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Conformal Ambient Metric Construction

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ABSTRACT. In this paper we relate the Fefferman–Graham ambient metric construction for conformal manifolds to the approach to conformal geometry via the canonical Cartan connection. We show that from any ambient metric that satisfies a weakening of the usual normalisation condition, one can construct the conformal standard tractor bundle and the normal standard tractor connection, which are equivalent to the Cartan bundle and the Cartan connection. This result is applied to obtain a procedure to get tractor formulae for all conformal invariants that can be obtained from the ambient metric construction. We also get information on ambient metrics which are Ricci flat to higher order than guaranteed by the results of Fefferman–Graham.

1. INTRODUCTION

It is an old result of E. Cartan that conformal manifolds of dimension ≥ 3 admit a canonical normal Cartan connection. While this solves the equivalence problem for conformal structures, the problem of a complete conformal invariant theory and the related problem of constructing conformally invariant differential operators remain very difficult. Based on similar ideas for CR–structures, Ch. Fefferman and C.R. Graham initiated a new approach to these problems in 1985, see [8]. Viewing a pseudo–conformal structure of signature (p,q) on a manifold M as a ray bundle $S^2 T^* M \supset \mathcal{Q} \to M$ the idea of this approach is to associate to the given conformal structure a pseudo–Riemannian metric of signature $(p+1, q+1)$ on $\mathcal{Q} \times (-1,1)$, the so–called ambient metric. This metric is required to be homogeneous and compatible with the conformal structure in a rather obvious sense, while the more subtle condition is that it should be Ricci flat up to some order along $\mathcal{Q}$. In odd dimensions, ambient metrics which are Ricci flat to infinite order along $\mathcal{Q}$ exist and are essentially unique, while in even dimensions there is an obstruction at finite order, but up to that order the ambient metric is again essentially unique.

This immediately leads to a construction for conformal invariants, since any Riemannian invariant of an ambient metric satisfying these Ricci conditions, which is of low enough order in the even–dimensional case, gives rise to a conformal invariant of the underlying conformal structure. Moreover, the ambient metric construction has been applied in [14] to construct conformally invariant powers of the Laplacian. This constructions gives arbitrarily

high powers in odd dimensions and powers up to some critical order in even dimensions. This is complemented by C.R. Graham's result in [13] that there is no conformally invariant third power of the Laplacian in dimension four, which strongly suggests that the obstruction to the ambient metric construction in even dimensions is of fundamental nature. It should also be remarked here that the ambient metric construction has recently received renewed interest because of its relation to the so–called Poincaré metric and via that to scattering theory and the AdS/CFT–correspondence in physics, see [20, 16, 17, 15, 9] and references therein.

Over the past few years the Cartan approach to conformal geometry and a more general class of geometric structures called parabolic geometries has been significantly developed. One development, whose origins can be traced back to the work of T. Thomas in the 1920's and 1930's [18, 19], is the concept of tractor bundles, which give an equivalent description of the Cartan bundle and the Cartan connection in terms of linear connections on certain vector bundles. These then lead to an efficient calculus, which has been successfully applied to the study of conformal invariants and conformally invariant differential operators, see e.g. [1, 7, 11].

The purpose of this paper is to relate precisely the ambient metric construction to the conformal standard tractor bundle and its canonical linear connection. We first construct a standard tractor bundle and a tractor connection on that bundle from a very general class of ambient metrics. Then we prove that normality of this tractor connection is equivalent to vanishing of the tangential components of the Ricci curvature of the ambient metric along $\mathcal{Q}$. Hence, we obtain the normal standard tractor bundle and tractor connection from any metric produced by the ambient metric construction. This is done in section 2.

In section 3, we express some basic elements of tractor calculus in terms of ambient data. This is then used to show that for any ambient metric which satisfies the Ricci conditions of [8], there is an algorithm to compute all covariant derivatives of the curvature (up to the critical order in even dimensions) from the tractor curvature. Since any local scalar conformal invariant obtained from the ambient metric construction is a complete contraction of a tensor product of such covariant derivatives, we obtain an algorithm to compute a tractor formula for any of these invariants. This is an important achievement, since in contrast to the situation of the ambient metric construction, converting tractor formulae into formulae in terms of metrics from the conformal class is a purely mechanical procedure and in particular does not involve solving any equations. In fact it is straightforward to write software for these expansions, see [12].

Our results also cover the case of metrics which are Ricci flat to higher order than the ones that can be obtained from the ambient metric construction. While such metrics do not exist on general conformal manifolds, studying the cases in which they do exist is of considerable interest in conformal geometry. What we can prove in this case is that all higher covariant derivatives of the curvature can be obtained from the tractor curvature and one "critical" covariant derivative.

It should be pointed out that formally, our results are completely independent from the results of [8] on the ambient metric construction. In particular, Theorem 3.4 goes a long way towards an independent proof of the uniqueness of the ambient metric. In fact, it contains enough information on the uniqueness to show that the ambient metric construction leads to conformal invariants. We believe that our results can be extended to a complete proof of uniqueness of the ambient metric, which is of entirely different nature than the one in [8]. The reason why we do not go further in that direction here is that we believe that the ideas we develop also can be used for existence proofs for ambient metrics and an analysis for the obstruction to the existence of an ambient metric which is Ricci flat to infinite order in even dimensions, and we will take up this whole circle of problems elsewhere.

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2. The ambient construction of conformal standard tractors

2.1. Conformal structures. Let M be a smooth manifold of dimension $n \geq 3$. A *conformal structure* on M of signature (p, q) (with $p + q = n$) is an equivalence class of smooth pseudo–Riemannian metrics of signature (p, q) on M, with two metrics being equivalent if and only if one is obtained from the other by multiplication with a positive smooth function.

For a point $x \in M$, and two metrics g and $\hat{g}$ from the conformal class, there is an element $s \in \mathbb{R}_+$ such that $\hat{g}_x = sg_x$. Thus, we may equivalently view the conformal class as being given by a smooth ray subbundle $\mathcal{Q} \subset S^2 T^* M$, whose fibre at x is formed by the values of g_x for all metrics g in the conformal class. By construction, $\mathcal{Q}$ has fibre $\mathbb{R}_+$ and the metrics in the conformal class are in bijective correspondence with smooth sections of $\mathcal{Q}$.

Denoting by $\pi : \mathcal{Q} \to M$ the restriction to $\mathcal{Q}$ of the canonical projection $S^2 T^* M \to M$, we can view this as a principal bundle with structure group $\mathbb{R}_+$. The usual convention is to rescale a metric g to $\hat{g} = f^2 g$. This corresponds to a principal action given by $\rho^s(g_x) = s^2 g_x$ for $s \in \mathbb{R}_+$ and $g_x \in \mathcal{Q}_x$, the fibre of $\mathcal{Q}$ over $x \in M$.

Having this, we immediately get a family of basic real line bundles $\mathcal{E}[w] \to M$ for $w \in \mathbb{R}$ by defining $\mathcal{E}[w]$ to be the associated bundle to $\mathcal{Q}$ with respect to the action of $\mathbb{R}_+$ on $\mathbb{R}$ given by $s{\cdot}t := s^{-w}t$. The usual correspondence between sections of an associated bundle and equivariant functions on the total space of a principal bundle then identifies the space $\Gamma(\mathcal{E}[w])$ of smooth sections of $\mathcal{E}[w]$ with the space of all smooth functions $f : \mathcal{Q} \to \mathbb{R}$ such that $f(\rho^s(g_x)) = s^w f(g_x)$ for all $s \in \mathbb{R}_+$.

Although the bundle $\mathcal{E}[w]$ as we defined it depends on the choice of the conformal structure, it is naturally isomorphic to a density bundle (which is independent of the conformal structure). Recall that the bundle of α–densities is associated to the full linear frame bundle of M with respect to the 1–dimensional representation $A \mapsto |\det(A)|^{-\alpha}$ of the group $GL(n, \mathbb{R})$. In particular, 1–densities are exactly the geometric objects that may be integrated in a coordinate–independent way on non–orientable manifolds, while in the orientable case they may be canonically identified with n–forms.

To obtain the identification, recall that any pseudo–Riemannian metric g on M determines a nowhere vanishing 1–density, the volume density $\mathrm{vol}(g)$. In local coordinates, this density is given by $\sqrt{|\det(g_{ij})|}$, which immediately implies that for a positive function f we get $\mathrm{vol}(f^2 g) = f^n \, \mathrm{vol}(g)$.

Consequently, any 1–density φ determines a smooth function $\mathcal{Q} \to \mathbb{R}$ by mapping g_x to $\varphi(x)/\mathrm{vol}(g)(x)$ and obviously this function is homogeneous of degree $-n$. This gives an identification of the basic density bundle with $\mathcal{E}[-n]$ and thus an identification of $\mathcal{E}[w]$ with the bundle of $(-\frac{w}{n})$–densities on M. Hence if we have not fixed a conformal structure in the sequel, we will switch the point of view and consider $\mathcal{E}[w]$ as being defined as the bundle of $(-\frac{w}{n})$–densities and a choice of a conformal structure providing an identification of this density bundle with an associated bundle to $\pi : \mathcal{Q} \to M$.

We will follow the convention that adding the expression $[w]$ to the name of any bundle indicates the tensor product of that bundle with $\mathcal{E}[w]$, so for example $TM[-1] = TM \otimes \mathcal{E}[-1]$. Clearly, sections of such weighted tensor bundles may be viewed as equivariant sections of pullback bundles. For example, smooth sections of $TM[w]$ are in bijective correspondence with smooth sections ξ of $\pi^* TM$ such that $\xi(s^2 g_x) = s^w \xi(g_x)$. (Recall that the fibres of $\pi^* TM$ in g_x and $s^2 g_x$ may be canonically identified, so this equation makes sense.) In particular, we may consider the tautological inclusion of $\mathcal{Q}$ into $\pi^* S^2 T^* M$ as a canonical section of $S^2 T^* M[2]$ describing the conformal class, which gives another equivalent description of a conformal structure.

Of course, homogeneity along $\mathcal{Q}$ may as well be characterised infinitesimally. For this, let X be the fundamental vector field for the $\mathbb{R}_+$–action on $\mathcal{Q}$, i.e. $X(g_x) = \frac{d}{dt}|_{t=0} \rho^{e^t}(g_x) = \frac{d}{dt}|_{t=0}(e^{2t} g_x)$. For a function $f : \mathcal{Q} \to \mathbb{R}$ and $w \in \mathbb{R}$, the equation $f(s^2 g_x) = s^w f(x)$ is then clearly equivalent to $X \cdot f = wf$. Similarly, a tensor field t on $\mathcal{Q}$ is called *homogeneous of degree* $w \in \mathbb{R}$ if and only if $(\rho^s)^* t = s^w t$, which is equivalent to $\mathcal{L}_X t = wt$, where $\mathcal{L}$ denotes the Lie derivative. Using this, we may for example interpret the space of smooth sections of $TM[w]$ as the quotient of the space $\{\xi \in \mathfrak{X}(\mathcal{Q}) : [X, \xi] = w\xi\}$ by the subspace consisting of those elements whose values in each point are proportional to X.

2.2. **Tractor description of conformal structures.** It is a result that goes back to E. Cartan that conformal structures admit a canonical normal Cartan connection, see [3]. More precisely, consider $\mathbb{V} = \mathbb{R}^{n+2}$ equipped with a non–degenerate inner product $\langle \, , \, \rangle$ of signature $(p+1, q+1)$. Now put $G := O(\mathbb{V})$, the orthogonal group of $\mathbb{V}$, so $G \cong O(p+1, q+1)$. Furthermore, we define $P \subset G$ to be the stabiliser of a fixed null line in $\mathbb{V}$. It then turns out that $P \subset G$ is a parabolic subgroup, which may be nicely described explicitly, see e.g. [7]. The relation of this pair to conformal geometry can be described as follows: Let $\mathcal{C}$ be the cone of nonzero null vectors in $\mathbb{V}$ and let N be its image in the projectivisation of $\mathcal{P}(\mathbb{V}) \cong \mathbb{R}P^{n+1}$. Then it is easy to see that $\langle \, , \, \rangle$ induces a conformal structure of signature (p, q) on N and G acts transitively by conformal isometries. Moreover, it turns out that this conformal structure is flat and while G does not act effectively on N (essentially since id and $-$ id both act as the identity on projective space) it is a two–fold covering of the group of conformal automorphisms of N. Thus $N \cong G/P$ is the homogeneous flat model of conformal geometry.

A slight generalisation of Cartan's result may be expressed as follows. A choice of a conformal structure on a smooth manifold M gives rise to a canonical principal P–bundle $\mathcal{G} \to M$ which is endowed with a uniquely determined Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, where $\mathfrak{g} = \mathfrak{o}(\mathbb{V})$ is the Lie algebra of G and ω satisfies a normalisation condition to be discussed below. To see this in more detail (see e.g. [4, 2.2]) note that the parabolic subgroup $P \subset G$ is actually related to a grading of the Lie algebra $\mathfrak{g}$ of the form $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Denoting by $G_0 \subset P$ the subgroup of all elements whose adjoint action preserves this grading, then it is elementary to verify (as outlined in [4, 2.3]) that this group consists of $(1 + n + 1) \times (1 + n + 1)$–block matrices of the form $\begin{pmatrix} c & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & c^{-1} \end{pmatrix}$ with $0 \neq c \in \mathbb{R}$ and $C \in O(p, q)$. The action of such an element on $\mathfrak{g}_{-1} \cong \mathbb{R}^n$ is given by the standard action of $c^{-1}C$. Now one immediately verifies that $(c, C) \mapsto (c/|c|, c^{-1}C)$ induces an isomorphism $G_0 \to \mathbb{Z}_2 \times CO(p, q)$, where $CO(p, q)$ denotes the (pseudo–) conformal group. The inverse isomorphism is given by $(\varepsilon, A) \mapsto (\varepsilon| \det(A)|^{-1/n}, \varepsilon| \det(A)|^{-1/n}A)$. This isomorphism intertwines the adjoint action of G_0 on $\mathfrak{g}_{-1}$ with the product of the trivial action of $\mathbb{Z}_2$ and the standard action of $CO(p, q)$ on $\mathbb{R}^{p+q}$. In particular, this implies that a (first order) G_0–structure is the same thing as a $CO(p, q)$–structure and hence a conformal structure on the manifold. Now the procedure of [6] applies to produce a normal Cartan connection. (Of course in this special case there are much simpler direct constructions of the Cartan bundle and the normal Cartan connection.) The Cartan bundle and its normal Cartan connection are uniquely determined by the underlying conformal structure up to isomorphism.

While this Cartan connection is convenient from the point of view of the equivalence problem, it is rather difficult to use it for problems like finding invariants of conformal structures or conformally invariant differential operators. To deal with such problems, it is often more efficient to switch to the description of conformal structures via the so–called standard tractor bundle and its canonical linear connection. Starting from the Cartan bundle and the Cartan connection, the standard tractor bundle $\mathcal{T} \to M$ is simply the associated bundle $\mathcal{G} \times_P \mathbb{V}$. By construction, this bundle carries a canonical metric of signature $(p + 1, q + 1)$. The distinguished null line in $\mathbb{V}$ used to define P leads to a subbundle $\mathcal{T}^1 \subset \mathcal{T}$ whose fibres are null lines and which is easily seen to be isomorphic to $\mathcal{E}[-1]$. Furthermore, it turns out that the Cartan connection ω induces a linear connection ∇ on $\mathcal{T}$, the so–called normal standard tractor connection, see [5]. Having these data at hand, one may then compute the fundamental D–operator on $\mathcal{T}$, see [5, section 3], which in turn leads to the so called tractor D–operator, see [10] or [4, section 3]. These operators have been successfully applied to the construction of conformally invariant differential operators, conformal invariants and other topics in conformal geometry, see e.g. [1, 2, 11]. In summary, having an explicit knowledge of the standard tractor bundle, the tractor metric and the normal standard tractor connection, one immediately gets a large number of tools for dealing with problems in conformal geometry.

It is an idea going back to the work of T. Thomas in the 20's to use the standard tractor bundle and its canonical connection as an alternative approach to conformal geometry. The precise relation between these data and the Cartan bundle and Cartan connection was completely clarified (in a much more general setting) in [5]. Specialised to conformal standard tractors, this goes as follows: Suppose that M is a smooth manifold, and that $\mathcal{T} \to M$ is a real rank $n + 2$ vector bundle endowed with a bundle metric h of signature $(p + 1, q + 1)$ and an injective bundle map $\mathcal{E}[-1] \to \mathcal{T}$, whose image $\mathcal{T}^1$ is null with respect to h. Suppose further that the $\mathcal{T}$ admits a tractor connection ∇ in the sense of [5, 2.5]. By definition, this means that ∇ is a non–degenerate $\mathfrak{o}(\mathbb{V})$–connection. The condition that ∇ is a $\mathfrak{o}(\mathbb{V})$ connection is easily seen to be equivalent to ∇ preserving the bundle metric h. On the other hand, $\mathcal{T}$ has a filtration of the form $\mathcal{T} \supset \mathcal{T}^0 \supset \mathcal{T}^1$, where $\mathcal{T}^0 := (\mathcal{T}^1)^\perp$. This immediately implies that for connections preserving h the non–degeneracy condition from [5, 2.5] is equivalent to the condition that for any $x \in M$ and $\xi \in T_x M$ there is a smooth section σ of $\mathcal{T}^1$ such that $\nabla_\xi \sigma(x) \notin \mathcal{T}_x^1$.

Given the data $(\mathcal{T}, h, \nabla)$ as above, we can now recover an underlying conformal structure of signature (p, q) on M: First, let σ_0 be a locally non–vanishing section of $\mathcal{T}^1$. Then $h(\sigma_0, \sigma_0) = 0$ and thus $0 = \xi \cdot h(\sigma_0, \sigma_0) = 2h(\nabla_\xi \sigma_0, \sigma_0)$ for all $\xi \in \mathfrak{X}(M)$. This immediately implies that for any smooth function f we get $h(\nabla_\xi(f\sigma_0), \sigma_0) = 0$, and since locally any smooth section of $\mathcal{T}^1$ can be written in the form $f\sigma_0$, we conclude that $\nabla_\xi \sigma \in \Gamma(\mathcal{T}^0)$ for all $\xi \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(\mathcal{T}^1)$. Now consider the map which maps (ξ, σ) to the class of $\nabla_\xi \sigma$ in $\Gamma(\mathcal{T}^0/\mathcal{T}^1)$. This is obviously bilinear over smooth functions, and thus induced by a bundle map $TM \otimes \mathcal{E}[-1] \to \mathcal{T}^0/\mathcal{T}^1$, which by the non–degeneracy assumption is injective on each fibre, so since both bundles have rank n, we obtain a bundle isomorphism $\mathcal{T}^0/\mathcal{T}^1 \cong TM[-1]$. On the other hand, since the restriction of h to $\mathcal{T}^0$ is degenerate with null space $\mathcal{T}^1$, h induces a non–degenerate bundle metric of signature (p, q) on $\mathcal{T}^0/\mathcal{T}^1$, and thus gives rise to a section of $S^2 T^* M[2]$, i.e. a conformal structure on M. We say $(\mathcal{T}, h, \nabla)$ is a standard tractor bundle corresponding to this conformal structure. Conversely beginning with a conformal structure on M there are ways (see e.g. [1, 4]) to directly construct standard tractor bundles for the given conformal structure.

Next, one may recover the Cartan bundle from the standard tractor bundle: For $x \in M$ define $\mathcal{G}_x$ to be the set of all orthogonal maps $\mathbb{V} \to \mathcal{T}_x$ which in addition map the distinguished null line to $\mathcal{T}_x^1$. By assumption, such maps exist, and composition from the right defines a transitive free right action of P on $\mathcal{G}_x$. Now the union $\mathcal{G} := \cup_{x \in M} \mathcal{G}_x$ may be naturally viewed as a subbundle of the frame bundle of $\mathcal{T}$, whence it obtains its smooth structure and the P–action from above makes it into a P–principal bundle. By construction, we have $\mathcal{T} = \mathcal{G} \times_P \mathbb{V}$ and the metric h and the subbundle $\mathcal{T}^1$ are obtained by carrying over the respective data from $\mathbb{V}$. In the language of [5, section 2], this means that $\mathcal{T} \to M$ is a standard tractor bundle and $\mathcal{G}$ is an adapted frame bundle for $\mathcal{T}$. (The adjoint tractor bundle lurking in the background is the bundle $\mathfrak{o}(\mathcal{T})$ of skew symmetric endomorphisms of $\mathcal{T}$).

Now by [5, Theorem 2.7] there is a bijective correspondence between tractor connections on $\mathcal{T}$ and Cartan connections on the adapted frame bundle $\mathcal{G}$. To recognise the normal tractor connection among all tractor connections, one notes that by [5, Proposition 2.9], the curvature R of a tractor connection ∇ is given by the action of the curvature of the corresponding Cartan connection. Now in the special case of conformal structures, the general Lie theoretic normalisation condition on Cartan connections used in [5] can be simplified considerably. First of all, any normal Cartan connection in the conformal case is torsion free, which simply means that the action of $R(\xi, \eta)$ on the standard tractor bundle preserves the subbundle $\mathcal{T}^1 \subset \mathcal{T}$ for all ξ and η. If this condition is satisfied, then $R(\xi, \eta)$ induces an endomorphism $W(\xi, \eta)$ of $\mathcal{T}^0/\mathcal{T}^1 \cong TM[-1]$, so we may as well view W as a section of $\Lambda^2 T^*M \otimes L(TM, TM)$. Using this and taking into account the description of ∂^* in the proof of [5, Proposition 4.3] and the formula for the algebraic bracket in the conformal case in [4, 2.3] one concludes that the normalisation condition on the curvature of a standard tractor connection is equivalent to vanishing of the Ricci–type contraction of W. Of course, uniqueness of the Cartan bundle and Cartan connection implies that $\mathcal{T}$ together with the subbundle $\mathcal{T}^1$, the metric h and the normal standard tractor connection ∇ is uniquely determined by the underlying conformal structure up to isomorphism. Summarising, we obtain

Theorem. *(1) Let M be a smooth manifold of dimension $n \geq 3$. Suppose that $\mathcal{T} \to M$ is a rank $n+2$ real vector bundle endowed with a bundle metric h of signature $(p+1, q+1)$, an injective bundle map $\mathcal{E}[-1] \to \mathcal{T}$ with null image $\mathcal{T}^1 \subset \mathcal{T}$ and a linear connection ∇ such that $\nabla h = 0$ and, for any $x \in M$ and any $\xi \in T_x M$, there is a smooth section $\sigma \in \Gamma(\mathcal{T}^1)$ such that $\nabla_\xi \sigma(x) \notin \mathcal{T}_x^1$. Then $(\mathcal{T}^1)^\perp/\mathcal{T}^1 \cong TM[-1]$, and $(\mathcal{T}, h, \nabla)$ is a standard tractor bundle for the conformal structure defined by the restriction of h to $(\mathcal{T}^1)^\perp/\mathcal{T}^1 \times (\mathcal{T}^1)^\perp/\mathcal{T}^1$.*
(2) The tractor connection ∇ on $\mathcal{T}$ is normal if and only if its curvature R has the properties that $R(\xi, \eta)(\mathcal{T}^1) \subset \mathcal{T}^1$ and the Ricci–type contraction of the element $W \in \Gamma(\Lambda^2 T^*M \otimes L(TM, TM))$, as described above, vanishes. If this is the case, then $(\mathcal{T}, \mathcal{T}^1, h, \nabla)$ is uniquely determined by the underlying conformal structure up to isomorphism.

2.3. Ambient manifolds and ambient metrics.

In [8], Ch. Fefferman and C.R. Graham have initiated a project to study conformal structures using the so–called ambient metric construction. The idea of that construction is to mimic the flat metric on the vector space $\mathbb{V}$ in the case of the homogeneous model as described in 2.2 above. The null cone $\mathcal{C}$ may be viewed as the image of the inclusion $\mathcal{E}[-2] \to S^2 T^*M$ provided by the conformally flat structure, so half of this null cone may be identified with the bundle $\mathcal{Q}$ of metrics in the conformal class. Now one starts with the bundle $\mathcal{Q} \to M$ of metrics defining an arbitrary conformal structure. Then in [8] it is shown that there is a Riemannian metric of signature $(p+1, q+1)$ on $\mathcal{Q} \times (-1, 1)$ (defined locally around $\mathcal{Q}$) whose Ricci curvature vanishes to a certain order (depending on the dimension) along $\mathcal{Q}$. Moreover, the corresponding jet of this metric along $\mathcal{Q}$ is unique in a certain sense. This construction has been applied in [14] to prove the existence of certain conformally invariant powers

of the Laplacian. Using $\mathcal{Q} \times (-1, 1)$ is slightly misleading, one could equally consider a germ along $\mathcal{Q}$ of an (unspecified) ambient manifold endowed with a free $\mathbb{R}_+$–action. Moreover, as we shall see later on, a much weaker normalisation condition on an ambient metric than the one used in [8] is sufficient to get the relation to standard tractors. Thus, we will start our discussion with a general version of ambient manifolds and ambient metrics.

Note first, that on any manifold endowed with a free action of $\mathbb{R}_+$, one has the notion of homogeneity of tensor fields as described in 2.2 above, which can equivalently be characterised infinitesimally.

Definition. Let $\pi : \mathcal{Q} \to M$ be a conformal structure. An *ambient manifold* is a smooth $(n+2)$-manifold $\tilde{M}$ endowed with a free $\mathbb{R}_+$–action ρ on $\tilde{M}$ and a $\mathbb{R}_+$–equivariant embedding $\iota : \mathcal{Q} \to \tilde{M}$.

If $\iota : \mathcal{Q} \to \tilde{M}$ is an ambient manifold, then an *ambient metric* is a pseudo–Riemannian metric $\boldsymbol{h}$ of signature $(p+1, q+1)$ on $\tilde{M}$ such that the following condition hold:
(i) The metric $\boldsymbol{h}$ is homogeneous of degree 2 with respect to the $\mathbb{R}_+$–action, i.e. if $\boldsymbol{X} \in \mathfrak{X}(\tilde{M})$ denotes the fundamental field generating the $\mathbb{R}_+$–action and $\mathcal{L}_{\boldsymbol{X}}$ denotes the Lie derivative by $\boldsymbol{X}$, then we have $\mathcal{L}_{\boldsymbol{X}} \boldsymbol{h} = 2\boldsymbol{h}$.
(ii) For $u = g_x \in \mathcal{Q} \subset \tilde{M}$ and $\xi, \eta \in T_u\mathcal{Q}$, we have $\boldsymbol{h}(\xi, \eta) = g_x(T\pi \cdot \xi, T\pi \cdot \eta)$.

Since the action of $\mathbb{R}_+$ on $\tilde{M}$ extends the action on $\mathcal{Q}$, we will denote both actions by the symbol ρ and we use $\boldsymbol{X}$ to denote the fundamental field for both actions. Moreover, we will usually view $\mathcal{Q}$ as a submanifold of $\tilde{M}$ and suppress the embedding ι.

Since we will frequently have to deal with the question of vanishing of tensor fields along $\mathcal{Q}$ to some order, we collect some information on that. A tensor field t on $\tilde{M}$ is said to vanish along Q to order $\ell \geq 1$ if and only if $t|_Q = 0$ and for any integer $k < \ell$ and arbitrary vector fields $\xi_1, \ldots, \xi_k \in \mathfrak{X}(\tilde{M})$ the iterated Lie derivative $\mathcal{L}_{\xi_k} \cdots \mathcal{L}_{\xi_1} t$ vanishes along $\mathcal{Q}$. Equivalently, one may require all iterated covariant derivatives $\nabla_{\xi_k} \cdots \nabla_{\xi_1} t$ to vanish along $\mathcal{Q}$. The tensor field t is said to vanish to infinite order along $\mathcal{Q}$ if it vanishes to order ℓ for all $\ell \in \mathbb{N}$. If we choose any defining function r for $\mathcal{Q}$, i.e. a smooth real valued function defined locally around $\mathcal{Q}$ such that $\mathcal{Q} = r^{-1}(0)$ and dr does not vanish in any point of $\mathcal{Q}$, then any tensor field t that vanishes along $\mathcal{Q}$ may be written as $t = rt'$ for some tensor field t' of the same type as t. Inductively, one sees that t vanishes to order ℓ along $\mathcal{Q}$ if and only if $t = r^\ell t'$ for some tensor field t'. Thus, we will use the notation $t = O(r^\ell)$ to indicate that t vanishes to order ℓ along Q.

There are some points we should make that are particular to the case of sections of $\otimes^s T^*M$. This is especially relevant for the case of differential forms. On the one hand, in this case vanishing to order ℓ along $\mathcal{Q}$ can be equivalently expressed as vanishing of $\xi_k \cdots \xi_1 \cdot t(\eta_1, \ldots, \eta_s)$ for arbitrary $k < \ell$ and vector fields ξ_i and η_j. On the other hand, for tensor fields t of that type there is the weaker condition that *tangential components of* t vanish to some order along $\mathcal{Q}$. One says that the tangential components of t vanish along $\mathcal{Q}$ if $\iota^* t = 0$, where $\iota : \mathcal{Q} \to \tilde{M}$ is the inclusion. Equivalently, $t(\eta_1, \ldots, \eta_s)$ has to vanish along $\mathcal{Q}$ if for any $u \in Q$ and any j one has $\eta_j(u) \in T_u\mathcal{Q} \subset T_u\tilde{M}$. We say that the tangential components of t vanish

to order ℓ if and only if for $k < \ell$, arbitrary vector fields $\xi_1, \ldots, \xi_k \in \mathfrak{X}(\tilde{M})$ and vector fields $\eta_1, \ldots, \eta_s \in \mathfrak{X}(\tilde{M})$ such that each $\eta_j|_{\mathcal{Q}}$ is tangent to $\mathcal{Q}$, the function $\xi_k \cdots \xi_1 \cdot t(\eta_1, \ldots, \eta_s)$ vanishes along $\mathcal{Q}$. In this case, however, one may not replace this by a condition on Lie derivatives or covariant derivatives, since a Lie derivative or covariant derivative of a vector field whose restriction to $\mathcal{Q}$ is tangent to $\mathcal{Q}$ in general does not have the same property.

The normalisation conditions on ambient metrics used in [8] are based on the Ricci curvature of the ambient metric $\boldsymbol{h}$. However, to get the relation to standard tractors, we need a different condition. We shall show in 2.6 that this condition is a consequence of the weakest possible condition on the Ricci curvature. The condition we need is based on the one–form α dual to the generator $\boldsymbol{X}$ of the $\mathbb{R}_+$–action, i.e. $\alpha(\xi) = \boldsymbol{h}(\boldsymbol{X}, \xi)$. Notice that since $T\pi \cdot \boldsymbol{X} = 0$, condition (ii) in the definition of an ambient metric implies that $\iota^* \alpha = 0$. Thus, we also have $0 = d\iota^* \alpha = \iota^* d\alpha$, so the tangential components of $d\alpha$ vanish along $\mathcal{Q}$.

Expanding the homogeneity condition $\mathcal{L}_{\boldsymbol{X}} \boldsymbol{h} = 2\boldsymbol{h}$, we get $\boldsymbol{X} \cdot \boldsymbol{h}(\xi, \eta) - \boldsymbol{h}([\boldsymbol{X}, \xi], \eta) - \boldsymbol{h}(\xi, [\boldsymbol{X}, \eta]) = 2\boldsymbol{h}(\xi, \eta)$, and rewriting the Lie brackets in terms of covariant derivatives, we obtain

$$\boldsymbol{h}(\boldsymbol{\nabla}_\xi \boldsymbol{X}, \eta) + \boldsymbol{h}(\xi, \boldsymbol{\nabla}_\eta \boldsymbol{X}) = 2\boldsymbol{h}(\xi, \eta),$$

which says that $\boldsymbol{X}$ is a conformal Killing field of dilation type. On the other hand, by definition of the exterior derivative, we get $d\alpha(\xi, \eta) = \xi \cdot \boldsymbol{h}(\boldsymbol{X}, \eta) - \eta \cdot \boldsymbol{h}(\boldsymbol{X}, \xi) - \boldsymbol{h}(\boldsymbol{X}, [\xi, \eta])$, and expanding the right hand side of this in terms of covariant derivatives gives

$$\boldsymbol{h}(\boldsymbol{\nabla}_\xi \boldsymbol{X}, \eta) - \boldsymbol{h}(\xi, \boldsymbol{\nabla}_\eta \boldsymbol{X}) = d\alpha(\xi, \eta),$$

which just expresses the fact that the exterior derivative of a differential form is obtained by alternating the covariant derivative. Putting these two equations together, we obtain

$$(1) \qquad \boldsymbol{h}(\boldsymbol{\nabla}_\xi \boldsymbol{X}, \eta) = \boldsymbol{h}(\xi, \eta) + \frac{1}{2} d\alpha(\xi, \eta).$$

This equation shows that $d\alpha = O(r^\ell)$ implies $\boldsymbol{\nabla}_\xi \boldsymbol{X} = \xi + O(r^\ell)$ for any vector field ξ. On the other hand, if ξ is homogeneous of degree w, then $w\xi = [\boldsymbol{X}, \xi] = \boldsymbol{\nabla}_{\boldsymbol{X}} \xi - \boldsymbol{\nabla}_\xi \boldsymbol{X}$, which shows that $\boldsymbol{\nabla}_{\boldsymbol{X}} \xi = (w+1)\xi + O(r^\ell)$ provided that $d\alpha = O(r^\ell)$. Suppose next that $i_{\boldsymbol{X}} d\alpha|_{\mathcal{Q}} = 0$, where i denotes the insertion operator. Since α is obviously homogeneous of degree two, this implies that $2\alpha|_{\mathcal{Q}} = \mathcal{L}_{\boldsymbol{X}} \alpha|_{\mathcal{Q}} = di_{\boldsymbol{X}} \alpha|_{\mathcal{Q}}$. Thus we see that assuming $i_{\boldsymbol{X}} d\alpha|_{\mathcal{Q}} = 0$, the function $r := \frac{1}{2} \boldsymbol{h}(\boldsymbol{X}, \boldsymbol{X})$ satisfies $dr|_{\mathcal{Q}} = \alpha|_{\mathcal{Q}}$, so we get a canonical defining function in this case.

2.4. **The standard tractor bundle and connection induced by an ambient metric.** Let $\pi : \mathcal{Q} \to M$ be a conformal structure, $\tilde{M} \supset \mathcal{Q}$ an ambient manifold and $\boldsymbol{h}$ and ambient metric on $\tilde{M}$. Throughout this subsection we assume that $\boldsymbol{h}$ has the property that the one–form α dual to $\boldsymbol{X}$ satisfies $d\alpha|_Q = 0$.

Consider the restriction $T\tilde{M}|_{\mathcal{Q}}$ of the ambient tangent bundle to $\mathcal{Q}$ and define an action of $\mathbb{R}_+$ on this space by $s \cdot \xi := s^{-1} T\rho^s \cdot \xi$. This is compatible with the $\mathbb{R}_+$ action on $\mathcal{Q}$, so defining $\mathcal{T}$ to be the quotient $(T\tilde{M}|_{\mathcal{Q}})/\mathbb{R}_+$,

we immediately see that this is a vector bundle over $\mathcal{Q}/\mathbb{R}_+ = M$, and the fibre dimension of this bundle is $n + 2$. Moreover, by construction, there is a bijective correspondence between the space $\Gamma(\mathcal{T})$ of smooth sections of $p : \mathcal{T} \to M$ and the space of ambient vector fields along $\mathcal{Q}$ (i.e. sections of $T\tilde{M}|_{\mathcal{Q}} \to \mathcal{Q}$) which are homogeneous of degree -1, or equivalently satisfy $[\boldsymbol{X}, \xi] = -\xi$.

The fact that the ambient metric $\boldsymbol{h}$ is homogeneous of degree 2 immediately implies that for vector fields ξ and η on $\tilde{M}$ which are homogeneous of degree w and w', respectively, the function $\boldsymbol{h}(\xi, \eta)$ is homogeneous of degree $w + w' + 2$. In particular, applying $\boldsymbol{h}$ to the vector fields corresponding to two sections of $\mathcal{T}$, the resulting function on $\mathcal{Q}$ is constant on $\mathbb{R}_+$ orbits, and thus descends to a smooth function on M. Hence $\boldsymbol{h}$ descends to a smooth bundle metric h of signature $(p + 1, q + 1)$ on $\mathcal{T}$.

The bundle metric h immediately leads to a filtration of the bundle $\mathcal{T}$: Since the vertical tangent bundle of $\pi : \mathcal{Q} \to M$ is stable under the $\mathbb{R}_+$–action, it gives rise to a distinguished line bundle $\mathcal{T}^1 \subset \mathcal{T}$. By construction, sections of this subbundle correspond to ambient vector fields along $\mathcal{Q}$, which are of the form $f\boldsymbol{X}$ for some smooth function $f : \mathcal{Q} \to \mathbb{R}$, and in order that this is a section of $\mathcal{T}$, the function f must be homogeneous of degree -1. Thus, mapping f to $f\boldsymbol{X}$ defines an isomorphism $\mathcal{E}[-1] \cong \mathcal{T}^1$. On the other hand, we have already observed that $\boldsymbol{h}(\boldsymbol{X}, \boldsymbol{X}) = 0$ along $\mathcal{Q}$. Hence, defining $\mathcal{T}^0$ to be the orthogonal complement of $\mathcal{T}^1$ with respect to $\boldsymbol{h}$, we see that $\mathcal{T}^0 \subset \mathcal{T}$ is a smooth subbundle of rank $n + 1$ and $\mathcal{T}^1 \subset \mathcal{T}^0$. To identify the quotient $\mathcal{T}/\mathcal{T}^0$, we observe that for any section $s \in \Gamma(\mathcal{T})$ with corresponding vector field ξ along $\mathcal{Q}$, we get a function $\boldsymbol{h}(\xi, \boldsymbol{X})$, which is homogeneous of degree one. By construction, this vanishes if and only if s has values in $\mathcal{T}^0$, so it induces an isomorphism $\mathcal{T}/\mathcal{T}^0 \cong \mathcal{E}[1]$ of vector bundles.

Finally, assume that ξ is a vector field on M and $f \in \Gamma(\mathcal{E}[-1])$ is a smooth section, i.e. a function $\mathcal{Q} \to \mathbb{R}$ homogeneous of degree -1. Then we may lift ξ to an ambient vector field $\tilde{\xi}$ along $\mathcal{Q}$, which is homogeneous of degree zero, and this lift is unique up to adding fields of the form $\varphi\boldsymbol{X}$ with $\varphi : \mathcal{Q} \to \mathbb{R}$ homogeneous of degree zero. Then $f\tilde{\xi}$ is a section of $\mathcal{T}$ and by property (ii) of $\boldsymbol{h}$ we have $\boldsymbol{h}(f\tilde{\xi}, \boldsymbol{X}) = 0$, whence $f\tilde{\xi} \in \Gamma(\mathcal{T}^0)$. Moreover, the class of $f\tilde{\xi}$ in $\mathcal{T}^0/\mathcal{T}^1$ is independent of the choice of the lift $\tilde{\xi}$. Hence we obtain a bundle map $TM[-1] \to \mathcal{T}^0/\mathcal{T}^1$, which is obviously injective in each fibre, so since both bundles have the same rank, we conclude that $\mathcal{T}^0/\mathcal{T}^1 \cong TM[-1]$, which implies that $p : \mathcal{T} \to M$ is a candidate for a conformal standard tractor bundle. Notice that up to now we have not used the assumption that $d\alpha$ vanishes along $\mathcal{Q}$.

Next, let $\boldsymbol{\nabla}$ be the Levi–Civita connection of $\boldsymbol{h}$. The fact that $\boldsymbol{\nabla}$ is torsion free and $\boldsymbol{\nabla}\boldsymbol{h} = 0$ imply the well know global formula

$$2\boldsymbol{h}(\boldsymbol{\nabla}_\xi \eta, \zeta) = \xi \cdot \boldsymbol{h}(\eta, \zeta) + \eta \cdot \boldsymbol{h}(\xi, \zeta) - \zeta \cdot \boldsymbol{h}(\xi, \eta)$$
$$+ \boldsymbol{h}([\xi, \eta], \zeta) - \boldsymbol{h}([\xi, \zeta], \eta) - \boldsymbol{h}([\eta, \zeta], \xi)$$

for all vector fields $\xi, \eta, \zeta \in \mathfrak{X}(\tilde{M})$. Observe that if $\xi \in \mathfrak{X}(\tilde{M})$ is homogeneous of degree w and $f : \tilde{M} \to \mathbb{R}$ is homogeneous of degree w', then the equation $\boldsymbol{X} \cdot \xi \cdot f = [\boldsymbol{X}, \xi] \cdot f + \xi \cdot \boldsymbol{X} \cdot f$ shows that the function $\xi \cdot f$ is homogeneous of degree $w + w'$. Hence choosing the three vector fields in the

above formula to be homogeneous of degrees w, w', and w'', we see that any summand on the right hand side is homogeneous of degree $w + w' + w'' + 2$, which immediately implies that $\boldsymbol{\nabla}_\xi \eta$ is homogeneous of degree $w + w'$. In particular, if ξ is invariant, i.e. homogeneous of degree zero, then $\boldsymbol{\nabla}_\xi \eta$ has the same homogeneity as η.

On the other hand, we have already observed in the end of 2.3 above, that $d\alpha|_Q = 0$, i.e. $d\alpha = O(r)$ implies that $\boldsymbol{\nabla}_\xi \boldsymbol{X} = \xi + O(r)$, and $\boldsymbol{\nabla}_{\boldsymbol{X}} \xi = (w + 1)\xi + O(r)$ if ξ is homogeneous of degree w. In particular, $\boldsymbol{\nabla}_{\boldsymbol{X}} \xi|_Q = 0$ for ξ homogeneous of degree -1. Using this, we can now show that $\boldsymbol{\nabla}$ descends to a linear connection ∇ on the bundle $\mathcal{T}$. Suppose that $s \in \Gamma(\mathcal{T})$ is a section corresponding to the ambient vector field $\tilde{\eta}$ along Q and that $\xi \in \mathfrak{X}(M)$ is a vector field. As before, we may lift ξ to an ambient vector field $\tilde{\xi}$ along Q, which is unique up to adding terms of the form $\varphi \boldsymbol{X}$ with φ homogeneous of degree zero. Extend $\tilde{\eta}$ to a homogeneous field on $\tilde{M}$ and observe that since the flow lines of $\tilde{\xi}$ are contained in Q, it follows that, along Q, the ambient vector field $\boldsymbol{\nabla}_{\tilde{\xi}} \tilde{\eta}$ is independent of the extension of $\tilde{\eta}$. Since $\boldsymbol{\nabla}_{\boldsymbol{X}} \tilde{\eta}|_Q = 0$, we conclude that $\boldsymbol{\nabla}_{\tilde{\xi}} \tilde{\eta}$ depends only on ξ and not on the lift $\tilde{\xi}$. Moreover, from above we know that $\boldsymbol{\nabla}_{\tilde{\xi}} \tilde{\eta}$ is homogeneous of degree -1, so it corresponds to a section of $\mathcal{T}$, which we denote by $\nabla_\xi s$. One immediately verifies that this defines a linear connection ∇ on $\mathcal{T}$, which by construction is compatible with the bundle metric h.

To verify that ∇ is a tractor connection on $\mathcal{T}$, we thus only have to verify the non–degeneracy condition from 2.2, which is very easy: For a section $s \in \Gamma(\mathcal{T}^1)$, the corresponding ambient field is of the form $f \boldsymbol{X}$ with $f : Q \to \mathbb{R}$ homogeneous of degree -1. For a lift $\tilde{\xi}$ of a vector field $\xi \in \mathfrak{X}(M)$ as above, we get $\boldsymbol{\nabla}_{\tilde{\xi}} f \boldsymbol{X} = (\tilde{\xi} \cdot f) \boldsymbol{X} + f \boldsymbol{\nabla}_{\tilde{\xi}} \boldsymbol{X}$ and the second summand equals $f \tilde{\xi}$ along Q. In particular, we see that $\nabla_\xi s \in \Gamma(\mathcal{T}^0)$, and the image of this section in $\Gamma(\mathcal{T}^0/\mathcal{T}^1)$ is simply the element $f\xi$, which implies that ∇ is a tractor connection on $\mathcal{T}$. Thus we have proved:

Proposition. *Let $\pi : Q \to M$ be a conformal structure on a smooth manifold M, $\tilde{M}$ an ambient manifold and $\boldsymbol{h}$ an ambient metric, and let $\alpha \in \Omega^1(\tilde{M})$ be the one–form dual to the infinitesimal generator of the $\mathbb{R}_+$– action on $\tilde{M}$. Then for the $\mathbb{R}_+$ action on $T\tilde{M}|_Q$ defined above, $\boldsymbol{h}$ descends to a bundle metric h on $\mathcal{T} := (T\tilde{M}|_Q)/\mathbb{R}_+$. If $d\alpha|_Q = 0$, then the Levi–Civita connection of $\boldsymbol{h}$ descends to a tractor connection on $\mathcal{T}$ which preserves h. This together with the filtration induced by the vertical subbundle means $\mathcal{T}$ is a conformal standard tractor bundle.*

2.5. **The normalisation condition.** Let us assume that $\boldsymbol{h}$ is an ambient metric on an ambient manifold $\tilde{M}$ for a given conformal structure on M such that the one–form α dual to the infinitesimal generator $\boldsymbol{X}$ of the $\mathbb{R}_+$– action on $\tilde{M}$ has the property that $d\alpha|_Q = 0$. Then we have the induced conformal standard tractor bundle $(\mathcal{T}, h, \nabla)$. Now it is almost obvious that the curvature $\boldsymbol{R}$ of $\boldsymbol{h}$ descends to curvature R of the tractor connection ∇. Indeed, choosing invariant lifts $\tilde{\xi}$ and $\tilde{\eta}$ for vector fields $\xi, \eta \in \mathfrak{X}(M)$, and considering the (homogeneous of degree -1) ambient vector field ζ along Q corresponding to a section $s \in \Gamma(\mathcal{T})$, the ambient vector field $\boldsymbol{\nabla}_\xi \boldsymbol{\nabla}_\eta \zeta$

corresponds to the section $\nabla_\xi \nabla_\eta s \in \Gamma(\mathcal{T})$. Moreover, $[\tilde{\xi}, \tilde{\eta}]$ is an invariant lift of $[\xi, \eta]$, which immediately implies that $\boldsymbol{R}(\tilde{\xi}, \tilde{\eta})\zeta$ corresponds to the section $R(\xi, \eta)s$ of $\mathcal{T}$.

Hence to understand the tractor curvature R, we have to analyse the ambient curvature $\boldsymbol{R}$. For later use, we work in a more general setting and prove more specific results than are required for the verification of the normalisation condition.

Proposition. *Let $\boldsymbol{h}$ be an ambient metric on $\tilde{M}$, $\boldsymbol{R}$ the Riemann curvature of $\boldsymbol{h}$, $\boldsymbol{X} \in \mathfrak{X}(\tilde{M})$ the infinitesimal generator of the $\mathbb{R}_+$–action on $\tilde{M}$ and $\alpha \in \Omega^1(\tilde{M})$ the one–form dual to $\boldsymbol{X}$. Then we have:*

$$\boldsymbol{h}(\boldsymbol{R}(\xi, \eta)\boldsymbol{X}, \zeta) = -\boldsymbol{h}(\boldsymbol{R}(\xi, \eta)\zeta, \boldsymbol{X}) = \boldsymbol{h}(\boldsymbol{R}(\boldsymbol{X}, \zeta)\xi, \eta) = -\tfrac{1}{2}(\boldsymbol{\nabla}\, d\alpha)(\zeta, \xi, \eta),$$

for all vector fields $\xi, \eta, \zeta \in \mathfrak{X}(\tilde{M})$. In particular, if $d\alpha = O(r^\ell)$ for some $\ell \geq 1$, then this expression is $O(r^{\ell-1})$ and it is $O(r^\ell)$ if either ζ or both ξ and η have the property that the restriction to $\mathcal{Q}$ is tangent to $\mathcal{Q}$. Hence, if $d\alpha = O(r^\ell)$, then tangential components of $d\alpha$ vanish to order $\ell + 1$ along $\mathcal{Q}$.

Proof. The equality of the first three expressions follows from standard symmetries of the curvature of a pseudo–Riemannian metric. Now we compute

$$\boldsymbol{h}(\boldsymbol{\nabla}_\xi \boldsymbol{\nabla}_\eta \boldsymbol{X}, \zeta) = \xi \cdot \boldsymbol{h}(\boldsymbol{\nabla}_\eta \boldsymbol{X}, \zeta) - \boldsymbol{h}(\boldsymbol{\nabla}_\eta \boldsymbol{X}, \boldsymbol{\nabla}_\xi \zeta),$$

and inserting formula (1) from 2.3 in both summands, we obtain

$$\boldsymbol{h}(\boldsymbol{\nabla}_\xi \eta, \zeta) + \tfrac{1}{2}\xi \cdot d\alpha(\eta, \zeta) - \tfrac{1}{2}d\alpha(\eta, \boldsymbol{\nabla}_\xi \zeta).$$

Taking the alternation of this in ξ and η and subtracting $\boldsymbol{h}(\boldsymbol{\nabla}_{[\xi,\eta]}\boldsymbol{X}, \zeta) = \boldsymbol{h}([\xi, \eta], \zeta) + \tfrac{1}{2}d\alpha([\xi, \eta], \zeta)$ we may expand the Lie bracket into covariant derivatives which implies that all terms involving $\boldsymbol{h}$ cancel, and we are left with the expression

$$\tfrac{1}{2}\left(\xi \cdot d\alpha(\eta, \zeta) - \eta \cdot d\alpha(\xi, \zeta) - d\alpha([\xi, \eta], \zeta) - d\alpha(\eta, \boldsymbol{\nabla}_\xi \zeta) + d\alpha(\xi, \boldsymbol{\nabla}_\eta \zeta)\right)$$

for $\boldsymbol{h}(\boldsymbol{R}(\xi, \eta)\boldsymbol{X}, \zeta)$. Expanding the equation $0 = d(d\alpha)(\xi, \eta, \zeta)$ we may rewrite $\xi \cdot d\alpha(\eta, \zeta) - \eta \cdot d\alpha(\xi, \zeta) - d\alpha([\xi, \eta], \zeta)$ as $-\zeta \cdot d\alpha(\xi, \eta) - d\alpha([\xi, \zeta], \eta) + d\alpha([\eta, \zeta], \xi)$, and expressing the Lie brackets as commutators of covariant derivatives, we arrive at the claimed formula for $\boldsymbol{h}(\boldsymbol{R}(\xi, \eta)\boldsymbol{X}, \zeta)$.

By definition, $(\boldsymbol{\nabla}\, d\alpha)(\zeta, \xi, \eta) = \zeta \cdot d\alpha(\xi, \eta) - d\alpha(\boldsymbol{\nabla}_\zeta \xi, \eta) - d\alpha(\xi, \boldsymbol{\nabla}_\zeta \eta)$, and if $d\alpha = O(r^\ell)$, the the last two terms visibly are $O(r^\ell)$, while the first is $O(r^{\ell-1})$. If in addition $\zeta|_\mathcal{Q}$ is tangent to $\mathcal{Q}$, the equation $\zeta' \cdot \zeta \cdot d\alpha(\xi, \eta) = \zeta \cdot \zeta' \cdot d\alpha(\xi, \eta) + [\zeta', \zeta] \cdot d\alpha(\xi, \eta)$ shows that the first summand is $O(r^\ell)$, too. On the other hand, if both $\xi|_\mathcal{Q}$ and $\eta|_\mathcal{Q}$ are tangent to $\mathcal{Q}$, then by the Bianchi identity, we get $\boldsymbol{h}(R(\xi, \eta)\zeta, \boldsymbol{X}) = -\boldsymbol{h}(R(\zeta, \xi)\eta, \boldsymbol{X}) - \boldsymbol{h}(R(\eta, \zeta)\xi, \boldsymbol{X})$, and from above we know that both terms of the right hand side are $O(r^\ell)$. Turning around the argument, we see now that if $\xi|_\mathcal{Q}$ and $\eta|_\mathcal{Q}$ are tangent to $\mathcal{Q}$, then for any vector field ζ, the function $\zeta \cdot d\alpha(\xi, \eta)$ is $O(r^\ell)$, whence tangential components of $d\alpha$ vanish to order $\ell + 1$ along $\mathcal{Q}$. $\square$

Using these results we can now prove:

Theorem. *Let $\boldsymbol{h}$ be an ambient metric such that the one–form α dual to the infinitesimal generator of the $\mathbb{R}_+$–action satisfies $d\alpha|_{\mathcal{Q}} = 0$. Then the standard tractor bundle $(\mathcal{T}, h, \nabla)$ induced by $\boldsymbol{h}$ is normal if and only if the tangential components of the Ricci curvature $\mathrm{Ric}(\boldsymbol{h})$ vanish along $\mathcal{Q}$.*

Proof. From the above Proposition we see that $d\alpha = O(r)$ implies that if $\tilde{\xi}|_{\mathcal{Q}}$ and $\tilde{\eta}|_{\mathcal{Q}}$ are tangent to $\mathcal{Q}$, then $\boldsymbol{h}(\boldsymbol{R}(\tilde{\xi},\tilde{\eta})\boldsymbol{X}, \zeta)$ and $\boldsymbol{h}(\boldsymbol{R}(\tilde{\xi},\tilde{\eta})\zeta, \boldsymbol{X})$ vanish along $\mathcal{Q}$. Applied to invariant lifts of vector fields $\xi, \eta \in \mathfrak{X}(M)$, the first equation exactly means that $R(\xi,\eta)$ vanishes on $\mathcal{T}^1$, while the second equation says that $R(\xi,\eta)$ maps $\mathcal{T}$ to $\mathcal{T}^0$. In particular, $R(\xi,\eta) : \mathcal{T} \to \mathcal{T}$ is filtration preserving for all ξ,η.

Hence $\boldsymbol{R}(\xi,\eta)$ induces an endomorphism of $\mathcal{T}^0/\mathcal{T}^1$, which we may as well view as an endomorphism of TM. From 2.2 we know that normality of the tractor connection is equivalent to vanishing of the Ricci–type contraction of the resulting operator $W \in \Gamma(\Lambda^2 T^*M \otimes L(TM, TM))$. The value of this contraction at a point $x \in M$ on tangent vectors $\xi, \eta \in T_x M$ can be computed as $\sum_{i=1}^n \varphi_i(W(\xi_i, \xi)\eta)$, for a basis $\{\xi_1, \ldots, \xi_n\}$ of $T_x M$ with dual basis $\{\varphi_1, \ldots, \varphi_n\}$ of $T_x^* M$.

Choosing a point u in $\mathcal{Q}$ over x and lifts $\tilde{\xi}_i$, $\tilde{\xi}$ and $\tilde{\eta}$ of the vector fields involved, we may compute φ_i as $\boldsymbol{h}(\_, \tilde{\eta}_i)$, where the tangent vectors $\tilde{\eta}_1, \ldots, \tilde{\eta}_n \in T_u \mathcal{Q} \subset T_u \tilde{M}$ are defined by $\boldsymbol{h}(\tilde{\eta}_i, \tilde{\xi}_j) = \delta_{ij}$. (Note that $\tilde{\eta}_i \in T_u \mathcal{Q}$ implies $\boldsymbol{h}(\tilde{\eta}_i, \boldsymbol{X}) = 0$). Hence our contraction applied to ξ and η corresponds to

$$\sum_{i=1}^n \boldsymbol{h}\big(\boldsymbol{R}(\tilde{\xi}_i, \tilde{\xi})\tilde{\eta}, \eta_i\big).$$

Now let $Y \in T_u \mathcal{Q}$ be the unique null tangent vector such that $\bar{h}(\boldsymbol{X}, Y) = 1$ and $\boldsymbol{h}(\tilde{\xi}_i, Y) = 0$ for all i. Then clearly $\{\boldsymbol{X}, \tilde{\xi}_1, \ldots, \tilde{\xi}_n, Y\}$ is a basis of $T_u \tilde{M}$ with dual basis (with respect to $\boldsymbol{h}$) given by $\{Y, \tilde{\eta}_1, \ldots, \tilde{\eta}_n, \boldsymbol{X}\}$. But from the above proposition, we know that $\boldsymbol{h}\big(\boldsymbol{R}(\boldsymbol{X}, \tilde{\xi})\tilde{\eta}, Y\big)$ and $\boldsymbol{h}\big(\boldsymbol{R}(Y, \tilde{\xi})\tilde{\eta}, \boldsymbol{X}\big)$ vanish along $\mathcal{Q}$, since the restrictions of $\tilde{\xi}$ and $\tilde{\eta}$ to $\mathcal{Q}$ are tangent to $\mathcal{Q}$. Adding these two summands to the above sum, we by definition get $\mathrm{Ric}(\boldsymbol{h})(\tilde{\xi}, \tilde{\eta})(u)$, where Ric denotes the Ricci curvature of $\boldsymbol{h}$, which implies the result. $\square$

Note that it follows immediately from the theorem that the normal tractor curvature R is induced by the curvature of any ambient metric $\boldsymbol{h}$ which has the property that $d\alpha$ and tangential components of $\mathrm{Ric}(\boldsymbol{h})$ vanish along $\mathcal{Q}$.

2.6. We next want to show that an ambient metric $\boldsymbol{h}$, such that the tangential components of $\mathrm{Ric}(\boldsymbol{h})$ vanish along $\mathcal{Q}$, automatically satisfies $d\alpha|_{\mathcal{Q}} = 0$, where α is the one–form dual to the infinitesimal generator $\boldsymbol{X}$ of the $\mathbb{R}_+$–action. In particular, this implies that any ambient metric satisfying the Ricci condition can be used to construct the normal standard tractor bundle.

Let us again start with an arbitrary ambient metric $\boldsymbol{h}$ on an ambient manifold $\tilde{M}$ for $\pi : \mathcal{Q} \to M$. Let us first choose appropriate dual frames defined locally around a point in $\mathcal{Q}$. Note that along $\mathcal{Q}$, the tangent spaces of $\mathcal{Q}$ are orthogonal to $\boldsymbol{X}$. Thus, locally around a point $u_0 \in \mathcal{Q}$, we may choose ambient vector fields $\xi_i \in \mathfrak{X}(\tilde{M})$ for $i = 1, \ldots, n$, which are homogeneous of degree -1, satisfy $\boldsymbol{h}(\boldsymbol{X}, \xi_i) = 0$ and have the property that

$\{\boldsymbol{X}(u), \xi_1(u), \ldots, \xi_n(u)\}$ is a basis for $T_u\mathcal{Q} \subset T_u\tilde{M}$ for $u \in \mathcal{Q}$ close to u_0. Further, choose an ambient vector field Y such that $\boldsymbol{h}(\boldsymbol{X}, Y) = 1$ (which forces Y to be homogeneous of degree -2) and $\boldsymbol{h}(Y, \xi_i)|_{\mathcal{Q}} = 0$ for all i. Adding an appropriate multiple of $\boldsymbol{X}$, we may assume that $\boldsymbol{h}(Y, Y)|_{\mathcal{Q}} = 0$. Clearly, the fields $\boldsymbol{X}$, ξ_i and Y form a frame for $T\tilde{M}$ in a neighbourhood of u_0 in $\mathcal{Q}$, and thus locally around u_0. Let $\{\tilde{Y}, \eta_i, \tilde{X}\}$ be the local frame dual to $\{\boldsymbol{X}, \xi_i, Y\}$. Then by construction $\tilde{Y}|_{\mathcal{Q}} = Y|_{\mathcal{Q}}$ and $\tilde{X}|_{\mathcal{Q}} = \boldsymbol{X}|_{\mathcal{Q}}$, but this is not true off $\mathcal{Q}$, since as we shall see immediately the weakest possible assumption on Ricci flatness implies that $\boldsymbol{h}(\boldsymbol{X}, \boldsymbol{X})$ is nonzero off $\mathcal{Q}$.

Theorem. *Let $\boldsymbol{h}$ be an ambient metric and let α be the one–form dual to the infinitesimal generator $\boldsymbol{X}$ of the $\mathbb{R}_+$–action. Then we have:*
(1) $\mathrm{Ric}(\boldsymbol{h})(\boldsymbol{X}, \boldsymbol{X})|_{\mathcal{Q}} = 0$ if and only if $i_{\boldsymbol{X}} d\alpha|_{\mathcal{Q}} = 0$. If this is the case, then $r := \frac{1}{2}\boldsymbol{h}(\boldsymbol{X}, \boldsymbol{X})$ is a defining function for $\mathcal{Q}$ such that $Y \cdot r = 1 + O(r)$.
(2) The tangential components of $\mathrm{Ric}(\boldsymbol{h})(\boldsymbol{X}, \_)$ vanish along $\mathcal{Q}$ if and only if $d\alpha|_{\mathcal{Q}} = 0$. In particular, for any ambient metric $\boldsymbol{h}$ such that the tangential components of $\mathrm{Ric}(\boldsymbol{h})$ vanish along $\mathcal{Q}$, the procedure from 2.4 can be used to obtain a normal standard tractor bundle.

Proof. From Proposition 2.5 we get $\boldsymbol{h}(\boldsymbol{R}(\boldsymbol{X}, \zeta)\xi, \eta) = -\frac{1}{2}(\boldsymbol{\nabla} d\alpha)(\zeta, \xi, \eta)$. Thus, we may compute $2\,\mathrm{Ric}(\boldsymbol{h})(\boldsymbol{X}, \xi)$ by taking the trace over ζ and η in $(\boldsymbol{\nabla} d\alpha)(\zeta, \xi, \eta)$, i.e. by inserting the elements of dual frames and summing up, and we use dual frames as introduced above. We only have to consider the case that $\xi|_{\mathcal{Q}}$ is tangent to $\mathcal{Q}$, and since we are only interested in the restriction of the result to $\mathcal{Q}$, we may as well replace $\tilde{X}$ by $\boldsymbol{X}$ and $\tilde{Y}$ by Y. The term with $\zeta = \boldsymbol{X}$ and $\eta = Y$ never contributes since $(\boldsymbol{\nabla} d\alpha)(\boldsymbol{X}, \xi, Y) = -2\boldsymbol{h}(\boldsymbol{R}(\boldsymbol{X}, \boldsymbol{X})\xi, Y) = 0$ by Proposition 2.5.

Let us next look at the terms with $\zeta = \xi_i$ and $\eta = \eta_i$. By definition tangential components of α vanish along $\mathcal{Q}$, so the same holds for $d\alpha$. In particular, $d\alpha(\xi, \eta_i)$ vanishes along $\mathcal{Q}$ and since ξ_i is tangent to $\mathcal{Q}$ also $\xi_i \cdot d\alpha(\xi, \eta_i)$ vanishes along $\mathcal{Q}$. Moreover, for any vector field ζ, the restriction of $\zeta - \alpha(\zeta)Y$ to $\mathcal{Q}$ is tangent to $\mathcal{Q}$, which implies $d\alpha(\xi, \zeta)|_{\mathcal{Q}} = \alpha(\zeta)d\alpha(\xi, Y)|_{\mathcal{Q}}$. Applying the same argument with ξ replaced by η_i, we see that $d\alpha(\boldsymbol{\nabla}_{\xi_i}\xi, \eta_i)|_{\mathcal{Q}} = \alpha(\boldsymbol{\nabla}_{\xi_i}\xi)d\alpha(Y, \eta_i)|_{\mathcal{Q}}$. Since $\xi|_{\mathcal{Q}}$ is tangent to $\mathcal{Q}$ and thus $\boldsymbol{h}(\xi, \boldsymbol{X})|_{\mathcal{Q}} = 0$, we see that, along $\mathcal{Q}$, we have $\alpha(\boldsymbol{\nabla}_{\xi_i}\xi) = \boldsymbol{h}(\boldsymbol{\nabla}_{\xi_i}\xi, \boldsymbol{X}) = -\boldsymbol{h}(\xi, \boldsymbol{\nabla}_{\xi_i}\boldsymbol{X})$. Since both $\xi|_{\mathcal{Q}}$ and $\xi_i|_{\mathcal{Q}}$ are tangent to $\mathcal{Q}$ and tangential components of $d\alpha$ vanish, formula (1) from 2.3 implies that this restricts to $-\boldsymbol{h}(\xi, \xi_i)$ on $\mathcal{Q}$, so

$$d\alpha(\boldsymbol{\nabla}_{\xi_i}\xi, \eta_i)|_{\mathcal{Q}} = \boldsymbol{h}(\xi, \xi_i)d\alpha(\eta_i, Y).$$

Similarly, $d\alpha(\xi, \boldsymbol{\nabla}_{\xi_i}\eta_i) = \alpha(\boldsymbol{\nabla}_{\xi_i}\eta_i)d\alpha(\xi, Y)$, and $\alpha(\boldsymbol{\nabla}_{\xi_i}\eta_i)|_{\mathcal{Q}} = -\boldsymbol{h}(\eta_i, \xi_i) = -1$. Together with the above, this implies that for any vector field ξ such that $\xi|_{\mathcal{Q}}$ is tangent to $\mathcal{Q}$, we obtain

$$2\,\mathrm{Ric}(\boldsymbol{h})(\boldsymbol{X}, \xi)|_{\mathcal{Q}} = (\boldsymbol{\nabla} d\alpha)(Y, \xi, \boldsymbol{X})|_{\mathcal{Q}} + n\,d\alpha(\xi, Y)|_{\mathcal{Q}} - \sum_i \boldsymbol{h}(\xi, \xi_i)d\alpha(\eta_i, Y)|_{\mathcal{Q}}$$

(1) Putting $\xi = \boldsymbol{X}$ in the above formula, we see that the first summand vanishes since $\boldsymbol{\nabla} d\alpha$ is skew symmetric in the last two entries. On the other hand, the last sum vanishes since $\boldsymbol{h}(\boldsymbol{X}, \xi_i) = 0$ by construction. Thus, we are left with $2\,\mathrm{Ric}(\boldsymbol{h})(\boldsymbol{X}, \boldsymbol{X})|_{\mathcal{Q}} = n\,d\alpha(\boldsymbol{X}, Y)|_{\mathcal{Q}}$, and since tangential components of $d\alpha$ vanish, the vanishing of $d\alpha(\boldsymbol{X}, Y)|_{\mathcal{Q}}$ is equivalent to

$i_{\boldsymbol{X}}d\alpha|_{\mathcal{Q}} = 0$. We have already verified in 2.3 that the latter condition implies that $r = \frac{1}{2}\boldsymbol{h}(\boldsymbol{X},\boldsymbol{X})$ is a defining function for $\mathcal{Q}$ since $dr|_{\mathcal{Q}} = \alpha|_{\mathcal{Q}}$. The last statement obviously implies $Y\cdot r = 1 + O(r)$.

(2) We may assume that the equivalent conditions of (1) are satisfied and show that vanishing of the rest of $\mathrm{Ric}(\boldsymbol{h})(\boldsymbol{X},\_)$ is equivalent to $d\alpha = O(r)$. Using the above formula for $2\,\mathrm{Ric}(\boldsymbol{h})(\boldsymbol{X},\xi)$, we first note that since $i_{\boldsymbol{X}}d\alpha$ vanishes along $\mathcal{Q}$, we get

$$(\boldsymbol{\nabla}d\alpha)(Y,\xi,\boldsymbol{X})|_{\mathcal{Q}} = Y\cdot d\alpha(\xi,\boldsymbol{X})|_{\mathcal{Q}} - d\alpha(\xi,\boldsymbol{\nabla}_Y\boldsymbol{X})|_{\mathcal{Q}}.$$

The first term in the right hand side may be written as $-Y\cdot(i_{\boldsymbol{X}}d\alpha(\xi))|_{\mathcal{Q}}$, and since $i_{\boldsymbol{X}}d\alpha|_{\mathcal{Q}} = 0$ and $\xi|_{\mathcal{Q}}$ is tangent to $\mathcal{Q}$, this equals $di_{\boldsymbol{X}}d\alpha(\xi,Y) = \mathcal{L}_{\boldsymbol{X}}d\alpha(\xi,Y)$. By construction, α is homogeneous of degree two, so the same holds for $d\alpha$, whence this gives $2d\alpha(\xi,Y)$. For the second summand, we get $d\alpha(\xi,\boldsymbol{\nabla}_Y\boldsymbol{X})|_{\mathcal{Q}} = \alpha(\boldsymbol{\nabla}_Y\boldsymbol{X})d\alpha(\xi,Y)$ as above, and clearly $\boldsymbol{h}(\boldsymbol{\nabla}_Y\boldsymbol{X},\boldsymbol{X}) = \frac{1}{2}Y\cdot\boldsymbol{h}(\boldsymbol{X},\boldsymbol{X})$, which restricts to 1 on $\mathcal{Q}$ by part (1).

Finally, by construction $\sum_i \boldsymbol{h}(\xi,\xi_i)\eta_i$ coincides with ξ up to addition of a multiple of $\boldsymbol{X}$, so since $i_{\boldsymbol{X}}d\alpha|_{\mathcal{Q}} = 0$ we obtain $\sum_i \boldsymbol{h}(\xi,\xi_i)d\alpha(\eta_i,Y) = d\alpha(\xi,Y)$.

Collecting our results, we see that (assuming $\mathrm{Ric}(\boldsymbol{h})(\boldsymbol{X},\boldsymbol{X})|_{\mathcal{Q}} = 0$) we get $2\,\mathrm{Ric}(\boldsymbol{h})(\boldsymbol{X},\xi) = nd\alpha(\xi,Y)$ for any ξ such that $\xi|_{\mathcal{Q}}$ is tangent to $\mathcal{Q}$, which immediately implies the result. $\qquad\square$

3. An application

In this section, we show how our results can be applied to the study of conformal invariants obtained from the ambient metric construction. Some of these ideas were sketched in [12] but here we generalise the setting considerably. In particular, we derive an algorithm that can be used to compute a tractor expression for any conformal invariant which can be obtained from the ambient metric construction. Our results are however more general than that, since they also deal with the case of ambient metrics which are Ricci flat to higher order than those whose existence is proved by Fefferman and Graham. While the existence of such metrics is obstructed on general conformal manifolds, we believe studying these "better" metrics in the cases when they do exist is very interesting.

It should be remarked at this point that another line of applications of the results derived in this paper can be found in [12], where they are applied to the study of conformally invariant powers of the Laplacian and Q–curvatures.

3.1. To carry out some computations, we introduce abstract index notation. Given an ambient manifold $\tilde{M}$ and an ambient metric $\boldsymbol{h}$ for a conformal structure $\mathcal{Q} \to M$, we write $\tilde{\mathcal{E}}(w)$ for the space smooth functions on $\tilde{M}$ which are homogeneous of degree w, i.e. $\tilde{f} \in \tilde{\mathcal{E}}(w)$ means $\boldsymbol{X}\cdot\tilde{f} = w\tilde{f}$. We will write $\tilde{\mathcal{E}}^A = \tilde{\mathcal{E}}^A(0)$ $(\tilde{\mathcal{E}}_{\mathcal{Q}}^A = \tilde{\mathcal{E}}_{\mathcal{Q}}^A(0))$ to denote the space of sections of $T\tilde{M}$ $(T\tilde{M}|_{\mathcal{Q}})$ which are homogeneous of degree -1. (We adopt this convention since sections of $\tilde{\mathcal{E}}_{\mathcal{Q}}^A$ correspond to sections of the standard tractor bundle.) Then finally we will write $\tilde{\mathcal{E}}^{AB}(w)$ $(\tilde{\mathcal{E}}_{\mathcal{Q}}^{AB}(w))$ to mean $\tilde{\mathcal{E}}^A \otimes \tilde{\mathcal{E}}^B \otimes \tilde{\mathcal{E}}(w)$

$(\tilde{\mathcal{E}}_{\mathcal{Q}}^{A} \otimes \tilde{\mathcal{E}}_{\mathcal{Q}}^{B} \otimes \tilde{\mathcal{E}}_{\mathcal{Q}}(w)$ respectively) and so forth. For lower indices, the appropriate convention is that $\tilde{\mathcal{E}}_{A} = \tilde{\mathcal{E}}_{A}(0)$ denotes the space of ambient one–forms, which are homogeneous of degree 1, since then $\boldsymbol{h}_{AB}$ (which is homogeneous of degree 2) induces an isomorphism $\tilde{\mathcal{E}}^{A} \to \tilde{\mathcal{E}}_{A}$. The extensions to multiple lower and mixed indices, as well as the notation for sections along $\mathcal{Q}$ is done as above. In this context we will refer to w as the conformal *weight* (to distinguish it from the homogeneity degree). This means that for an ambient tensor field, the conformal weight equals the homogeneous degree plus the number of upper indices minus the number of lower indices. We raise and lower indices using the ambient metric $\boldsymbol{h}_{AB}$ and its inverse $\boldsymbol{h}^{AB}$. We also adopt the usual conventions that round brackets (square brackets) around indices indicate a symmetrisation (antisymmetrisation) of the enclosed indices, except indices between vertical lines, and that the same index occurring twice indicates a trace.

We start with some general results for pseudo–Riemannian metrics:

Proposition. *Let $\boldsymbol{h}$ be a pseudo–Riemannian metric with curvature $\boldsymbol{R} = \boldsymbol{R}_{AB}{}^{C}{}_{D}$ and Ricci curvature $\boldsymbol{Ric} = \boldsymbol{Ric}_{AB} = \boldsymbol{R}_{CA}{}^{C}{}_{B}$. Then we have:*
(1) $\nabla^{E} \boldsymbol{R}_{EABC} = 2\nabla_{[B} \boldsymbol{Ric}_{C]A}$.
(2) $\Delta \boldsymbol{R}_{ABCD} = 2(\nabla_{A}\nabla_{[C}\boldsymbol{Ric}_{D]B} - \nabla_{B}\nabla_{[C}\boldsymbol{Ric}_{D]A}) + \Psi_{ABCD}$, where Δ denotes the ambient Laplacian and Ψ_{ABCD} is a linear combination of partial contractions of $\boldsymbol{R} \otimes \boldsymbol{R}$.
(3) Let $\Phi_{A...B} \in \mathcal{E}_{A...B}(w)$ be any section. Then the commutator of the Laplacian Δ with a covariant derivative ∇ acts as

$$
\begin{aligned}
[\Delta, \nabla_C]\Phi_{A...B} = & -2h^{EF}\left(\Phi_{E...B}\nabla_{[F}\boldsymbol{Ric}_{A]C} + \cdots + \Phi_{A...E}\nabla_{[F}\boldsymbol{Ric}_{B]C}\right) \\
& - 2\left(\boldsymbol{R}_{FC}{}^{E}{}_{A}\nabla^{F}\Phi_{E...B} + \cdots + \boldsymbol{R}_{FC}{}^{E}{}_{B}\nabla^{F}\Phi_{A...E}\right) \\
& + \boldsymbol{Ric}_{CE}\nabla^{E}\Phi_{A...B},
\end{aligned}
$$

where in the two sums there is one summand for each index of Φ, and E is contracted into that index.

Proof. (1) The algebraic Bianchi identity $0 = \boldsymbol{R}_{[EA|B|C]}$ together with the usual symmetries of the Riemann curvature gives us $\boldsymbol{R}_{EABC} = -\boldsymbol{R}_{ABEC} + \boldsymbol{R}_{ACEB}$. The differential Bianchi identity $0 = \nabla_{[F}\boldsymbol{R}_{AB]EC}$ together with the symmetries of $\boldsymbol{R}$ leads to $\nabla_{F}\boldsymbol{R}_{ABEC} = -\nabla_{B}\boldsymbol{R}_{FAEC} + \nabla_{A}\boldsymbol{R}_{FBEC}$, and similarly we get $\nabla_{F}\boldsymbol{R}_{ACEB} = -\nabla_{C}\boldsymbol{R}_{FAEB} + \nabla_{A}\boldsymbol{R}_{FCEB}$. Contracting with $\boldsymbol{h}^{EF}$ the claim now follows from symmetry of $\boldsymbol{Ric}$.
(2) By definition $(\Delta \boldsymbol{R})_{ABCD} = \nabla^{E}\nabla_{E}\boldsymbol{R}_{ABCD}$. Using the differential Bianchi identity and curvature symmetries, we may write $\nabla_{E}\boldsymbol{R}_{ABCD}$ as $-\nabla_{B}\boldsymbol{R}_{EACD} + \nabla_{A}\boldsymbol{R}_{EBCD}$. Now the commutator of two covariant derivatives is given by the algebraic action of the curvature, so $-\nabla^{E}\nabla_{B}\boldsymbol{R}_{EACD}$ may be written as the sum of $-\nabla_{B}\nabla^{E}\boldsymbol{R}_{EACD}$ and a sum of partial contractions of $\boldsymbol{R} \otimes \boldsymbol{R}$. Similarly, $-\nabla^{E}\nabla_{A}\boldsymbol{R}_{EBCD}$ is the sum of $-\nabla_{A}\nabla^{E}\boldsymbol{R}_{EBCD}$ and a sum of partial contractions of $\boldsymbol{R} \otimes \boldsymbol{R}$. Now the result immediately follows from (1).
(3) Let us compute $\Delta\nabla_{C}\Phi_{A...B} = h^{EF}\nabla_{E}\nabla_{F}\nabla_{C}\Phi_{A...B}$. The definition of the curvature reads as $[\nabla_{A}, \nabla_{B}]V^{C} = \boldsymbol{R}_{AB}{}^{C}{}_{D}V^{D}$, and thus $[\nabla_{A}, \nabla_{B}]V_{C} =$

$-\boldsymbol{R}_{AB}{}^D{}_C V_D$. Using this, we get

$$\nabla_F \nabla_C \Phi_{A...B} = \nabla_C \nabla_F \Phi_{A...B} - (\boldsymbol{R}_{FC}{}^I{}_A \Phi_{I...B} + \cdots + \boldsymbol{R}_{FC}{}^I{}_B \Phi_{A...I}),$$

with one summand for each index of Φ in the sum in brackets. Hitting that sum with ∇^F, each summand splits into a sum of two terms, one in which ∇^F acts on $\boldsymbol{R}$ and one in which ∇^F acts on Φ. Using (1) we see that the terms in which ∇^F acts on $\boldsymbol{R}$ exactly give the terms in the first sum of the claimed formula for $[\Delta, \nabla_C]\Phi_{A...B}$. On the other hand, the terms in which ∇^F acts on Φ exactly give half of the second sum in the claimed formula.

Again swapping covariant derivatives, we may write $\nabla_E \nabla_C \nabla_F \Phi_{A...B}$ as the sum of $\nabla_C \nabla_E \nabla_F \Phi_{A...B}$ (which after contraction with $\boldsymbol{h}^{EF}$ gives $\nabla_C \Delta \Phi_{A...B}$) and

$$-\boldsymbol{R}_{EC}{}^I{}_F \nabla_I \Phi_{A...B} - (\boldsymbol{R}_{EC}{}^I{}_A \nabla_F \Phi_{I...B} + \cdots + \boldsymbol{R}_{EC}{}^I{}_B \nabla_F \Phi_{A...I}),$$

again with one summand for each index of Φ in the sum in brackets. Contracting with $\boldsymbol{h}^{EF}$, the the sum in brackets gives the second half of the second sum in our claimed formula, while the other summand gives the last term in the claimed formula. $\qquad\square$

Remark. Of course, in the proof of part (2), it is no problem to compute an explicit formula for the sum of partial contractions Ψ_{ABCD} of $\boldsymbol{R} \otimes \boldsymbol{R}$ (see [12]).

3.2. To proceed, we next specialise to an ambient metric $\boldsymbol{h}$ such that tangential components of the ambient Ricci curvature $\boldsymbol{Ric}$ vanish along $\mathcal{Q}$. By part (2) of Theorem 2.6 the one form α dual to the infinitesimal generator $\boldsymbol{X}$ of the $\mathbb{R}_+$–action then satisfies $d\alpha = O(r)$ and the procedure of 2.4 can be applied to construct a normal standard tractor bundle $(\mathcal{T}, h, \nabla)$ from $(\tilde{M}, \boldsymbol{h})$.

We may regard the ambient curvature $\boldsymbol{R}$ as 2-form taking values in $\mathrm{End}(T\tilde{M})$. We have observed in 2.5 above that if $\tilde{\xi}$ and $\tilde{\eta}$ are invariant lifts of vector fields $\xi, \eta \in \mathfrak{X}(M)$, then $\boldsymbol{R}(\tilde{\xi}, \tilde{\eta})$ is precisely the homogeneous degree 0 section of $\mathrm{End}(T\tilde{M}|_{\mathcal{Q}})$ corresponding to the section $R(\xi, \eta)$ of $\mathrm{End}\,\mathcal{T}$. Thus the homogeneous $\mathrm{End}(T\tilde{M})$–valued three form $\alpha \wedge \boldsymbol{R}$ is, along $\mathcal{Q}$, uniquely determined by the tractor curvature. Similarly, the Levi-Civita connection ∇ is determined by its action on vector fields homogeneous of degree -1, so along $\mathcal{Q}$ covariant derivatives in tangential directions are determined by the underlying tractor connection.

Since $d\alpha|_{\mathcal{Q}} = 0$, we know from 2.3 that $r := \frac{1}{2}\boldsymbol{h}(\boldsymbol{X}, \boldsymbol{X})$ satisfies $\alpha = dr + O(r)$, and hence r is a smooth defining function for $\mathcal{Q}$. Notice that the ambient vector field $\boldsymbol{X} = \boldsymbol{X}^A$ has conformal weight 1 and since the ambient covariant derivative is compatible with homogeneities, the ambient differential operator ∇_A has conformal weight -1. By definition, the ambient one–form α_A is given by $\boldsymbol{h}_{AB} \boldsymbol{X}^B$, so we will also denote this form by $\boldsymbol{X}_A$. (So $\boldsymbol{X}$ will mean either a 1-form or a vector field according to index placement and/or context.) For example $\alpha \wedge \boldsymbol{R}$ is $3\boldsymbol{X}_{[A} \boldsymbol{R}_{BC]}{}^D{}_E \in \tilde{\mathcal{E}}_{[ABC]}{}^D{}_E(-1)$. Note that by definition $\boldsymbol{X}^A \boldsymbol{X}_A = \boldsymbol{h}_{AB} \boldsymbol{X}^A \boldsymbol{X}^B = 2r$.

To compute efficiently in the sequel, we have to determine the commutators of covariant derivatives with r and $\boldsymbol{X}_A$, viewed as multiplication operators. Since $d\alpha = O(r)$, we get $i_{\boldsymbol{X}} d\alpha = 2(\alpha - dr) = O(r)$, so $\alpha = dr + r\beta$ for some ambient one–form β. Moreover, $d\alpha = dr \wedge \beta + r d\beta$ and this being $O(r)$ implies vanishing of the tangential components of β along $\mathcal{Q}$, whence $\beta = \varphi\alpha + r\tilde{\gamma}$ for some ambient smooth function φ and one–form $\tilde{\gamma}$, and hence $\alpha = dr + \varphi r\alpha + r^2\tilde{\gamma}$. In particular, $dr = (1 - \varphi r)\alpha + O(r^2)$, and viewing r as a multiplication operator, this implies the commutation formula $[\boldsymbol{\nabla}_A, r] = (1 - \varphi r)\boldsymbol{X}_A + O(r^2)$. Further, the above equations immediately imply $dr \wedge \alpha = O(r^2)$ and $d\alpha = r\gamma \wedge \alpha + O(r^2)$, where $\gamma := d\varphi - 2\tilde{\gamma}$, which in index notaiton reads as $\boldsymbol{\nabla}_{[A}\boldsymbol{X}_{B]} = r\gamma_{[A}\boldsymbol{X}_{B]} + O(r^2)$. On the other hand, equation (1) from 2.3 gives us $\boldsymbol{\nabla}_A\boldsymbol{X}_B = \boldsymbol{h}_{AB} + r\gamma_{[A}\boldsymbol{X}_{B]} + O(r^2)$. Viewing X_B as a multiplication operator, we thus get the commutator formula $[\boldsymbol{\nabla}_A, \boldsymbol{X}_B] = \boldsymbol{h}_{AB} + r\gamma_{[A}\boldsymbol{X}_{B]} + O(r^2)$.

The next step is to compute two basic tractor operators in ambient terms. The first obvious operator to consider is $\alpha \wedge \boldsymbol{\nabla}$, which, along $\mathcal{Q}$, obviously only needs derivatives in tangential directions, and may be written as $\tilde{\mathbb{D}}_{AB} := 2\boldsymbol{X}_{[A}\boldsymbol{\nabla}_{B]}$. Since $[\boldsymbol{\nabla}_A, r] = (1 - \varphi r)\boldsymbol{X}_A + O(r^2)$, one immediately concludes that $[\tilde{\mathbb{D}}_{AB}, r] = O(r^2)$, which in particular means that for any ambient tensor field V, the restriction $(\tilde{\mathbb{D}}_{AB}V)|_{\mathcal{Q}}$ depends only on $V|_{\mathcal{Q}}$. Hence for arbitrary indices, we get a well defined operator $\tilde{\mathbb{D}}_{AB} : (\tilde{\mathcal{E}}_{\mathcal{Q}})^{C...D}_{E...F}(w) \to (\tilde{\mathcal{E}}_{\mathcal{Q}})^{C...D}_{[AB]E...F}(w)$, that clearly can be computed in terms of the underlying standard tractor bundle. It is easy to identify this operator: By definition, the adjoint tractor bundle of M is the bundle of skew symmetric endomorphisms of the standard tractor bundle $\mathcal{T}$. Since sections of $\mathcal{T}$ may be identified with sections in $\tilde{\mathcal{E}}^A_{\mathcal{Q}}$, we obtain, using the inverse $\boldsymbol{h}^{AB}$ of the ambient metric, an identifiction of smooth sections of the adjoint tractor bundle $\mathcal{A}$ and sections in $\tilde{\mathcal{E}}^{[AB]}_{\mathcal{Q}}$. Thus $\tilde{\mathbb{D}}_{AB}$ determines a conformally invariant operator $\mathbb{D}$ on M which goes between $\mathcal{A} \otimes \mathcal{F} \otimes \mathcal{E}[w]$ and $\mathcal{F} \otimes \mathcal{E}[w]$, where $\mathcal{F}$ is any tensor power of $\mathcal{T}$. There is a natural projection $\mathcal{A} \to TM$. Under the identification of $\Gamma(\mathcal{A})$ with $\tilde{\mathcal{E}}^{[AB]}_{\mathcal{Q}}$ this is explicitly given by mapping Φ^{AB} to $\boldsymbol{X}_A\Phi^{AB} - \boldsymbol{X}_A\Phi^{BA}$ modulo multiples of $\boldsymbol{X}^B$. Using this, one immediately verifies that on the standard tractor bundle, $\mathbb{D}$ coincides with the composition of the tractor connection with this projection. On density bundles one obtains a similar composition of Levi-Civita connection with the projection invariantly combined with the canonical action of an adjoint tractor on the density bundle; in total the fundamental D–operator, see [4, section 3]. The obvious compatibility of $\tilde{\mathbb{D}}_{AB}$ with tensor products then implies that it is exactly the operator obtained by twisting the fundamental D on the density bundle with the tractor connection on the tractor bundle. This is precisely the "double-D" operator of [10, 11] (and see also [4, section 3]).

Now we can follow these sources to obtain the tractor D–operator: Consider the operator $\boldsymbol{h}^{AB}\tilde{\mathbb{D}}_{A(Q}\tilde{\mathbb{D}}_{|B|P)_0}$, which by construction acts tangentially on tensor fields along $\mathcal{Q}$. Here $(\cdots)_0$ indicates the trace-free symmetrisation over enclosed indices (excluding any in the $|\cdots|$). Using the commutator

formulae from above, one immediately verifies directly that

$$4h^{AB}X_{[A}\nabla_{Q]}X_{[B}\nabla_{P]} = -nX_Q\nabla_P - h_{PQ}X^A\nabla_A + X_QX_P\Delta$$
$$- X_QX^A\nabla_A\nabla_P - X_PX^A\nabla_Q\nabla_A + O(r).$$

The definition of the Riemann curvature implies that on any tensor bundle we may rewrite $\nabla_Q\nabla_A$ as $\nabla_A\nabla_Q - R_{AQ}\#$, where $R_{AQ}\#$ denotes the tensorial action of $R_{AQ}{}^E{}_F$ (via the indices E and F) on the given tensor field. Thus we conclude that $h^{AB}\tilde{\mathbb{D}}_{A(Q}\tilde{\mathbb{D}}_{|B|P)_0} = -X_{(Q}D_{P)_0} + O(r)$, where

$$D_A := n\nabla_A + 2X^B\nabla_B\nabla_A - X_A\Delta - X^BR_{BA}\#,$$

where the $\#$ as above indicates a tensorial action. Since the map $\xi_P \mapsto X_{(Q}\xi_{P)_0}$ is an injection of $T^*\tilde{M}|_{\mathcal{Q}}$ into $\otimes^2 T^*\tilde{M}|_{\mathcal{Q}}$ we see that this construction determines D_A as a well defined operator between $\tilde{\mathcal{E}}_{\mathcal{Q}}^A \otimes \tilde{\mathcal{E}}_{\mathcal{Q}}^\Psi(w)$ and $\tilde{\mathcal{E}}_{\mathcal{Q}}^\Psi(w-1)$, where $\tilde{\mathcal{E}}_{\mathcal{Q}}^\Psi$ is any tensor power of $\tilde{\mathcal{E}}_{\mathcal{Q}}^B$. Thus D determines an operator D between weighted tractor bundles on M. By its construction from $\tilde{\mathbb{D}}$ it is clear that this action of D_A is determined by the underlying standard tractor bundle and its connection. In fact this construction of D is exactly the interpretation on $\mathcal{Q}$ of the construction of the standard tractor D operator from $\mathbb{D}$ as in [10] and [4, 3.2]. For which it follows immediately that D is the standard tractor D operator.

We can easily verify explicitly that D_A acts tangentially on homogeneous ambient tensor fields along $\mathcal{Q}$. For an ambient tensor T of conformal weight w, the covariant derivative ∇T has conformal weight $w-1$. Whence we get $D_A = (n+2w-2)\nabla_A - X_A\Delta - X^BR_{BA}\#$ on tensor fields of conformal weight w. Moreover, from above we know that $\nabla_A r = (1-\varphi r)X_A + O(r^2)$. Hitting this with ∇^A, we immediately conclude that $\Delta r = (n+2) + O(r)$. More generally, if V is any ambient tensor field of conformal weight w, then $\Delta(rV) = (\Delta r)V + 2h^{AB}(\nabla_A r)\nabla_B V + r\Delta V = (n+2w+2)V + O(r)$. Since rV has conformal weight $w+2$, together with the above formula for the action of D_A on tensor fields of fixed conformal weight this implies that $D_A rV = O(r)$, so D_A indeed acts tangentially along $\mathcal{Q}$.

3.3. The simple relation between the operator D_A and the ambient covariant derivative together with the fact that D_A acts tangentially leads to very remarkable consequences. The point here is that, along $\mathcal{Q}$, D_A depends only on the underlying standard tractor bundle, and for any ambient tensor field Φ the restriction of $D_A\Phi$ to $\mathcal{Q}$ depends only on the restriction of Φ to $\mathcal{Q}$. In particular, we have noted in 3.2 that $X_{[A}R_{BC]EF}$ is a section of $\tilde{\mathcal{E}}_{[ABC][EF]}(-1)$ whose restriction to $\mathcal{Q}$ depends only on the tangential components of the ambient curvature. These tangential components are determined by the curvature of the normal standard tractor connection. Consequently, $3D^AX_{[A}R_{BC]EF}$ is a section of $\tilde{\mathcal{E}}_{[BC][EF]}(-2)$ whose restriction to $\mathcal{Q}$ can be computed from the tractor curvature (and thus depends only on the underlying conformal structure).

To compute this explicitly, we need the commutator of the Laplacian with X_A. From section 3.2 above, we have $\nabla_B X_A = h_{BA} + r\gamma_{[B}X_{A]} + X_A\nabla_B + O(r^2)$. Hitting this with ∇^B, we obtain $\Delta X_A = 2\nabla_A + \frac{1}{2}X^B\gamma_B X_A + X_A\Delta + O(r)$. Since $X^B\gamma_B$ and the curvature R_{BCEF} both have conformal

weight -2, this allows us to write $\boldsymbol{D}_G \boldsymbol{X}_{[A} \boldsymbol{R}_{BC]EF}$ as $(n-2)\boldsymbol{\nabla}_G \boldsymbol{X}_{[A} \boldsymbol{R}_{BC]EF} +$
$(\tfrac{1}{2}\boldsymbol{X}^J \gamma_J - \boldsymbol{\Delta})\boldsymbol{X}_G \boldsymbol{X}_{[A}\boldsymbol{R}_{BC]EF} - \boldsymbol{X}^Q \boldsymbol{R}_{QG}\#\boldsymbol{X}_{[A}\boldsymbol{R}_{BC]EF} + O(r)$. Taking into
account the conformal weights and using the formulae derived above, we can
now expand this expression explicitly and contracting with $\boldsymbol{h}^{GA}$ we obtain

$$
\begin{aligned}
3\boldsymbol{D}^A \boldsymbol{X}_{[A}\boldsymbol{R}_{BC]EF} = \\
(n-2)\,[(n-4)\,\boldsymbol{R}_{BC\,EF} + 2\boldsymbol{X}_{[B}\boldsymbol{\nabla}^A \boldsymbol{R}_{C]AEF}] \\
+(\boldsymbol{X}^J \gamma_J - 2\boldsymbol{\Delta})(\boldsymbol{X}^A \boldsymbol{X}_{[B}\boldsymbol{R}_{C]AEF}) \\
-3\boldsymbol{h}^{AG}(\boldsymbol{X}^Q \boldsymbol{R}_{QG}\#\boldsymbol{X}_{[A}\boldsymbol{R}_{BC]EF}) + O(r).
\end{aligned}
$$

(2)

In particular, in dimensions $\neq 4$, the curvature of the ambient metric
shows up in this formula. In these dimensions, existence of an ambient
metric $\boldsymbol{h}$ such that $\mathrm{Ric}(\boldsymbol{h}) = O(r)$ and such that the tangential components
of $\mathrm{Ric}(\boldsymbol{h})$ vanish to second order along $\mathcal{Q}$ has been proved in [8]. Assuming
that we deal with such a metric we may simplify the formula considerably.
We first claim that the Ricci condition implies $d\alpha = O(r^2)$. We know that
$d\alpha = O(r)$, so $d\alpha = r\beta$ for some two–form β, which is homogeneous of
degree 0, and thus has conformal weight -2. From 3.2 we conclude that
$\boldsymbol{\Delta}(d\alpha) = \boldsymbol{\Delta}(r\beta) = (n-2)\beta + O(r)$, so it suffices to prove $\boldsymbol{\Delta}(d\alpha) = O(r)$
in order to conclude $d\alpha = O(r^2)$. In index notation, proposition 2.5 reads
as $\boldsymbol{\nabla}_A (d\alpha)_{BC} = 2\boldsymbol{X}^E \boldsymbol{R}_{EABC}$. Consequently, we may compute $\boldsymbol{\Delta}(d\alpha)$ as
$\tfrac{1}{2}\boldsymbol{\nabla}_A \boldsymbol{X}_E \boldsymbol{R}^{EA}{}_{BC}$. From 3.2 we know that up to an $O(r)$ we may replace
$\boldsymbol{\nabla}_A \boldsymbol{X}_E$ by $\boldsymbol{h}_{AE} + \boldsymbol{X}_E \boldsymbol{\nabla}_A$, so the fact that the curvature is skew in the
first two indices implies $\boldsymbol{\Delta}(d\alpha) = \tfrac{1}{2}\boldsymbol{X}^E \boldsymbol{\nabla}^A \boldsymbol{R}_{EABC} + O(r)$. By part (1)
of proposition 3.1, this may be rewritten as $-\boldsymbol{X}^E \boldsymbol{\nabla}_{[B}\boldsymbol{Ric}_{C]E}$. Since we
assume $\mathrm{Ric}(\boldsymbol{h}) = O(r)$ and that the tangential components of $\mathrm{Ric}(\boldsymbol{h})$ vanish
to second order, we may write $\boldsymbol{Ric}_{CE} = r(\boldsymbol{X}_C K_E + \boldsymbol{X}_E K_C) + O(r^2)$ for an
appropriate K. Hitting this with $\boldsymbol{\nabla}_B$, we obtain $\boldsymbol{X}_B \boldsymbol{X}_C K_E + \boldsymbol{X}_B \boldsymbol{X}_E K_C +$
$O(r)$, and skewing in B and C the first summand vanishes. Thus we see
that $\boldsymbol{\nabla}_{[B}\boldsymbol{Ric}_{C]E} = \boldsymbol{X}_E K_{[C} \boldsymbol{X}_{B]} + O(r)$. Contracting this with $\boldsymbol{X}^E$, we get
an $O(r)$ term, which completes the proof that $d\alpha = O(r^2)$.

In particular, vanishing of the Ricci and vanishing of its tangential com-
ponents to second order implies that $\boldsymbol{X}^E \boldsymbol{R}_{EABC} = O(r)$, so the last line
in formula (2) becomes an $O(r)$. Next, $\boldsymbol{X}^A \boldsymbol{R}_{CAEF} = -\tfrac{1}{2}\boldsymbol{\nabla}_C (d\alpha)_{EF}$ imme-
diately implies $\boldsymbol{X}^A \boldsymbol{X}_{[B}\boldsymbol{R}_{C]AEF} = -\tfrac{1}{4}\tilde{\mathbb{D}}_{BC}(d\alpha)_{EF}$. Since $d\alpha = O(r^2)$, the
same is true for $\tilde{\mathbb{D}}(d\alpha)$, and from 3.2 we know that applying $\boldsymbol{\Delta}$ to an $O(r^2)$
term we get an $O(r)$ term, so the last but one line in (2) also contributes an
$O(r)$ only. On the other hand, using part (1) of Proposition 3.1 we get

$$
\boldsymbol{X}_{[B}\boldsymbol{\nabla}^A \boldsymbol{R}_{C]AEF} = (-\boldsymbol{X}_B \boldsymbol{\nabla}_{[E}\boldsymbol{Ric}_{F]C} + \boldsymbol{X}_C \boldsymbol{\nabla}_{[E}\boldsymbol{Ric}_{F]B})
$$

Writing $\boldsymbol{Ric}_{FC} = r(\boldsymbol{X}_F K_C + \boldsymbol{X}_C K_F) + O(r^2)$ as above, we get $\boldsymbol{\nabla}_{[E}\boldsymbol{Ric}_{F]C} =$
$\boldsymbol{X}_C K_{[F} \boldsymbol{X}_{E]} + O(r)$, which immediately implies $\boldsymbol{X}_{[B}\boldsymbol{\nabla}^A \boldsymbol{R}_{C]AEF} = O(r)$.
Collecting our results, we see that for metrics with $\mathrm{Ric}(\boldsymbol{h}) = O(r)$ and with
tangential components of $\mathrm{Ric}(\boldsymbol{h})$ vanishing to second order along $\mathcal{Q}$, formula
(2) boils down to $3\boldsymbol{D}^A \boldsymbol{X}_{[A}\boldsymbol{R}_{BC]EF}|_{\mathcal{Q}} = (n-2)(n-4)\boldsymbol{R}_{BCEF}|_{\mathcal{Q}}$. In partic-
ular, the curvature of such metrics is intrinsic to the conformal structure,
provided that $n \neq 4$.

3.4. Assuming higher order vanishing of the Ricci curvature, we can obtain a similar result for certain covariant derivatives of the ambient curvature, except for a single critical order in even dimensions. As we have noted above, the restriction of $\boldsymbol{R}$ to $\mathcal{Q}$ may be viewed as a section of $(\tilde{\mathcal{E}}_{ABCD})_{\mathcal{Q}}(-2)$, and similarly for any $\ell > 0$ the restriction of $\boldsymbol{\nabla}^\ell \boldsymbol{R} = (\boldsymbol{\nabla} \circ \ldots \circ \boldsymbol{\nabla})\boldsymbol{R}$ to $\mathcal{Q}$ defines a section of $(\tilde{\mathcal{E}}_{A_1 \ldots A_{\ell+4}})_{\mathcal{Q}}(-2-\ell)$.

Lemma. *Let $\tilde{M}$ be an ambient manifold for a conformal structure $\mathcal{Q} \to M$ on an n–dimensional manifold M, and let $\boldsymbol{h}$ be an ambient metric on $\tilde{M}$ with curvature $\boldsymbol{R}$ and Ricci curvature $\boldsymbol{Ric}$. If for some $k > 0$ we have $\boldsymbol{Ric}_{AB} = O(r^{k+1})$ and tangential components of $\boldsymbol{Ric}$ vanish to order $k + 2$ along $\mathcal{Q}$, and if $n \neq 2k + 4$, then there is a universal formula that computes $\boldsymbol{\nabla}^k \boldsymbol{R}|_{\mathcal{Q}}$ from $\boldsymbol{R}|_{\mathcal{Q}}$ and $\boldsymbol{\nabla}^\ell \boldsymbol{R}|_{\mathcal{Q}}$ for $\ell < k$.*

Proof. By assumption, we always have $\mathrm{Ric}(\boldsymbol{h}) = O(r)$ and tangential components of $\mathrm{Ric}(\boldsymbol{h})$ vanish to second order along $\mathcal{Q}$, so from 3.3 and 3.2 we conclude that $\boldsymbol{D}_A = (n + 2w - 2)\boldsymbol{\nabla}_A - \boldsymbol{X}_A \boldsymbol{\Delta}$ on ambient tensor fields of conformal weight w. The section $\boldsymbol{\nabla}^{k-1}\boldsymbol{R}$ has conformal weight $-k - 1$, so $\boldsymbol{D}_A \boldsymbol{\nabla}^{k-1}\boldsymbol{R} = (n - 2k - 4)\boldsymbol{\nabla}_A \boldsymbol{\nabla}^{k-1}\boldsymbol{R} - \boldsymbol{X}_A \boldsymbol{\Delta}\boldsymbol{\nabla}^{k-1}\boldsymbol{R}$, and by assumption $n - 2k - 4 \neq 0$. Since $\boldsymbol{D}_A \boldsymbol{\nabla}^{k-1}\boldsymbol{R}|_{\mathcal{Q}}$ depends only on $\boldsymbol{\nabla}^{k-1}\boldsymbol{R}|_{\mathcal{Q}}$ it suffices to compute $\boldsymbol{\Delta}\boldsymbol{\nabla}^{k-1}\boldsymbol{R}$ from the restrictions of the sections $\boldsymbol{\nabla}^\ell \boldsymbol{R}$ for $\ell < k$ to $\mathcal{Q}$. Now one immediately verifies inductively that

$$[\boldsymbol{\Delta}, \boldsymbol{\nabla}^{k-1}] = \sum_{\ell=0}^{k-2} \boldsymbol{\nabla}^\ell [\boldsymbol{\Delta}, \boldsymbol{\nabla}]\boldsymbol{\nabla}^{k-2-\ell}.$$

Our assumptions on $\boldsymbol{Ric}$ together with part (3) of proposition 3.1 imply that for any ambient tensor field $\Phi_{A\ldots B}$, we can write $[\boldsymbol{\Delta}, \boldsymbol{\nabla}_C]\Phi_{A\ldots B}$ as

$$-2\left(\boldsymbol{R}_{EC}{}^F{}_A \boldsymbol{\nabla}^E \Phi_{F\ldots B} + \cdots + \boldsymbol{R}_{EC}{}^F{}_B \boldsymbol{\nabla}^E \Phi_{A\ldots F}\right) + O(r^k).$$

Inserting $\Phi = \boldsymbol{\nabla}^{k-2-\ell}\boldsymbol{R}$ for some $\ell = 0, \ldots, k - 2$, we get an expression for $[\boldsymbol{\Delta}, \boldsymbol{\nabla}]\boldsymbol{\nabla}^{k-2-\ell}\boldsymbol{R}$ in terms of $\boldsymbol{R}$ and $\boldsymbol{\nabla}^{k-1-\ell}\boldsymbol{R}$ up to some $O(r^k)$. Applying $\boldsymbol{\nabla}^\ell$ and restricting to $\mathcal{Q}$, the $O(r^k)$ cannot contribute, and we only get covariant derivatives of order at most $k - 1$ of $\boldsymbol{R}$. Thus we obtain a universal formula which expresses $[\boldsymbol{\Delta}, \boldsymbol{\nabla}^{k-1}]\boldsymbol{R}|_{\mathcal{Q}}$ in terms of $\boldsymbol{R}|_{\mathcal{Q}}$ and $\boldsymbol{\nabla}^\ell \boldsymbol{R}|_{\mathcal{Q}}$ for $\ell < k$.

To complete the proof it hence suffices to analyse $\boldsymbol{\nabla}^{k-1}\boldsymbol{\Delta}\boldsymbol{R}$, and we have computed $\boldsymbol{\Delta}\boldsymbol{R}$ in part (2) of Proposition 3.1. Now we claim that the term $\boldsymbol{\nabla}_A \boldsymbol{\nabla}_{[C}\boldsymbol{Ric}_{D]B} - \boldsymbol{\nabla}_B \boldsymbol{\nabla}_{[C}\boldsymbol{Ric}_{D]A}$ showing up in that formula is $O(r^k)$. Indeed, by assumption $\boldsymbol{Ric} = O(r^{k+1})$ and tangential components of $\boldsymbol{Ric}$ vanish to order $k + 2$ along $\mathcal{Q}$, which implies that $\boldsymbol{Ric}_{AB} = r^{k+1}(\boldsymbol{X}_A K_B + \boldsymbol{X}_B K_A) + O(r^{k+2})$ for some ambient one–form K_B. As in 3.3, we then obtain $\boldsymbol{\nabla}_{[C}\boldsymbol{Ric}_{D]B} = (k + 1)r^k \boldsymbol{X}_B K_{[D}\boldsymbol{X}_{C]} + O(r^{k+1})$. Hitting this with $\boldsymbol{\nabla}_A$ we get $k(k + 1)r^{k-1}\boldsymbol{X}_A \boldsymbol{X}_B K_{[D}\boldsymbol{X}_{C]} + O(r^k)$, and skewing over A and B the claim follows.

But then it follows from part (2) of Proposition 3.1 that $\boldsymbol{\nabla}^{k-1}\boldsymbol{\Delta}\boldsymbol{R}|_{\mathcal{Q}} = \boldsymbol{\nabla}^{k-1}\Psi_{ABCD}|_{\mathcal{Q}}$, and since Ψ_{ABCD} is a partial contraction of $\boldsymbol{R} \otimes \boldsymbol{R}$ we conclude that $\boldsymbol{\nabla}^{k-1}\Psi_{ABCD}|_{\mathcal{Q}}$ can be expressed by a universal formula in terms of $\boldsymbol{R}|_{\mathcal{Q}}$ and $\boldsymbol{\nabla}^\ell \boldsymbol{R}|_{\mathcal{Q}}$ for $\ell \leq k - 1$. $\qquad\square$

Together with the tractor formula computing the ambient curvature, this now leads to

Theorem. *Let $\tilde{M}$ be an ambient manifold for a conformal structure $\mathcal{Q} \to M$ on an n–dimensional manifold M, and let $\boldsymbol{h}$ be an ambient metric on $\tilde{M}$ with curvature $\boldsymbol{R}$ and Ricci curvature $\boldsymbol{Ric}$. Assume that $\mathrm{Ric}(\boldsymbol{h}) = O(r^{k+1})$ and tangential components of $\mathrm{Ric}(\boldsymbol{h})$ vanish to order $k+2$ along $\mathcal{Q}$ for some $k \geq 0$. Then using the convention that $\boldsymbol{\nabla}^0 \boldsymbol{R} = \boldsymbol{R}$ we have:*
(1) If n is odd or n is even and $k < \frac{n-4}{2}$, then for each $0 \leq \ell \leq k$ there is a universal tractor formula that computes $\boldsymbol{\nabla}^\ell \boldsymbol{R}$ from the tractor curvature of the underlying standard tractor bundle.
(2) If n is even and $\frac{n-4}{2} < \ell \leq k$, then there is a universal tractor formula that computes $\boldsymbol{\nabla}^\ell \boldsymbol{R}$ from the tractor curvature of the underlying standard tractor bundle and $(\boldsymbol{\nabla}^{\frac{n-4}{2}} \boldsymbol{R})|_\mathcal{Q}$.

Proof. Since our assumptions on $\mathrm{Ric}(\boldsymbol{h})$ imply that at least $\mathrm{Ric}(\boldsymbol{h}) = O(r)$ and tangential components of $\mathrm{Ric}(\boldsymbol{h})$ vanish to second order, we may can compute $\boldsymbol{R}$ via $\boldsymbol{R}_{BCEF}|_\mathcal{Q} = \frac{3}{(n-2)(n-4)} \boldsymbol{D}^A \boldsymbol{X}_{[A} \boldsymbol{R}_{BC]EF}|_\mathcal{Q}$ provided that $n \neq 4$. Thus we get (1) for $\ell = 0$. Iterated application of the lemma then leads to a universal formula for $\boldsymbol{\nabla}^\ell \boldsymbol{R}$ in terms of $\boldsymbol{R}$ provided that in no step we get $n = 2\ell + 4$, and so (1) follows.

For (2), we first note that by (1) we get a universal formula for $\boldsymbol{\nabla}^i \boldsymbol{R}$ in terms of the tractor curvature for $i < \frac{n-4}{2}$. But using this, the result again follows from iterated application of the lemma. $\square$

Remarks. Part (1) of this theorem ties in nicely with the results on existence and uniqueness of ambient metrics in [8]. It is shown in that paper that in odd dimensions there exists an infinite order power series solution along $\mathcal{Q}$ for an ambient metric $\boldsymbol{h}$ such that $\mathrm{Ric}(\boldsymbol{h})$ vanishes to infinite order along $\mathcal{Q}$ and this solution is unique up to the action of an equivariant diffeomorphism fixing $\mathcal{Q}$. On the other hand, in the case of even dimensions the Fefferman-Graham construction is obstructed at finite order. More precisely, if n is even then there exist an ambient metric $\boldsymbol{h}$ such that $\mathrm{Ric}(\boldsymbol{h}) = O(r^{\frac{n-4}{2}})$ and tangential components of $\mathrm{Ric}(\boldsymbol{h})$ vanish to order $\frac{n-2}{2}$. Moreover, this solution is unique up to the action of an equivariant diffeomorphism fixing $\mathcal{Q}$ and addition of terms which vanish to order $\frac{n}{2}$ along $\mathcal{Q}$. In fact it is elementary to show that adding terms vanishing to order $\frac{n}{2}$ along $\mathcal{Q}$ one may even obtain an ambient metric such that $\mathrm{Ric}(\boldsymbol{h}) = O(r^{\frac{n-2}{2}})$. The uniqueness stetement for such metrics is slightly more complicated. There is an obstruction to the existence of an ambient metric $\boldsymbol{h}$ such that, in addition to these conditions, the tangential components of $\mathrm{Ric}(\boldsymbol{h})$ vanish to order $\frac{n}{2}$ along $\mathcal{Q}$.

The uniqueness result on the ambient metric in particular implies that for $n \neq 4$ the curvature of such an ambient metric as well as those of its covariant derivatives that are covered in part (1) of the theorem are intrinsic to the underlying conformal structure. Since the tractor curvature is intrinsic to the underlying conformal structure, part (1) of the theorem provides an alternative proof for this fact, which clearly comes close to an alternative proof for uniqueness of Fefferman–Graham metrics. It seems to us, however,

that the ideas developed in this paper should also have applications to the question of existence of ambient metrics and the nature of the obstruction in the Fefferman–Graham construction, so we will take up this whole circle of problems elsewhere. Below we will show how our results can be applied to obtain explicit descriptions of ambient Weyl invariants.

On the other hand, part (2) of the theorem goes significantly beyond the results in [8], since it analyses the cases in which the obstruction in the Fefferman–Graham construction vanishes. It shows that in these cases the only essential new ingredient is the critical covariant derivative $\nabla^{\frac{n-4}{2}} R$ of the ambient curvature, which then determines all higher covariant derivatives by universal tractor formulae. It should also be remarked here that large parts of this critical covariant derivative are again determined by the underlying conformal structure, since covariant derivatives in tangential directions are determined by the tractor connection.

3.5. Applications to the study of ambient Weyl invariants.
One of the main applications of the conformal ambient metric construction is that it allows a systematic construction of conformal invariants. It is well known that by Weyl's classical invariant theory any scalar valued polynomial Riemannian invariant can be written as a linear combination of complete contractions of tensor powers of iterated covariant derivatives of the Riemann curvature tensor. Consider an arbitrary Riemannian invariant in odd dimensions or an invariant depending only on covariant derivatives up to order less than $\frac{n-4}{2}$ in even dimensions $n \neq 4$. Applied to a Fefferman–Graham metric, one can consider the restriction of the resulting function to $\mathcal{Q}$. Since parts of different homogeneity of this function must be individually invariant, we may without loss of generality assume that our function is homogeneous of some degree and hence may be interpreted as a density on M. Now the fact that we deal with a Riemannian invariant exactly eliminates the diffeomorphism freedom in the ambient metric, while the freedom of adding terms that vanish along $\mathcal{Q}$ to the appropriate order has already been taken care of. Consequently, the resulting density on M is conformally invariant. Invariants obtained in that way are called ambient Weyl–invariants.

Theorem 3.4 not only provides an alternative proof for the fact the the construction outlined above leads to conformal invariants, but also provides a way to compute explicit formulae for ambient Weyl invariants, which otherwise is a difficult problem. On the one hand, one directly obtains an iterative way to compute tractor expressions for ambient Weyl invariants. For many purposes this is already sufficient, since one obtains genuine formulae for the given invariant and many qualitative features of the invariant can be appreciated in this compact form. On the other hand, converting tractor expressions into formulae in terms of a metric representing the conformal class, its Levi-Civita connection and curvature is a completely mechanical procedure which even may be left to a computer. Since we do not want to introduce too much tractor calculus here, we only roughly analyse a simple case below. More involved applications in a similar direction can be found in [12].

For the first step in this analysis, we assume $n \neq 4$, and we are dealing with an ambient metric $\boldsymbol{h}$ such that $\mathrm{Ric}(\boldsymbol{h}) = O(r)$ and tangential components of $\mathrm{Ric}(\boldsymbol{h})$ vanish to second order. Then we have observed above that $3\boldsymbol{D}^A\boldsymbol{X}_{[A}\boldsymbol{R}_{BC]EF}|_{\mathcal{Q}} = (n-2)(n-4)\boldsymbol{R}_{BCEF}|_{\mathcal{Q}}$. Further, $\boldsymbol{X}_{[A}\boldsymbol{R}_{BC]EF}|_{\mathcal{Q}}$ depends only on the tangential components (in the indices B and C) of the ambient curvature which are given by the tractor connection. Otherwise put, extending the tractor curvature κ_{bcEF} in any way to a tractor field Ω_{BCEF} and forming $\boldsymbol{X}_{[A}\Omega_{BC]EF}$, one will obtain the same result. In the general setting of tractor calculus, the expression $W_{BC}{}^E{}_F := \frac{3}{n-2}D^A X_{[A}\Omega_{BC]}{}^E{}_F$ has been known for some time (see [10, 11]) as a natural conformally invariant tractor extension of the tractor curvature (which itself is an extension of the Weyl curvature or the Cotton–York tensor in dimension 3). Thus we see that $W_{BC}{}^E{}_F$ is the tractor field equivalent to $(n-4)\boldsymbol{R}_{BC}{}^E{}_F|_{\mathcal{Q}}$.

To describe a formula for $W_{BC}{}^E{}_F$ we have to introduce some basic elements of tractor calculus, see [1, 10, 11, 12]. We use lower case letter for tensor indices and upper case letters for standard tractor indices and also ambient indices. We use the tractor metric and its inverse to raise and lower tractor indices. Choosing a metric g from the conformal class, we may raise and lower tensor indices using g (but taking into account that this changes the weight), and the standard tractor bundle $\mathcal{E}^A$ splits as $\mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$. This can be conveniently encoded by adding to the natural section $X^A \in \mathcal{E}^A[1]$ (which represents the natural inclusion $\mathcal{E}[-1] \to \mathcal{E}^A$) tractor sections $Y^A \in \mathcal{E}^A[-1]$ and $Z^{aA} \in \mathcal{E}^{aA}[-1]$ which represent the other two inclusions that depend on the choice of the metric g. Basic properties of the tractor metric imply that $Y^A X_A = 1$, $X^A X_A = Y^A Y_A = X^A Z_A^a = Y^A Z_A^a = 0$ and $Z^{aA}Z^b{}_A = g^{ab}$. Further, we denote by W_{abcd} the Weyl–curvature, by S the scalar curvature of g, by $\mathsf{P}_{ab} = \frac{1}{n-2}(\mathrm{Ric}_{ab} - \frac{1}{2(n-1)}Sg_{ab})$ the rho–tensor of g and by $B_{eb} := \nabla^q\nabla^p W_{peqb} + (n-3)\mathsf{P}^{qp}W_{peqb}$ the Bach tensor. Using the formulae in [10, 11] one verifies that W_{ABCE} is given by

$$(n-4)Z_A{}^a Z_B{}^b Z_C{}^c Z_E{}^e W_{abce} - 4(n-4)Z_A{}^a Z_B{}^b X_{[C}Z_{E]}{}^e\nabla_{[a}\mathsf{P}_{b]e}$$

$$-4(n-4)X_{[A}Z_{B]}{}^b Z_C{}^c Z_E{}^e\nabla_{[c}\mathsf{P}_{e]b} + \frac{4}{n-3}X_{[A}Z_{B]}{}^b X_{[C}Z_{E]}{}^e B_{eb}$$

This is a general tractor formula not related to the ambient metric in any way, so in particular, it also holds in dimension $n = 4$. In that case, only the last term survives, which shows that B_{ab} is a conformal invariant in dimension 4. However, our interest here is in the case $n \neq 4$, and the main fact that we need from the above formula is that since $X^A Z_A^a = 0$ and $Z^{aA}Z^b{}_A = g^{ab}$ one immediately concludes that any complete contraction of $W_{ABCD} \otimes \cdots \otimes W_{IJKL}$ gives the complete contraction of $W_{abcd} \otimes \cdots \otimes W_{ijkl}$ with the same pairing of indices. Applying this to the case of ambient Weyl invariants, we obtain an alternative proof of [8, Proposition 3.2] that any complete contraction of a tensor power of the ambient curvature gives the "same" complete contraction of the corresponding tensor power of the Weyl–curvature. Of course, these are exactly the invariants one knew about in advance, so to get more interesting invariants one has to consider covariant derivatives of the ambient curvature.

The simplest ambient Weyl invariant involving covariant derivatives is $\|\boldsymbol{\nabla R}\|^2 := (\boldsymbol{\nabla}^A \boldsymbol{R}^{BCEF})\boldsymbol{\nabla}_A \boldsymbol{R}_{BCEF}$. In order that this is well defined, we have to assume that $n \neq 4, 6$ and that we are dealing with an ambient metric $\boldsymbol{h}$ such that $\mathrm{Ric}(\boldsymbol{h}) = O(r^2)$ and tangential components of $\mathrm{Ric}(\boldsymbol{h})$ vanish to third order along $\mathcal{Q}$. To apply our method, we first note following the proof of Lemma 3.4 that $\boldsymbol{R}$ has conformal weight -2, and thus

$$\boldsymbol{D}_A \boldsymbol{R}_{BCEF} = (n-6)\boldsymbol{\nabla}_A \boldsymbol{R}_{BCEF} - \boldsymbol{X}_A \boldsymbol{\Delta R}_{BCEF}.$$

From part (2) of Proposition 3.1 we next conclude that our assumptions on $\mathrm{Ric}(\boldsymbol{h})$ imply that the restriction to $\mathcal{Q}$ of $\boldsymbol{\Delta R}_{BCEF}$ coincides with a partial contraction of $\boldsymbol{R} \otimes \boldsymbol{R}$. Thus we see that up to a nonzero factor and the addition of complete contractions of tensor powers of the Weyl curvature, the ambient Weyl invariant $\|\boldsymbol{\nabla R}\|^2$ coincides with $(D^A W^{BCEF})D_A W_{BCEF}$. Using the standard formulae for the tractor D–operator and the formulae for W_{BCEF} above, this can be expanded into formula in terms a chosen representative of the conformal class. While this is straightforward, it is quite tedious since there are many components in $D_A W_{BCEF}$ and most of them do not contribute to the final invariant.

A more efficient way to proceed is to rewrite the original invariant as follows: Applying the differential Bianchi identity to the first term, we see that $(\boldsymbol{\nabla}_A \boldsymbol{R}_{BCEF})\boldsymbol{\nabla}^A \boldsymbol{R}^{BCEF}$ may be written as $2(\boldsymbol{\nabla}_B \boldsymbol{R}_{ACEF})\boldsymbol{\nabla}^A \boldsymbol{R}^{BCEF}$. On the other hand, consider $\boldsymbol{\nabla}^A \boldsymbol{\nabla}_B \boldsymbol{R}_{ACEF}$. Switching the covariant derivatives can be compensated by adding partial contractions of $\boldsymbol{R} \otimes \boldsymbol{R}$, and using part (1) of Proposition 3.1 and the proof of Lemma 3.4, we see that $\boldsymbol{\nabla}^A \boldsymbol{R}_{ACEF}$ can be written as $4r \boldsymbol{X}_C K_{[E} \boldsymbol{X}_{F]} + O(r^2)$ for an appropriate tractor field K_E. Hitting this with $\boldsymbol{\nabla}_B$, we obtain $4\boldsymbol{X}_B \boldsymbol{X}_C K_{[E} \boldsymbol{X}_{F]} + O(r)$, which vanishes upon contraction into $\boldsymbol{R}^{BCEF}$. The upshot of this is that, along $\mathcal{Q}$ and up to adding complete contractions of tensor powers of $\boldsymbol{R}$, we may rewrite $\|\boldsymbol{\nabla R}\|^2$ as $2\boldsymbol{\nabla}_A \boldsymbol{\nabla}_B \boldsymbol{R}^{ACDE} \boldsymbol{R}^B{}_{CDE}$.

Following the same idea as for the single covariant derivative above, one next shows that

$$\boldsymbol{D}_A \boldsymbol{D}_B \boldsymbol{R}^{ACDE} \boldsymbol{R}^B{}_{CDE} = (n-6)(n-8)\boldsymbol{\nabla}_A \boldsymbol{\nabla}_B \boldsymbol{R}^{ACDE} \boldsymbol{R}^B{}_{CDE},$$

up to complete contractions of tensor powers of $\boldsymbol{R}$. Thus the conformal invariant given by $\|\boldsymbol{\nabla R}\|^2$ is, up to the addition of complete contractions of tensor powers of the Weyl curvature, exactly

$$\frac{2}{(n-4)^2(n-6)(n-8)}D_A D_B W^{ACDE}W^B{}_{CDE}.$$

in dimensions other than $4, 6$. Note that we know in advance that the expression $D_A D_B W^{ACDE}W^B{}_{CDE}$ will yield $(n-8)$ times a conformal invariant because $\|\boldsymbol{\nabla R}\|^2$ is well defined in dimension 8. This is then easily expanded out using formulae for the tractor connection as in [4] or [12] to yield (once again modulo complete contractions of tensor powers of W),

$$\frac{2}{n-6}\Box\|W\|^2 + \frac{n-10}{(n-6)}[\|\nabla W\|^2 + 8h(W, U) - 4(n-6)\|C\|^2],$$

where $\Box$ is the tractor Laplacian, $C_{abd} = 2\nabla_{[a}\mathsf{P}_{b]d}$ is the Cotton–York tensor and $U_{abcd} := \nabla_a C_{cdb} + \mathsf{P}_a{}^e W_{ebcd}$. This shows explicitly how the invariant

simplifies in dimension 10. Alternatively we can re-express in the more compact form

$$\|\nabla W\|^2 + 16(W, U) - 4(n - 10)\|C\|^2.$$

It is readily verified that this precisely agrees with the result obtained by Fefferman-Graham in [8]. (In fact we have borrowed some notation from that source to simplify the comparison.) Note that although $\|\boldsymbol{\nabla R}\|^2$ is not well defined in dimensions 4 and 6, the last display does give an invariant in these dimensions. This is immediate from the fact that in arbitrary dimension n, $D_A D_B W^{ACDE} W^B{}_{CDE}$ may be written as the sum of $\frac{(n-4)^2(n-6)(n-8)}{2}(\|\nabla W\|^2 + 16h(W, U) - 4(n - 10)\|C\|^2)$ and complete contractions of tensor powers of W, and that the transformation formulae for the scalars $\|\nabla W\|^2$, $h(W, U)$ and $\|C\|^2$, under a change of metric from the conformal class, is of polynomial type.

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