is a consequence of Eqs. (23), and that the compatibility of Eqs. (22.b, 22.c) with Eqs. (23) leads to an algebraic relation between μ and W:

$$\alpha^2 = 1 \,, \tag{24}$$

and

$$\mu + \frac{W^2 - 1}{r} = \pm 1 \,. \tag{25}$$

Then the first order system (23) is solved by a single first order equation,

$$W' = - \frac{rW}{\mp r - 1 + W^2} \,, \tag{26}$$

while g_{00} becomes

$$g_{00} = A^2 \mu^2 = \exp \left[2 \int \frac{W^2 - 1}{(\pm r - W^2 + 1) r} \, dr \right] . \tag{27}$$

Eq. (26) is analyzed briefly as follows: (assuming $W > 0$). For the plus sign (W_+) Eq. (26) has two singular points, namely $p_1 = (W = 0,\ r = 1)$ and $p_2 = (W = 1,\ r = 0)$. p_1 is a hyperbolic point with indices $\lambda = \pm 1$, while p_2 is a focal point with complex indices $\lambda = (1 \pm i\sqrt{7})/2$. For the minus sign ($W_-$) there is only one singular point p_2. Infinitely many solutions start from the singular point p_2 but become soon double-valued and leave the physical region. In the first case there are two separatrices in the (r, W_+) plane; $W_+ \equiv 0$ and a second one which cuts the $r = 0$ axis at the point $W_+(0) = 1.2888\ldots$. Solutions with initial value larger then $W_+(0)$ are defined for all r. As $W_+(0) \neq 1$ all these solutions are singular at $r = 0$. They differ substantially from zero for $r < 1$ and decay exponentially. We present some solutions of Eq. (26) obtained numerically on Fig. 1.

For $W_+ = 0$ one obtains the extreme Reissner - Nordstrøm metric $g_{00} = (1 - 1/r)^2$, the only solution with a horizon. Linearizing around $W \equiv 0$ shows that, in the region $r > 1$, the the gauge field behaves as

$$W \sim \frac{e^{-r}}{r} \frac{1}{1 - \frac{1}{r}} \tag{28}$$

and the metric tends to the extreme Reissner - Nordstrøm form,

$$g_{00} \sim \big(1 - \frac{1}{r}\big)^2 \times \exp\big[-c \int^r \frac{e^{-2\rho}}{(\rho - 1)^4}\, d\rho\big]. \tag{29}$$

For completeness, we discuss briefly the first order eqs. in the presence of the dilaton. Since in the coordinate system (19) the algebraic equation corresponding to (25) is rather complicated it is more convenient to use the isotropic coordinate system (19), used also in refs. [8], [9]. (In the absence of the dilaton field Eq. (26) is simpler since it is only first order.) In the spherically symmetric case the metric is $V^2 dt^2 - V^{-2}(dR^2 + R^2 d\Omega)$, $V = V(R)$. Eqs. (5) can be written as

$$(\ln V)' = V \frac{W^2 - 1}{R^2} . \qquad W = V W' , \tag{30}$$

Eliminating V, a single second order eq. for W is obtained. Then Eqs. (23) are replaced by:

$$V = e^{\sigma/2\gamma} , \qquad e^{\gamma\sigma/2} V W' = W . \qquad \frac{1 + \gamma^2}{2\gamma} \sigma' = e^{\sigma\gamma/2} V \frac{(W^2 - 1)}{R^2} . \tag{31}$$

Eliminating V and σ from Eq. (31) leads to the same second order eq. for W as in the case without the dilaton. As discussed in Ref. [9], Eqs. (31) do not have regular solutions.

Acknowledgements.
P. F. and Z. H. would like to acknowledge the receipt of a French government grant and to thank Tours University for hospitality extended to them as well as the Hungarian National Science and Research Foundation (Grant No. 2177) for a partial financial support. P. F. would also like to thank the Erwin Schrödinger Institute for hospitality extended to him.

References

[1] R. Bartnik and J. McKinnon, *Phys. Rev. Lett.* **61** (1988) 141
H.P. Künzle and A.K.M. Masood-ul-Alam, *J. Math. Phys.* **31** (1990) 928
N. Straumann and Z.H. Zhou, *Phys. Lett.* **B237** (1990) 353; ibid. B243 (1990) 33
M.S. Volkov, *JETP Lett.* **50** (1989) 346
P. Bizon, *Phys. Rev. Lett.* **64** (1990) 2844

[2] P. Breitenlohner, P. Forgács, and D. Maison, *Nucl. Phys.* **B383** (1992) 357
K. Lee, V.P. Nair and E.J. Weinberg, *Phys. Rev.* **D45** (1992) 2751
M. Ortiz, *Phys. Rev.* **D45** (1992) 2586
P.R. Greene, S.D. Mathur and C.M. O'Neil, *Phys. Rev.* **D47** (1993) 2242
P.C. Aichelburg and P. Bizon, *Phys. Rev.* **D48** (1993) 607

[3] B. Linet, *Phys. Lett.* **A124** (1987) 240
A. Comtet and G. Gibbons, *Nucl. Phys.* **B299** (1988) 719
J.D. Edelstein, G. Lozano and F.A. Schaposnik, *Mod. Phys. Lett.* **A8** (1993) 3665

[4] E.B. Bogomolny, *Yad. Fiz.* **24** (1976) 8561 [*Sov. J. Nucl. Phys.* **24** (1976) 449]
H. De Vega and F.A. Schaposnik, *Phys. Rev.* **D14** (1976) 1100

[5] A. Comtet, P. Forgács and P.A. Horváthy, *Phys. Rev.* **D30** (1984) 468

[5] C. Callan, J. Harvey, and A. Strominger *Nucl. Phys.* **B359** (1991) 611; ibid. **367** (1991) 60

[7] G.W. Gibbons, D. Kastor, L.A.J. London, P.K. Townsend and J. Traschen, *Nucl. Phys.* **B416** (1994) 850

[8] B.S. Balakrishna and K.C. Wali, *Phys. Rev.* **D46** (1992) R5228

[9] Y.M. Cho, J.W. Kim, D.H. Park and J.H. Yoon, *Phys. Lett.* **B308** (1993) 23

[10] A. Comtet, *Ann. Inst. H. Poincaré* **32** (1980) 283

Figure caption

Fig.1 Numerical solutions of Eq. (26).