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# (3+1)-Dimensional Schwinger Terms and Non-commutative Geometry

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## Abstract

We discuss 2-cocycles of the Lie algebra  $Map(M^3; \underline{g})$  of smooth, compactly supported maps on 3-dimensional manifolds  $M^3$  with values in a compact, semi-simple Lie algebra  $\underline{g}$ . We show by explicit calculation that the Mickelsson-Faddeev-Shatashvili cocycle  $\frac{i}{24\pi^2} \int \text{tr}(A[dX, dY])$  is cohomologous to the one obtained from the cocycle given by Mickelsson and Rajeev for an abstract Lie algebra  $\underline{g}_2$  of Hilbert space operators modeled on a Schatten class in which  $Map(M^3; \underline{g})$  can be naturally embedded. This completes a rigorous field theory derivation of the former cocycle as Schwinger term in the anomalous Gauss' law commutators in chiral QCD(3+1) in an operator framework. The calculation also makes explicit a direct relation of Connes' non-commutative geometry to (3+1)-dimensional gauge theory and motivates a novel calculus generalizing integration of  $\underline{g}$ -valued forms on 3-dimensional manifolds to the non-commutative case.

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**1. Introduction.** Infinite dimensional Lie algebras  $Map(M^d; \underline{g})$  of smooth, compactly supported maps from a  $d$ -dimensional manifold  $M^d$  to a compact, semi-simple Lie algebra  $\underline{g}$  (e.g.  $\underline{g} = \mathfrak{su}(N)$ ) are closely related to  $(d+1)$ -dimensional quantum field theory (QFT). One strong motivation for studying projective representations of these algebras is the hope that they could lead to progress in the understanding of the non-perturbative structure of associated QFT models. This has been indeed so for  $d = 1$ : the by-now well-understood representation theory of the loop algebras  $Map(S^1; \underline{g})$  has played a crucial role in recent spectacular progress in  $(1+1)$ -dimensional QFT (e.g. conformal QFT(1+1); for a recent construction of QCD(1+1) with massless quarks based on the representation theory of loop algebras see [1]).

One natural interpretation of  $Map(M^d; \underline{g})$  is as Lie algebra of the group of static gauge transformations of a Yang-Mills gauge theory on space-time  $M^d \times \mathbb{R}$ . It is then natural to consider also the set  $\mathcal{A}(M^d)$  of all (static) Yang-Mills field configurations  $A$  on  $M^d$ , i.e.  $A$  are the  $\underline{g}$ -valued, compactly supported 1-forms on  $M^d$ . For  $d = 3$  an extension of  $Map(M^3; \underline{g})$  is given by the Mickelsson-Faddeev-Shatashvili cocycle [2, 3]

$$c_{MFS}(X, Y; A) = \frac{i}{24\pi^2} \int_{M^3} \text{tr}(A[dX, dY]) \quad (1)$$

( $X, Y \in Map(M^3; \underline{g})$ ,  $A \in \mathcal{A}(M^3)$ ; we use the standard notation for forms on  $M^3$  ( $d$  is the exterior derivative etc.) suppressing the wedge product, and implicitly assume a representation of  $\underline{g}$  in some  $\mathfrak{gl}(N)$  (algebra of complex  $N \times N$  matrices) acting on  $\mathbb{C}^N = \mathbb{C}_{color}^N$  where  $\text{tr}$  is the usual trace of  $N \times N$  matrices).

There are “big” abstract Lie algebras  $\underline{g}_p$  of operators on a Hilbert space modeled on Schatten classes which play a central role in the mathematical investigation of  $Map(M^d; \underline{g})$ . The motivation for introducing  $\underline{g}_p$  is that it naturally contains  $Map(M^d; \underline{g})$  for *any* ‘nice’<sup>1</sup>  $d$ -dimensional manifold  $M^d$  if  $p = (d + 1)/2$ , and that it is possible to develop the representation theory of  $\underline{g}_p$  as a whole and obtain the ones of  $Map(M^d; \underline{g})$  by restriction from that [4, 5]. Actually, for  $p > 1$  the representation theory of  $\underline{g}_p$  requires to introduce another “big” set of operators  $Gr_p$  — the so-called Grassmannian — modeled on the same Schatten class as  $\underline{g}_p$ . From a physical point of view this is quite natural as one can naturally embed the sets  $\mathcal{A}(M^d)$  of Yang-Mills configurations in  $Gr_p$  if  $p = (d + 1)/2$ , and there is a natural action of  $\underline{g}_p$  on  $Gr_p$  gen-

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<sup>1</sup> $C^\infty$  manifold with a Riemannian- and a spin structure

eralizing the gauge transformations by which elements of  $Map(M^d; \underline{g})$  act on  $\mathcal{A}(M^d)$ . It is also interesting to note that these very Lie algebras  $\underline{g}_p$  play a fundamental role in Connes' non-commutative geometry [6].

To define  $\underline{g}_p$  and  $Gr_p$  one considers a separable Hilbert space  $h$  which is decomposed in a direct sum of two infinite dimensional, orthogonal subspaces,  $h = h_+ \oplus h_-$  (we recall that abstractly, all such Hilbert spaces  $h$  are essentially — up to unitary equivalence — the same). Such a decomposition is uniquely determined by the operator  $\varepsilon$  on  $h$  which is  $+1$  on  $h_+$  and  $-1$  on  $h_-$ ,  $h_{\pm} = \frac{1}{2}(1 \pm \varepsilon)h$ . Then  $\underline{g}_p$  is defined as the Lie algebra of all bounded operators on  $h$  such that  $([\varepsilon, u]^*[\varepsilon, u])^p$  is trace class ( $*$  is the Hilbert space adjoint; we recall that an operator  $a$  on  $h$  is *trace class* if  $\sum_n | \langle f_n, a f_n \rangle |$  is finite for any complete orthonormal basis  $\{f_n\}$  in  $h$ , and then its *Hilbert space trace*  $\text{Tr}(a) \equiv \sum_n \langle f_n, a f_n \rangle$  exists, i.e. it is finite and independent of  $\{f_n\}$  [7]).

To explain the embeddings of  $Map(M^d; \underline{g})$  in  $\underline{g}_p$  and  $\mathcal{A}(M^d)$  in  $Gr_p$  we consider chiral fermions on space-time  $M^d \times \mathbb{R}$  coupled to an external Yang-Mills field  $A \in \mathcal{A}(M^d)$ . Then the Gauss' law generators  $G(X)$  implementing the infinitesimal gauge transformations  $X \in Map(M^d; \underline{g})$  in the physical Hilbert space of the fermions should obey equal-time commutators of the following form,

$$[G(X), G(Y)]_{\text{ETC}} = G([X, Y]) + S_{d+1}(X, Y; A) \quad (2)$$

with a Schwinger terms  $S_{d+1}$  satisfying a 2-cocycle relation due to the Jacobi identity for the equal-time commutator [3]. For  $d = 3$  cohomological arguments suggest that this Schwinger term should be equal (up to a boundary) to the MFS cocycle (1) [3].

To explicitly construct these Gauss' law generators, one can start with the Hilbert space  $h$  of 1-particle states of the chiral fermions, i.e.  $h = L^2(M^d) \otimes V_{spin} \otimes \mathbb{C}_{color}^N$  with  $V_{spin}$  a vector space carrying the spin structure. Then the 1-particle time evolution of the fermions is determined by the Weyl operator  $\mathcal{D}_A$  in the external field  $A$ . This is a self-adjoint operator on  $h$ , and it provides a natural splitting of  $h$  in positive- and negative energy states,  $h = h_+^A \otimes h_-^A$  with  $\mathcal{D}_A \geq 0$  ( $< 0$ ) on  $h_+^A$  ( $h_-^A$ ). It is this splitting which determines the physical Hilbert space of the fermions (one has to fill up the Dirac sea corresponding to the negative energy states). Then  $F_A$  is defined to be the operator which is  $\pm 1$  on  $h_{\pm}^A$ , and  $\varepsilon = F_0$  (no external field). Note that the mapping  $A \mapsto F_A$  is continuous along gauge orbits (it has discontinuities only for

those configurations  $A$  where an eigenvalue of  $\mathcal{D}_A$  crosses zero).

Infinitesimal gauge transformations  $X \in \text{Map}(M^d; \underline{g})$  naturally correspond to self-adjoint operators on  $h$ ,  $(Xf)(\vec{x}) = X(\vec{x})f(\vec{x})$  for all  $f \in h$  (to simplify notation we use the same symbol for  $X \in \text{Map}(M^d; \underline{g})$  and the corresponding operator on  $h$ ). The basic result implying the embedding referred to above is that  $F_A \in Gr_p$  and  $X \in \underline{g}_p$  for all  $A \in \mathcal{A}(M^d)$  and  $X \in \text{Map}(M^d; \underline{g})$  if  $p \geq (d+1)/2$ , see e.g. [4].

From an abstract point of view, every  $u \in \underline{g}_p$  corresponds to an infinitesimal fermion transformation and every  $F \in Gr_p$  to a fermion Dirac sea, and it is natural to consider implementors  $G(u)$  for all  $u \in \underline{g}_p$  and  $F \in Gr_p$  satisfying

$$[G(u), G(v)]_{\text{ETC}} = G([u, v]) + c_p(u, v; F) \quad (3)$$

where  $c_p$  is a 2-cocycle. Indeed, the very definitions of  $\underline{g}_p$  and  $Gr_p$  characterize a certain degree of divergence and thus determine a regularization procedure adequate for this type of divergence [5, 8]. Moreover, this regularization procedure is uniquely determined by the 2-cocycle  $c_p$  up to a coboundary  $\delta b$ ,

$$(\delta b)(u, v; F) \equiv b([u, v]; F) - \mathcal{L}_u b(v; F) + \mathcal{L}_v b(u; F) \quad (4a)$$

with the Lie derivative  $\mathcal{L}_u$  acting on functions  $f(F)$  as

$$\mathcal{L}_u f(F) \equiv \left. \frac{1}{i} \frac{\partial}{\partial t} f(e^{-iut} F e^{iut}) \right|_{t=0}. \quad (4b)$$

The change  $c_p \rightarrow c_p + \delta b$  corresponds to a *finite* (i.e. trivial) change of the regularization. It is worth pointing out that this abstract construction is not only convenient mathematically but also natural from the physical point of view: Besides the infinitesimal gauge transformation,  $\underline{g}_p$  contains also other operators of interest for  $(d+1)$ -dimensional gauge theories with fermions (see e.g. [5]), and the mathematical construction of the algebra (3) should therefore provide a general procedure adequate for (ultra-violet) divergences in the matter sector of such theories.

For  $p = 2$  (corresponding to  $d = 3$ ) the natural extension of  $\underline{g}_2$  is given by the Mickelsson-Rajeev cocycle [4]

$$c_{MR}(u, v; F) = -\frac{1}{8} \text{Tr}_C ((F - \varepsilon)[[\varepsilon, u], [\varepsilon, v]]) \quad (5)$$

( $u, v \in \underline{g}_2, F \in Gr_2$ ) where we introduced the conditional trace

$$\text{Tr}_C(a) \equiv \frac{1}{2} \text{Tr}(a + \varepsilon a \varepsilon) \quad (6)$$

which exists and is finite for all operator  $a$  on  $h$  so that  $a + \varepsilon a \varepsilon$  is trace class . We call such operators *conditionally trace class*. Note that  $\text{Tr}_C(a) = \text{Tr}(a)$  for all trace class operators  $a$ . (We note that the operator  $(F - \varepsilon)[[\varepsilon, u], [\varepsilon, v]]$  for  $F \in Gr_2$ ,  $u, v \in \underline{g}_2$  is *not* trace class but only conditionally trace class in general [5], hence only its conditional trace exists. The necessity to use  $\text{Tr}_C$  and not  $\text{Tr}$  in the formula for the MR cocycle has not been made sufficiently clear in [4, 5].)

In [4] a cohomological argument was given that the MFS-cocycle should be equivalent to the MR-cocycle, i.e. given the natural embeddings of  $Map(M^3; \underline{g})$  in  $\underline{g}_2$ , and  $A \mapsto F_A$  of  $\mathcal{A}(M^d)$  in  $Gr_p$ ,

$$c_{MR}(X, Y; F_A) = c_{MFS}(X, Y; A) + (\delta b)(X, Y; A) \quad (7)$$

for some boundary  $\delta b$ .

In this paper we prove by explicit calculation that this is true. To avoid technicalities we restrict ourselves to the simplest case  $M^3 = \mathbb{R}^3$  (we recall that all mappings  $X, Y, A$  considered have compact support). The extension of our result to arbitrary manifolds  $M^3$  can then be done using basic results on symbol calculus on manifolds [9].

We believe that this calculation is interesting for two reasons. Firstly, in combination with the results in [5] it provides a *rigorous* derivation of the MFS-cocycle in the anomalous commutators of the Gauss' law generators in chiral QCD(3+1) in an operator framework, the Yang-Mills field being treated as external, non-quantized field. (A different solution to this problem was recently explained in [10].) Though several field theory derivations of this result exist in the literature (using the BJL-limit to define equal time commutators, e.g. [11], or Berry's phase, e.g. [12], none of these is very satisfactory from a more mathematical point of view<sup>2</sup>. Secondly (as we discuss in more detail in the final paragraph), the calculation shows very explicitly a natural relation of non-commutative geometry (NCG) [6] to (3+1)-dimensional Yang-Mills gauge theory and motivates a new generalization of the integration calculus of forms to the non-commutative case. It has been repeatedly pointed out by Connes that NCG should provide an appropriate mathematical framework for formulating and studying *quantum* gauge theory without perturbation theory. To our knowledge this

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<sup>2</sup>An earlier indication of non-vanishing Schwinger terms was obtained in a perturbative computation of vacuum expectation values of hadronic currents in external U(1) gauge field [13]

program has not yet lead to many new results (one result in this direction is Rajeev's universal Yang-Mills theory [14]). We therefore believe that the study of Lie algebras  $Map(M^d; \underline{g})$  by extending to operator algebras  $\underline{g}_2$  provides a very interesting example where the NCG point of view is successfully used for getting deeper insight in QFT divergences arising in a gauge theory. The present paper can be regarded as an attempt to bridge the gap between this abstract, mathematical approach and more standard particle physics methods for the physically relevant case  $d = 3$ .

**2. Calculation. a.** Our Hilbert space is  $h = L^2(\mathbb{R}^3) \otimes \mathbb{C}_{spin}^2 \otimes \mathbb{C}_{color}^N$ , and the free Weyl operator can be represented as  ${}^3\mathcal{D}_0 = (-i)\partial_i\sigma_i$  where  $\sigma_i$  are the Pauli spin matrices acting on  $\mathbb{C}_{spin}^2$  and  $\partial_i = \partial/\partial x^i$ . For our calculation we need some basic facts about symbol calculus [9]. We recall that every pseudodifferential operator (PDO<sup>4</sup>)  $a$  on  $h$  can be represented by its *symbol*  $\sigma(a)(\vec{p}, \vec{x})$  which is a  $\mathfrak{gl}(2)_{spin} \otimes \mathfrak{gl}(N)_{color}$ -valued function on phase space  $\mathbb{R}^3 \times \mathbb{R}^3$  and defined such that for any  $f \in h$ ,

$$(af)(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} e^{-i\vec{p}\vec{x}} \sigma(a)(\vec{p}, \vec{x}) \hat{f}(\vec{p}) \quad (8)$$

where  $\hat{f}(\vec{p}) = \int \frac{d^3x}{(2\pi)^3} e^{i\vec{p}\vec{x}} f(\vec{x})$  denotes the Fourier transform of  $f$ . It follows then that

$$\sigma(ab)(\vec{p}, \vec{x}) = \int_{\mathbb{R}^3} \frac{d^3q}{(2\pi)^3} \int_{\mathbb{R}^3} d^3y e^{i(\vec{x}-\vec{y})(\vec{p}-\vec{q})} \sigma(a)(\vec{q}, \vec{x}) \sigma(b)(\vec{p}, \vec{y}), \quad (9)$$

and for  $a$  trace-class,

$$\text{Tr}(a) = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \int_{\mathbb{R}^3} d^3x \text{tr}'(\sigma(a)(\vec{p}, \vec{x})) \quad (10)$$

where  $\text{tr}' = \text{tr}_{spin} \text{tr}_{color}$ . Especially,  $\sigma(\varepsilon)(\vec{p}, \vec{x}) = \not{p} \equiv \varepsilon(\vec{p})$  where  $\not{p} \equiv p_i \sigma_i$  and  $p \equiv |\vec{p}|$ , and  $\sigma(X)(\vec{p}, \vec{x}) = X(\vec{x})$  for all  $X \in Map(\mathbb{R}^3; \underline{g})$ .

All operators  $a$  of interest to us allow an asymptotic expansion  $\sigma(a) \sim \sum_{j=0}^{\infty} \sigma_{-j}(a)$  where  $\sigma_{-j}(a)(\vec{p}, \vec{x})$  is homogeneous of degree  $-j$  in  $\vec{p}$ ,<sup>5</sup> and it goes to zero like  $p^{-j}$  for  $p \rightarrow \infty$ . We write

$$\sigma(a)(\vec{p}, \vec{x}) = \sum_{j=0}^n \sigma_{-j}(a)(\vec{p}, \vec{x}) + O(p^{-n-1}) \quad (11)$$

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<sup>3</sup>repeated indices  $i \in \{1, 2, 3\}$  are summed over throughout

<sup>4</sup>all operators of interest to us are PDOs

<sup>5</sup>i.e.  $\sigma_{-j}(a)(s\vec{p}, \vec{x}) = s^{-j} \sigma_{-j}(a)(\vec{p}, \vec{x})$  for all  $s > 0$

for all integers  $n$ . Moreover, eq. (9) has an asymptotic expansion in powers of  $p^{-1}$ ,

$$\sigma(ab)(\vec{p}, \vec{x}) \sim \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \frac{\partial^n \sigma(a)(\vec{p}, \vec{x})}{\partial p_{i_1} \cdots \partial p_{i_n}} \frac{\partial^n \sigma(b)(\vec{p}, \vec{x})}{\partial x_{i_1} \cdots \partial x_{i_n}}. \quad (12)$$

This allows to determine the asymptotic expansion of  $\sigma(ab)$  from the ones of  $\sigma(a)$  and  $\sigma(b)$ . Especially if  $\sigma(a)$  is  $O(p^{-n})$  and  $\sigma(b)$   $O(p^{-m})$  then  $\sigma(ab)$  is  $O(p^{-(n+m)})$ .

In our calculation we shall only need the leading terms of the asymptotic expansion of the symbols of  $[\varepsilon, X]$  for  $X \in \text{Map}(\mathbb{R}^3; \underline{g})$  and  $F_A - \varepsilon$  for  $A \in \mathcal{A}(\mathbb{R}^3)$ ,

$$\sigma([\varepsilon, X])(\vec{p}, \vec{x}) = (-i) \frac{\partial \varepsilon(\vec{p})}{\partial p_i} \partial_i X(\vec{x}) + O(p^{-2}) \quad (13a)$$

$$\sigma(F_A - \varepsilon)(\vec{p}, \vec{x}) = \frac{\partial \varepsilon(\vec{p})}{\partial p_i} A_i(\vec{x}) + O(p^{-2}). \quad (13b)$$

(Eq. (13a) immediately follows from (12). An elementary argument proving (13a) is as follows. One writes  $F_A = \not{D}_A / \sqrt{\not{D}_A^2}$ . Then with  $\sigma(\not{D}_A)(\vec{p}, \vec{x}) = \not{p} + \not{A}(\vec{x})$  one gets  $\sigma(\not{D}_A^2)(\vec{p}, \vec{x}) = p^2 \left(1 + \frac{\not{p}\not{A}(\vec{x})}{p^2} + \frac{\not{A}(\vec{x})\not{p}}{p^2} + O(p^{-2})\right)$ , and using (12),

$$\sigma(1/\sqrt{\not{D}_A^2})(\vec{p}, \vec{x}) = \frac{1}{p} \left(1 - \frac{1}{2} \frac{\not{p}\not{A}(\vec{x})}{p^2} - \frac{1}{2} \frac{\not{A}\not{p}(\vec{x})}{p^2} + O(p^{-2})\right)$$

implying

$$\sigma(F_A)(\vec{p}, \vec{x}) = \frac{1}{p} \left(\not{p} + \not{A}(\vec{x}) - \frac{1}{2} \frac{\not{p}^2 \not{A}(\vec{x})}{p^2} - \frac{1}{2} \frac{\not{p}\not{A}(\vec{x})\not{p}}{p^2}\right) + O(p^{-2}).$$

Noting that  $\varepsilon(\vec{p}) = \frac{\not{p}}{p}$  and  $\frac{\partial \varepsilon(\vec{p})}{\partial p_i} = \frac{1}{p} \left(\sigma_i - \frac{\not{p} p_i}{p^2}\right)$ , eq. (13a) follows from the properties of the Pauli matrices  $\sigma_i$ .)

We will have to evaluate (regularized) traces only for operators with symbols having compact support in  $\vec{x}$ . Note that such an operator  $a$  is trace class if and only if its symbol is  $O(p^{-4})$ , and if it is not trace class we still can define its *regularized trace* by introducing a momentum cut-off  $\Lambda > 0$  for the divergent parts of its symbol,

$$\begin{aligned} \text{Tr}_\Lambda(a) &= \int_{p \leq \Lambda} \frac{d^3 p}{(2\pi)^3} \int_{\mathbb{R}^3} d^3 x \text{tr}'(\sigma(a)(\vec{p}, \vec{x})) + \\ &\int_{p > \Lambda} \frac{d^3 p}{(2\pi)^3} \int_{\mathbb{R}^3} d^3 x \text{tr}' \left( \left( \sigma(a) - \sum_{j=0}^3 \sigma_{-j}(a) \right) (\vec{p}, \vec{x}) \right). \end{aligned} \quad (14)$$

Using the rules for symbol calculus above one can easily convince oneself that

$$\text{Tr}_\Lambda(a) = \text{Tr}_\Lambda(\varepsilon a \varepsilon) \quad (15)$$

for all bounded PDOs  $a$  with symbols having compact support in  $\vec{x}$ . Obviously  $\text{Tr}_\Lambda(a) = \text{Tr}(a)$  independent of  $\Lambda > 0$  for trace class operators  $a$ , and this implies

$$\text{Tr}_C(a) = \text{Tr}_\Lambda(a) \quad \forall \Lambda > 0 \text{ if } a \text{ is conditionally trace class.} \quad (16)$$

**b.** For trace class operators  $u, v$ , the MR cocycle [4] is trivial and can be represented as (see e.g. [5]; this statement will be also verified during our calculation below)

$$c_{MR}(u, v; F) = (\delta b)(u, v; F) \quad (17)$$

with

$$\begin{aligned} b &= b_1 + b_2 \\ b_1(u; F) &= -\frac{1}{2}\text{Tr}(u\varepsilon) \quad \text{independent of } F \\ b_2(u; F) &= \frac{1}{16}\text{Tr}([\varepsilon, F][\varepsilon, u]). \end{aligned} \quad (18)$$

The boundary operation  $\delta$  is defined in (4a,b). Defining  $c_{MR}^\Lambda$  as in (5) with  $\text{Tr}_C$  replaced by  $\text{Tr}_\Lambda$  and similarly  $b^\Lambda$ ,  $b_1^\Lambda$  and  $b_2^\Lambda$ , we introduce

$$\Delta c_{MR}^\Lambda \equiv c_{MR}^\Lambda - \delta b^\Lambda. \quad (19)$$

From (16) it follows that  $c_{MR}(u, v; F) = c_{MR}^\Lambda(u, v; F)$  for  $F \in Gr_2$ ,  $u, v \in \underline{g}_2$ , hence we can write

$$c_{MR}(u, v; F) = \Delta c_{MR}^\Lambda(u, v; F) + (\delta b^\Lambda)(u, v; F). \quad (20)$$

Eq. (17) implies that for  $u, v$  trace class we should get that  $\Delta c_{MR}^\Lambda(u, v; F) = 0$ . We therefore expect that it should be possible to make this explicit and represent  $\Delta c_{MR}^\Lambda(u, v; F)$  as a sum of terms of the form  $\text{Tr}_\Lambda([a, b])$  for some operators  $a, b$  (note that  $\text{Tr}_\Lambda$  is not cyclic:  $\text{Tr}_\Lambda(ab) \neq \text{Tr}_\Lambda(ba)$  in general!).

Indeed, it is not difficult to find such a representation: We write

$$\begin{aligned} c_{MR}^\Lambda &= c_1^\Lambda + c_2^\Lambda \\ c_1^\Lambda(u, v; F) &= \frac{1}{8}\text{Tr}_\Lambda(\varepsilon[[\varepsilon, u], [\varepsilon, v]]) \\ c_2^\Lambda(u, v; F) &= -\frac{1}{8}\text{Tr}_\Lambda(F[[\varepsilon, u], [\varepsilon, v]]). \end{aligned}$$

We first calculate the  $F$ -independent part of  $\Delta c_{MR}^\Lambda$  and obtain by a straightforward calculation using (15)

$$c_1^\Lambda(u, v; F) - (\delta b_1^\Lambda)(u, v; F) = \frac{1}{4}\text{Tr}_\Lambda([u, \varepsilon v] - (u \leftrightarrow v)).$$

To calculate the  $F$  dependent part, we first observe that due to (15), we can write  $b_2^\Lambda$  as

$$b_2^\Lambda(u; F) = -\frac{1}{8}\text{Tr}_\Lambda (F\varepsilon[\varepsilon, u])$$

and that  $(\delta b_2^\Lambda)(u, v; F) = b_2^\Lambda([u, v]; F) - b_2^\Lambda(v; [F, u]) + b_2^\Lambda(u; [F, v])$  as  $b_2^\Lambda(u; F)$  is linear in  $F$ ; due to the Jacobi identity,  $[\varepsilon, [u, v]] = [[\varepsilon, u], v] - (u \leftrightarrow v)$ , hence

$$(\delta b_2^\Lambda)(u, v; F) = -\frac{1}{8}\text{Tr}_\Lambda (F\varepsilon[[\varepsilon, u], v] + [F, v]\varepsilon[\varepsilon, u] - (u \leftrightarrow v))$$

Writing now

$$c_2^\Lambda(u, v; F) = \frac{1}{8}\text{Tr}_\Lambda (F[\varepsilon, v][\varepsilon, u] - (u \leftrightarrow v))$$

we see that, using the Jacobi identity for the commutator twice, we can write

$$c_2^\Lambda(u, v; F) - (\delta b_2^\Lambda)(u, v; F) = \frac{1}{8}\text{Tr}_\Lambda ([F\varepsilon[\varepsilon, u], v] - (u \leftrightarrow v)).$$

As  $\text{Tr}_\Lambda ([[ \varepsilon, u ], v] - (u \leftrightarrow v)) = \text{Tr}_\Lambda ([\varepsilon, [u, v]]) = 0$  (we used the Jacobi identity and (15)), replacing  $F$  in this expression by  $(F - \varepsilon)$  does not have any effect.

Collecting terms, we therefore obtain

$$\begin{aligned} \Delta c_{MR}^\Lambda(u, v; F) &= \frac{1}{4}\text{Tr}_\Lambda ([u, \varepsilon v] - (u \leftrightarrow v)) \\ &+ \frac{1}{8}\text{Tr}_\Lambda ([ (F - \varepsilon)\varepsilon[\varepsilon, u], v ] - (u \leftrightarrow v)). \end{aligned} \quad (21)$$

which now is of the form we were after.

We now claim: For  $X, Y \in \text{Map}(\mathbb{R}^3; \underline{g})$ ,  $F_A = \text{sign}(\not{D}_A)$  with  $A \in \mathcal{A}(\mathbb{R}^3)$ , we have

$$\begin{aligned} \Delta c^\Lambda(X, Y; F_A) &= \frac{i}{24\pi^2} \int_{\mathbb{R}^3} d^3x \text{tr} (\epsilon_{ijk} A_i(\vec{x}) (\partial_j u(\vec{x}) \partial_k v(\vec{x}) - \partial_j v(\vec{x}) \partial_k u(\vec{x}))) \\ &= c_{MFS}(X, Y; A) \end{aligned} \quad (22)$$

where  $c_{MFS}$  is the MFS cocycle (1) ( $\epsilon_{ijk}$  is the antisymmetric tensor with  $\epsilon_{123} = 1$ ).

**c.** To evaluate the l.h.s. of (22) we use symbol calculus.

As  $\text{tr}_{spin}(\sigma_i) = 0$ , we obviously have  $\text{Tr}_\Lambda([u, \varepsilon v]) = 0$ , and the  $F$ -independent term in (21) does not contribute to the l.h.s. of (22). Moreover,

$$\sigma([ (F_A - \varepsilon)\varepsilon[\varepsilon, X], Y ])(\vec{x}, \vec{p}) = (-i)^2 \frac{\partial}{\partial p_i} \left( \frac{\partial \varepsilon(\vec{p})}{\partial p_j} A_j(\vec{x}) \varepsilon(\vec{p}) \frac{\partial \varepsilon(\vec{p})}{\partial p_k} \partial_k X(\vec{x}) \right) \partial_i Y(\vec{x}) + O(p^{-4}). \quad (23)$$

Under  $\text{Tr}_\Lambda$  the  $O(p^{-4})$ -term does not contribute, hence we get

$$\Delta c^\Lambda(X, Y; F_A) = \frac{1}{8} J_{ijk}^\Lambda \int_{\mathbb{R}^3} d^3x \text{tr} (A_j(\vec{x}) (\partial_j X(\vec{x}) \partial_k Y(\vec{x}) - (X \leftrightarrow Y))) \quad (24)$$

where

$$J_{ijk}^\Lambda = - \int_{p \leq \Lambda} \frac{d^3 p}{(2\pi)^3} \text{tr}_{spin} \left( \frac{\partial}{\partial p_i} \left( \frac{\partial \varepsilon(\vec{p})}{\partial p_j} \varepsilon(\vec{p}) \frac{\partial \varepsilon(\vec{p})}{\partial p_k} \right) \right). \quad (25)$$

To evaluate the last integral, we note that  $\partial \varepsilon(\vec{p}) / \partial p_i = P_{il} \sigma_l / p$  with  $P_{il} = (\delta_{il} - p_i p_l / p^2)$ , hence with  $\text{tr}_{spin}(\sigma_i \sigma_j \sigma_k) = 2i \epsilon_{ijk}$  we get after a simple calculation

$$J_{ijk}^\Lambda = - \int_{p \leq \Lambda} \frac{d^3 p}{(2\pi)^3} \frac{\partial}{\partial p_i} \left( 2i \epsilon_{jlk} \frac{p_l}{p^3} \right)$$

and are left with the elementary integral

$$I_{il}^\Lambda = \int_{p \leq \Lambda} \frac{d^3 p}{(2\pi)^3} \frac{\partial}{\partial p_i} \left( \frac{p_l}{p^3} \right) = \frac{\delta_{il}}{6\pi^2}.$$

(The latter equality can be seen by an elementary calculation ( $ijk$ ) is a cyclic permutation of  $(1, 2, 3)$ , and  $(p_j, p_k) = (q \cos(\varphi), q \sin(\varphi))$  polar coordinates as usual):

$$\begin{aligned} I_{il}^\Lambda &= \frac{1}{(2\pi)^3} \int_0^\Lambda dq q \int_0^{2\pi} d\varphi \int_{-\sqrt{\Lambda^2 - q^2}}^{\sqrt{\Lambda^2 - q^2}} dp_i \frac{\partial}{\partial p_i} \left( \frac{p_l}{p^3} \right) \\ &= \frac{1}{(2\pi)^3} \int_0^\Lambda dq q 2\pi 2\delta_{il} \frac{\sqrt{\Lambda^2 - q^2}}{\Lambda^3} = \frac{\delta_{il}}{6\pi^2}; \end{aligned}$$

note that  $I_{ik}^\Lambda$  does not depend on  $\Lambda$ .) With that we get

$$J_{ijk}^\Lambda = \frac{i}{3\pi^2} \epsilon_{ijk} \quad (26)$$

independent of  $\Lambda$  which together with eq. (24) proves the assertion.

The infrared singularity of the symbols (the pole at  $p = 0$ ) in the calculation above is essential. If one ignores it, the result is zero: If one first performs the angular integration in momentum space in eq. (25) the trace  $\text{tr}_{spin}$  vanishes. In the above calculation of  $J_{ijk}^\Lambda$  we have respected the distributional nature of the momentum space derivatives of  $\varepsilon(\vec{p})$ . Another check for the computation is obtained if one replaces  $\varepsilon$  by a smooth (non-singular) function  $\tilde{\varepsilon}(\vec{p})$  such that  $\tilde{\varepsilon}(\vec{p}) = \varepsilon(\vec{p})$  far away from the origin. The above computation, when repeated for  $\tilde{\varepsilon}(\vec{p})$ , shows that the trace is a boundary integral in momentum space and *in no way depends on the choice of the smoothing  $\tilde{\varepsilon}(\vec{p})$  near the origin*. Thus the commutator anomaly is a result of a nontrivial interplay of the ultraviolet behavior and the infrared properties of gauge currents.

**3. Final Remarks.** Given a Hilbert space  $h$  and a grading operator  $\varepsilon$  on  $h$ , one basic object of NCG is the graded differential complex  $\Omega_p = \bigoplus_{n=0}^\infty \Omega_p^{(n)}$  where

$\Omega_p^{(0)} = \underline{g}$  and  $\Omega_p^{(n)}$  is generated by linear combinations of operators of the form

$$\omega_n = u[\varepsilon, v_1] \dots [\varepsilon, v_n] \quad u, v_1, \dots, v_n \in \underline{g}_p. \quad (27)$$

Then

$$\hat{d}\omega_n = \begin{cases} i[\varepsilon, \omega_n] & \text{if } n \text{ is even} \\ i\{\varepsilon, \omega_n\} & \text{if } n \text{ is odd} \end{cases} \quad (28)$$

defines a derivation on  $\Omega$  satisfying  $\hat{d}^2 = 0$  and which is supposed to generalize the exterior derivative  $d$  acting on  $\underline{g}$ -valued forms on a  $d$ -dimensional manifold  $M^d$ , [6].

In noncommutative geometry one replaces the classical  $\mathfrak{gl}(N)$ -valued forms on  $M^d$  in  $\Omega_p$  by operators by setting

$$\begin{aligned} \underline{g}\text{-valued form on } M^d &\rightarrow \Omega_p \\ X dY_1 \dots dY_n &\rightarrow i^n X[\varepsilon, Y_1] \dots [\varepsilon, Y_n] \\ \forall X, Y_1, \dots, Y_n &\in \text{Map}(M^d; \mathfrak{gl}(N)) \subset \underline{g}_p. \end{aligned} \quad (29a)$$

(Note that the arrow above is not really a well-defined map because a vanishing linear combination of the classical forms on the left could lead to a non-vanishing operator form on the right.) With this in mind, the equations (13a,b) for the symbols of operators look very suggestive: it seems that the leading powers of operator symbols exactly realize the embedding (29a) for  $p = 2$  by assigning

$$dx^i \rightarrow \frac{\partial \varepsilon(\vec{p})}{\partial p_i}. \quad (29b)$$

Moreover it seems that  $\mathcal{A}(M^3) \ni A \rightarrow (F_A - \varepsilon)$  is just a special case of the embedding (29a) for 1-forms. The latter is not true, however: if it was true,  $\{\varepsilon, F_A - \varepsilon\} + (F_A - \varepsilon)^2 = F_A^2 - 1$  should correspond to the magnetic field  $B \equiv (-i)dA + A^2$ , but  $F_A^2 - 1 = 0$  always. One can, however, find for every  $A \in \mathcal{A}(M^3)$  an operator  $\Phi_A$  in  $\Omega_2^{(2)}$  whose symbol is identical to the one of  $F_A$  to leading- and next-to-leading order in  $p^{-1}$ , but which is such that  $\Phi_A^2 - 1$  is non-zero in general and naturally represents  $B$ ,

$$\sigma(\Phi_A^2 - 1)(\vec{p}, \vec{x}) = \frac{\partial \varepsilon(\vec{p})}{\partial p_i} \frac{\partial \varepsilon(\vec{p})}{\partial p_j} (\partial_i A_j(\vec{x}) - \partial_j A_i(\vec{x}) + i[A_i(\vec{x}), A_j(\vec{x})]) + \mathcal{O}(p^{-3})$$

(this was pointed out already by Rajeev [14]).

The formula (5) for the MR cocycle can therefore be regarded just as the non-commutative generalization of the MFS cocycle (1) if one replaces

$$\frac{i}{3\pi^2} \int_{M^3} \rightarrow \text{Tra}_C, \quad (29c)$$

i.e. one regards the conditional trace as a non-commutative generalization of integration of 3-forms on  $M^3$ . This is indeed very natural as one can prove that [16]

$$\frac{i}{3\pi^2} \int_{M^3} \text{tr}(X dY_1 dY_2 dY_3) = i^3 \text{Tr}_C(X[\varepsilon, Y_1][\varepsilon, Y_2][\varepsilon, Y_3])$$

$$\forall X, Y_1, Y_2, Y_3 \in \text{Map}(M^3; \text{gl}(N)). \quad (30)$$

(There is an analog relation for arbitrary dimension  $d$  [16]. Note that the resulting non-commutative integration calculus is different from the one suggested in [17].) This implies especially that the MFS cocycle is *identical* to the MR cocycle in case  $A$  is a pure gauge. In general, one has an exact formula

$$c_{MFS}(X, Y, A) = c_{MR}(X, Y; \Phi_A) \quad (31)$$

for a suitable choice of  $\Phi_A$  as discussed above.

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## References

- [1] Langmann E. and Semenoff G. W., QCD(1+1) with massless quarks and gauge covariant Sugawara construction, [hep-th/9404159](#).
- [2] Mickelsson J., *Lett. Math. Phys.* **7**, 45 (1983); *Commun. Math. Phys.* **97**, 361 (1985).
- [3] Faddeev L. D., *Phys. Lett.* **145 B**, 81 (1984); Faddeev L. D. and Shatashvili S. L., *Theor. Math. Phys.* **60**, 770 (1984).
- [4] Mickelsson J. and Rajeev S. G., *Commun. Math. Phys.* **116**, 400 (1988).
- [5] Langmann E., *Commun. Math. Phys.* **162**, 1 (1994).
- [6] Connes A., *Publ. Math. IHES* **63**, 257 (1985).

- [7] Reed R., and Simon B.: *Methods of Modern Mathematical Physics I. Functional Analysis*, Academic Press, New York (1968); *II. Fourier Analysis, Self-Adjointness*, Academic Press, New York (1975).
- [8] Mickelsson J., *Commun. Math. Phys.* **127**, 285 (1990).
- [9] Hörmander L.: *The Analysis of Linear Partial Differential Operators III*. Springer-Verlag, Berlin (1985).
- [10] Mickelsson J., Regularization of current algebra, [hep-th/9306133](#); Wodzicki residue and anomalies of current algebras, [hep-th/9404093](#); *Lett. Math. Phys.* **28**, 97 (1993).
- [11] Jo S.-G., *Phys. Lett.* **163B**, 353 (1985); Alekseev A., Madaichik Ya., Faddeev L. D., and Shatashvili S. L., *Theor. Math. Phys.* **73**, 1149 (1988).
- [12] Niemi A. J. and Semenoff G. W., *Phys. Rev. Lett.* **55**, 927 (1985); Sonoda H., *Nucl.Phys. B* **B266**, 410 (1986).
- [13] Jackiw R. and Johnson K., *Phys. Rev.* **182**, 1459 (1969).
- [14] Rajeev S. G., *Phys. Rev. D* **42**, 2779 (1990); *Phys. Lett.* **209B**, 53 (1988).
- [15] Rajeev S. G., *Phys. Rev. D* **44**, 1836 (1991).
- [16] Langmann E. and Mickelsson J., in preparation.
- [17] Connes A., *Commun. Math. Phys.* **117**, 117 (1988).