

Spin Foam Models for Quantum Gravity

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Vienna, Preprint ESI 1199 (2002)

September 5, 2002

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via <http://www.esi.ac.at>

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September 20, 2002

Abstract

In this article we review the present status of the spin foam formulation of non-perturbative (background independent) quantum gravity. The article is divided in two parts. In the first part we present a general introduction to the main ideas emphasizing their motivation from various perspectives. Riemannian 3-dimensional gravity is used as a simple example to illustrate conceptual issues and main goals of the approach. The main features of the various existing models for 4-dimensional gravity are also presented here. We conclude with a discussion of important questions to be addressed in four dimensions (gauge invariance, discretization independence, etc.).

In the second part we concentrate on the definition of the Barrett-Crane model. We present the main results obtained in this framework from a critical perspective. Finally we review the combinatorial formulation of spin foam models based on the dual group field theory technology. We present the Barrett-Crane model in this framework and review the finiteness results obtained for both its Riemannian as well as its Lorentzian variants.

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1 Introduction

Quantum Gravity, the fundamental theory expected to reconcile the principles of quantum mechanics and general relativity remains a major challenge (for a review of the history of quantum gravity see [1]). The fully dynamical nature of spacetime geometry in general relativity claims for a background independent formulation of quantum gravity. This amounts to a definition of a quantum field theory without an underlying fixed metric structure. The necessity of such formulation is by now generally recognized; there is however debate about the way it should be realized.

The difference between the main viewpoints can be traced back to the interpretation of the non-renormalizability of perturbative quantum gravity. According to the (background dependent) view from standard QFT[2], non-renormalizability signals the inconsistency of the theory at high energies that has to be corrected by a more fundamental theory in the UV regime. A classical example of this is Fermi's four fermions theory as an effective description of the weak interaction. According to this view different approaches to quantum gravity have been defined in terms of modifications of general relativity based on supersymmetry, higher dimensions, strings, etc. The finiteness properties of the perturbative expansions (which are background dependent from the onset) are improved in these theories; however, the question of how a background independent formulation can be obtained from these remains mysterious.

The approach of non-perturbative quantum gravity is based on a different interpretation of the infinities in perturbative quantum gravity: it is precisely the perturbative (background dependent) techniques which are inconsistent with the fundamental nature of gravity. This view is strongly suggested by the prediction of discreteness of space at Planck scale by the background independent formulation of loop quantum gravity. Loop quantum gravity (LQG) is a non-perturbative formulation of quantum gravity based on the connection formulation of general relativity (for an reviews on the subject see [3, 4, 5, 6]). From the background independent perspective it is the assumption of a non-dynamical background which is smooth at all scales what fails near the Planck scale in perturbative quantum general relativity. In addition, LQG successfully incorporates interaction between quantum geometry and quantum matter in a way that is completely free of divergencies [7] (the quantum nature of space appears as a physical regulator for the other interactions).

There are however technical difficulties in addressing dynamics in LQG. These difficulties are expected to be partly due to the breaking of manifest 4-diffeomorphism invariance introduced by the 3+1 splitting of the canonical formulation. Consequently, there has been growing interest in trying to define dynamics in the theory from a 4-dimensional covariant perspective. This has given rise to the so-called spin foam approach to quantum gravity. The fundamental idea behind the spin foam approach is the search of a rigorous definition of the path integral for gravity using the deep insights we have gained about quantum geometry from LQG.

The underlying discreteness discovered in LQG is crucial: in spin foam models the formal Misner-Hawking functional integral for gravity is replaced by a sum over combinatorial objects (colored 2-complexes called spin foams). States of the theory –corresponding

to boundary data in the path integral— are given by the 3-geometry states of LQG (spin network states). General covariance implies the absence of a meaningful notion of time and transition amplitudes are to be interpreted as defining the physical scalar product.

While the construction can be explicitly carried out in three dimensions there are additional technical difficulties in four dimensions. Various models have been proposed. One of the questions is whether the infinite sums over geometries defining transition amplitudes would converge. In fact, there is no UV problem due to the fundamental discreteness and potential divergencies are associated to the IR regime. There are recent results in the context of the Barrett-Crane model showing that amplitudes are well defined when the topology of the histories is restricted in a certain way.

The aim of this article is to provide a comprehensive review of the progress that has been achieved in the spin foam approach over the last few years. The article is divided into two fundamental parts. In the first part we present a general introduction to the subject including a brief summary of LQG in Section 2. We introduce the spin foam formulation from different perspectives in Section 3. In Section 4 we present a simple example of spin foam model: Riemannian 3-dimensional gravity. We use this example as the basic tool to introduce the main ideas and to illustrate various conceptual issues. We review the different proposed models for 4-dimensional quantum gravity in Section 5. Finally, in Section 6 we conclude the first part by analyzing the various conceptual issues that arise in the approach. The first part is thought as a general introduction to the formalism, it is self contained and could be read independently.

One of the simplest and most studied spin foam model for 4-dimensional gravity is the Barrett-Crane model[8, 9]. The main purpose of the second part is to present a critical survey of the different results that have been obtained in this framework and its combinatorial generalizations[10, 11, 12] based on the dual group field theory (GFT) formulation. In Section 7.1 we present the spin foam quantization of 4-dimensional $Spin(4)$ BF theory which is of relevance for the later analysis and interpretation of the Barrett-Crane model. In Section 7 we present the derivation[13, 14] of the Barrett-Crane model (some auxiliary material is included in the first two subsection which is useful for the analysis of the model). Spin foams can be thought of as Feynman diagrams. In fact a wide class of spin foam models can be derived from the perturbative (Feynman) expansion of certain dual group field theories (GFT)[15, 16]. A brief review of the main ideas involved is presented in Section 8. We conclude the second part by studying the GFT formulation of the Barrett-Crane model for both Riemannian and Lorentzian geometry. The general arguments that establish this duality are reviewed in Section 8. The definition of the actual models and the sketch of the corresponding finiteness proofs[17, 18, 19] are given in Section 9.

2 Loop Quantum Gravity and Quantum Geometry

Loop quantum gravity is a rigorous realization of the quantization program established in the 60's by Dirac, Wheeler, De-Witt, among others (for recent reviews see [3, 4, 20]). The technical difficulties of Wheeler's 'geometrodynamics' are circumvented by the use of

connection variables instead of metrics[21, 22, 23]. At the cinemactical level, the formulation is similar to that of standard gauge theories. The fundamental difference is however the absence of any non-dynamical background field in the theory.

The configuration variable is an $SU(2)$ -connection A_a^i on a 3-manifold Σ representing space. The canonical momenta are given by the densitized triad E_i^a . The latter encode the (fully dynamical) Riemannian geometry of Σ and are the analog of the ‘electric fields’ of Yang-Mills theory.

In addition to diffeomorphisms there is the local freedom of $SU(2)$ -rotating the triad and gauge transforming the connection respectively. According to Dirac, gauge freedoms result in constraints among the phase space variables which conversely are the generating functionals of infinitesimal gauge transformations. In terms of connection variables the constraints are

$$\mathcal{G}_i = \mathcal{D}_a E_i^a = 0, \quad \mathcal{C}_a = E_k^b F_{ba}^k = 0, \quad \mathcal{S} = \epsilon^{ijk} E_i^a E_j^b F_{abk} + \dots = 0, \quad (1)$$

where \mathcal{D}_a is the covariant derivative and F_{ba}^k is the curvature of A_a^i . \mathcal{G}_i is the familiar Gauss constraint –analogous to the Gauss law of electromagnetism–generating infinitesimal $SU(2)$ gauge transformations, \mathcal{C}_a is the vector constraint generating spacial diffeomorphism, and \mathcal{S} is the scalar constraint generating ‘time’ reparameterization.

Loop quantum gravity is defined using Dirac quantization. One first represents (1) as operators in an auxiliary Hilbert space \mathcal{H} and then solves the constraint equations

$$\hat{\mathcal{G}}_i \Psi = 0, \quad \hat{\mathcal{C}}_a \Psi = 0, \quad \hat{\mathcal{S}} \Psi = 0. \quad (2)$$

The Hilbert space of solutions is the so-called physical Hilbert space \mathcal{H}_{phys} . In a generally covariant system quantum dynamics is fully governed by constraint equations. In the case of loop quantum gravity they represent *quantum Einstein’s equations*.

States in the auxiliary Hilbert space are represented by wave functionals of the connection $\Psi(A)$ which are square integrable with respect to a natural diffeomorphism invariant measure, the Ashtekar-Lewandowski measure [24] (we denote it $\mathcal{L}^2[\mathcal{A}]$ where \mathcal{A} is the space of (generalized) connections). This space can be decomposed into a direct sum of orthogonal subspaces $\mathcal{H} = \bigoplus_{\gamma} \mathcal{H}_{\gamma}$ labelled by a graph γ in Σ . The fundamental excitations are given by the holonomy $h_{\ell}(A) \in SU(2)$ along a path ℓ in Σ :

$$h_{\ell}(A) = \mathcal{P} \exp \int_{\ell} A. \quad (3)$$

Elements of \mathcal{H}_{γ} are given by functions

$$\Psi_{f,\gamma}(A) = f(h_{\ell_1}(A), h_{\ell_2}(A), \dots, h_{\ell_n}(A)), \quad (4)$$

where h_{ℓ} is the holonomy along the links $\ell \in \gamma$ and $f : SU(2)^n \rightarrow \mathbb{C}$ is (Haar measure) square integrable. They are called *cylindrical functions* and represent a dense set in \mathcal{H} denoted *Cyl*.

Gauge transformations generated by the Gauss constraint act non-trivially at the end-points of the holonomy, i.e., at nodes of graphs. The Gauss constraint (in (1)) is solved by looking at $SU(2)$ gauge invariant functionals of the connection ($\mathcal{L}^2[\mathcal{A}]/\mathcal{G}$). The fundamental gauge invariant quantity is given by the holonomy around closed loops. An orthonormal basis of the kernel of the Gauss constraint is defined by the so called spin network states [25, 26, 27] $\Psi_{\gamma, \{j_\ell\}, \{\iota_n\}}(A)$. Spin-networks are defined by a graph γ in Σ , a collection of spins $\{j_\ell\}$ –unitary irreducible representations of $SU(2)$ – associated with links $\ell \in \gamma$ and a collection of $SU(2)$ intertwiners $\{\iota_n\}$ associated to nodes $n \in \gamma$ (see Figure 1). The spin-network gauge invariant wave functional $\Psi_{\gamma, \{j_\ell\}, \{\iota_n\}}(A)$ is constructed by first associating an $SU(2)$ matrix in the j_ℓ -representation to the holonomies $h_\ell(A)$ corresponding to the link ℓ , and then contracting the representation matrices at nodes with the corresponding intertwiners ι_n , namely

$$\Psi_{\gamma, \{j_\ell\}, \{\iota_n\}}(A) = \prod_{n \in \gamma} \iota_n \prod_{\ell \in \gamma} j_\ell[h_\ell(A)], \quad (5)$$

where $j_\ell[h_\ell(A)]$ denotes the corresponding j_ℓ -representation matrix evaluated at corresponding link holonomy and the matrix index contraction is left implicit.

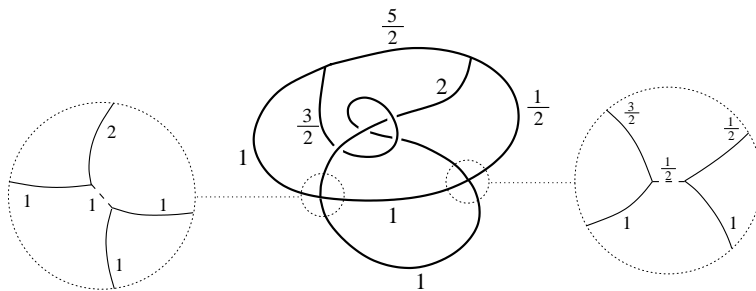


Figure 1: Spin-network state: At 3-valent nodes the intertwiner is uniquely specified by the corresponding spins. At 4 or higher valent nodes an intertwiner has to be specified. Choosing an intertwiner corresponds to decompose the n -valent node in terms of 3-valent ones adding new virtual links (dashed lines) and their corresponding spins. This is illustrated explicitly in the figure for the two 4-valent nodes.

The solution of the vector constraint is more subtle[24]. One uses group averaging techniques and the existence of the diffeomorphism invariant measure. The fact that zero lies in the continuous spectrum of the diffeomorphism constraint implies solutions to correspond to generalized states. These are not in \mathcal{H} but correspond to elements of the topological dual Cyl^* . Intuitively, they represent equivalence classes of spin-network states up to diffeomorphism. This can be interpreted as an indication suggesting the existence of a more combinatorial picture (a formulation of the canonical theory based on piecewise linear manifolds has been defined in [28]). This motivates the use of algebraic topological structures in spin foams.

¹Recall the triple contention $Cyl \subset \mathcal{H} \subset Cyl^*$.

Quantum geometry

The generalized states described above solve all the constraints (1) but the scalar constraint and are regarded as quantum states of the Riemannian geometry on Σ . They define the cinemematical sector of the theory known as *quantum geometry*.

Geometric operators acting in this cinemematical Hilbert space can be defined in terms of the fundamental triad \hat{E}_i^a operators. The simplest of such operators is the area of a surface S classically given by

$$A_S(E) = \int_S dx^2 \sqrt{\text{Tr}[n_a n_b E^a E^b]} \quad (6)$$

where n is a conormal. The geometric operator $\hat{A}_S(E)$ can be rigourously represented in the kinematical Hilbert [29, 30]. The area operator gives a clear geometrical interpretation to spin-network states: the fundamental 1-dimensional excitations defining a spin-network state can be thought of as quantized ‘flux lines’ of area. More precisely, if the surface $S \subset \Sigma$ is punctured by a spin-network link carrying a spin j , this state is an eigenstate of $\hat{A}_S(E)$ with eigenvalue $8\pi\iota_P^2 \sqrt{j(j+1)}$. In the generic sector –where no nodes lie on the surface– the spectrum takes the simple form

$$a_S(\{j\}) = 8\pi\iota_P^2 \sum_i \sqrt{j_i(j_i+1)}, \quad (7)$$

where i labels punctures. $a_S(\{j\})$ is the sum of single puncture contributions. The general form of the spectrum including the cases where nodes lie on S has been computed in closed form[30].

The spectrum of the volume operator is also discrete [29, 33, 34]. If we define the volume operator $\hat{V}_\sigma(E)$ of a 3-dimensional region $\sigma \subset \Sigma$ then non vanishing eigenstates are given by spin-networks containing n -valent nodes in σ for $n > 3$. Volume is concentrated in nodes.

Quantum dynamics

In contrast to the Gauss and vector constraints the scalar constraint does not have a simple geometrical meaning. This makes its quantization more involved. Regularization choices has to be made and the result is not unique. After Theimann’s first rigorous quantization[35] other well defined possibilities have been found [36, 37, 38]. This ambiguity affects dynamics governed by

$$\hat{S}\Psi = 0. \quad (8)$$

The difficulty in dealing with the scalar constraint is not surprising. The vector constraint –generating space diffeomorphism– and the scalar constraint – generating time reparameterizations– arise from the underlying 4-diffeomorphism invariance of gravity. In

²The quantity ι is a free parameter in the theory known as the Immirzi parameter [31]. This ambiguity is purely quantum mechanical (it disappears in the classical limit). It has to be fixed in terms of physical predictions. The computation of BH entropy in LQG fixes the value of ι (see [32]).

the canonical formulation the $3 + 1$ splitting breaks the manifest 4-dimensional symmetry. The prize paid is the complexity of the time re-parameterization constraint \mathcal{S} . The situation is somewhat reminiscent of that in standard quantum field theory where manifest Lorentz invariance is lost in the Hamiltonian formulation ³.

From this perspective, there has been growing interest in approaching the problem of dynamics defining a covariant formulation of quantum gravity. The idea is that (as in the QFT case) one can keep manifest 4-dimensional covariance in the path integral formulation. Spin foam approach is an attempt to define the path integral quantization of gravity using what we have learn from LQG.

In standard quantum mechanics path integrals provide the solution of dynamics as a device to compute the time evolution operator. Similarly, in the generally covariant context it provides a tool to find solutions to the constraint equations (this has been emphasized formally in various places: in the case of gravity see for example [39], for a detailed discussion of this in the context of quantum mechanics see [40]) We will come back to this issue later.

Lets finish by stating som properties of \hat{S} that do not depend on the ambiguities mentioned above. One is the discovery that smooth loop states naturally solve the scalar constraint operator[41, 42]. This set of states is clearly to small to represent the physical Hilbert space (e.g., they span a zero volume sector). However, this implies that \hat{S} acts only on spin network nodes. Its action modifies spin networks at nodes by creating new links according Figure 2 ⁴. This is crucial in the construction of the spin foam approach of the next section.

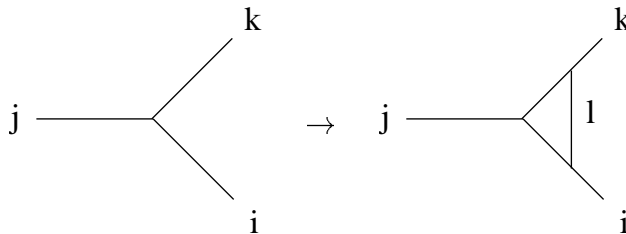


Figure 2: A typical transition generated by the action of the scalar constraint

3 Spin Foams and the path integral for gravity

The possibility of defining quantum gravity using Feynman's Path integral approach has been considered long ago by Misner and later extensively studied by Hawking, Hartle and others. Given a 4-manifold \mathcal{M} with boundaries Σ_1 and Σ_2 , and denoting by G the space of

³There is however an additional complication here: the canonical constraint algebra does not reproduce the 4-diffeomorphism Lie algebra. This complicates the geometrical meaning of S .

⁴This is not the case in all the available definitions of the scalar constraints as for example the one defined in [37, 38].

metrics on \mathcal{M} , the transition amplitude between $[[q_{ab}]]$ on Σ_1 and $[[q'_{ab}]]$ on Σ_2 is formally

$$\langle [[q_{ab}]] | [[q'_{ab}]] \rangle = \int_{[g]} \mathcal{D}[g] e^{iS([g])}, \quad (9)$$

where the integration on the right is performed over all spacetime metrics up to 4-diffeomorphisms $[g] \in G/Diff(\mathcal{M})$ with fixed boundary values up to 3-diffeomorphisms $[q_{ab}], [q'_{ab}]$, respectively.

There are various difficulties associated with (9). Technically there is the problem of defining the functional integration over $[g]$ on the RHS. This is partially because of the difficulties in defining infinite dimensional functional integration beyond the perturbative framework. In addition, there is the issue of having to deal with the space $G/Diff(\mathcal{M})$, i.e., how to characterize the diffeomorphism invariant information in the metric. This gauge problem (3-diffeomorphisms) is also present in the definition of the boundary data: there is no well defined notion of cinemematical state $[[q_{ab}]]$ standard metric variables.

We can be more optimistic in the framework of loop quantum gravity. The notion of quantum state of 3-geometry is rigorously defined in terms of spin-network states. They carry the diff-invariant information of the Riemannian structure of Σ . In addition, and very importantly, these states are intrinsically discrete (colored graphs on Σ) suggesting a possible solution to the functional measure problem, i.e., the possibility of constructing a notion of Feynman ‘path integral’ in a combinatorial manner involving sums over spin network worldsheets amplitudes. Heuristically, ‘4-geometries’ are to be represented by ‘histories’ of quantum states of 3-geometries or spin network states. These ‘histories’ involve a series of transitions between spin network states (Figure 3), and define a foam-like structure (a ‘2-graph’ or 2-complex) where its components inherit the spin representations from the underlying spin networks. These spin network worldsheets are the so-called spin foams.

The precise definition of spin foams was introduced by Baez in [13] emphasizing their role as morphisms in the category defined by spin networks. A spin foam $\mathcal{F} : s \rightarrow s'$ – representing a transition from the spin-network $s = (\gamma, \{j_\ell\}, \{\iota_n\})$ into $s' = (\gamma', \{j_{\ell'}\}, \{\iota_{n'}\})$ – is defined by a 2-complex⁵ \mathcal{J} bordered by the graphs of γ and γ' respectively, a collection of spins $\{j_f\}$ associated with faces $f \in \mathcal{J}$ and a collection of intertwiners $\{\iota_e\}$ associated to edges $e \in \mathcal{J}$. Both spins and intertwiners of exterior faces and edges match the boundary values defined by the spin networks s and s' respectively. Spin foams $\mathcal{F} : s \rightarrow s'$ and $\mathcal{F}' : s' \rightarrow s''$ can be composed into $\mathcal{F}\mathcal{F}' : s \rightarrow s''$ by gluing together the two corresponding 2-complexes at s' . A spin foam model is an assignment of amplitudes $A[\mathcal{F}]$ which is consistent with this composition rule in the sense that

$$A[\mathcal{F}\mathcal{F}'] = A[\mathcal{F}]A[\mathcal{F}']. \quad (10)$$

Transition amplitudes between spin network states are defined by

$$\langle s, s' \rangle_{phys} = \sum_{\mathcal{F}:s \rightarrow s'} A[\mathcal{F}], \quad (11)$$

⁵In most of the paper we use the concept of piecewise linear 2-complexes as in [13] in Section 8 we shall study a formulation of spin foam in terms of certain combinatorial 2-complexes.

where the notation anticipates the interpretation of such amplitudes as defining the physical scalar product. The domain of the previous sum is left unspecified at this stage. We shall discuss this question further in Section 6. This last equation is the spin foam counterpart of equation (9). This definition remains formal until we specify what the set of allow spin foams in the sum are and define the corresponding amplitudes.

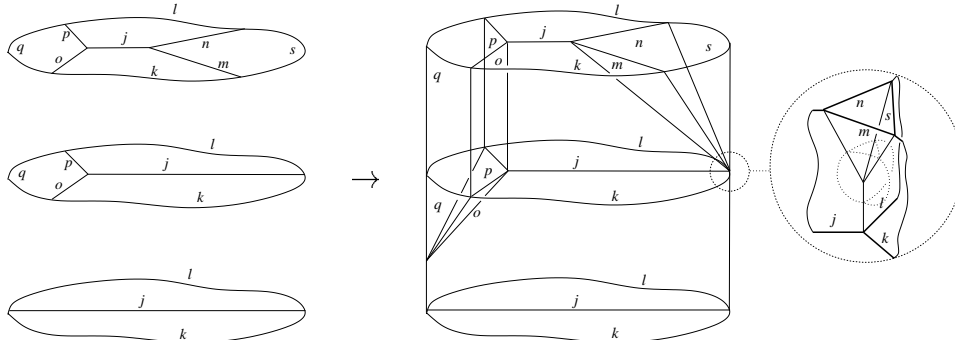


Figure 3: A typical path in a path integral version of loop quantum gravity is given by a series of transitions through different spin-network states representing a state of 3-geometries. Nodes and links in the spin network evolve into 1-dimensional edges and faces. New links are created and spins are reassigned at vertices (emphasized on the right). The ‘topological’ structure is provided by the underlying 2-complex while the geometric degrees of freedom are encoded in the labelling of its elements with irreducible representations and intertwiners.

In standard quantum mechanics the path integral is used to compute the matrix elements of the evolution operator $U(t)$. It provides in this way the solution for dynamics since for any cinematical state Ψ the state $U(t)\Psi$ is a solution to Schroedinger’s equation. Analogously, in a generally covariant theory the path integral provides a device for constructing solutions to the quantum constraints. Transition amplitudes represent the matrix elements of the so-called generalized ‘projection’ operator P (Sections 3.1 and 6.3) such that $P\Psi$ is a physical state for any kinematical state Ψ . As in the case of the vector constraint the solutions of the scalar constraint correspond to distributional states (zero is in the continuum part of its spectrum). Therefore, \mathcal{H}_{phys} is not a proper subspace of \mathcal{H} and the operator P is not a projector (P^2 is ill defined)⁶. In Section 4 we explicitly show an explicit example of this construction.

The non-perturbative character of spin foam is manifest. The 2-complex can be thought of as representing ‘spacetime’ while the boundary graphs as representing ‘space’. They do not carry any geometrical information in contrast with the standard concept of a lattice. Geometry is encoded in the spin labelling which represent the degrees of freedom of the gravitational field.

⁶In the notation of the previous section they correspond to elements of Cyl^* .

3.1 Spin foams and the projection operator into \mathcal{H}_{phys}

Spin foams naturally arise in the formal definition of the exponentiation of the scalar constraint as studied by Reisenberger and Rovelli in [43] and Rovelli [44]. The basic idea consists of constructing the ‘projection’ operator P providing a definition of the formal expression

$$P = \prod_{x \in \Sigma} \delta(\hat{\mathcal{S}}(x)) = \int \mathcal{D}[N] e^{i\hat{\mathcal{S}}[N]}, \quad (12)$$

where $\hat{\mathcal{S}}[N] = \int dx^3 N(x) \hat{\mathcal{S}}(x)$, $N(x)$ is a lapse function. P defines the *physical* scalar product according to

$$\langle s, s' \rangle_{phys} = \langle sP, s' \rangle, \quad (13)$$

where the RHS is defined using the kinematical scalar product. Reisenberger and Rovelli make progress toward a definition of (12) by constructing a truncated version P_Λ (where Λ can be regarded as an infrared cutoff). One of the main ingredients is Rovelli’s definition of a diffeomorphism invariant measure $\mathcal{D}[N]$ generalizing techniques of [45].

The starting point is the expansion of the exponential in (12) in powers

$$\langle sP_\Lambda, s' \rangle = \int_{|N(x)| \leq \Lambda} \mathcal{D}[N] \left\langle s \sum_{n=0}^{\infty} \frac{i^n}{n!} (\mathcal{S}[N])^n, s' \right\rangle. \quad (14)$$

The construction works for a generic form of quantum scalar constraint as long as it acts locally on spin network nodes both creating and destroying links (this local action generates a vertex of the type emphasized in Figure 2). The action of $\hat{\mathcal{S}}[N]$ depends on the value of the lapse at nodes. Integration over the lapse can be performed and the final result is given by a power series in the cutoff Λ , namely

$$\begin{aligned} \langle sP_\Lambda, s' \rangle &= \sum_{n=0}^{\infty} \frac{i^n \Lambda^n}{n!} \sum_{\mathcal{F}_n : s \rightarrow s'} A[\mathcal{F}_n] \\ &= \sum_{n=0}^{\infty} \frac{i^n \Lambda^n}{n!} \sum_{\mathcal{F}_n : s \rightarrow s'} \prod_v A_v(\rho_v, \iota_v), \end{aligned} \quad (15)$$

where $\mathcal{F}_n : s \rightarrow s'$ are spin foams generated by n actions of the scalar constraint, i.e., spin foams with n vertices. The spin foam amplitude $A[\mathcal{F}_n]$ factorizes in a product of vertex contributions $A_v(\rho_v, \iota_v)$ depending of the spins j_v and ι_v neighboring faces and edges. The spin foam shown in Figure 3 corresponds in this context to two actions of \mathcal{S} and would contribute to the amplitude in the order Λ^2 .

Physical observables can be constructed out of cinematical operators using P . If O_{kin} represents an operator commuting with all but the scalar constraint then $O_{phys} = PO_{kin}P$ defines a physical observable. Its expectation value is

$$\langle O_{phys} \rangle = \frac{\langle sPO_{kin}P, s \rangle}{\langle sP, s \rangle} := \lim_{\Lambda \rightarrow \infty} \frac{\langle sP_\Lambda O_{kin} P_\Lambda, s \rangle}{\langle sP_\Lambda, s \rangle}, \quad (16)$$

where the limit of the ratio of truncated quantities is expected to converge for suitable operators O_{kin} . Issues of convergence have not been studied and they would be clearly regularization dependent.

3.2 Spin foams from lattice gravity

Spin foam models naturally arise in lattice-discretizations of the path integral of gravity and generally covariant gauge theories. This was originally studied by Reisenberger[25]. The spacetime manifold is replaced by a lattice given by a cellular complex. The discretization allows for the definition of the functional measure reducing the number of degrees of freedom to finitely many. The formulation is similar to that of standard lattice gauge theory. However, the nature of this truncation is fundamentally different: background independence implies that it can not be simply interpreted as a UV regulator (we will be more explicit in the sequel).

We present a brief outline of the formulation for details see [25, 46, 47]. Start from the action of gravity in some first order formulation ($S(e, A)$) the formal path integral takes the form

$$Z = \int \mathcal{D}[e] \mathcal{D}[A] e^{iS(e,A)} = \int \mathcal{D}[A] e^{iS_{\text{eff}}(A)}, \quad (17)$$

where in the second line we have formally integrated over the tetrad e obtaining an effective action S_{eff} ⁷. From this point on the derivation is analogous to that of generally covariant gauge theory. The next step is to define the previous equation on a ‘lattice’.

As for Wilson’s action for standard lattice gauge theory the relevant structure for the discretization is a 2-complex \mathcal{J} . We assume the 2-complex to be defined in terms of the dual 2-skeleton \mathcal{J}_Δ of a simplicial complex Δ . Denoting the edges $e \in \mathcal{J}_\Delta$ and the plaquette or faces $f \in \mathcal{J}_\Delta$ one discretizes the connection by assigning a group element g_e to edges

$$A \rightarrow \{g_e\}.$$

The Haar measure on the group is used to represent the connection integration:

$$\mathcal{D}[A] \rightarrow \prod_{e \in \mathcal{J}_\Delta} dg_e.$$

The action of gravity depends on the connection A only through the curvature $F(A)$ so that upon discretization the action is expressed as a function of the holonomy around faces g_f corresponding to the product of the g_e ’s which we denote $g_f = g_e^{(1)} \cdots g_e^{(n)}$:

$$F(A) \rightarrow \{g_f\}.$$

⁷This is a simplifying assumption in the derivation. One could put the full action in the lattice[46] and then integrate over the discrete e to obtain the discretized version of the quantity on the right of (17). This is what we will do in Section 4.

In this way, $S_{\text{eff}}(A) \rightarrow S_{\text{eff}}(\{g_f\})$. So that the lattice path integral becomes

$$Z = \int \prod_{e \in \mathcal{J}_\Delta} dg_e \exp[iS_{\text{eff}}(\{g_f\})]. \quad (18)$$

Reisenberger assumes that $S_{\text{eff}}(\{g_f\})$ is local in the sense that the amplitude of any piece of the 2-complex obtained as its intersection with a ball depends only on the boundary value of the connection. Degrees of freedom communicate through the lattice connection on the boundary. One can compute amplitudes of pieces of \mathcal{J}_Δ (at fixed boundary data) and then obtain the full \mathcal{J}_Δ amplitude by gluing the pieces together integrating out the mutual boundary connections along common boundaries. The boundary of a portion of \mathcal{J}_Δ is a graph. The boundary value is an assignment of group elements to its links. The amplitude is a function of the boundary connection, i.e., an element of Cyl . In the case of a cellular 2-complex there is a maximal splitting corresponding to cutting out a neighborhood around each vertex. If the discretization is based on the dual of a triangulation these elementary building blocks are all alike and denoted *atoms*. Such an atom in four dimensions is represented in Figure 4.

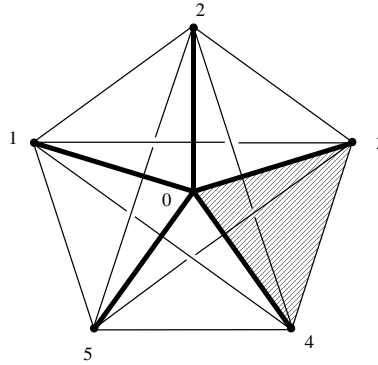


Figure 4: A fundamental *atom* is defined by the intersection of a dual vertex in \mathcal{J}_Δ (corresponding to a 4-simplex in Δ) with a 3-sphere. The thick lines represent the internal edges while the thin lines the intersections of the internal faces with the boundary. They define the boundary graph denoted γ_5 below. One of the faces has been emphasized.

The atom amplitude depends on the boundary data given by the value of the holonomies on the ten links of the pentagonal boundary graph γ_5 shown in the figure. This amplitude can be represented by a function

$$V(\alpha_{ij}) \quad \text{for } \alpha_{ij} \in G \quad \text{and } i \neq j = 1, \dots, 5 \quad (19)$$

where α_{ij} represents the boundary lattice connection along the link ij in Figure 4. Gauge invariance ($V(\alpha_{ij}) = V(g_i \alpha_{ij} g_j^{-1})$) implies that the function can be spanned in terms of spin networks functions $\Psi_{\gamma_5, \{\rho_{ij}\}, \{\iota_i\}}(\alpha_{ij})$ based on the pentagonal graph γ_5 , namely

$$V(\alpha_{ij}) = \sum_{\rho_{ij}} \sum_{\iota_i} \tilde{V}(\{\rho_{ij}\}, \{\iota_i\}) \Psi_{\gamma_5, \{\rho_{ij}\}, \{\iota_i\}}(\alpha_{ij}) \quad (20)$$

where $\tilde{V}(\{\rho_{ij}\}, \{\iota_i\})$ is the atom amplitude in ‘momentum’ space depending on ten spins ρ_{ij} labelling the faces and five intertwiners ι_i labelling the edges. Gluing the atoms together the integral over common boundaries is replaced by the sum over common values of spin labels and intertwiners⁸. The total amplitude becomes

$$Z[\mathcal{J}_\Delta] = \sum_{\mathcal{C}_f: \{f\} \rightarrow \{\rho_f\}} \sum_{\mathcal{C}_e: \{e\} \rightarrow \{\iota_e\}} \prod_{f \in \mathcal{J}_\Delta} A_f(\rho_f, \iota_f) \prod_{v \in \mathcal{J}_\Delta} \tilde{V}(\rho_v, \iota_v), \quad (21)$$

where $\mathcal{C}_e : \{e\} \rightarrow \{\iota_e\}$ denotes the assignment of intertwiners to edges, $\mathcal{C}_f : \{\rho_f\} \rightarrow \{f\}$ de assignment of spins ρ_f to faces, and $A_f(\rho_f, \iota_f)$ is the face amplitude arising in the integration over the lattice connection. The lattice definition of the path integral for gravity and covariant gauge theories becomes a discrete sum of spin foam amplitudes!

3.3 Spin foams for gravity from BF theory

The integration over the tetrad we formally performed in (17) is not always possible in practice. There is however a type of generally covariant theory for which the analog integration is trivial. This is the case of a class of theories called BF theory. General relativity in three dimensions is of this type. Consequently, BF theory can be quantized along the lines of the previous section. BF spin foam amplitudes are simply given by certain invariants in the representation theory of the gauge group. We shall study in some detail the case of 3-dimensional Riemannian quantum gravity in the next section.

The relevance of BF theory is its close relation to general relativity in four dimensions. In fact, general relativity can be described by certain BF theory action plus Lagrange multiplier terms imposing certain algebraic constraints on the fields[48]. This is the starting point for the definition of various of the models we will present in this article: a spin foam model for gravity can be defined by imposing restriction on the spin foams that enter in the partition function of the BF theory. These restrictions are essentially the translation into representation theory of the constraints that reduce BF theory to general relativity.

3.4 Spin foams as Feynman diagrams

As already pointed out in [13] spin foams can be interpreted in close analogy to Feynman diagrams. Standard Feynman graphs are generalized to 2-complexes and the labelling of propagators by momenta to the assignment of spins to faces. Finally, momentum conservation at vertices in standard feynmanology is now represented by spin-conservation at edges, ensured by the assignment of the corresponding intertwiners. In spin foam models the non-trivial content of amplitudes is contained in the vertex amplitude as pointed out

⁸ This is a consequence of the basic orthogonality of unitary representations, namely

$$\int j^* [g]_{\alpha\beta} k [g]_{\gamma\mu} dg = \frac{1}{2j+1} \delta_{\alpha\gamma} \delta_{\beta\mu}.$$

in Sections 3.1 and 3.2 which in the language of Feynman diagrams can be interpreted as an interaction. We shall see that this analogy is indeed realized in the formulation of spin foam models in terms of a group field theory (GFT) [15, 16].

4 Spin foams for 3-dimensional gravity

Three dimensional gravity is an example of BF theory to which the spin foam approach can be implemented in a rather simple way. Despite its simplicity the theory allows for the study of many of the conceptual issues to be addressed in four dimensions. In addition, as we mentioned in Section 3.3, spin foams for BF theory are the basic building block of 4-dimensional gravity models. For a beautiful presentation of BF theory and its relation to spin foams see [49]).

4.1 The classical theory

Euclidean gravity in 3 dimensions is a theory with no local degrees of freedom, i.e., no gravitons. Its action is given by

$$S(e, A) = \int_{\mathcal{M}} Tr(e \wedge F(A)), \quad (22)$$

where the field A is an $SU(2)$ -connection and the triad e is a Lie algebra valued 1-form. The local symmetries of the action are $SU(2)$ gauge transformations

$$\delta e = [e, \omega], \quad \delta A = d_A \omega, \quad (23)$$

where ω a $\mathfrak{su}(2)$ -valued 0-form, and ‘topological’ gauge transformation

$$\delta e = d_A \eta, \quad \delta A = 0, \quad (24)$$

where d_A denotes the covariant exterior derivative and η is a $\mathfrak{su}(2)$ -valued 0-form. The first invariance is manifest from the form of the action, while the second is a consequence of the Bianchi identity, $d_A F = 0$. The gauge freedom is so big that the theory has only global degrees of freedom. This can be checked directly by writing the equations of motion

$$F(A) = d_A A = 0, \quad d_A B = 0. \quad (25)$$

The first implies that the connection is flat which in turn means that it is locally gauge ($A = d_A \omega$). The solutions of the second equation are also locally gauge as any closed form is locally exact ($B = d_A \eta$).⁹ This very simple theory can be quantized in a direct manner.

⁹One can easily check that the infinitesimal diffeomorphism gauge action $\delta e = \mathcal{L}_v e$, and $\delta A = \mathcal{L}_v A$, where \mathcal{L}_v is the Lie derivative in the v direction, is a combination of (23) and (24) for $\omega = v^a A_a$ and $\eta_b = v^a e_{ab}$, respectively, acting on the space of solutions, i.e. when (25) holds.

4.2 Canonical quantization

Assuming $\mathcal{M} = \Sigma \times \mathbb{R}$ where Σ is a Riemann surface representing space, the phase space of the theory is parameterized by the spacial connection A and its conjugate momentum E . The constraints that result from the gauge freedoms (23) and (24) are

$$D_a E_i^a = 0, \quad F(A) = 0. \quad (26)$$

The first is the familiar Gauss constraint; the second we call curvature constraint. There are 6 independent constraints for the 6 components of the connection, i.e., no local degrees of freedom. The kinematical setting is analogous to that of 4-dimensional gravity and the quantum theory is defined along similar lines. The kinematical Hilbert space (quantum geometry) is spanned by $SU(2)$ spin-network states on Σ which automatically solve the Gauss constraint. ‘‘Dynamics’’ is governed by the curvature constraint. The physical Hilbert space is obtained by restricting the connection to be flat and the physical scalar product is defined by a natural measure in the space of flat connections[49]. The distributional character of the solutions of the curvature constraint is manifest here. Different spin network states are physically equivalent when they differ by a null state (states with vanishing physical scalar product with all spin network states). This happens when the spin networks are related by certain skein relations. One can reconstruct \mathcal{H}_{phys} directly from the skein relations which in turn can be found by studying the covariant spin foam formulation of the theory.

4.3 Spin foam quantization of 3d gravity

Here we apply the general framework of Section 3.2. This has been studied by Iwasaki in [50, 51]. The partition function, \mathcal{Z} , is formally given by

$$\mathcal{Z} = \int \mathcal{D}[e] \mathcal{D}[A] e^{i \int_{\mathcal{M}} \text{Tr}[e \wedge F(A)]}, \quad (27)$$

where for the moment we assume \mathcal{M} to be a compact and orientable. Integrating over the e field in (27) we obtain

$$\mathcal{Z} = \int \mathcal{D}[A] \delta(F(A)). \quad (28)$$

The partition function \mathcal{Z} corresponds to the ‘volume’ of the space of flat connections on \mathcal{M} .

In order to give a meaning to the formal expressions above, we replace the 3-dimensional manifold \mathcal{M} with an arbitrary cellular decomposition Δ . We also need the notion of the associated dual 2-complex of Δ denoted by \mathcal{J}_Δ . The dual 2-complex \mathcal{J}_Δ is a combinatorial object defined by a set of vertices $v \in \mathcal{J}_\Delta$ dual 3-cells in Δ , edges $e \in \mathcal{J}_\Delta$ dual 2-cells in Δ , and faces $f \in \mathcal{J}_\Delta$ dual to 1-cells in Δ .

The fields e and A have support on these discrete structures. The $su(2)$ -valued 1-form field e is represented by the assignment of a $e \in su(2)$ to each 1-cell in Δ . The connection field A is represented by the assignment of group elements $g_e \in SU(2)$ to each edge in \mathcal{J}_Δ .

The partition function is defined by

$$\mathcal{Z}(\Delta) = \int \prod_{f \in \mathcal{J}_\Delta} de_f \prod_{e \in \mathcal{J}_\Delta} dg_e e^{i\text{Tr}[e_f U_f]}, \quad (29)$$

where de_f is the regular Lebesgue measure on $su(2) = \mathbb{R}^3$, dg_e is the Haar measure on $SU(2)$, and U_f denotes the holonomy around faces, i.e., $U_f = g_e^1 \dots g_e^N$ for N being the number of edges bounding the corresponding face. An arbitrary orientation is assigned to faces when computing U_f . We use the fact that faces in \mathcal{J}_Δ are in one-to-one correspondence with 1-cells in Δ and label e_f with a face subindex.

Integrating over e_f , we obtain

$$\mathcal{Z}(\Delta) = \int \prod_{e \in \mathcal{J}_\Delta} dg_e \prod_{f \in \mathcal{J}_\Delta} \delta(g_e^1 \dots g_e^N), \quad (30)$$

where δ corresponds to the delta distribution defined on $\mathcal{L}^2(SU(2))$. Notice that the previous equation corresponds to the discrete version of equation (28).

The integration over the discrete connection ($\prod_e dg_e$) can be performed expanding first the delta function in the previous equation using the Peter-Weyl decomposition

$$\delta(g) = \sum_{j \in \text{irrep}(SU(2))} \Delta_j \text{Tr}[j(g)], \quad (31)$$

where $\Delta_j = 2j + 1$ denotes the dimension of the unitary representation j , and $j(g)$ is the corresponding representation matrix. Using equation (31), the partition function (30) becomes

$$\mathcal{Z}(\Delta) = \sum_{c: \{j\} \rightarrow \{f\}} \int \prod_{e \in \mathcal{J}_\Delta} dg_e \prod_{f \in \mathcal{J}_\Delta} \Delta_{j_f} \text{Tr}[j_f(g_e^1 \dots g_e^N)], \quad (32)$$

where the sum is over coloring of faces in the notation of (21).

Going from equation (29) to (32) we have replaced the continuous integration over the e 's by the sum over representations of $SU(2)$. Roughly speaking, the degrees of freedom of e are now encoded in the representation being summed over in (32).

Now it remains to integrate over the lattice connection $\{g_e\}$. If an edge $e \in \mathcal{J}_\Delta$ bounds n faces there are n traces of the form $\text{Tr}[j_f(\dots g_e \dots)]$ in (32) containing g_e in the argument. The relevant formula is

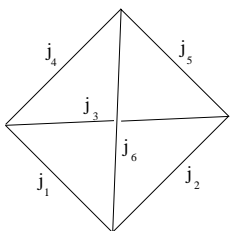
$$\int dg j_1(g) \otimes j_2(g) \otimes \dots \otimes j_n(g) = \sum_{\iota} C_{j_1 j_2 \dots j_n}^{\iota} C_{j_1 j_2 \dots j_n}^{*\iota}, \quad (33)$$

i.e., the projector onto $\text{Inv}[j_1 \otimes j_2 \otimes \dots \otimes j_n]$. On the RHS we have chosen an orthonormal basis of invariant vectors (intertwiners) to express the projector. Notice that the assignment of intertwiners to edges is a consequence of the integration over the connection. This is not

a particularity of this example but rather a general property of local spin foams as pointed out in Section 3.2. Finally (30) can be written as a sum over spin foam amplitudes

$$\mathcal{Z}(\Delta) = \sum_{\mathcal{C}:\{j\}\rightarrow\{f\}} \sum_{\mathcal{C}:\{v\}\rightarrow\{e\}} \prod_{f \in \mathcal{J}_\Delta} \Delta_{j_f} \prod_{v \in \mathcal{J}_\Delta} A_v(\iota_v, j_v), \quad (34)$$

where $A_v(\iota_v, j_v)$ is given by the appropriate trace of the intertwiners ι_v corresponding to the edges bounded by the vertex and j_v are the corresponding representations. This amplitude is given in general by an $SU(2)$ $3Nj$ -symbols corresponding to the flat evaluation of the spin network defined by the intersection of the corresponding vertex with a 2-sphere. When Δ is a simplicial complex all the edges in \mathcal{J}_Δ are 3-valent and vertices are 4-valent (one such vertex is emphasized in Figure 3, the intersection with the surrounding S^2 is shown in dotted lines). Consequently, the vertex amplitude is given by the contraction of the corresponding four 3-valent intertwiners, i.e., a $6j$ -symbol. In that case the partition function takes the familiar Ponzano-Regge[52] form

$$\mathcal{Z}(\Delta) = \sum_{\mathcal{C}:\{j\}\rightarrow\{f\}} \prod_{f \in \mathcal{J}_\Delta} \Delta_{j_f} \prod_{v \in \mathcal{J}_\Delta} \text{Diagram}, \quad (35)$$


were the sum over intertwiners disappears since $\dim(\text{Inv}[j_1 \otimes j_2 \otimes j_3]) = 1$ for $SU(2)$ and there is only one term in (33). Ponzano and Regge originally defined the amplitude (35) from the study of the asymptotic properties of the $6j$ -symbol.

4.3.1 Discretization independence

A crucial property of the partition function (and transition amplitudes in general) is that it does not depend on the discretization Δ . Given two different cellular decompositions Δ and Δ' (not necessarily simplicial)

$$\tau^{-n_0} Z(\Delta) = \tau^{-n'_0} Z(\Delta'), \quad (36)$$

where n_0 is the number of 0-simplexes in Δ (hence the number of bubbles in \mathcal{J}_Δ), and $\tau = \sum_j 2j + 1$ is clearly divergent which makes discretization independence a formal statement without a suitable regularization.

The sum over spins in (35) is typically divergent, as indicated by the previous equation. Divergences occur due to infinite volume factors corresponding to the topological gauge freedom (24)¹⁰. The factor τ in (36) represents such volume factor. It can also be interpreted as a $\delta(0)$ coming from the existence of redundant delta function in (30). One

¹⁰ For simplicity we concentrate on the Abelian case $G = U(1)$. The analysis can be extended to the

can partially gauge fix this freedom at the level of the discretization. This has the effect of eliminating bubbles from the 2-complex.

In the case of simply connected Σ the gauge fixing is complete. One can eliminate bubbles and compute finite transition amplitudes. The result is equivalent to the physical scalar product defined in the canonical picture in terms of the delta measure¹¹.

In the case of gravity with cosmological constant the state-sum generalizes to the Turaev-Viro model[54] defined in terms of $SU_q(2)$ with $q^n = 1$ where the representations are finitely many. Heuristically, the presence of the cosmological constant introduces a physical infrared cutoff. Equation (36) has been proven in this case for the case of simplicial decompositions in [54], see also [55, 56]. The generalization for arbitrary cellular decomposition was obtained in [57].

4.3.2 Transition amplitudes

Transition amplitudes can be defined along similar lines using a manifold with boundaries. Given Δ , then \mathcal{J}_Δ defines graphs on the boundaries. Consequently, spin foams induce spin networks on the boundaries. The amplitudes have to be modified concerning the boundaries to have the correct composition property (10). This is achieved by changing the face amplitude from (Δ_{j_f}) to $(\Delta_{j_t})^{1/2}$ on external faces.

non-Abelian case. Writing $g \in U(1)$ as $g = e^{i\theta}$ the analog of the gravity simplicial action is

$$S(\Delta, \{e_f\}, \{\theta_e\}) = \sum_{f \in \mathcal{J}_\Delta} e_f F_f(\{\theta_e\}), \quad (37)$$

where $F_f(\{\theta_e\}) = \sum_{e \in \mathcal{J}_f} \theta_e$. Gauge transformations corresponding to (23) act at the end points of edges $e \in \mathcal{J}_\Delta$ by the action of group elements $\{\beta\}$ in the following way

$$\begin{aligned} B_f &\rightarrow B_f, \\ \theta_e &\rightarrow \theta_e + \beta_s - \beta_t, \end{aligned} \quad (38)$$

where the sub-index s (respectively t) labels the source vertex (respectively target vertex) according to the orientation of the edge. The gauge invariance of the simplicial action is manifest. The gauge transformation corresponding to (24) act on vertices of the triangulation Δ and is given by

$$\begin{aligned} B_f &\rightarrow B_f + \eta_s - \eta_t, \\ \theta_e &\rightarrow \theta_e. \end{aligned} \quad (39)$$

According to the discrete analog of Stokes theorem

$$\sum_{f \in \text{Bubble}} F_f(\{\theta_e\}) = 0,$$

which implies the invariance of the action under the transformation above. The divergence of the corresponding spin foam amplitudes is due to this last freedom. Alternatively, one can understand it from the fact that Stokes theorem implies a redundant delta function in (30) per bubble in \mathcal{J}_Δ [53].

¹¹If $\mathcal{M} = S^2 \times [0, 1]$ one can construct a cellular decomposition interpolating any two graphs on the boundaries without having internal bubbles and hence no divergences.

The crucial property of this spin foam model is that the amplitudes are independent of the chosen cellular decomposition [55, 57]. This allows for computing transition amplitudes between any spin network states $s = (\gamma, \{j\}, \{\iota\})$ and $s' = (\gamma', \{j'\}, \{\iota'\})$ according to the following rules¹²:

- Given $\mathcal{M} = \Sigma \times [0, 1]$ (piecewise linear) and spin network states $s = (\gamma, \{j\}, \{\iota\})$ and $s' = (\gamma', \{j'\}, \{\iota'\})$ on the boundaries –for γ and γ' piecewise linear graphs in Σ – choose any cellular decomposition Δ such that the dual 2-complex \mathcal{J}_Δ is bordered by the corresponding graphs γ and γ' respectively (existence can be shown easily).
- Compute the transition amplitude between s and s' by summing over all spin foam amplitudes (rescaled as in (36)) for the spin foams $F : s \rightarrow s'$ defined on the 2-complex \mathcal{J}_Δ .

4.3.3 The generalized projector

We can compute the transition amplitudes between any element of the kinematical Hilbert space \mathcal{H} ¹³. Transition amplitudes define the physical scalar product by reproducing the skein relations of the canonical analysis. We can construct the physical Hilbert space by considering equivalence classes under states with zero transition amplitude with all the elements of \mathcal{H} , i.e., null states.

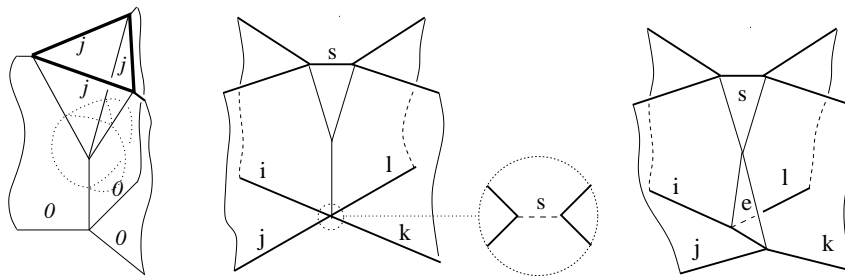


Figure 5: Elementary spin foams used to prove skein relations.

Here we explicitly construct a few examples of null states. For any contractible Wilson loop in the j representation the state

$$\psi = 2j + 1 \text{ } s - \left(\text{ } j \text{ } \right) \otimes_{\text{phys}} s = 0, \quad (40)$$

for any spin network state s , has vanishing transition amplitude with any element of \mathcal{H} . This can be easily checked by using the rules stated above and the portion of spin foam

¹²Here we are ignoring various technical issues in order to emphasize the relevant ideas. The most delicate is that of the divergencies due to gauge factors mentioned above. For a more rigorous treatment see [58].

¹³The sense in which this is achieved should be apparent from our previous definition of transition amplitudes. For a rigorous statement see [58].

illustrated in Figure 5 to show that the two terms in the previous equation have the same transition amplitude (with opposite sign) to any spin-network state in \mathcal{H} . Using the second elementary spin foam in Figure 5 one can similarly show that

$$\begin{array}{c} \text{i} \quad \text{l} \\ \diagdown \quad \diagup \\ \text{---} \text{s} \text{---} \\ \diagup \quad \diagdown \\ \text{j} \quad \text{k} \end{array} - \begin{array}{c} \text{i} \quad \text{l} \\ \diagdown \quad \diagup \\ \text{s} \\ \diagup \quad \diagdown \\ \text{j} \quad \text{k} \end{array} \stackrel{\text{phys}}{=} 0, \quad (41)$$

or the re-coupling identity using the elementary spin foam on the right of Figure 5

$$\begin{array}{c} \text{i} \quad \text{l} \\ \diagdown \quad \diagup \\ \text{s} \\ \diagup \quad \diagdown \\ \text{j} \quad \text{k} \end{array} - \sum_e \sqrt{2s+1} \sqrt{2e+1} \left\{ \begin{array}{ccc} i & j & s \\ & k & l & e \end{array} \right\} \begin{array}{c} \text{i} \quad \text{l} \\ \diagdown \quad \diagup \\ \text{e} \\ \diagup \quad \diagdown \\ \text{j} \quad \text{k} \end{array} \stackrel{\text{phys}}{=} 0, \quad (42)$$

where the quantity in brackets represents an $SU(2)$ $6j$ -symbol. All skein relations can be found in this way. The transition amplitudes imply the skein relations that define the physical Hilbert space! The spin foam quantization is equivalent to the canonical one.

4.3.4 The continuum limit

Recently Zapata [58] formalized the idea of a continuum spin foam description of 3-dimensional gravity using projective techniques inspired by those utilized in the canonical picture [24]. The heuristic idea is that due to the discretization invariance one can define the model in an ‘infinitely’ refined cellular decomposition that contains any possible spin network state on the boundary (this intuition is implicit in our rules for computing transition amplitudes above). Zapata concentrates on the case with non-vanishing cosmological constant and constructs the continuum extension of the Turaev-Viro model.

4.4 Conclusions

We have illustrated the general notion of the spin foam quantization in the simple case of 3 dimensional Riemannian gravity (for the generalization to the Lorentzian case see [59]). The main goal of the approach is to provide a definition of the physical Hilbert space. The example of this section sets the guiding principles of what one would like to realize in four dimensions. However, as it should be expected there are various new issues that make the task by far more involved.

5 Spin foams for 4-dimensional quantum gravity

In this section we briefly describe the various spin foam models for quantum gravity in the literature.

5.1 The Reisenberger model

According to Plebanski [48] action of self dual Riemannian gravity can be written as a constrained $SU(2)$ BF theory

$$S(B, A) = \int \text{Tr}[B \wedge F(A)] - \psi_{ij} \left[B^i \wedge B^j - \frac{1}{3} \delta^{ij} B^k \wedge B_k \right], \quad (43)$$

where the symmetric Lagrange multiplier ψ_{ij} imposes the constraints

$$\Omega^{ij} = B^i \wedge B^j - \frac{1}{3} \delta^{ij} B^k \wedge B_k = 0. \quad (44)$$

when B is non degenerate the constraints are satisfied if and only if $B^i = \pm(e^0 \wedge e^i + \frac{1}{2} \epsilon^i_{jk} e^j \wedge e^k)$ which reduces the previous action to that self-dual general relativity. Reisenberger studied the simplicial discretization of this action in [46] as a preliminary step toward the definition of the corresponding spin foam model. The consistency of the simplicial action is argued by showing that the simplicial theory converges to the continuum formulation when the triangulation is refined: both the action and its variations (equations of motion) converge to those of the continuum theory.

In reference [47] Reisenberger constructs a spin foam model for this simplicial theory by imposing the constraints Ω^{ij} directly on the $SU(2)$ BF amplitudes. The spin foam path integral for BF theory is obtained as in Section 4. The constraints are imposed by promoting the B^i to operators \mathcal{J}^i (combinations of left/right invariant vector fields) acting on the discrete connection¹⁴. The model is defined as

$$Z_{GR} = \int \underbrace{\prod_{e \in \mathcal{J}_\Delta} dg_e}_{\int \mathcal{D}[A]} \underbrace{\delta(\hat{\Omega}^{ij})}_{\int \mathcal{D}[\psi] e^{i\psi_{ij}\Omega^{ij}}} \underbrace{\sum_{\mathcal{C}:\{j\} \rightarrow \{f\}} \prod_{f \in \mathcal{J}_\Delta} \Delta_{j_f} \text{Tr}[j_f(g_e^1 \dots g_e^N)]}_{\int \mathcal{D}[B] e^{i \int \text{Tr}[B \wedge F(A)]}}, \quad (46)$$

where $\hat{\Omega} = \mathcal{J}^i \wedge \mathcal{J}^j - \frac{1}{3} \delta^{ij} \mathcal{J}^k \wedge \mathcal{J}_k$ and we have emphasized the correspondence of the different terms with the continuum formulation. The previous equation is rather formal for the rigorous implementation see [47]. Reisenberger uses locality so that constraints are implemented on a single 4-simplex amplitude. He defines a one parameter family of models by inserting the operator

$$e^{-\frac{1}{2z^2} \hat{\Omega}^2} \quad (47)$$

instead of the delta function above. In the limit $z \rightarrow \infty$ the constraints are sharply imposed. The properties of the kernel of $\hat{\Omega}$ has not been studied in detail.

¹⁴Notice that (for example) the right invariant vector field $\mathcal{J}^i(U) = \sigma^i_B{}^A U_C^B \partial / \partial U_C^A$ has a well defined action at the level of equation (32) and acts as a B operator at the level of (29) since

$$-i\mathcal{J}^i(U) \left[e^{i\text{Tr}[BU]} \right] |_{U \sim 1} = \text{Tr}[\sigma^i U B] e^{i\text{Tr}[BU]} |_{U \sim 1} \sim B^i e^{i\text{Tr}[BU]}, \quad (45)$$

where σ^i are Pauli matrices.

5.2 The Freidel-Krasnov prescription

In reference [60] Freidel and Krasnov define a general framework to construct spin foam models corresponding to theories whose action has the general form

$$S(B, A) = \int \text{Tr}[B \wedge F(A)] + \Phi(B), \quad (48)$$

where the first term is the BF action while $\Phi(B)$ is certain polynomial function of the B field. The formulation is constructed for compact internal groups. The definition based on the formal equation

$$\int \mathcal{D}[B]\mathcal{D}[A] e^{i \int \text{Tr}[B \wedge F(A)] + \Phi(B)} := e^{i \int \Phi(\frac{\delta}{\delta J})} Z[J] \Big|_{J=0}, \quad (49)$$

where the *generating functional* $Z[J]$ is defined as

$$Z[J] := \int \mathcal{D}[B]\mathcal{D}[A] e^{i \int \text{Tr}[B \wedge F(A)] + \text{Tr}[B \wedge J]}, \quad (50)$$

where J is an algebra valued 2-form field. They provide a rigorous definition of the generating functional by introducing a discretization of \mathcal{M} in the same spirit of the other spin foam models discussed here. Their formulation can be used to describe various theories of interest such as BF theories with cosmological terms, Yang-Mills theories (in 2 dimensions) and Riemannian self-dual gravity. In the case of self dual gravity B and A are valued in $su(2)$, while

$$\Phi(B) = \int \psi_{ij} \left[B^i \wedge B^j - \frac{1}{3} \delta^{ij} B^k \wedge B_k \right] \quad (51)$$

according to equation 43. The model obtained in this way is very similar to Reisenberger's one. There are however some technical differences. One of the more obvious one is that the non-commutative invariant vector fields \mathcal{J}^i representing B^i are replaced here by the commutative functional derivatives $\delta/\delta J^i$. The explicit properties of these models have not been studied further.

5.3 The Iwasaki model

Iwasaki defines a spin foam model of self dual Riemannian gravity ¹⁵ by a direct lattice discretization of the continuous Ashtekar formulation of general relativity. This model constitutes an example of the general prescription of Section 3.2. The action

$$S(e, A) = \int dx^4 \epsilon^{\mu\nu\lambda\sigma} \left[2 e_{[\mu}^0 e_{\nu]i} + \epsilon^0_{ijk} e_{\mu}^j e_{\nu}^k \right] \left[2 \partial_{[\lambda} A_{\sigma]}^i + \epsilon^{0i}_{lm} A_{\lambda}^l A_{\sigma}^m \right], \quad (52)$$

¹⁵Iwasaki defines another model involving multiple cellular complexes to provide a simpler representation of wedge products in the continuum action. A more detail presentation of this model would require the introduction of various technicalities at this stage so we refer the reader to [61].

where A_μ^i is an $SU(2)$ connection. The fundamental observation of [62] is that one can write the discrete action in a very compact form if we encode part of the degrees of freedom of the tetrad in an $SU(2)$ group element. More precisely, if we take $g_{\mu\mu} = e^i e_\mu^j \delta_{ij} = 1$ we can define $\mathbf{e}_\mu := e_\mu^0 + i\sigma_i e_\mu^i \in SU(2)$ where σ_i are the Pauli matrices. In this parameterization of the ‘angular’ components of the tetrad and using a hypercubic lattice the discrete action becomes

$$S_\Delta = -\beta \sum_{v \in \Delta} \epsilon^{\mu\nu\lambda\sigma} r_\mu r_\nu \text{Tr} [\mathbf{e}_\mu^\dagger \mathbf{e}_\nu U_{\lambda\sigma}], \quad (53)$$

where $r_\mu := (\beta^{1/2} \ell_p)^{-1} \epsilon \sqrt{g_{\mu\mu}}$, $U_{\mu\nu}$ is the holonomy around the $\mu\nu$ -plaquette, ϵ the lattice constant and β is a cutoff for r_μ used as a regulator ($r_\mu \leq \beta^{1/2} \ell_p \epsilon^{-1}$). The lattice path integral is defined by using the Haar measure both for the connection and the ‘spherical’ part of the tetrad \mathbf{e} ’s and the radial part $dr_\mu := dr_\mu r_\mu^3$. The key formula to obtain an expression involving spin foams is

$$e^{ix \text{Tr}[U]} = \sum_j (2j+1) \frac{J_{2j+1}(2x)}{x} \chi_j(U) \quad (54)$$

Iwasaki writes down an expression for the spin foam amplitudes in which the integration over the connection and the \mathbf{e} ’s can be computed explicitly. Unfortunately, the integration over the radial variables r involves products of Bessel functions and its behavior is not analyzed in detailed. In 3 dimensions the radial integration can be done and the corresponding amplitudes coincide the results of Section (4.3).

5.4 The Barrett-Crane model

The appealing feature of the previous models is the clear connection to the continuum theory. Moreover, since they correspond to self dual gravity they are well suited for studying the relationship with loop quantum gravity (boundary states are $SU(2)$ -spin networks). The drawback is the lack of closed simple expressions for the amplitudes which complicates their analysis. There is however a simple model defined in a spirit which shares some similarities with the quantization of $SO(4)$ Plebanski’s action (see next subsection). This model was introduced by Barrett and Crane in [8] and further motivated by Baez in [13]. The basic idea behind the definition is that of the *quantum tetrahedron* introduced by Barbieri in [63] and generalized to 4d in [14]. The beauty of the model resides in its remarkable simplicity. This has stimulated a great deal of explorations and produced many interesting results. We will review most of these in Section 7.

5.5 Markopoulou-Smolin causal spin networks

Using the kinematical setting of LQG with the assumption of the existence of a micro-local (in the sense of Planck scale) causal structure Markopoulou and Smolin define a general class of (causal) spin foam models for gravity [64, 65] (see also [66]). The elementary transition amplitude $A_{s_I \rightarrow s_{I+1}}$ from an initial spin network s_I to s_{I+1} is defined by a set of

simple combinatorial rules based on a definition of causal propagation of the information at nodes. The rules and amplitudes have to satisfy certain causal restriction (motivated by the standard concepts in classical Lorentzian physics). These rules generate surface-like excitations of the same kind we encounter in the more standard spin foam model but endow the foam with a notion of causality. Spin foams $\mathcal{F}_{s_i \rightarrow s_f}^N$ are labelled by the number of times this elementary transitions take place. Transition amplitudes are defined as

$$\langle s_i, s_f \rangle = \sum_N \prod_{I=0}^{N-1} A_{s_I \rightarrow s_{I+1}}. \quad (55)$$

The models are not related to any continuum action. The only guiding principles are the restrictions imposed by causality, simplicity and the requirement of the existence of a non-trivial critical behavior that would reproduce general relativity at large scales. Some indirect evidence of a possible non trivial continuum limit has been obtained in some versions of the model in $1 + 1$ dimensions.

5.6 Gambini-Pullin model

In reference [67] Gambini and Pullin introduced a very simple model obtained by modification of the BF theory skein relations. As we argued in Section 4 skein relations defining the physical Hilbert space of BF theory are implied by the spin foam transition amplitudes. These relations reduce the big cinemactical Hilbert space of BF theory (analogous to that of quantum gravity) to a physical Hilbert space corresponding to the quantization of a finite number of degrees of freedom. Gambini and Pullin define a model by modifying these amplitudes so that some of the skein relations are now forbidden. This simple modification of frees local excitations of a field theory. A remarkable feature is that the corresponding physical states are (in a certain sense) solutions to various regularizations of the scalar constraint for (Riemannian) LQG. The fact that physical states of BF theory solve the scalar constraint is well known [68], since roughly $F(A) = 0$ implies $EEF(A) = 0$. The situation here is of a similar nature, and –as the authors argue– one should interpret this result as an indication that some ‘degenerate’ sector of quantum gravity might be represented by this model. The definition of this spin foam model is not explicit since the theory is directly defined by the physical skein relations.

5.7 Capovilla-Dell-Jacobson theory on the lattice

The main technical difficulty that we gain in going from 3-dimensional general relativity to the 4-dimensional one is that the integration over the e ’s becomes intricate. In the Capovilla-Dell-Jacobson[69, 70] formulation of general relativity this ‘integration’ is partially performed at the continuum level. The action

$$S(\eta, A) = \int \eta \text{Tr} [\epsilon \cdot F(A) \wedge F(A) \epsilon \cdot F(A) \wedge F(A)], \quad (56)$$

where $\epsilon \cdot F \wedge F := \epsilon^{abcd} F_{ab} F_{cd}$. Integration over η can be formally performed in the path integral and we obtain

$$Z = \int \prod_x \delta(\text{Tr}[\epsilon \cdot F(A) \wedge F(A) \epsilon \cdot F(A) \wedge F(A)]), \quad (57)$$

This last expression is of the form (17) and can be easily discretized along the lines of Section 3.2. The final expression (after integrating over the lattice connection) involves a sum over spin configurations with no implicit integrations.

6 Some conceptual issues

6.1 Anomalies and gauge fixing

As we mentioned in previous section and illustrated with the example of Section 4, spin foam path integral is meant to provide a device to reconstruct the physical Hilbert space. Spin foam transition amplitudes are not interpreted as defining propagation in time but rather as defining the physical scalar product. This interpretation of spin foam models appears as the only consistent one with diffeomorphism invariance and what we know about generally covariant systems from the canonical viewpoint. The formal arguments that motivates this interpretation rely on the gauge invariance of the path integral measure and that of the action. In turn this implies divergences due to the contributions of gauge equivalent configurations if no appropriate gauge fixing condition is provided. In this section we analyze these issues in the context of the spin foam approach.

Let us first address the question of the internal gauge. If this gauge group is compact then one can avoid gauge fixing by using appropriate variables and normalized measures. This is the case for the models of Riemannian gravity considered in this paper (internal gauge group $SO(4)$ or $SU(2)$) or in standard lattice gauge theory where one represents the connections in terms of group elements (holonomies) and uses the (normalized) Haar measure in the integration. In the Lorentzian sector (internal gauge group $SL(2, \mathbb{C})$) the internal gauge orbits have infinite volume and the lattice path integral would diverge without an appropriate gauge fixing condition. These gauge fixing conditions generally exist for the connection in spin foam models: it amounts to set to the identity redundant lattice connections. We will study an example in Section 9 (for a general treatment see [71]).

The remaining gauge freedom is diffeomorphisms. We concentrate on the case of a model defined according to Section 3.2, i.e., on a fixed discretization. In this case 4-diffeomorphism covariance is replaced by its discrete counterpart: finite group of symmetries of the discrete configurations. The path integral is defined on a fixed discretization Δ which replaces the notion of the manifold in the continuum. We can imagine Δ as embedded in a smooth manifold with some smooth field configuration. On the lattice one can represent such smooth configuration in an approximate manner (e.g., assuming for simplicity that we are dealing with a covariant gauge theory $S(A)$ by assigning the

corresponding holonomies). The discretization captures in this way the imprint of the continuum configuration (see the schematic representation on the left of Figure 6). Clearly, the discrete action will fail to be diffeomorphism invariant in a continuum sense¹⁶. The remanent of diffeomorphisms in the discretization are discrete symmetries. We require the discrete action to be invariant under this notion of discrete symmetries. An active diffeomorphism on a continuum configuration is replaced by a discrete symmetry represented on the right of Figure 6. This symmetry is to be incorporated in the definition of the spin foam model of a background independent theory. Clearly, in a background dependent theory (such as standard lattice gauge theory) the two configurations on the right of Figure 6 are different physical configurations for the lattice contains information about a background geometry (lattice scale). The spin foam model is to be defined as a sum over equivalence classes of spin foams¹⁷ or equivalently one should sum over all configurations and divide by the corresponding symmetry factors. The latter are finite for any finite discretization. According to this one expects spin foam models for gravity not to diverge on the ground of ‘diffeomorphism volume’ factors. No diffeomorphism gauge fixing is necessary since one has control on the discrete symmetries in a combinatorial way. The point of view is that this combinatorial equivalence among spin foams represent the fundamental symmetry that reproducing diffeomorphism invariance as the discretization dependence is removed in a suitable way (see next section).

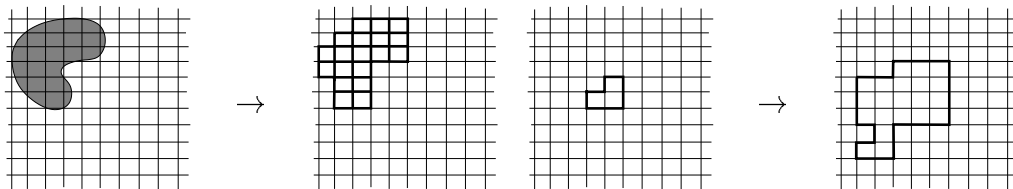


Figure 6: Diffeomorphisms are replaced by discrete symmetries in spin foams. In this simplified picture there are two possible colorings of 0 or 1, the latter represented by the darkening of the corresponding lattice element. The two configurations on the right as regarded as physically equivalent.

Gambini and Pullin[72, 73] are studying the canonical formulation of theories that live on a lattice from the onset. This provides a way to analyze the meaning of gauge symmetries directly à la Dirac. Their results indicate that diff-invariance is indeed broken by the discretization in the sense that there is no infinitesimal generator of diffeomorphism. This is consistent with our covariant picture of discrete symmetries above. In their formulation the canonical equations of motion fix the value of what where Lagrange multipliers in the continuum (e.g. lapse and shift). This is interpreted as a breaking of diffeomorphism invariance; however, the solutions of the multiplier equations are not unique. The ambiguity

¹⁶We could act with a diffeo that shrinks the smooth configuration shown in Figure 6 so that no edge on the lattice is excited changing in this way the value of the lattice action.

¹⁷In [13] this arises naturally as a requirement if one is to interpret spin foams as morphisms in the category whose objects are given by spin networks.

in selecting a particular solution corresponds to the remanent diffeomorphism invariance of the discrete theory.

Finally let us briefly recall the situation in 3-dimensional gravity. In three dimensions the discrete action is invariant under transformations that are in correspondence with the continuum gauge freedoms (23) and (24). As we mentioned in Section 4.3.1 spin foam amplitudes diverge due to (24). This is in no contradiction to our previous argument since (refgauge2) involves more than just diffeomorphisms in 3d.

Some of the spin foam models in Section 5 are defined from BF theory by implementing constraints that restore the graviton excitations of general relativity. Gauge divergencies of the BF amplitudes are expected to disappear in the constrained models since the implementation of constraints breaks the topological symmetry (24).

6.2 Discretization dependence

The spin foam models we have introduced so far are defined on a fixed cellular decomposition of \mathcal{M} . This is to be interpreted as an intermediate step toward the definition of the theory. The discretization reduces the infinite dimensional functional integral to a multiple integration over a finite number of variables. This cutoff is reflected by the fact that only a restrictive set of spin foams (spin network histories) is allowed in the path integral: those that can be obtained by all possible coloring of the underlying 2-complex. In addition it restricts the number of possible 3-geometry states (spin network states) on the boundary by fixing a finite underlying boundary graph. This represents a truncation in the allowed fluctuations and the set of states of the theory that can be interpreted as a regulator. However, the nature of this regulator is fundamentally different from the standard concept in the background independent framework: since geometry is encoded in the coloring (that can take any spin values) the configurations involve fluctuations all the way to Plank scale¹⁸. This scenario is different in lattice gauge theories where the lattice introduces an effective UV cutoff given by the lattice spacing. Transition amplitudes are however discretization dependent now. A consistent definition of the path integral using spin foams should include a prescription to eliminate this discretization dependence.

A special case is that of topological theories such as gravity in 3 dimensions. In this case, one can define the sum over spin foams with the aid of a fixed cellular decomposition Δ of the manifold. Since the theory has no local excitations (no gravitons), the result is independent of the chosen cellular decomposition. A single discretization suffices to capture the degrees of freedom of the topological theory.

In lattice gauge theory the solution to the problem is implemented through the so-called continuum limit. In this case the existence of a background geometry is crucial, since it allows to define the limit when the lattice constant (length of links) goes to zero. In addition the possibility of working in the Euclidean regime allows the implementation of statistical mechanical methods.

¹⁸Changing the label of a face from j to $j + 1$ amounts to changing an area eigenvalue by an amount of the order of Planck length squared according to (7).

None of these structures are available in the background independent context. The lattice (triangulation) contains only topological information and there is no geometrical meaning associated to its components. As we mentioned above this has the novel consequence that the truncation can not be regarded as an UV cutoff as in the background dependent context. This in turn represents a conceptual obstacle to the implementation of standard techniques. Moreover, no Euclidean formulation seem meaningful in a background independent scenario. New means to eliminate the truncation introduced by the lattice have to be developed.

This is a major issue where concrete results have not been obtained so far beyond the topological case. Here we explain the two main approaches to recover general covariance corresponding to the realization of the notion of ‘summing over discretizations’ of [44].

- *Refinement of the discretization:*

According to this idea topology is fixed by the simplicial decomposition. The truncation in the number of degrees of freedom should be removed by considering triangulations of increasing number of simplexes for that fixed topology. The flow in the space of possible triangulations is controlled by the Pachner moves. The formal idea is to take a limit in which the number of four simplexes goes to infinity together with the number of tetrahedra on the boundary. Given a 2-complex \mathcal{J}_2 which is a refinement of a 2-complex \mathcal{J}_1 then the set of all possible spin foams defined on \mathcal{J}_1 is naturally contained in those defined on \mathcal{J}_2 (taking into account the equivalence relations for spin foams mentioned in the previous section). The refinement process should also enlarge the space of possible 3-geometry states (spin networks) on the boundary recovering the full kinematical sector in the limit of infinite refinements. An example where this procedure is well defined corresponds to Zapata’s treatment of the Turaev-Viro model [58]. The key point in this case is that amplitudes are independent of the discretization (due to the topological character of the theory) so that the refinement limit is trivial. In the general case here is a great deal of ambiguity involved in the definition of refinement¹⁹. The hope is that the nature of the transition amplitudes would be such that these ambiguities will not affect the final result. The Turaev-Viro model is example of where this prescription works[58].

If the refinement limit is well defined one would expect that working with a ‘sufficiently refined’ but fixed discretization would serve as an approximation that could

¹⁹It is not difficult to define the refinement in the case of a hypercubic lattice. In the case of a simplicial complex a tentative definition can be attempted using Pachner moves. For example: given an initial triangulation of a manifold with boundary Δ_1 define Δ'_1 by implementing a 1 – 5 Pachner move to each 4-simplex in Δ_1 . Δ'_1 is a homogeneous refinement of Δ_1 ; however, the boundary triangulation remain unchanged so that they will support the same space of boundary data or spin networks. As mentioned above, in the refinement process one also wants to refine the boundary triangulation so that the corresponding dual graph will get refined and the space of possible boundary data will become bigger (all of Cyl in the limit). In order to achieve this we define Δ_2 by erasing from Δ'_1 all 4-simplexes sharing a tetrahedron with the boundary. This amounts to carrying out a 1 – 4 Pachner move on the boundary. This completes the refinement $\Delta_1 \rightarrow \Delta_2$. One would need to proof that spin foam amplitudes converge as the discretization is refined and that the limit is independent of the choice of initial triangulation.

be used to extract physical information within some quantifiable precision.

- *Spin foams as Feynman diagrams:*

This idea has been motivated by the generalized matrix models of Boulatov and Ooguri [74, 75]. The fundamental observation is that spin foams admit a dual formulation in terms of a field theory over a group manifold [10, 15, 16]. The duality holds in the sense that spin foam amplitudes correspond to Feynman diagram amplitudes of the GFT. The perturbative Feynman expansion of the GFT (expansion in a fiducial coupling constant λ) provides a definition of *sum over* discretizations which is fully combinatorial and hence independent of any manifold structure²⁰. The latter is most appealing feature of this approach. However, the convergence issues become clearly more involved. In addition it is not clear how the notion of diffeomorphism would be addressed in this framework. Diffeomorphism equivalent configurations (in the discrete sense described above) appear at all orders in the perturbation series.²¹ It is not clear how these ideas are to be implemented in the context of the GFT formulation. The GFT formulation has been however very useful in the definition of simple normalization of the Barret-Crane model and has simplified its generalizations to the Lorentzian sector. We will study this formulation in detail in Section 8.

6.3 Physical scalar product revisited

In this subsection we study the properties of the generalized projection operator P introduced before. The generalized projection operator is a linear map from the kinematical Hilbert space into the physical Hilbert space (states annihilated by the Scalar constraint). These states are not contained in the kinematical Hilbert space but are rather elements of the dual space Cyl^* . For this reason the operator P^2 is not well defined. We have briefly mentioned the construction of the generalized projection operator in the particular context of Rovelli's model of Section (3.1). After the above discussion of gauge and discretization dependence we revisit the construction of the P operator. We consider the case in which discretization independence is obtained by a refinement procedure.

The number of 4-simplexes N in the triangulation plays the role of cutoff. The matrix elements of the P operator is expected to be defined by the refinement limit of fixed discretization transition amplitudes ($\langle sP_N, s' \rangle$), namely

$$\langle s, s' \rangle_{phys} = \langle sP, s' \rangle = \lim_{N \rightarrow \infty} \langle sP_N, s' \rangle. \quad (58)$$

If we have a well defined operator P (formally corresponding to $\delta[\hat{\mathcal{S}}]$) the reconstruction of the physical Hilbert space goes along the lines of the GNS construction[78] where the

²⁰This is more than a 'sum over topologies' as many of the 2-complex appearing in the perturbative expansion cannot be associated to any manifold [76].

²¹The GFT formulation is clearly non trivial already in the case of topological theories. There has been attempts to make sense of the GFT formulation dual to BF theories in lower dimensions [77].

algebra is given by Cyl and the state $\omega(\Psi)$ for $\Psi \in Cyl$ is defined by P as

$$\omega(\Psi) = \frac{\langle 0P, \Psi \rangle}{\langle 0P, 0 \rangle},$$

In the spin foam language this corresponds to the transition amplitude from the state Ψ to nothing. The representation of this algebra implied by the GNS construction has been used in [79] to define observables in the theory.

In Section 3.1 we mentioned another way to obtain observables according to the definition

$$\langle sO_{phys}, s' \rangle = \langle sPO_{kin}P, s' \rangle, \quad (59)$$

where O_{kin} is a kinematical observable. This construction is expected to be well defined only for some suitable kinematical operators O_{kin} (e.g., the previous equation is clearly divergent for the kinematical identity). There kinematical operators which can be used as

regulators. They corresponds to Rovelli's ‘sufficiently localized in time’ operators [44]²². In [80] it is argued on physical grounds that when posing a physical questions one is lead to conditioning the path integral and in this way improving the convergence properties. Conditioning is represented here by the regulator operators (see previous footnote).

6.4 Contact with the low energy world

A basic test to any theory of quantum gravity is the existence of a well defined classical limit corresponding to general relativity. In the case of the spin foam approach this should be accomplished simultaneously with a limiting procedure that bridges the fundamental discrete theory with the smooth description of classical physics. That operation is sometimes referred to as the *continuum limit* but it should not be confused with the issues

²² Consider the following simple example. Take the torus $S^1 \times S^1$ represented by $[0, 2\pi] \times [0, 2\pi] \in \mathbb{R}^2$ with periodic boundary conditions. An orthonormal basis of states is given by the (‘spin network’) wave functions $\langle xy, nm \rangle = \frac{1}{2\pi} e^{inx} e^{imy}$ $n, m \in \mathbb{Z}$. Take the scalar constraint to be given by \hat{X} so that the generalized projection \hat{P} simply becomes

$$\langle nm, \hat{P} \mid = \sum_{\alpha} \langle \alpha m \mid.$$

The RHS is no longer in \mathcal{H}_{kin} since is not a normalizable: it is meaningful as a distributional state. The physical scalar product is defined as in (58), namely

$$\langle nm, n'm' \rangle_{phys} = \langle nmP, n'm' \rangle = \sum_{\alpha} \langle \alpha m, n'm' \rangle = \delta_{mm'}.$$

Therefore we have that $|nm\rangle \stackrel{phys}{=} |km\rangle \forall k$. We can define physical observables by means of the formula

$$\langle m, \hat{O}_{ph} \mid m' \rangle = \langle nm, \hat{P} \hat{O} \hat{P} \mid n'm' \rangle,$$

for suitable kinematic observables O . If for example we take $O = 1$ the previous equation would involve \hat{P}^2 which is not defined. However, take the following family of kinematical observables

$$|O_k, nm\rangle = \delta_{n,k} |km\rangle$$

then $\langle m, \hat{O}_{ph} \mid m' \rangle = \delta_{m,m'}$, i.e., \hat{O}_{ph} is just the physical identity operator for any value of k . The whole family of kinematic operators define a single physical operator. The key of the convergence of the previous definition is the fact that is a projection operator into an eigenstate of the operator \hat{P}_x canonically conjugate to the constraint \hat{X} . This illustrates what kind of operators will be suitable for the above definition to work. In particular any operator that decays sufficiently fast along going to $\pm\infty$ in the spectrum of \hat{p}_x will be. For example the Gaussian $\hat{O}_k |nm\rangle = e^{\frac{-(k-n)^2}{\epsilon}} |nm\rangle$ which also projects down to a multiple of the physical identity. Moreover, availability of such a kinematical operator can serve as a regulator of non-suitable operators, namely

$$\langle m, \hat{O}_{ph} \mid m' \rangle = \langle nm, \hat{P} \hat{O}_k \hat{O} \hat{O}_k \hat{P} \mid n'm' \rangle,$$

will converge even for $\hat{O} = 1$. The operator \hat{O}_k can be regarded as a quantum gauge fixing: it selects a point along the gauge orbit generated by the constraint.

analyzed in the previous subsection.

The general strategy (which is in fact motivated by our experience in background physics) is to setup a renormalization (à la Wilson) scheme where microscopic degrees of freedom are summed out to obtain a coarse grained effective description. There are interesting suggestions on how to implement such a program which include the use of novel mathematical techniques [81, 82]. There is little work done in trying to implement these ideas to concrete models. In addition, the whole conceptual framework has yet to be clarified. This is a major issue in the approach.

7 $SO(4)$ Plebanski's action, the quantum tetrahedron and the Barrett-Crane model

The Barrett-Crane model is one of the most extensively studied spin foam models for quantum gravity (for a specific review see [83]). We concentrate on its definition in the following presenting the results obtained in the context of its Riemannian version and its generalization to the Lorentzian sector.

The Barrett-Crane model is usually viewed as a spin foam quantization of $SO(4)$ Plebanski's formulation of general relativity. The general idea being analogous to the one implemented in the model described in Section 5.1. In order to analyze the sense in which such a viewpoint is applicable we start this section by reviewing the spin foam quantization of $SO(4)$ 4-dimensional BF theory and by giving a brief description of Plebanski's theory. In subsection (7.10), we analyze the connection between the Barrett-Crane model and Plebanski's formulation.

7.1 Quantum $SO(4)$ BF theory

Classical ($Spin(4)$) BF theory is defined by the action

$$S[B, A] = \int \text{Tr} [B \wedge F(A)], \quad (60)$$

where B_{ab}^{IJ} is a $Spin(4)$ Lie-algebra valued 2-form, A_a^{IJ} is a connection on a $Spin(4)$ principal bundle over \mathcal{M} . The theory has no local excitations. Its properties are very much analogous to the case of 3-dimensional gravity studied in Section 4.

A discretization of \mathcal{M} can be introduced along the same lines presented in Section 4. Upon integration over the B 's [84] the path integral becomes

$$Z(\Delta) = \int \prod_{e \in \Delta^*} dg_e \prod_{f \in \Delta^*} \delta(g_{e_1} \cdots g_{e_n}), \quad (61)$$

which is the analog of (30). Using Peter-Weyl's theorem one can integrate over the connection. When the discretization is defined in terms of a triangulation the final result

is

$$Z_{BF}(\Delta) = \sum_{\mathcal{C}_f: \{f\} \rightarrow \rho_f} \sum_{\mathcal{C}_e: \{e\} \rightarrow \{\iota_e\}} \prod_{f \in \Delta^*} \Delta_\rho \prod_{v \in \Delta^*} \text{Diagram}, \quad (62)$$

where the pentagonal diagram representing the vertex amplitude denotes the trace of the product of five intertwiners $C_{\rho_1 \rho_2 \rho_3 \rho_4}^\iota$. As in the model for 3-dimensional gravity, the vertex amplitude corresponds to the flat evaluation of the spin network state defined by the pentagonal diagram in (62), a $15j$ -symbol. Vertices $v \in \Delta^*$ are in one-to-one correspondence to 4-simplexes in the triangulation Δ . In addition we have $\mathcal{C}_e: \{e\} \rightarrow \{\iota_e\}$ representing the assignment of intertwiners to edges (this arises in the group integration as a consequence of the analog of equation (33)). The sum over the *coloring* of edges, \mathcal{C}_e , comes from (33).

The state sum (62) is generically divergent (due to the gauge freedom analogous to (24)). A regularized version defined in terms of $SU_q(2) \times SU_q(2)$ was introduced by Crane-Yetter [85, 86]. As in three dimensions (62) is topological invariant and the spin foam path integral is discretization independent.

7.2 Classical $SO(4)$ Plebanski action

Plebanski's Riemannian action depends on an $so(4)$ connection A , a Lie-algebra-valued 2-form B and Lagrange multiplier fields λ and μ . Writing explicitly the Lie-algebra indices, the action is given by

$$S[B, A, \lambda] = \int [B^{IJ} \wedge F_{IJ}(A) + \lambda_{IJKL} B^{IJ} \wedge B^{KL} + \mu \epsilon^{IJKL} \lambda_{IJKL}], \quad (63)$$

where μ is a 4-form and $\lambda_{IJKL} = -\lambda_{JLKL} = -\lambda_{IJLK} = \lambda_{KLIJ}$ is tensor in the internal space. Variation with respect to μ imposes the constraint $\epsilon^{IJKL} \lambda_{IJKL} = 0$ on λ_{IJKL} . λ_{IJKL} has then 20 independent components. Variation with respect to λ imposes 20 algebraic equations on the 36 components of B . Solving for μ they are

$$\epsilon^{\mu\nu\rho\sigma} B_{\mu\nu}^{IJ} B_{\rho\sigma}^{KL} = e \epsilon_{IJKL} \quad (64)$$

which is equivalent to

$$\epsilon_{IJKL} B_{\mu\nu}^{IJ} B_{\rho\sigma}^{KL} = e \epsilon_{\mu\nu\rho\sigma}, \quad (65)$$

for $e \neq 0$ where $e = \frac{1}{4!} \epsilon_{OPQR} B_{\mu\nu}^{OP} B_{\rho\sigma}^{QR} \epsilon^{\mu\nu\rho\sigma}$ [87]. The solutions to these equations are

$$B = \pm^*(e \wedge e), \quad \text{and} \quad B = \pm e \wedge e, \quad (66)$$

in terms of the 16 remaining degrees of freedom of the tetrad field e_a^I . If one substitutes the first solution into the original action one obtains Palatini's formulation of general relativity

$$S[e, A] = \int \text{Tr}[e \wedge e \wedge {}^*F(A)]. \quad (67)$$

7.3 Discretized Plebanski's constraints

In the previous section we have derived the spin foam model for $Spin(4)$ BF theory. We have seen that the sum over B -configurations corresponding spin foam model is encoded in the sum over unitary irreducible representations of $Spin(4)$. Can we restrict the spin foam configurations to those that satisfy Plebanski's constraints of the previous subsection? To answer this one first needs to translate the constraints of Plebanski's formulation into the simplicial framework. There are two ways in which we can approach the problem. One is by looking at the constraints in the form (63) the other is by using (65). We need to define a discrete notion of B field. The idea is to associate the algebra valued 2-forms $B_{\mu\nu}^{IJ}$ to an algebra valued B_t^{IJ} assigned to triangles $t \in \Delta$. One interprets B_t^{IJ} as the smearing of the 2-form on the 2-dimensional triangles [87]. The rule is then to replace the space-time indices of the 2-form by a triangle:

$$\mu\nu \rightarrow t = \text{triangle} \quad (68)$$

The first version of the constraints is of the type of constraints Ω^{ij} (recall equation 44) that appears in Plebanski's self dual version, i.e., it has free algebra indices. A model implementing these constraints could be defined along similar lines of the models presented in Sections 5.1 and 5.2. This possibility has not been explored and it would be interesting to see whether this proposal can provide a simple model.

The other possibility is to discretize (65). We will see that these constraints have the functional form of the ones implemented in the Barrett-Crane model. However, in the latter case the constraints are not implemented on BF configuration but using the auxiliary definition of the so-called quantum tetrahedron.

To translate these constraints to the simplicial framework it suffices to concentrate on a single 4-simplex. Using the rule (65) becomes

$$\epsilon_{IJKL} B_t^{IJ} B_{t'}^{KL} = 0, \quad (69)$$

whenever $t = t'$ or the triangles t and t' share an edge, i.e., whenever the two triangles belong to the same tetrahedron T , ($t, t' \in T$). The other case corresponds to the situation when t and t' do not lie on the same tetrahedron (hence they span a 4-dimensional region). If we label vertices in the 4-simplex $0 \cdots 4$ we can label the six triangles containing the vertex 0 simply as jk ($j, k = 1 \cdots 4$). Only these are needed to express the independent constraints

$$\text{Tr}[B_{12} \wedge B_{34}^*] = \text{Tr}[B_{13} \wedge B_{42}^*] = \text{Tr}[B_{14} \wedge B_{23}^*], \quad (70)$$

where $B_{IJ}^* := \epsilon_{IJKL} B^{KL}$.

7.4 The quantum tetrahedron in 4d and the Barrett-Crane intertwiner

The basic concept in the definition of the Barrett-Crane model is the *quantum tetrahedron*. Its definition is independent of the arguments presented in the previous subsections. It is

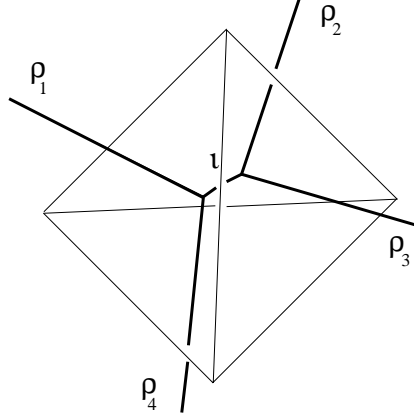


Figure 7: The quantum tetrahedron.

constructed applying geometric quantization to the classical notion of a tetrahedron: bi-vectors associated to triangles are promoted to ‘angular momentum’ operators. A rigorous account of the mathematics involved can be found in [14]. The basic idea is to associate to each triangle in the tetrahedron T a Hilbert space $\mathcal{H}_t := \bigoplus_{\rho} \mathcal{H}_{\rho}$, where \mathcal{H}_{ρ} defined by the $SO(4)$ -irreducible representation ρ . The Hilbert space of the quantum tetrahedron \mathcal{H}_T is defined as

$$\mathcal{H}_T = \mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \mathcal{H}_{t_3} \otimes \mathcal{H}_{t_4}. \quad (71)$$

The bi-vector B_t^{IJ} is associated with the infinitesimal generator of $SO(4)$ rotations J_t^{IJ} acting on the corresponding face-Hilbert space:

$$B_t^{IJ} \rightarrow J_t^{IJ}. \quad (72)$$

The classical condition that the sum of the oriented bivectors defined by the triangles should vanish for a tetrahedron ($B_1 + B_2 + B_3 + B_4 = 0$) translates (according to the rule (72)) into the restriction of the states of the quantum tetrahedron to rotationally invariant ones, i.e. those in

$$\bigoplus_{\rho_1 \rho_2 \rho_3 \rho_4} \text{Inv} [\mathcal{H}_{\rho_1} \otimes \mathcal{H}_{\rho_2} \otimes \mathcal{H}_{\rho_3} \otimes \mathcal{H}_{\rho_4}]. \quad (73)$$

We can choose a basis of invariant orthonormal vectors in each component $C_{\rho_1 \rho_2 \rho_3 \rho_4}^t \in \text{Inv} [\mathcal{H}_{\rho_1} \otimes \mathcal{H}_{\rho_2} \otimes \mathcal{H}_{\rho_3} \otimes \mathcal{H}_{\rho_4}]$. These are precisely the intertwiners corresponding to 4-valent spin network nodes! Tetrahedra are assigned with intertwiners in a similar way as it happens in the quantization of BF theory (this is illustrated in Figure 7).

Irreducible representations ρ of $Spin(4)$ are labelled by two half-integers (j_ℓ, j_r) . When $t = t'$ the constraints (69) become the $SO(4)$ -Casimir

$$\text{Tr}[J_t \hat{J}_t^*] = j_\ell(j_\ell + 1) - j_r(j_r + 1) = 0 \quad (74)$$

whose solutions are $j_\ell = j_r$ or $j_\ell = j_r^*$ ²³. The representations satisfying the previous constraint are called simple or spherical representations.

When triangles are different the quantum constraint becomes

$$\text{Tr}[J_t J_{t'}^*] = \frac{1}{2} \text{Tr}[(J_t + J_{t'})(J_t^* + J_{t'}^*)] = 0, \quad (75)$$

where the second equality is obtained assuming that (74) is solved. For triangles 1 and 2 in Fig. 7 the previous constraint corresponds to the Casimir operator (74) evaluated on the representation ι . The constraint is satisfied if ι is also simple, i.e., if $\iota = (\iota, \iota^*)$. However, the same constraint must also hold if we take triangles 1 and 3 or any other combination of triangles in T . There is a unique solution (up to proportionality) which can be decomposed in terms of only simple intertwiners in any decomposition [88]. This state was proposed by Barrett and Crane in [8] as the building block for a state-sum model for quantum gravity. It is explicitly given by

$$\Psi_{BC} = \sum_{\iota} \text{Diagram} \quad (76)$$

where we represent the $SO(4)$ -intertwiner of Figure 7 in terms of the corresponding (left/right) $SU(2)$ -intertwiners. Ψ_{BC} defines the so-called Barrett-Crane intertwiner.

Constraints that involve different tetrahedra in a given 4-simplex –corresponding to the classical equation (70)– are automatically satisfied as operator equations on the Barrett-Crane solutions.

There is a natural definition of the 4-simplex amplitude (i.e., the vertex amplitude A_v^{BC}) defined by the natural trace of the five tetrahedron wave functions Ψ_{BC} . Graphically, using

²³In [14] the second solution is chosen in order to resolve the ambiguity in (66). This can be regarded as the discrete analogy of the action of the $*$ operation on the curvature $F(A)$ in (67). This restriction (imposed by the so-called chirality constraint, Section 7.7) implies that the ‘fake tetrahedron’ configurations –corresponding to solutions of the constraints on the right of equation (66)– are dropped from the state sum.

(76):

$$A_v^{BC} = \sum_{\iota_1 \cdots \iota_5} \left(\begin{array}{c} \iota_2 \\ \text{---} j_{12} \text{---} \\ \text{---} j_{13} \text{---} \\ \text{---} j_{23} \text{---} \\ \text{---} j_{15} \text{---} \\ \text{---} j_{14} \text{---} \\ \text{---} j_{24} \text{---} \\ \text{---} j_{35} \text{---} \\ \text{---} j_{34} \text{---} \\ \text{---} j_{45} \text{---} \\ \iota_1 \quad \iota_3 \quad \iota_4 \quad \iota_5 \end{array} \right) \left(\begin{array}{c} \iota_2^* \\ \text{---} j_{12}^* \text{---} \\ \text{---} j_{13}^* \text{---} \\ \text{---} j_{23}^* \text{---} \\ \text{---} j_{15}^* \text{---} \\ \text{---} j_{14}^* \text{---} \\ \text{---} j_{24}^* \text{---} \\ \text{---} j_{35}^* \text{---} \\ \text{---} j_{34}^* \text{---} \\ \text{---} j_{45}^* \text{---} \\ \iota_1^* \quad \iota_3^* \quad \iota_4^* \quad \iota_5^* \end{array} \right) . \quad (77)$$

This amplitude depend on the ten representations labelling the ten faces in a 4-simplex and it is referred to as $10j$ -symbol. The Barrett-Crane state sum on a fixed triangulation is defined as

$$Z_{BC}(\mathcal{J}_\Delta) = \sum_{\mathcal{C}_f: \{f\} \rightarrow j_f} \prod_{f \in \mathcal{J}_\Delta} A_f \prod_{e \in \mathcal{J}_\Delta} A_e \prod_{v \in \mathcal{J}_\Delta} A_v^{BC}, \quad (78)$$

where A_f and A_e denote a possible edge and face amplitudes which are not determined by the Barrett-Crane construction.

7.5 An integral expression for the $10j$ -symbol

In Section 9 we present a discretization independent formulation of the Barrett-Crane model based on a GFT. The realization by Barrett [89] that the vertex amplitude (77) admit an integral representation is important in that construction and also in the computation of asymptotics of the next subsection. The key observation in the construction of an integral expression for (77) is that equation (76) has precisely the form of (33) where the $SU(2)$ intertwiners $C_{j_1 j_2 j_3 j_4}^{\iota}$ have been graphically represented. Therefore,

$$\Psi_{BC} = \int_{SU(2)} du \, j_1(u) \otimes j_2(u) \otimes j_3(u) \otimes j_4(u), \quad (79)$$

where $j(u)$ denote $SU(2)$ representation matrices in the representation j . In this way each one of the 5 pairs of intertwiners in (77) can be obtained as an integral over $SU(2)$. It is easy to verify that each of the 10 representation matrices j_{ik} ($i \neq k = 1 \cdots 5$) appears in two integrals. Contracting the matrix indices according to (77) these two representation matrices combine into a trace $\text{Tr} [j_{ik}(u_i u_k^{-1})]$. Parameterizing $SU(2)$ with spherical coordinates on the 3-sphere S^3 it can be shown that

$$\text{Tr} [j_{ik}(u_i u_k^{-1})] = \frac{\sin(2j_{ik} + 1)\Theta_{ik}}{\sin(\Theta_{ik})} := (2j_{ik} + 1)K_\rho(y_i, y_k), \quad (80)$$

where Θ_{ik} is the azimuthal angle between the points y_i, y_k on the sphere corresponding to u_i and u_k respectively. On the RHS of the previous equation we introduce the definition

of the kernel $K_\rho(y_i, y_k)$ in terms of which the Barret-Crane vertex amplitude (77) becomes

$$A_v(j_{ik}) = \int_{(S^3)^5} \prod_{i=1}^5 dy_i \prod_{i < k} (2j_{ik} + 1) K_{j_{ik}}(y_i, y_k). \quad (81)$$

Each of the five integration variables in S^3 can be regarded as a unit vectors in \mathbb{R}^4 . They can be interpreted as the unit normal vector to the 3-dimensional hyperplane spanned by the corresponding five tetrahedron. The angles Θ_{ik} is defined by $\cos \Theta_{ik} = y_i \cdot y_k$ and corresponds to the exterior angle between two hyperplanes (analogous to the dihedral angles of Regge calculus). These normals determine a 4-simplex in \mathbb{R}^4 up to translations and scaling[89].

7.6 The asymptotics for the vertex amplitude

The large spin behavior of the spin foam amplitudes provides information about the low ‘energy’ world and consequently the classical limit of the model ²⁴. Evidence showing a connection between the asymptotics of the Barrett-Crane vertex and the action of general relativity was found by Crane and Yetter in [91].

A complete computation of the asymptotic (large j) expression of the Barrett-Crane vertex amplitude for non-degenerate configurations was obtained by Barrett and Williams in [92]. They computed $A_v(j_{ik})$ for large j_{ik} by looking at the stationary phase approximation of the oscillatory integral (81). The large spin behavior of the vertex amplitude is given by

$$A_v(j_{ik}) \sim \sum_{\sigma} P(\sigma) \cos \left[S_{Regge}(\sigma) + \kappa \frac{\pi}{4} \right] + D \quad (82)$$

where there is a sum over geometric 4-simplexes σ whose face areas are fixed by the spins. The action in the argument of the cosine corresponds to Regge action which in four dimensions is defined by $S_{Regge}(\sigma) = \sum_{i < k} A_{ik} \Theta_{ik}(\sigma)$ where A_{ik} is the area of the ik -triangle. $P(\sigma)$ is a normalization factor which does not oscillate with the spins. D is the contribution of degenerate configurations, i.e. those for which some of the hyperplane normals defined above coincide. However, in a recent paper [93] Baez, Christensen and Egan show that the term D is in fact dominant in the previous equation, i.e. the leading order terms are contained in the set of degenerate configurations!

7.7 Area and Volume in the Barrett-Crane model

The other Casimir operator of $Spin(4)$ corresponds to

$$\text{Tr}[\hat{J}_t \hat{J}_t] = j_L(j_L + 1) + j_R(j_R + 1). \quad (83)$$

It is easy to check that this operator commutes with all the constraints and therefore is well defined on the Hilbert space of the quantum tetrahedron. Its geometrical meaning is

²⁴See [90].

clear if we recall that at the classical level J_t represent the bivector associate to the triangle $t \in \mathcal{T}$. Namely, it is proportional to the area squared of corresponding triangle. On the Hilbert space of the quantum tetrahedron $j_R = j_L = j$ and the area of the triangles have discrete eigenvalues a_j given by

$$a_j \propto \sqrt{j(j+1)} \quad (84)$$

in agreement with the result of quantum geometry (7)²⁵.

Can we define the analog of volume operator of quantum geometry? The candidate for the square of such operator are

$$U_{\pm}(J_1, J_2, J_3) = \epsilon_{ijk} [J_{R1}^i J_{R2}^j J_{R3}^k \pm J_{L1}^i J_{L2}^j J_{L3}^k], \quad (85)$$

where J_1, J_2 , and J_3 are the bivector operators corresponding to three different triangles in the tetrahedron. The operator U_+ vanishes identically on the solutions of the quantum constraints. In fact it can be expressed as the commutator of the constraints of the type (75). This is the chirality constraint of [14].

One would like to define the volume operator as the square root of U_- ; however, U_- is not a well defined operator on the Hilbert space of the quantum tetrahedron since it does not commute with the simplicity constraints. In other words the action of the volume operators map states out of the solution of the constraints. Only when the dimension of $\text{Inv}[j_1 \otimes j_2 \otimes j_3 \otimes j_4]$ is one the commutator vanishes (this is in fact a necessary and sufficient condition). The volume operator is well defined in this subspace but it vanishes. This degenerate subspace arises naturally in the prescription we describe in Section 7.10. There is no well define volume operator in the Barrett-Crane model.

7.8 Lorentzian generalization

A Lorentzian generalization of the Riemannian Barrett-Crane vertex amplitude was proposed in [9]. Using the spin foam model GFT duality of Section 8 a generalization of the full model (including face and edge amplitudes) was found in [12]. The relevant representations of the Lorentz group are the unitary ones. Unitary irreducible representations of $SL(2, \mathbb{C})$ are infinite dimensional and labelled by $n \in \mathbb{N}$ and $\rho \mathbb{R}^+$. The simplicity constraints select the representations for which $n = 0$. The triangle area spectrum is given by

$$a_\rho \propto \sqrt{\rho^2 + 1}. \quad (86)$$

There is still a minimum eigenvalue but the spectrum is continuous. We will study this model in Section 9.3 where we show that the Lorentzian extension is not unique. In fact a new Lorentzian model can be defined following Barrett-Crane prescription [94].

²⁵Given a triangulation Δ of \mathcal{M} and the induced triangulation Δ_Σ of a slice $\Sigma \subset \mathcal{M}$ any spin foam defined on \mathcal{J}_Δ induces a spin network state on a graph γ_Σ dual to Δ_Σ . Links of γ_Σ are dual to triangles in Δ_Σ . These triangles play the role of the surface S in the definition of the area operator (7).

7.9 Positivity of spin foam amplitudes

In reference [95] Baez and Christensen it is shown that spin foam amplitudes of the (Riemannian) Barrett-Crane model are positive for any closed 2-complex. For open spin foams they are real and (if not zero) its sign is given by $(-1)^{2J}$ where J is the sum of the spin labels of the edges of the boundary spin networks. This is a rather puzzling property since (as the authors point out) this seems to imply the absence of quantum interference. Positivity of the Lorentzian model seems to hold according to numerical evaluations of the vertex amplitude.

7.10 Does Barrett-Crane's model correspond to a spin foam quantization of Plebanski's formulation?

The $SO(4)$ Plebanski action of Section 7.2 corresponds to the $SO(4)$ BF action plus certain Lagrange multiplier terms imposing constraints on the B field. Therefore, one can formally quantize the theory restricting the BF-path-integral to paths that satisfy the B-field constraints. This is the main idea behind the models presented in Sections 5.1 and 5.2.

In the literature, there is an implicit assumption that the Barrett-Crane model corresponds to a realization of this idea. In other words, the definition of the quantum tetrahedron in 4d (giving rise to the BC model) is sometimes regarded as an alternative way to impose the required restrictions on the B-configurations in the discretization. In this section we argue on general grounds that this is not the case. A systematic analysis of this issue is studied in [96]. Here we shall quote the results.

The first observation is that the Barrett-Crane model does not correspond, in a strict sense, to the general structure of spin foam model presented in Section 3.2. More precisely, if an edge e connects two vertices v_1 and v_2 (in the dual picture: a tetrahedron is shared by 4-simplexes) there are two independent sums over intertwiners corresponding to the two vertex amplitudes A_{v_1} and A_{v_2} respectively (see (77)). There is no one-to-one correspondence between intertwiners and edges in the Barrett-Crane model while this the case in BF theory. The Barrett-Crane configurations are not contained in the BF state sum (62)!

There is however an non-empty intersection between the set of BF spin foams and the Barrett-Crane model. These are configurations for which

$$\text{Dim}(\text{Inv}[\rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \rho_4]) = 1 \tag{87}$$

at every tetrahedron. A new model based incorporating (87) corresponds to the implementation of (65) –a slight variation of Plebanski's constraints (63)– on $SO(4)$ BF theory and could be regarded as a quantization of Plebanski's action. However, (87) implies that all tetrahedron have zero volume. The corresponding quantum states of space correspond to degenerate geometries [96]. This represents a serious obstruction for the reconstruction of the classical limit of the model.

There is no way to interpret the Barrett-Crane model as a systematic quantization of Plebanski's formulation!

8 The GFT ansatz

In this section we present the motivation and main results of [15, 16] where the duality relation between spin foam models and group field theories is established. This is the formulation we referred to in Section 6.2 as one of the possible discretization independent definitions of spin foam models. The result is based on the generalization of matrix models introduced by Boulatov[74] and Ooguri[75] dual to BF theory in three and four dimensions respectively. This formulation was first proposed for the Riemannian Barrett-Crane model in [10] and then generalized to a wide class of spin foam models [15, 16].

Given a spin foam model defined on a fixed 2-complex \mathcal{J} (dual to a triangulation Δ) –thus the partition function $Z[\mathcal{J}]$ is of the form (21)– there exist a GFT such that the perturbative expansion of the field theory partition function generalizes (21) to a sum over 2-complexes represented by the Feynman diagrams of the field theory. These diagrams look ‘locally’ as dual to triangulations (vertices are 5-valent, edges are 4-valent) but they are no longer tight to any manifold structure[76].

We now motivate the duality using what we know of the model defined on a fixed simplicial decomposition. The action of the GFT is of the form

$$I[\phi] = I_0[\phi] + \frac{\lambda}{5!} \mathcal{V}[\phi], \quad (88)$$

where $I_0[\phi]$ is the ‘kinetic’ term quadratic in the field and $\mathcal{V}[\phi]$ denotes the interaction term. The field ϕ is defined below. The expansion in λ of the partition function takes the form

$$\mathcal{Z} = \int \mathcal{D}[\phi] e^{-I[\phi]} = \sum_{\mathcal{J}_N} \frac{\lambda^N}{\text{Sym}[\mathcal{J}]} Z[\mathcal{J}_N], \quad (89)$$

where \mathcal{J}_N is a Feynman diagram (2-complex) with N vertices and $Z[\mathcal{J}_N]$ is the one given by (21). In fact, the interaction term $\mathcal{V}[\phi]$ is fixed uniquely by the 4-simplex amplitude of the simplicial model while the kinetic term $I_0[\phi]$ is trivial as we argue below.

Expression (89) is taken as the discretization (and manifold) independent definition of the model. Transition amplitudes between spin network states on boundary graphs γ_1 and γ_2 are shown to be given by the correlation functions

$$\int \mathcal{D}[\phi] \underbrace{\phi \cdots \phi}_{\gamma_1} \underbrace{\phi \cdots \phi}_{\gamma_2} e^{-I[\phi]}, \quad (90)$$

where boundary graphs are determined by the arrangement of fields in the product. The field can be defined as operators creating four valent nodes of spin networks [97, 79]. In [79] correlation functions $\langle \phi \cdots \phi \rangle$ are interpreted as a complete family of gauge invariant observables for quantum gravity. They encode (in principle) the physical content of the theory and can be used to reconstruct the physical Hilbert space in a way that mimics Wightman’s procedure for standard QFT [98, 99].

How do we construct the GFT out of the spin foam model on a fixed 2-complex? In (89) the combinatoric of diagrams is completely fixed by the form of the action (88). To

construct the action of the GFT one starts from the spin foam model defined on a 2-complex \mathcal{J}_Δ dual to a simplicial complex Δ . The sum over 2-complexes in (90) contains only those 2-complexes that locally look like the dual of a simplicial decomposition.

Spin foams on such 2-complex have edges e with which is associated tensor product of 4 representations $\rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \rho_4$. If we want to think of the edge e as associated to the propagator of a field theory then such propagator should be a map

$$P : \rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \rho_4 \rightarrow \rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \rho_4. \quad (91)$$

According to Peter-Weyl theorem, elements of $\rho_1 \otimes \cdots \otimes \rho_4$ naturally appear in the mode expansion of a function $\phi(g_1, \cdots, g_4)$ for $g_i \in G$. Spin labels arise as ‘momentum’ variables in the field theory. Intertwiners assigned to edges in spin foams impose compatibility conditions on the representations. In the context of the GFT this is interpreted as ‘momentum conservation’ which is guaranteed by the requirement that

$$\phi(g_1, g_2, g_3, g_4) = \phi(g_1g, g_2g, g_3g, g_4g) \quad \forall g \in G. \quad (92)$$

One also requires ϕ to be invariant under permutations of its arguments. One can equivalently take the field ϕ to be arbitrary and impose translation invariance by acting with P defined as

$$P\phi(g_1, g_2, g_3, g_4) = \int dg \phi(g_1g, g_2g, g_3g, g_4g). \quad (93)$$

All the information of local spin foams is in the vertex amplitude. The kinetic term $I_0[\phi]$ is simply given by

$$I_0[\phi] = \frac{1}{2} \int d^4g [P\phi(g_1, g_2, g_3, g_4)]^2, \quad (94)$$

Since vertices are 5-valent in our discretization the interaction term should contain the product of five field operators. This is a function of 20 group elements. If we use the compact notation $\phi(g^i) := \phi(g_1, \cdots, g_4)$. The general ‘translation invariant’ form is

$$\mathcal{V}[\phi] = \int d^{20}g V(g_j^i [g_i^j]^{-1}) P\phi(g_i^1) P\phi(g_i^2) P\phi(g_i^3) P\phi(g_i^4) P\phi(g_i^5), \quad (95)$$

where V is a function of 10 variables evaluated on the ‘translation invariant’ combinations $\alpha_{ij} := g_j^i [g_i^j]^{-1}$. For local spin foams the function $V(\alpha_{ij})$ is in one-to-one correspondence with the fundamental 4-simplex (atom) amplitude (19) determined by the duality (90). The precise way in which 2-complexes are generated as Feynman diagrams of the GFT will be illustrated in the following section.

9 The Barrett-Crane model and its dual GFT-formulation

In addition to providing a discretization independent formulation of the Barrett-Crane model the GFT formulation provides a natural completion of the definition of the model.

In Section 7 (equation (78)) we noticed that the model on a fixed discretization is defined up to lower dimensional simplex amplitudes such as that for faces f (dual to triangles) and edges e (dual to tetrahedra). The GFT formulation presented here resolves this ambiguity in a natural way. This normalization of the Barrett-Crane model was also obtained in [100] using similar techniques but on a fixed triangulation.

9.1 The general GFT

In this section we introduce the general GFT action that can be specialized to define the various spin foam models described in the rest of the paper.

Consider the Lie group G corresponding to either $Spin(4)$ or $SL(2, \mathbb{C})$ – for the GFT dual to Riemannian or Lorentzian Barrett-Crane model respectively. The field $\phi(g_1, g_2, g_3, g_4)$ is denoted $\phi(g_l)$ where $l = 1 \dots 4$, and $g_l \in G$ symmetric under permutation of its arguments, i.e., $\phi(g_1, g_2, g_3, g_4) = \phi(g_{\sigma(1)}, g_{\sigma(2)}, g_{\sigma(3)}, g_{\sigma(4)})$, for σ any permutation of four elements. Define the projectors P and R as

$$P\phi(g_l) \equiv \int_G dg \phi(g_l g) \quad (96)$$

and

$$R\phi(g_l) \equiv \int_{U^4} du_l \phi(g_l u_l), \quad (97)$$

where $U \subset G$ is a fixed subgroup, and dg and du are the corresponding invariant measures. The projector P imposes the translation invariance property (92).

Different choices of the subgroup yields different interesting GFT. When $U = \{1\}$ ($R = 1$) the GFT is dual to BF theory[75]. The GFT is dual to the Riemannian BC model for $U = SU(2) \subset Spin(4)$. Similarly for the GFT dual to the Lorentzian models the subgroups $U \subset SL(2, \mathbb{C})$ are $U = SU(2)$ (leaving invariant time-like direction) or $U = SU(1, 1)$ (for an invariant space-like direction); they result in two different models.

The GFT action is

$$S[\phi] = \int_{G^4} dg_l [P\phi(g_l)]^2 + \frac{\lambda}{5!} \int_{G^{10}} dg_{ij} \prod_{i=1}^5 PRP\phi(g_{ij}), \quad (98)$$

where $i, j = 1 \dots 5$, $i \neq j$. For example, $\phi(g_{1j})$ denotes $\phi(g_{12}, g_{13}, g_{14}, g_{15})$.

Strictly speaking the operators P and R are projectors only when the corresponding groups are compact. Formally we have

$$P^2 = (G - \text{volume}) \times P, \quad \text{and} \quad R^2 = (U - \text{volume}) \times R, \quad (99)$$

where $G - \text{volume}$, and $U - \text{volume}$ denotes the volume of G and U , respectively. These volume factors can be taken to be one when the G and/or U are compact by using Haar

measures. When G and/or U are noncompact the factors are infinite. This is a rather simple technical problem with which we shall deal later. Essentially one must drop redundant projectors in the functional integral.

The partition function can be computed as a perturbative expansion in Feynman diagrams J , namely

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S[\phi]} = \sum_J \frac{\lambda^{n_0}}{\text{sym}(J)} A(J), \quad (100)$$

where N is the number of vertices in J and $\text{sym}(J)$ is the symmetry factor.

If we write the action as

$$\begin{aligned} S[\phi] &= \frac{1}{2} \int dg_i d\tilde{g}_i \phi(g_i) \mathcal{K}(g_i, \tilde{g}_i) \phi(\tilde{g}_i) \\ &+ \frac{\lambda}{5!} \int dg_{ij} \mathcal{V}(g_{ij}) \phi(g_{1j}) \phi(g_{2j}) \phi(g_{3j}) \phi(g_{4j}) \phi(g_{5j}), \end{aligned} \quad (101)$$

where $i \neq j$. The kinetic operator $\mathcal{K}(g_i, \tilde{g}_i)$ can be written as

$$\begin{aligned} \mathcal{K}(g_i, \tilde{g}_i) &= \sum_{\sigma} \int_{G^2} d\hat{g} d\tilde{g} \prod_{i=1}^4 \delta(g_i \hat{g} \tilde{g}^{-1} \tilde{g}_{\sigma(i)}^{-1}) \\ &= \sum_{\sigma} \int_{G^2} d\hat{g} dg \prod_{i=1}^4 \delta(g_i g \tilde{g}_{\sigma(i)}^{-1}) \\ &= (G - \text{volume}) \times \sum_{\sigma} \int_G dg \prod_{i=1}^4 \delta(g_i g \tilde{g}_{\sigma(i)}^{-1}), \end{aligned} \quad (102)$$

where the \tilde{g} and \hat{g} integrations correspond to the action of the projectors P in (98). Redefining the integration variables $g = \tilde{g} \hat{g}^{-1}$ we obtain the second line in the previous equation. The G -volume factor comes from (99). We regularize the kinetic operator by simply dropping one of the G -integrations in the previous expression, namely

$$\mathcal{K}(g_i, \tilde{g}_i) \equiv \sum_{\sigma} \int_G dg \prod_{i=1}^4 \delta(g_i g \tilde{g}_{\sigma(i)}^{-1}). \quad (103)$$

P acts by projecting the field ϕ into its ‘translation invariant’ part $P\phi(g_i)$. The action (98) depends only on the gauge invariant part of the field, namely $S[\phi] = S[P\phi]$ (recall (92)). The inverse of \mathcal{K} (in the subspace of right invariant fields) corresponds to itself; therefore the propagator of the theory is simply

$$\mathcal{P}(g_i, \tilde{g}_i) = \mathcal{K}(g_i, \tilde{g}_i). \quad (104)$$

The propagator is defined by 4 delta functions (plus the symmetrization and the integration over the group) and it can be represented as shown on the right diagram of Fig. 8.

The potential term can be written as

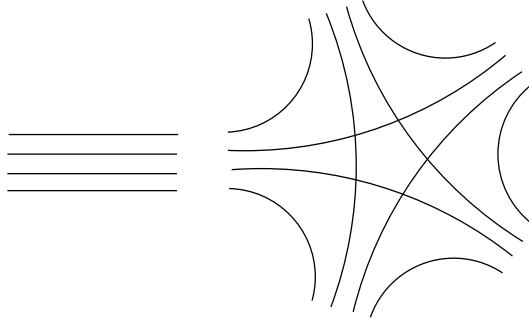


Figure 8: The structure of the propagator and the interaction vertex. Each line represents a delta function given by equation (103) in the case of the propagator and by (106) in the case of the vertex. For simplicity we represent one of the terms in the sum over permutation that define the propagator. All other permutations σ in (103) are obtained by diagrams including crossings.

$$\mathcal{V}(g_{ij}) = \frac{1}{5!} \int dg_i d\hat{g}_i du_{ij} \prod_{i < j} \delta(g_{ji}^{-1} \hat{g}_i u_{ij} g_i^{-1} g_j u_{ji} \hat{g}_j^{-1} g_{ij}), \quad (105)$$

where the $u_{ij} \in U$ correspond to the action of R in (98) and $g_i, \hat{g}_i \in G$ to the action of the corresponding P 's, respectively. It is easy to check that in the evaluation of a closed Feynman diagram the \hat{g}_i 's can be absorbed by redefining the g_i 's in the corresponding adjacent propagators. In this process, the \hat{g}_i 's drop out of the integrand and each integral over \hat{g}_i gives a G -volume factor. The second P projector in (98) is redundant in the computation of (100). The regularization is analogous to the one implemented in equation (103): we drop redundant P 's. The regularized vertex amplitude (for a vertex in the bulk of a diagram) is then defined as

$$\mathcal{V}(g_{ij}) = \frac{1}{5!} \int dg_i du_{ij} \prod_{i < j} \delta(g_{ji}^{-1} u_{ij} g_i^{-1} g_j u_{ji} g_{ij}). \quad (106)$$

In the case of open diagrams \hat{g} 's remain at external legs.

There is still a redundant g_i integration in (106) which introduces an infinite G -volume factor in the vertex amplitude. Notice that the (106) depends on the 'translational invariant' combinations $g_i^{-1} g_j$ so that one of the integrations is redundant. The regularization now consists of removing an arbitrary g_i from the expression of the vertex amplitude. This results in the regularization scheme proposed by Barrett and Crane in [9]. The regularization presented here can be applied to any non-compact group model on a lattice and can be regarded as a gauge fixing condition for the internal gauge invariance (recall Section 6.1). For notational simplicity we do not implement the regularization explicitly in (106). The structure of the vertex is represented on the left diagram in Fig. 8.

The Feynman diagrams of the theory are obtained by connecting the 5-valent vertices with propagators (see Fig. 8). At the open ends of propagators and vertices there are the

four group variables corresponding to the arguments of the field. For a fixed permutation σ in each propagator, one can follow the sequence of delta functions with common arguments across vertices and propagators. On a closed graph, each such sequence must close. By associating a surface to each such sequence of propagators, we construct a 2-complex J [10]. Thus, by expanding in Feynman diagrams and in the sum over permutations in (102), we obtain a sum over 2-complexes. Each 2-complex is given by a certain vertices-propagators topology plus a fixed choice of a permutation on each propagator.

9.1.1 Evaluation in configuration space

Combining (104) and (106) and integrating over internal configurations g_{ij} the sequence of delta functions associated to a face $f \in J$ reduces to a single delta function. Denoting $v_1 \dots v_N$ the ordered set of N vertices bounding f , and e_{ij} the edge connecting v_i with v_j , the delta function corresponding to the face f becomes

$$\delta \left([\hat{u}_1^v g_{v_1}^{-1} \hat{g}_{v_1} u_1^v] g_{e_{12}} [\hat{u}_2^v g_{v_2}^{-1} \hat{g}_{v_2} u_2^v] g_{e_{23}} \cdots g_{e_{N-1N}} [\hat{u}_N^v g_{v_N}^{-1} \hat{g}_{v_N} u_N^v] g_{e_{N1}} \right), \quad (107)$$

where the g 's $\in G$ and the u 's $\in U$. The product of group elements between brackets correspond to the vertex contribution to the face (see (106)) while the $g_{e_{ij}}$ comes from the corresponding propagators (104), also we have that $g_{e_{ij}} = g_{e_{ji}}^{-1}$. Using (107) the amplitude $A(J)$ of an arbitrary closed 2-complex becomes a multiple integral of the form

$$A(J) = \int \prod_e dg_e \prod_v dg_v d\hat{g}_v du^v d\hat{u}^v \prod_f \delta \left([\hat{u}_1^v g_{v_1}^{-1} \hat{g}_{v_1} u_1^v] g_{e_{12}} [\hat{u}_2^v g_{v_2}^{-1} \hat{g}_{v_2} u_2^v] g_{e_{23}} \cdots g_{e_{N-1N}} [\hat{u}_N^v g_{v_N}^{-1} \hat{g}_{v_N} u_N^v] g_{e_{N1}} \right). \quad (108)$$

9.1.2 The spin foam representation

The spin foam representation for the amplitude can be obtained by expanding the delta functions in terms of irreducible unitary representations of G , namely

$$\delta(g) = \sum_{\rho} \Delta_{\rho} \text{Tr} [\rho(g)], \quad (109)$$

where ρ is labels unitary irreducible representations and the rest of the notation is that of (31) when G is compact. In the noncompact case representations are infinite-dimensional so a formally equivalent expression holds where Δ_{ρ} correspond to the so-called Pancharel measure[94].

Using (109) in (107) we obtain

$$\sum_{\rho} \Delta_{\rho} \text{Tr} \left[\rho(\hat{u}_1^v) \cdot \rho(g_{v_1}^{-1} \hat{g}_{v_1}) \cdot \rho(u_1^v) \cdot \rho(g_{e_{12}}) \cdot \rho(\hat{u}_2^v) \cdot \rho(g_{v_2}^{-1} \hat{g}_{v_2}) \cdot \rho(u_2^v) \cdot \rho(g_{e_{23}}) \cdots \rho(\hat{u}_N^v) \cdot \rho(g_{v_N}^{-1} \hat{g}_{v_N}) \cdot \rho(u_N^v) \cdot \rho(g_{e_{N1}}) \right]. \quad (110)$$

The u 's appear just once per face so we can perform the u -integrations independently of other faces. Notice that $\int du \rho(u)$ is the projection onto the invariant subspace under the action of U in the Hilbert space \mathcal{H}_ρ . When $U = \{1\}$ the projector is the identity. In the other cases the subspace turns out to be 1-dimensional. Consequently the projector can be written as

$$\int_U du \rho(u) = |w_\rho\rangle\langle w_\rho|, \quad (111)$$

where $|w_\rho\rangle$ is the corresponding invariant vector. Equation (110) becomes

$$\sum_\rho \Delta_\rho \langle w_\rho | \rho(g_{v_1}^{-1} \hat{g}_{v_1}) w_\rho \rangle \langle w_\rho | \rho(g_{e_{12}}) w_\rho \rangle \langle w_\rho | \rho(g_{v_2}^{-1} \hat{g}_{v_2}) w_\rho \rangle \cdot \\ \langle w_\rho | \rho(u_2^v) w_\rho \rangle \langle w_\rho | \rho(g_{e_{23}}) w_\rho \rangle \cdots \langle w_\rho | \rho(g_{v_N}^{-1} \hat{g}_{v_N}) w_\rho \rangle \langle w_\rho | \rho(g_{e_{N1}}) w_\rho \rangle. \quad (112)$$

The group element g_e associated to a an edge $e \in J$ appears four times as there are four delta functions in the propagator (104). Integrals over such g 's have the general form

$$A_e(\rho_\epsilon^1, \dots, \rho_\epsilon^4) = \int_G \langle w_{\rho_1} | \rho^1(g) w_{\rho_1} \rangle \langle w_{\rho_2} | \rho^2(g) w_{\rho_2} \rangle \langle w_{\rho_3} | \rho^3(g) w_{\rho_3} \rangle \langle w_{\rho_4} | \rho^4(g) w_{\rho_4} \rangle, \quad (113)$$

and define the edge amplitude. If we define the *kernel* $K_\rho(g)$ as

$$K_\rho(g) \equiv \langle w_\rho | \rho(g) w_\rho \rangle, \quad (114)$$

then using the subgroup invariance of the $|w_\rho\rangle$'s the previous equation becomes

$$A_e(\rho_1, \dots, \rho_4) = (\text{U - volume}) \int_{G/U} dy K_{\rho_1}(g) K_{\rho_2}(y) K_{\rho_3}(y) K_{\rho_4}(y), \quad (115)$$

where the integration is over the homogeneous space G/U . Integration over the group elements associated to vertices are of the general form

$$A_v(\rho_{ik}) = (\text{U - volume})^4 \int_{(G/U)^4} \prod_{i=1}^5 dy_i \prod_{i < k} K_{\rho_{ik}}(y_i, y_k),$$

and correspond to the vertex amplitude. When U is non-compact the previous expressions have to be regularized as above. The amplitude of a 2-complex J is then given by

$$A(J) = \sum_{\mathcal{C}_f: \{f\} \rightarrow \{\rho\}} \prod_{f \in J} \Delta_{\rho_f} \prod_{e \in J} A_e(\rho_1, \dots, \rho_4) \prod_{v \in J} A_v(\rho_1, \dots, \rho_{10}) \quad (116)$$

As we shall see in the following the presence of R in (98) can be interpreted as imposing the Barrett-Crane constraints on the GFT dual to BF theory.

9.1.3 boundaries

The amplitude of an open diagram, that is, a diagram with a boundary, is a function of the variables on the boundary, as for conventional QFT Feynman diagrams. The boundary of a 2-complex is given by a graph. To start with, the amplitude of the open diagram is a function of 4 group arguments per each external leg. However, consider a surface of an open 2-complex and the link ab of the boundary graph that bounds it. Let a and b be the nodes on the boundary graph that bounds ab . The surface determines a sequence of delta functions that starts with one of the group elements in a , say g_a , and ends with one of the group elements in b , say g_b . By integrating internal variables, all these delta functions can be contracted to a single one of the form $\delta(g_a \cdots g_b^{-1}) = \delta(g_b^{-1} g_a \cdots)$. We can thus define the group variable $\rho_{ab} = g_b^{-1} g_a$, naturally associated to the link ab , and conclude that the amplitude of an open 2-complex is a function $A(\rho_{ab})$ of one group element per each link of its boundary graph. $A(\rho_{ab})$ is gauge invariant as it can be easily checked.

In ‘‘momentum space’’, boundary degrees of freedom are encoded in spin-network states. That is, if s is a spin network given by a coloring of the boundary graph, then

$$A(s) = \int d\rho_{ab} \bar{\psi}_s(\rho_{ab}) A(\rho_{ab}), \quad (117)$$

where $\psi_s(\rho_{ab})$ is the spin network function [26, 27]. The formula can be inverted

$$A(\rho_{ab}) = \sum_s A(s) \psi_s(\rho_{ab}), \quad (118)$$

where the sum is performed over all spin network states defined on the given boundary graph.

9.2 GFT dual to the Riemannian Barret-Crane model

The representation theory of $Spin(4)$ is particularly simple due to the fact that $Spin(4) = SU(2) \times SU(2)$. Unitary irreducible representations of ρ of $Spin(4)$ are labelled by two half-integers j_l and j_r , and are given by the tensor product of unitary irreducible representations of $SU(2)$, namely

$$\rho_{j_l j_r} = j_l \otimes j_r. \quad (119)$$

In terms of representation matrices we have

$$R^{j_l j_r}(g)_{mm' qq'} = D_{mm'}^{j_l}(g_r) D_{qq'}^{j_r}(g_l), \quad (120)$$

where $g = (g_l, g_r) \in Spin(4)$, and $D_{mm'}^{j_l}$ are $SU(2)$ -representation matrices. The subgroup U of previous section is taken as $U = SU(2) \subset Spin(4)$ defined by the diagonal action on $Spin(4)$ as follows

$$gu \equiv (g_l u, g_r u) \quad \text{for } u \in U. \quad (121)$$

The projector (111) becomes

$$\int_U du \rho_{j_l j_r}(u) = \int du D_{m m'}^j(u) D_{q q'}^{j_r}(u) = \frac{\delta_{j_l j_r^*}}{2j_l + 1} \delta_{m q} \delta_{m' q'} = |w_{j_l}\rangle \langle w_{j_l}|, \quad (122)$$

where we have used the orthonormality of $SU(2)$ representation matrices (Footnote 8). The previous equation confirms that the invariant subspace for each $\rho_{j_l j_r}$ is 1-dimensional as anticipated above. An immediate consequence of (122) is that the kernel K_ρ in (114) becomes

$$K_j(g) = \langle w_j | \rho_{j' j^*}(g) w_j \rangle = \frac{\delta_{j' j}}{2j + 1} \text{Tr} [D^j(g_l g_r)]. \quad (123)$$

The factor $\delta_{j' j}$ restricts the representation to the simple representations $j_l = j_r = j$ (the projector R in (98) imposes the Barrett-Crane constraints!). Balanced representations appear in the harmonic analysis of functions on the 3-sphere, S_3 . The fact that we encounter them here is not surprising since S_3 is the homogeneous space $S_3 = Spin(4)/SU(2)$ under the $SU(2)$ diagonal insertion (121). The presence of R in (98) projects out those modes that are not *spherical*. Indeed, $\langle w_j | \rho_{j' j^*}(g) w_j \rangle$ can be thought of as a function on S_3 : $\langle w_j | \rho_{j' j^*}(g) w_j \rangle$ depends on the product $g_l g_r \in SU(2)$ which is isomorphic to S_3 .

Finally, the invariant measure on $SU(2)$ can be written as a measure on S_3 induced by the isomorphism $SU(2) \rightarrow S_3$. We can parameterize $h \in SU(2)$ as $h = y_{(h)}^\mu \tau_\mu$ where $\tau_k = i\sigma_k$ for $k = 1, 2, 3$, $\{\sigma_k\}$ is the set of Pauli matrices, and $\tau_0 = 1$. $h \in SU(2)$ implies $y_{(h)}^\mu y_{(h)\mu} = 1$ (indexes are lowered and raised with δ_ν^μ), i.e., $y \in S_3$. In this parameterization the $SU(2)$ Haar measure becomes

$$dh \rightarrow dy = \frac{1}{\pi^2} dy^4 \delta(y^\mu y_\mu - 1), \quad (124)$$

We can simplify the measure using spherical coordinates for which an arbitrary point $y \in S^3$ is written

$$y = (\cos(\psi), \sin(\psi)\sin(\theta)\cos(\phi), \sin(\psi)\sin(\theta)\sin(\phi), \sin(\psi)\cos(\theta)), \quad (125)$$

where $0 \leq \psi \leq \pi$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$. The Haar measure becomes

$$dy = \frac{1}{2\pi^2} \sin^2(\psi) \sin(\theta) d\psi d\theta d\phi. \quad (126)$$

Using the known formula for the trace of $SU(2)$ representation matrices it is easy to check that (123) becomes

$$K_j(\psi_h) = \frac{\delta_{j' j}}{2j + 1} \frac{\sin((2j + 1)\psi_h)}{\sin(\psi_h)}, \quad (127)$$

where ψ_h is the value of the coordinate ψ in (125) corresponding to $h \in SU(2)$. Notice that we have rediscovered the kernel (80).

Accordingly, the general edge and vertex amplitudes ((115) and (116)) reduce to multiple integrals over S_3 . These can be interpreted as the evaluation of Feynman diagrams on

the sphere with propagator (127). $A_e(j_1, \dots, j_4)$ can be computed using (123) and (33), it follows that

$$A_e(j_1, \dots, j_4) = \frac{\dim[\text{Inv}(j_1 \otimes j_2 \otimes j_3 \otimes j_4)]}{\dim[j_1 \otimes j_2 \otimes j_3 \otimes j_4]}, \quad (128)$$

where $\dim[\text{Inv}(j_1 \otimes j_2 \otimes j_3 \otimes j_4)]$ denotes the dimension of the trivial component of $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4}$ and $\dim[j_1 \otimes j_2 \otimes j_3 \otimes j_4]$ the dimension of the full space. ²⁶ So finally,

$$A(J) = \sum_{\mathcal{C}_f: \{f\} \rightarrow \{j\}} \prod_{f \in J} (2j_f + 1)^2 \prod_{e \in J} A_e(j_1, \dots, j_4) \prod_{v \in J} A_v(j_1, \dots, j_{10}), \quad (129)$$

where the vertex amplitude is given by (116) using (127). This is the Barrett-Crane model (78) where the values of A_e and A_f have been fixed by the GFT formulation.

9.2.1 Finiteness

Lemma 9.1 ([17]). *For any subset of κ elements $j_1 \dots j_\kappa$ out of the corresponding four representations appearing in $A_e(j_1, \dots, j_4)$, the following bounds hold:*

$$|A_e(j_1, \dots, j_4)| \leq \frac{1}{\left(\prod_{i=1}^{\kappa} 2j_i + 1\right)^{\alpha_\kappa}}, \quad \text{where } \alpha_\kappa = \begin{cases} 1 & \text{for } \kappa \leq 3 \\ \frac{3}{4} & \text{for } \kappa = 4 \end{cases}.$$

The inequality for $\kappa \leq 3$ is sharp.

Lemma 9.2 ([17]). *For any 4 spins $j_1 \dots j_4$ labelling links converging at the same node in the relativistic spin-network corresponding to $A_v(j_1, \dots, j_{10})$ the following bounds hold:*

$$|A_v(j_1, \dots, j_{10})| \leq \frac{1}{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)(2j_4 + 1)},$$

from which follows

$$|A_v(j_1, \dots, j_{10})| \leq \frac{1}{((2j_1 + 1) \dots (2j_{10} + 1))^{2/5}}.$$

Definition 9.1. *A 2-complex J is said to be degenerate if it contains some faces bounded by only one or two edges.*

Theorem 9.1 ([17]). *The state sum $A(J)$, (??), converges for any nondegenerate 2-complex J .*

²⁶There was a mistake in the computation of the edge amplitude in [11] due to the propagation of a typo in [10]. The erroneous expression of the edge amplitude contained $\dim[j_1 \otimes j_2 \otimes j_3 \otimes j_4]^2$ in the denominator.

Proof. The amplitude (??) can be bounded in the following way:

$$|A(J)| \leq \prod_{f \in J} \sum_{j_f} ((2j_f + 1))^{2 - \frac{3}{4}n_f - \frac{4}{10}n_f} = \prod_{f \in J} \sum_{j_f} ((2j_f + 1))^{2 - \frac{23}{20}n_f}, \quad (130)$$

where n_f denotes the number of edges of the corresponding face, and we have used the fact the number of edges equals the number of vertices in a face of J . The term $(2j + 1)^2$ in (130) comes from the face amplitude, $(2j + 1)^{-\frac{3}{4}n_f}$ from Lemma 9.1 ($\kappa = 4$), and $(2j + 1)^{-\frac{4}{10}n_f}$ from Lemma (9.2) $\kappa = 10$. Notice that if the 2-complex contains faces with more than two edges the previous bound for the amplitude is finite, since the sum on the RHS of the previous equation converges for $n_f > 2$. \square

9.3 Lorentzian models

Unitary irreducible representations of $SL(2, \mathbb{C})$ appearing in the general expression (109) are the ones in the so-called *principal* series. They are labelled by a natural number n and a positive real number ρ . Unitary irreducible representations of $SL(2, \mathbb{C})$ are infinite dimensional. Those in the principal series are defined by their action on the linear space $\mathcal{D}_{n\rho}$ of homogeneous functions of degree $(\frac{n+i\rho}{2} - 1, \frac{-n+i\rho}{2} - 1)$ of two complex variables z_1 and z_2 . Due to the homogeneity properties of the elements $\mathcal{D}_{n\rho}$ they can be characterized by means of giving the value of functions on the sphere $|z_1|^2 + |z_2|^2 = 1$ which is isomorphic to $SU(2)$. The so called canonical basis is defined in terms of functions on $SU(2)$ and is well suited for studying the following model. The relevant facts about $SL(2, \mathbb{C})$ representation theory and the notation used in this section can be found in the appendix of reference [94] (for all about $SL(2, \mathbb{C})$ representation theory see [101]).

9.3.1 GFT dual to the Lorentzian Barrett-Crane model

Equation (111) becomes, in this case,

$$\int_U du \rho_{n\rho}(u) = \int_{SU(2)} du D_{\ell m \ell' m'}^{n\rho}(u) = \delta_{n0} \delta_{\ell 0} \delta_{\ell' 0} = |w_{n\rho}\rangle \langle w_{n\rho}|, \quad (131)$$

in terms of the canonical basis (see appendix in [94]). The $SU(2)$ invariant vectors $|w_{n\rho}\rangle$ are given by $\langle z_1, z_2 | w_{n\rho}\rangle = \delta_{n0} (|z_1|^2 + |z_2|^2)^{\frac{1}{2}\rho - 1}$ which is indeed a homogeneous function of degree $(\frac{i\rho}{2} - 1, \frac{i\rho}{2} - 1)$.

An immediate consequence of (131) is that the kernel K_ρ in (114) becomes

$$K_{n\rho}(g) = \langle w_{n\rho} | \rho_{n\rho}(g) w_{n\rho} \rangle = \delta_{n0} D_{0000}^{n\rho}(u). \quad (132)$$

The kernel $\langle w_{0\rho} | \rho_{0\rho}(g) w_{0\rho} \rangle$ can be thought of as a function on $H^+ = SL(2, \mathbb{C})/SU(2)$ (the upper sheet of 2-sheeted hyperboloid in Minkowski space). An arbitrary point $y \in H^+$ can be written in hyperbolic coordinates as

$$y = (\cosh(\eta), \sinh(\eta)\sin(\theta)\cos(\phi), \sinh(\eta)\sin(\theta)\sin(\phi), \sinh(\eta)\cos(\theta)), \quad (133)$$

where $0 \leq \eta \leq \infty$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$. The invariant measure in these coordinates is

$$dy = \frac{1}{2\pi^2} \sinh^2(\eta) \sin(\theta) d\eta d\theta d\phi. \quad (134)$$

In these coordinates, (132) becomes

$$K_\rho(\eta_y) = \frac{2 \sin(\frac{\rho}{2}\eta_y)}{\rho \sinh(\eta_y)}, \quad (135)$$

where η_y is the value of the coordinate η corresponding to $y \in H^+$ (this the generalization of (refkernito) proposed in [9]). Finally, the amplitude of an arbitrary diagram (116) becomes

$$A(J) = \int_{\mathcal{C}_f: \{f\} \rightarrow \{\rho\}} \prod_{f \in J} \rho_f^2 d\rho_f \prod_{e \in J} A_e(\{\rho_e\}) \prod_{v \in J} A_v(\{\rho_v\}), \quad (136)$$

where the formal sum in the general expression (116) becomes a multiple integral over the coloring $\mathcal{C}_f : \{f\} \rightarrow \{\rho\}$ of faces, and the weight Δ_ρ now is given by the Pancherel measure, $\rho^2 d\rho$, of $SL(2, \mathbb{C})$. The vertex amplitude was proposed in [9] the previous normalization was obtained in [12].

9.3.2 Finiteness

The amplitude of nondegenerate 2-complex turns out to be finite as in the Euclidean case. We state and prove the main result. Some of the following lemmas are stated without proof; the reader interested in the details is referred to the references.

Lemma 9.3 ([18, 19]). *For any subset of κ elements $\rho_1 \dots \rho_\kappa$ out of the corresponding four representations appearing in $A_e(\rho_1, \dots, \rho_4)$, the following bounds hold:*

$$|A_e(\rho_1, \dots, \rho_4)| \leq \frac{C_\kappa}{\left(\prod_{i=1}^{\kappa} \rho_i\right)^{\alpha_\kappa}}, \quad \text{where } \alpha_\kappa = \begin{cases} 1 & \text{for } \kappa \leq 3 \\ \frac{3}{4} & \text{for } \kappa = 4 \end{cases}.$$

for some positive constant C_κ .

Lemma 9.4 [[102]]. *$A_e(\rho_1, \dots, \rho_4)$ and $A_v(\rho_1, \dots, \rho_{10})$ are bounded by a constant independent of the ρ 's.*

Lemma 9.5 ([18, 19]). *For any subset of κ elements $\rho_1 \dots \rho_\kappa$ out of the corresponding ten representations appearing in $A_v(\rho_1, \dots, \rho_{10})$ the following bounds hold:*

$$|A_v(\rho_1, \dots, \rho_{10})| \leq \frac{K_\kappa}{\left(\prod_{i=1}^{\kappa} \rho_i\right)^{\frac{3}{10}}}$$

for some positive constant K_κ .

Theorem 9.2 ([18, 19]). *Given a non singular triangulation, the state sum partition function \mathcal{Z} is well defined, i.e., the multiple integral in (??) converges.*

Proof. We divide each integration region \mathbb{R}^+ into the intervals $[0, 1)$, and $[1, \infty)$ so that the multiple integral decomposes into a finite sum of integrations of the following types:

- i. All the integrations are in the range $[0, 1)$. We denote this term $T(F, 0)$, where F is the number of 2-simplexes in the triangulation. This term in the sum is finite by Lemma 9.4.
- ii. All the integrations are in the range $[1, \infty)$. This term $T(0, F)$ is also finite since, using Lemmas 9.3 and 9.5 for $\kappa = 4$, and $\kappa = 10$ respectively, we have

$$T(0, F) \leq \prod_f \int_{\rho_f=1}^{\infty} d\rho_f \rho_f^{2-\frac{3}{4}n_e-\frac{3}{10}n_v} \leq \left(\int_{\rho_f=1}^{\infty} d\rho_f \rho_f^{-\frac{46}{40}} \right)^F < \infty.$$

- iii. m integrations in $[0, 1)$, and $F-m$ in $[1, \infty)$. In this case $T(m, F-m)$ can be bounded using Lemmas (9.3) and (9.5) as before. The idea is to choose the appropriate subset of representations in the bounds (and the corresponding values of κ) so that only the $m - F$ representations integrated over $[1, \infty)$ appear in the corresponding denominators. Since this is clearly possible, the $T(m, F)$ terms are all finite.

We have bounded \mathcal{Z} by a finite sum of finite terms which concludes the proof. □

9.3.3 A new Lorentzian model

In the Euclidean case there was only one way of selecting the subgroup U of group elements leaving invariant a fixed direction in Euclidean spacetime. In the Lorentzian case there are two possibilities. The case in which that direction is space-like was treated in the previous section. When the direction is time-like the relevant subgroup is $U = SU(1, 1) \times \mathbb{Z}_2$.

This case is more complicated due to the non-compactness of U . Consequently, one has to deal with additional infinite volume factors of the form U -volume in (99). Another consequence is that the invariant vectors defined in (111) are now distributional and therefore no longer normalizable. All this makes more difficult the convergence analysis performed in the previous models and the issue of finiteness remains open.

On the other hand, the model is very attractive as its state sum representation contains simple representations in both the continuous and discrete series. As pointed out in [9] and discussed in the following section one would expect both types of simple representations to appear in a model of Lorentzian quantum gravity.

The difficulties introduced by the non-compactness of U make the calculation of the relevant kernels (114) more involved. No explicit formulas are known and they are defined by integral expressions. We will not derive these expressions here. A complete derivation can be found in [94]. The idea is to use harmonic analysis on the homogeneous space

$SL(2, \mathbb{C})/SU(1, 1) \times \mathbb{Z}_2$ which can be realized as the one-sheeted hyperboloid $y^\mu y^\nu \eta_{\mu\nu} = -1$ where y and $-y$ are identified (imaginary Lobachevskian space, from now on denoted H^-). The kernels correspond to eigenfunctions of the massless wave equation on that space.

The corresponding kernels are given by the following expression:

$$K_{\rho,n}(x, y) = \int_{C^+} d\omega (\delta_{n,0} |y^\nu \xi_\nu|^{i\frac{\rho}{2}-1} |x^\nu \xi_\nu|^{-i\frac{\rho}{2}-1} + \delta(\rho) \delta_{n,4k} \frac{32 \pi e^{-2ik[\Theta(x,y)]}}{k} \delta(x^\nu \xi_\nu) \delta(y^\nu \xi_\nu)), \quad (137)$$

where $x, y \in H^-$ and $\xi \in C^+$ is a normalized future pointing null vector in Minkowski spacetime. The integration is performed on the unit sphere defined by these vectors with the standard invariant measure $d\omega$.

As in the previous cases the expression for the edge and vertex amplitudes (equations (115) and (116)) reduce to integrals on the hyperboloid H^- . The expression for the amplitude (116) of an arbitrary diagram is [94]

$$A(J) = \sum_{n_f} \int_{\rho_f} d\rho_f \prod_f (\rho_f^2 + n_f^2) \prod_e A_e(\{n_e\}, \{\rho_e\}) \prod_v A_v(\{n_v\}, \{\rho_v\}),$$

where now there is a summation over the discrete representations n and an multiple integral over the continuous representations ρ . For the discrete representations the triangle area spectrum takes the form (84) if we define $k = j+1/2$. The finiteness properties of this model have not been studied so far. There is a relative minus sign between the continuous and discrete eigenvalues of the area squared operators that has been interpreted as providing a notion of micro causality in [94]. It would be interesting to study this in connection with the models of Section 5.5.

10 Discussion

Let's conclude this review with some remarks and a discussion of recent results and future perspectives in the subject.

1. Normalization

It is important to point out that the way in which the normalization (129) of the Barrett-Crane model is defined has its natural character in the context of the GFT formulation or using similar arguments[100, 103]. One would like to be able to derived it directly from the continuous action ($SO(4)$ Plebanski's theory?). However, as we emphasized in Section 7.10, contrary to the general belief, there is no relation between the Barrett-Crane model and $SO(4)$ Plebanski's theory. This fact opens the question of whether our 'natural' normalization of the model has any special character. This ambiguity seems to be present in the other models obtained from BF theory. A

possible answer to the issue of the normalization –or how to fix the ambiguities in A_f and A_e – could come in these models from a systematic analysis of the corresponding continuum action. Investigation in this direction is underway [104]. A very attractive property the normalization (129) is however the finiteness of transition amplitudes on a fixed discretization. This opens possibilities of concrete computations, e.g. numerical calculations or the study of the refinement limit of Section 6.3.

2. Numerical results

An important step in this is the development of efficient algorithms for the computation of the $10j$ -symbols [105]. In [80] Baez and Christensen study using numerical calculations different versions of the Riemannian Barrett-Crane model. They show that the sum over spin foams in the finite version of the Barrett-Crane model converges very fast so that amplitudes are dominated by spin foams where most of the faces are labelled by zero spin. The leading contribution comes from spin foams made up from isolated ‘bubbles of geometry’. They propose modifications of the normalization which are finite but avoid this puzzling feature. This problem is however not present the Lorentzian version (136).

3. The connection with the canonical picture

Can we establish a rigorous connection between the spin foam models presented here and LQG? In all the models (for Riemannian gravity) introduced in Section 5 with the exception of the Barrett-Crane model this connection is naively manifest since the boundary states are given by $SU(2)$ -spin networks. This kinematic connection exists by construction in these cases.

The Riemannian BC model is defined in terms of $Spin(4)$ so naturally one could expect the boundary data to be given by $Spin(4)$ spin networks (labelled by two half-integers (j_ℓ, j_r)). The simplicity constraints impose $j_\ell = j_r = j$ which could be interpreted as the existence of an underlying $SU(2)$ connection. However, it can be shown that the boundary data of the model can not be interpreted as given by any connection (this can be seen from the fact that the BC model is not of the general form derived in Section 3.2).

In the Lorentzian Barrett-Crane model boundary states could be naively related to $SL(2, \mathbb{C})$ spin networks; however similar considerations as in the Riemannian sector show that the simplicity constraints (applied à la Barrett-Crane) reduce the boundary states in a way that cannot be interpreted as a boundary connection. Another difficulty in making contact with LQG is the fact that operators associated to triangle-areas have continuous spectrum in contrast with the canonical result (7). From this perspective one is compelled to study the model we have described in Section 9.3.3 for which part of the area spectrum coincides with (7).

In the canonical framework the compact $SU(2)$ formulation is achieved by means of the introduction of the Immirzi parameter ι which defines a one parameter family of theories whose geometric operators are modulated by ι (see (7) for example). Can

the spin foam approach say something about ι ? There is no conclusive answer to this question so far. The spin foam quantization of generalizations of Plebanski's action including the ι ambiguity[106] are analyzed in [107, 108].

One of the main questions motivating the spin foam formulation of quantum gravity was the attempt to circumvent the difficulties associated to the regularization ambiguities appearing in the canonical picture. An open question is whether one can select a preferred regularization of the scalar constraint using the spin foam approach. An explicit analysis of the relation between the covariant (spin foam) and canonical formulation of BF theory is presented in [109]. The reconstruction of the corresponding scalar constraint operator is studied and carried out explicitly in simple models.

4. *Discretization dependence*

In Section 4 we have seen that the discretization dependence is trivial in three dimensions. In particular this has been nicely formalized in the definition of continuum spin foams by Zapata [58]. The refinement limit discussed in Section 6.2 should be investigated for the models with local excitations.

5. *The GFT theory*

The GFT formulation is very attractive since it provides a discretization independent formulation of spin foam from the outset. Also it has been very useful for the definition of the Lorentzian models of Sections 9.2 and 9.3 as a device for formal manipulations. However, the mathematical consistency of this definition depends on whether one can make sense of the expansion in λ of equation (89). Suggestions on how the series could be summable by complexification of the coupling λ can be found in [15]. The anomaly question seems however very difficult in this context. As we mentioned in Section 6.2 the notion of diffeomorphisms (equivalent spin foams) appear entangled in a complicated way with the perturbative series. This looks very much as a diffeomorphism anomaly as equivalent spin foams (in the sense of Section 6.1) have different contributions. Such anomaly would be a serious inconsistency from the point of view of Section 6.3.

6. *Coupling with matter*

In a recent paper Oriti and Pfeiffer[110] proposed a model which couples the Riemannian Barrett-Crane model with Yang-Mills theory. Their construction is likely to be generalizable to other models.

In the context of the GFT formulation of spin foam models Mikovic proposes a way to include matter by adding the appropriate fields into the GFT-action. In [111, 112] the author generalizes the GFT construction described in Section 8 allowing for the inclusion of spinor fields in finite dimensional representations of $SO(2)$ and $SO(3)$ representing matter corresponding to fermions and gauge fields. States in the theory are given by spin networks with open links which is consistent with results obtained in the canonical approach [113, 114, 115, 7].

More radical and very appealing possibilities are suggested by Crane. In [116] the author proposes a topological QFT of the type of BF theory as fundamental theory. The gravitational degrees of freedom are represented by subset of (constrained) representations while those of matter are encoded in the remaining ones. The full spin foam model is topological (no local degrees of freedom). In order to recover the low energy world (with local excitations) the author appeals to certain “symmetry breaking” of topological invariance. Crane’s other proposal consists of interpreting topological (conical) singularities naturally arising in the structure of the Feynman diagrams of the GFT theory as representing matter degrees of freedom [117].

7. *Alternatives to the Barrett-Crane model*

The fact that degenerate configurations dominate the Barrett-Crane amplitudes[93] or that the modifications of the model in compliance with the principles of Plebanski’s formulation contain only degenerate configurations[96] pose serious difficulties in interpreting the Barrett-Crane model as physically correct. It would be interesting to understand whether this feature is inherent to the way in which the constraints are implemented and can be corrected (e.g., implementing (64) instead of (65)). It is also important to understand what the situation is in the case of other proposals. For example, one would like to study in detail the kernel of Reisenberger’s constraint operator Ω^{ij} defined in (44) and evaluate the geometric properties of the solutions.

The suggestion that quantum gravity should be described in terms of discrete combinatorial structures can be traced all the way back to Einstein [118]. Spin foam models appears as a beautiful realization of this idea. There are many difficult open questions but we hope that new ideas and hard work will continue to contribute to their resolution in the near future.

11 Acknowledgments

I thank Abhay Ashtekar, John Baez, Martin Bojowald, Dan Christensen, Rodolfo Gambini, Amit Ghosh, Jorge Pullin, Michael Reisenberger and Carlo Rovelli for discussions. I thank Martin and Carlo for the careful reading of the manuscript. I specially thank Carlo Rovelli for his strong support. This work was supported in part by NSF Grant PHY-0090091, and Eberly Research Funds of Penn State.

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