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SLE_κ growth processes and conformal field theories

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Abstract

SLE_κ stochastic processes describe growth of random curves which, in some cases, may be identified with boundaries of two dimensional critical percolating clusters. By generalizing SLE_κ growths to formal Markov processes on the central extension of the 2d conformal group, we establish a connection between conformal field theories with central charges $c_\kappa = \frac{1}{2}(3\kappa - 8)(\frac{6}{\kappa} - 1)$ and zero modes – observables which are conserved in mean – of the SLE_κ stochastic processes.

Critical phenomena are characterized by large scale fluctuations. Conformal field theories [1] are powerful tools for analyzing their multifractal properties, leading for instance to exact results concerning scaling behavior of geometrical models, see eg. refs.[5]. An alternative probabilistic approach has recently been introduced [2]. It consists in formulating conformally covariant processes, so-called SLE_κ processes, based on Loewner's equation. Among the many conformal field theory results, including crossing percolation probabilities [6] or multifractal distributions of electrostatic potential near conformally invariant fractal boundaries [7], some have been rederived in the SLE_κ framework, see eg refs.[3, 4] for a review. However, the precise connection between SLE_κ models and conformal field theories in the sense

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of [1] remains elusive, although SLE_6 has been identified with 2d percolation, with central charge $c = 0$, SLE_2 with loop erased random walks, with $c = -2$, and $SLE_{8/3}$ with self-avoiding random walks, with $c = 0$ also. But see ref.[8].

The aim of this letter is to establish a direct relation between $c \leq 1$ conformal field theories and SLE_κ models by extending them to processes in the Virasoro group. This link goes through the construction (using conformal field theory) of observables which are conserved in mean during the SLE_κ growth. It bears some analogy with turbulent passive advections [9, 10].

SLE_κ processes. Stochastic Loewner evolution SLE_κ are growth processes [2] defined via conformal maps which are solutions of Loewner's equation:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t} \ , \quad g_{t=0}(z) = z \tag{1}$$

When ξ_t is a smooth real-valued function, the map $g_t(z)$ is the uniformizing map for a simply connected domain D_t of the upper half plane $\mathbf{H}$, $\mathrm{Im} z > 0$. The map $g_t(z)$, normalized by $g_t(z) = z + 2t/z + \cdots$ at infinity, is well-defined up to the time τ_z for which $g_{\tau_z}(z) = \xi_{\tau_z}$. Notice that $\mathrm{Im} g_t(z)$ is a decreasing function of time on $\mathbf{H}$. Following refs.[2, 3, 4], define $K_t = \{z \in \mathbf{H} : \ \tau_z \leq t\}$. They form an increasing sequence of sets, $K_{t'} \subset K_t$ for $t' < t$, and for smooth enough driving source ξ_t, they are simple curves staggering along $\mathbf{H}$. The domain D_t is $\mathbf{H} \setminus K_t$.

SLE_κ processes are defined [2] by choosing a Brownian motion as driving parameter in the Loewner's equation: $\xi_t = \sqrt{\kappa}\, B_t$ with B_t a two-sided normalized Brownian motion and κ a parameter. The growth processes are then that of the sets K_t.

Lifted SLE_κ processes. Let L_n be the generators of the Virasoro algebra vir – the central extension of the Lie algebra of conformal transformations – with commutation relations $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}$. Let $\mathcal{V}ir$ be the formal group obtained by exponentiating elements of vir. We define a (formal) stochastic Markov process on $\mathcal{V}ir$ by the first order stochastic differential equation generalizing random walks on Lie groups:

$$G_t^{-1}\, dG_t = -2dt\, L_{-2} + d\xi_t\, L_{-1} \tag{2}$$

where $d\xi_t \equiv w_t dt$, so that w_t is Gaussian with white-noice covariance $\langle w_t\, w_s \rangle = \kappa \delta(t-s)$. With initial condition $G_{t=0} = 1$, the elements G_t belong to the formal group $\mathcal{V}ir_-$ obtained by exponentiating the generators L_n, $n<0$, of negative grades of the Virasoro algebra. We may order factors in $\mathcal{V}ir_-$ according to their grades n, so that group elements G are presented in the form $\cdots e^{x_2 L_{-2}} e^{x_1 L_{-1}}$. Eq.(2) turns into a family of ordinary stochastic differential equations for the x_k's, with a probabilistic measure induced by that of the Brownian motion.

The connection with the previously introduced SLE_κ growth process emerges from the action induced by the above flows on conformal fields. Recall from ref.[1], that conformal primary fields $\phi_\Delta(z)$ are z-dependent operators acting on appropriate representations of the Virasoro algebra and transforming as forms of weight Δ under conformal transformations. They satisfy the following intertwining relations: $[L_n, \phi_\Delta(z)] = \ell_n^\Delta \cdot \phi_\Delta(z)$ with $\ell_n^\Delta \equiv z^{n+1}\partial_z + (n+1)\Delta z^n$. As a consequence, the group $\mathcal{V}ir$ acts on primary fields by conformal transformations and, in particular for the flows (2), one has:

$$G_t^{-1}\, \phi_\Delta(z)\, G_t = [\partial_z \hat{g}_t(z)]^\Delta\, \phi_\Delta(\hat{g}_t(z))$$

where $\hat{g}_t(z) \equiv g_t(z) - \xi_t$ with $g_t(z)$ solution of the Loewner's equation. This means that the conformal transformations induced by the lifted SLE_κ flows on the primary fields reduce to that of the conformal maps of the SLE_κ processes.

Alternatively, the SLE_κ stochastic equation may be represented as $\hat{g}_t(z) = H_t\, z\, H_t^{-1}$ with $H_t^{-1} dH_t = 2dt\, \ell^0_{-2} - d\xi_t\, \ell^0_{-1}$. Indeed, equation (1) translates into $\partial_t \hat{g}_t(z) = 2/\hat{g}_t(z) - w_t$ for $\hat{g}_t(z) = g_t(z) - \xi_t$ which is then equivalent to the commutation relation $[H_t^{-1}\partial_t H_t, z] = 2/z - w_t$ among differential operators. The group $\mathcal{V}ir_-$ may be presented as the group of germs of meromorphic functions with a pole at infinity such that $f(z) = z(1 + a_1/z + a_2/z^2 + \cdots)$ with a_k as coordinates, similar to Fock space coordinates. The Virasoro generators are then differential operators in the a_k's.

Time evolution and Fokker-Planck equations. Any observable of the random process G_t may be thought of as function on $\mathcal{V}ir$, and we use formal rules extending those valid in finite dimensional Lie groups. We denote by $\mathbf{E}[F(G_t)]$ the expectation value of the observable $F(G_t)$.

Our main result is the following representation of their time evolution:

$$\partial_t\, \mathbf{E}[F(G_t)] = \mathbf{E}[-2\nabla_{-2} F(G_t) + \frac{\kappa}{2}\nabla_{-1}^2 F(G_t)] \tag{3}$$

where ∇_n are the left invariant vector fields associated to the elements L_n in vir defined by $(\nabla_n F)(G) = \frac{d}{du} F(G\, e^{uL_n})|_{u=0}$ for any appropriate function F on $\mathcal{V}ir$.

Hints for the proof of eq.(3) are as follows. By definition of the lifted SLE_κ flows, the mean value of any test function of such flows evolves according to $\partial_t \mathbf{E}[F(G_t)] = \mathbf{E}[-2\nabla_{-2}F(G_t) + w_t\nabla_{-1}F(G_t)]$. Since w_t is a Gaussian variable with white-noice covariance, the last term may be identified with $\kappa\mathbf{E}[\nabla_{-1}\delta F(G_t)/\delta w_t]$. So one has to compute $\delta F(G_t)/\delta w_t$. Small variation of eq.(2) implies that $\frac{d}{dt}(\frac{\delta G_t}{\delta w_s}\, G_t^{-1}) = (G_s L_{-1} G_s^{-1})\delta(t-s)$. As a consequence, we get $\delta F(G_t)/\delta w_t = \frac{1}{2}\nabla_{-1}F(G_t)$ and eq.(3).

As for any Markov process, the time evolution of the probability distribution functions (Pdf) of the lifted SLE_κ processes are governed by Fokker-Planck equations whose hamiltonians are the generators of the semi-groups specifying the processes. These Pdf's, denoted $\mathcal{P}_t(G)$, may be written as the averages of the point Dirac measures localized on G: $\mathcal{P}_t(G) \equiv \mathbf{E}[\delta_G(G_t)]$. Their time evolution is:

$$\partial_t \mathcal{P}_t(G) = \mathcal{H}\cdot\mathcal{P}_t(G), \quad \mathcal{H} \equiv 2\nabla_{-2} + \frac{\kappa}{2}\nabla_{-1}^2 \tag{4}$$

This follows from eq.(3) with $F(G_t) = \delta_G(G_t)$, using the fact that the Lie derivatives of $\delta_G(G_t)$ with respect to G_t are the opposite of its Lie derivatives with respect to G.

By eq.(4), the Pdf's are time transported by the Fokker-Planck hamiltonian $\mathcal{H}$ so that we expect $\mathcal{P}_t(G) = \exp t\mathcal{H}\cdot\mathcal{P}_{t=0}(G)$. Alternatively, the probability transitions from G_0 at time t_0 to G at time $t > t_0$ are the kernels of the operator $\Big(\exp(t-t_0)\mathcal{H}\Big)_{G,G_0}$. A similar derivation remains valid if we consider the stochastic flows $x_t = G_t^{-1}\cdot x_0$ induced by eq.(2) on any representation of $\mathcal{V}ir$. Of course, these stochastic processes are calling for a more mathematical precise description, along the lines of refs.[2, 3, 4].

Zero modes and null vectors. Since left invariant Lie derivatives form a representation of vir, equation (3) may be written as:

$$\partial_t\, \mathbf{E}[F(G_t)] = \mathbf{E}[\mathcal{H}^T\cdot F(G_t)], \quad \mathcal{H}^T \equiv -2L_{-2} + \frac{\kappa}{2}\, L_{-1}^2 \tag{5}$$

with the Virasoro generators L_n acting on the appropriate representation space. By definition a zero modes F_ω is an eigenvector of $\mathcal{H}^T$ with zero eigenvalue: $\mathcal{H}^T\cdot F_\omega = 0$. As a consequence a zero mode is an observable conserved in mean.

To construct such conserved quantities we look for such F_ω's among zero modes annihilated by the L_n, $n > 0$, ie. among the so-called highest weight vectors, $L_nF_\omega = 0$, $n > 0$, with given conformal dimension h, $L_0F_\omega = hF_\omega$.

We now demand under which conditions $\mathcal{H}^T F_\omega$ is again a highest weight vector. We have

$$[L_n, \mathcal{H}^T] = (-2(n+2) + \frac{\kappa}{2}n(n+1))L_{n-2} + \kappa(n+1)L_{-1}L_{n-1} - c\delta_{n,2}$$

Hence, $L_n\mathcal{H}^T F_\omega = 0$ for any $n \geq 3$, but demanding $L_1\mathcal{H}^T F_\omega = 0$ requires $2\kappa h = 6 - \kappa$, whereas $L_2\mathcal{H}^T F_\omega = 0$ imposes $c = h(3\kappa - 8)$.

Thus, for these values of c and h, $\mathcal{H}^T F_\omega$ is a highest weight vector, a so-called null vector, which can be consistently set to zero, see eg.[1]. In other words, F_ω is such that $\mathcal{H}^T F_\omega \simeq 0$ if and only if the Virasoro central charge is adjusted to

$$c_\kappa = \frac{1}{2}\Big(3\kappa - 8\Big)\Big(\frac{6}{\kappa} - 1\Big) = 1 - 6\Big(\frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}\Big)^2 \tag{6}$$

and the conformal weight to $h_\kappa = \frac{1}{2}(\frac{6}{\kappa} - 1)$. Furtheremore, the stationary property of the mean of F_ω is a consequence of the existence of a null vector in the corresponding Virasoro Verma module. The analogy with conserved modes in turbulent passive advection [9, 10] is particularly striking.

Such zero modes may be constructed by considering the orbit of the highest weight vector ω of conformal dimension h_κ in the c_κ conformal field theory and defining $F_\omega(G_t) = G_t \cdot \omega$ so that

$$\partial_t\, \mathbf{E}\,[G_t \cdot \omega] = 0$$

All components of the vector $\mathbf{E}\,[G_t \cdot \omega]$ are conserved. Similarly, since the conformal field $\phi_x(z)$ associated to any state x in the Virasoro module with highest weight vector ω depends linearly on x, the matrix elements of $\mathbf{E}\,[\phi_{G_t\cdot\omega}(z)]$ are also conserved.

In conclusion, we have established a direct link between observables conserved in mean and conformal null vectors. This opens a route to the study of probabilistic properties of growth processes using conformal field theory. The above central charge coincides with that of the conformal field theory expected to be associated with the SLE_κ process. It is invariant under the duality transformation $\kappa \to 16/\kappa$, it vanishes for $\kappa = 6$ as expected for percolation, and the self-dual point $\kappa = 4$ corresponds to the $c_\kappa = 1$ free field theory.

More on the algebraic and geometrical aspects as well as on applications of this construction will be described elsewhere [11].

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