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Vienna, Preprint ESI 121 (1994) August 16, 1994

Supported by Federal Ministry of Science and Research, Austria
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ON SCHWARZ PROBLEM FOR THE $\widehat{su}_2$ KNIZHNIK–ZAMOLODCHIKOV EQUATION

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Abstract. We study the monodromy representations $\mathcal{B}^{k,I}$ of the mapping class group $\mathcal{B}_4$ acting on 4–point blocks satisfying the Knizhnik–Zamolodchikov equation for the level k su_2 current algebra. We classify all irreducible $\mathcal{B}^{k,I}$ which are realized by finite groups; we also display finite irreducible components for the reducible representations corresponding to $k = 10$.

1. Introduction

The discovery by V. Pasquier of the close interrelationship between V. Jones theory of subfactors and a family of 2–dimensional (critical) lattice models led to introducing the ADE models labelled by Dynkin diagrams and to the ensuing ADE classification of su_2 current algebra and minimal conformal theories [1]. The parallel with the classification of finite subgroups of SU_2 has not been fully understood in the 7 years since this publication. The present note reports on the first results of an attempt to relate the properties of rational $2D$ conformal field theories to the cases in which the associated monodromy representations of the braid group are finite matrix groups. We answer on the way the question when does the Knizhnik–Zamolodchikov (KZ) equation [2] for chiral 4 point blocks have an algebraic solution. Here we restrict our attention to conformal blocks of chiral vertex operators carrying the same isospin I and conformal weight $\Delta_I = \frac{I(I+1)}{k+2}$.

More than 120 years ago Schwarz [3] solved a similar problem for Gauss hypergeometric equation. Only comparatively recently his study was pushed further to incorporate the n–th order equation for the generalized hypergeometric function ${}_nF_{n-1}$ [4] and the classification of algebraic Appell–Lauricella functions [5].

The possibility to address Schwarz's problem for the KZ equation has been prepared in previous work [6, 7, 8] where an explicit form for the generators of the associated monodromy representation of the braid group $\mathcal{B}_4$ on 4 strands has been worked out. The role of braid invariance in constructing $2D$ correlation functions

Supported by the Federal Ministry of Science and Research, Austria

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has been already recognized in the pioneer work of Belavin, Polyakov and Zamolodchikov [9] on minimal conformal models. The monodromy representations of the braid group acting on chiral vertex operators of the su_2 conformal current algebra have been first studied by Tsuchiya and Kanie [10].

The results of the paper can be summarized as follows.

1. We construct a family of matrix representations of the mapping class group $\mathcal{B}_4$ of the 2–sphere with 4 punctures (which can be viewed as the braid group on 4 strands with 3 additional relations – Section 2A). Their matrix elements appear as polynomials with integer coefficients in a variable q or $q^{1/2}$ (depending on whether $2I$ is even or odd) satisfying

$$q^h = -1, \quad h = k + 2, \quad k = 1, 2, \ldots , \tag{1.1}$$

where h is the height of a level k representation of the $\widehat{s}u_2$ Kac–Moody algebra [11]. They thus belong to the cyclotomic field $Q(q^{1/2})$ (Section 2C). The resulting family of matrix groups provides a representation of the abstract groups studied in [12] - see Remark at the end of Section 2C.

2. A braid invariant hermitean form A, whose entries belong to the same cyclotomic field, is constructed and diagonalized (Section 3A). It is positive definite whenever the quantum dimensions of the intermediate states are positive:

$$[2\lambda + 1] > 0 \text{ for } \lambda = 0, 1, \ldots , m = \min(2I, k - 2I). \tag{1.2}$$

3. The main result of the paper is contained in Lemma 3.2 and Theorem 3.3 of Section 3B. The Lemma says that the matrix group $\mathcal{B}^{k,I}$ (consisting of $(m + 1) \times (m + 1)$ braid matrices that leave the non–degenerate hermitian form A invariant) is finite whenever A is totally positive: i.e., whenever it is positive together with all its Galois transforms, corresponding to the substitutions

$$q \rightarrow q^n \text{ for } (2h, n) = 1 \tag{1.3}$$

($(2h, n) = 1$ indicating that n is coprime with $2h$). Conversely, if the representation $\mathcal{B}^{k,I}$ is irreducible (or, equivalently, if the invariant form A is unique), the condition of total positivity is also necessary for the monodromy representation $\mathcal{B}^{k,I}$ of the mapping class group to be a finite group. As a result we end up essentially with three finite groups of 2×2 matrices (which can be identified within the original Schwarz classification – see also [13]) one group of 3×3 matrices and an infinite series of (1–dimensional) cyclic groups (of the type $\mathbb{Z}_4, \mathbb{Z}_2$ and $\{1\}$).

2. Monodromy representations of the mapping class group of the sphere with four punctures

2A. The group $\mathcal{B}_4$. Basic relations. We shall list the properties of the abstract group $\mathcal{B}_4$ in order of increasing specialization.

To begin with, $\mathcal{B}_4$ is a group of three generators $B_i, i = 1, 2, 3$, satisfying the relations for the braid group on four strands:

$$B_1 B_3 = B_3 B_1, \quad B_i B_{i+1} B_i = B_{i+1} B_i B_{i+1}, \quad i = 1, 2. \tag{2.1}$$

They further display the characteristic properties of a (cyclic) central extension of the braid group on the sphere $\mathbb{S}^2$:

$$B_1 B_2 B_3^2 B_2 B_1 = c^2 = B_3 B_2 B_1^2 B_2 B_3, c^2 B_i = B_i c^2. \tag{2.2}$$

(a finite power of the central element c being equal to 1). Finally, they obey the relation

$$(B_1 B_2 B_3)^4 = c^4 \tag{2.3}$$

which singles out a central extension of the mapping class group of the sphere with four punctures (the mapping class group proper corresponding to $c = 1$). The above definition yields the following

2.1. Corollary. The relations (2.1)-(2.3) imply

$$(B_1 B_2)^3 = c^2 = (B_2 B_3)^3 \tag{2.4}$$

and $B_1^2 = B_3^2$. Moreover, all generators belong to the same conjugacy class, since $(B_1 B_2 B_1) B_1 = B_2 (B_1 B_2 B_1)$.

2B. General characteristics of the monodromy representations (k, I) of $\mathcal{B}_4$. For each level $k (= 1, 2, \ldots)$ of the $\widehat{su}_2$ Kac–Moody algebra [11] there exists a finite family of chiral vertex operators V_I carrying isospin I in the range

$$2I = 0, 1, \ldots, k. \tag{2.5}$$

The 4 point blocks of V_I span a $2I+1$ dimensional space $\tilde{\mathcal{H}}_I$ of solutions of the KZ equation which carry a monodromy representation $(\tilde{2I})$ of $\mathcal{B}_4$. This representation is found to have the following properties [10, 6, 7].

(1) The two commuting generators of $\mathcal{B}_4$ (which according to Corollary 2.1 have the same squares – i.e. yield the same monodromy) are actually equal

$$B_1 = B_3 \tag{2.5}$$

so that $\mathcal{B}_4$ has just two generators, B_1 and B_2.

(2) The central element c^2 of (2.2) is an integer power of a primitive root of -1

$$c^2 = \bar{q}^{4I(I+1)}, \quad q^h = -1, h = k + 2, \bar{q} = q^{-1}. \tag{2.6}$$

(3) There exists a positive semidefinite $\mathcal{B}_4$ invariant hermitean form $(,)$ which vanishes on a $4I - k$ dimensional subspace of $\tilde{\mathcal{H}}_I$ for $k \leq 4I (\leq 2k)$. The representation $(\tilde{I})$ of $\mathcal{B}_4$ in $\tilde{\mathcal{H}}_I$ induces a representation (k, I) in the factor space

$$\mathcal{H}_{kI} = \tilde{\mathcal{H}}_I /_{(v,v)=0} \tag{2.7}$$

of dimension

$$\dim \mathcal{H}_{kI} = \min(2I + 1, k - 2I + 1) = m + 1, \tag{2.8a}$$

where

$$m = m(k, I) = \min(2I, k - 2I) = \frac{1}{2}(k - |k - 4I|). \tag{2.8b}$$

For $4I \leq k$ the invariant form has no kernel and the two $\mathcal{B}_4$ modules $\tilde{\mathcal{H}}_I$ and $\mathcal{H}_{kI}$ coincide.

(4) The generators $B_i = B_i^{k,I}$ of the (factor) representation (k, I) are cyclic elements of finite order:

$$B_i^h = c^h, \quad c = c_I = (-1)^{2I} \bar{q}^{2I(I+1)}, \quad (\bar{q}^h = -1), \tag{2.9}$$

so B_i^h is always a multiple of the unit matrix (in particular, it is a central element of $\mathcal{B}_4^{I,k}$); moreover, for integer isospin $B_i^{4h} = 1$, for half integer isospin $B_i^{4h} = 1$.

2C. Explicit realization. We shall exhibit a realization $\mathcal{B}_4^{(I)}$ of the (unfactored) representation $(\tilde{I})$ of $\mathcal{B}_4$ with the following properties [6, 7]:

$$B_1 B_2 B_1 = B_2 B_1 B_2 = c_I F, \quad F^2 = 1 \tag{2.10}$$

where the phase factor c_I is given in (2.9) and

$$F_{\lambda\mu} = \delta^{2I}_{\lambda+\mu}, \quad \lambda, \mu = 0, 1, \ldots, 2I, \quad (\det F = (-1)^{I(2I+1)}); \tag{2.11}$$

B_1 is, on the other hand, an upper triangular matrix, B_2 is a lower triangular one. These conditions still leave the freedom for a rescaling of the type

$$B_{\lambda\mu} \rightarrow g_\lambda g_\mu^{-1} B_{\lambda\mu} \text{ with } g_\lambda = g_{2I-\lambda} (\neq 0). \tag{2.12}$$

The choice we shall adopt here differs from the one made in [6, 7] by just such a rescaling. Taking a suitably normalized basis of $2I$-fold parametric integral solutions of the KZ equation, we find

$$(B_1)_{\lambda\mu} = (-1)^{2I-\mu} q^{\mu(\lambda+1)-2I(I+1)} \begin{bmatrix} 2I - \lambda \\ \mu - \lambda \end{bmatrix}. \tag{2.13}$$

Here $\begin{bmatrix} n \\ m \end{bmatrix}$ stand for the (real) q-binomial coefficients (vanishing for $n < m$):

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}, [n]! = [n][n-1]!, [0]! = 1, [n] = \frac{q^n - \bar{q}^n}{q - \bar{q}}. \tag{2.14}$$

This realization of the $(2I + 1) \times (2I + 1)$ matrix group $\mathcal{B}_4^{(I)}$ has two remarkable properties which will be exploited in Section 3 below.

The generators B_1 and $B_2 = F B_1 F$ are inverted by complex conjugation:

$$\bar{B}_i = B_i^{-1} (= B_i(q^{-1}))(\text{ also } \bar{F} = F^{-1} = F). \tag{2.15}$$

Clearly, this property is not preserved by products of non-commuting generators like $B_1 B_2$. It does, however, have an implication for any element of our matrix group.

2.2. Corollary. If $A(= A^{(*)})$ is an invariant hermitean form, i.e. if

$$B^* AB = A \text{ for all } B \in \mathcal{B}_4^{(I)} \tag{2.16}$$

then in the realization (2.11) (2.13) it also satisfies

$$^t BA = AB \text{ for } (^t B)_{\lambda\mu} = B_{\mu\lambda}. \tag{2.17}$$

The second property is an arithmetic one: the matrix elements of any $B \in \mathcal{B}_4^{(I)}$ are elements of the cyclotomic field $\mathbb{Q}(q^{1/2})$. In fact, they are polynomials in $q^{1/2}$ with integer coefficients (and $q^h = -1$). The elements of the commutator subgroup are ploynomials in q (rather than $q^{1/2}$) also for half integer isospins.

Remark on the connection with the work of Howie and Thomas [12]**.**

If we multiply the generators B_i of $\mathcal{B}_4^{(I)}$ by $q^{\frac{2}{3}I(I+1)}$ we can relate the resulting group $\hat{\mathcal{B}}^{(I)}$ of matrices of determinant ± 1 with a family of groups considered in [12]. Indeed $\hat{\mathcal{B}}^{(I)}$ is generated by F and

$$\omega = q^{\frac{2}{3}I(I+1)} B_1 B_2 \tag{2.18}$$

satisfying the relations

$$F^2 = \omega^3 = 1 = (F\omega)^{N(I,h)} \tag{2.19}$$

where $N(I, h)$ is an appropriate divisor of $6h$. If $\hat{\mathcal{B}}^{(I)}$ is finite then the group commutator should be of finite order

$$(F\omega F\omega^2)^n = 1 \text{ for some } n \in \mathbb{N}. \tag{2.20}$$

Thus, $\hat{\mathcal{B}}^{(I)}$ appears as a representation of the abstract group $(2, 3; N(I, h), n)$ considered in [12]. Consequently, in all cases (given by Theorem B of [12]) in which the above abstract group is finite, our matrix group $\hat{\mathcal{B}}^{(I)}$ is also finite. The converse is however, not true since our $\hat{\mathcal{B}}^{(I)}$ involves additional relations. (2.19) and (2.20) do not guarantee, for instance, that the monodromy element $\omega^2(F\omega)^3\omega F$ is of finite order. The result of Theorem 3.3 below shows in fact, that none of the interesting cases of finite mapping class groups is covered by the list of finite abstract groups of [12].

3. Invariant forms and irreducible finite groups

3A. The generic invariant symmetric form.

Proposition 3.1. *For any $q \neq 0$ there exists a diagonalizable $\hat{\mathcal{B}}_4^{(I)}$ invariant symmetric form*

$$A = {}^t SDS. \tag{3.1}$$

Here D is a diagonal matrix with non–zero elements

$$D_{\lambda\lambda} \equiv D_{\lambda}^{(k,I)} = \left\{ \frac{[\lambda]![2I+1+\lambda]!}{[2I+1]![2\lambda]!} \right\}^2 \frac{1}{[2\lambda+1]}, \lambda = 0, 1, \ldots, m, \tag{3.2}$$

where the upper limit $m = m(k, I)$ of indices λ for which $D_{\lambda\lambda}$ is non–zero is given by (2.8b). S is an upper triangular matrix satisfying

$$S_{\lambda\mu} = \delta_{\lambda\mu} \text{ for } \lambda, \mu \geq m+1 (\text{ if } 4I > k) \tag{3.3a}$$

$$S_{\lambda\mu} = (-1)^{\mu-\lambda} \begin{bmatrix} \mu \\ \lambda \end{bmatrix} \frac{[2I-\lambda]![2\lambda+1]!}{[2I-\mu]![\lambda+\mu+1]!} \text{ for } \lambda \leq \mu \leq m \tag{3.3b}$$

and ${}^tS (= S^*)$ is its transposed (equal to its hermitean conjugate).

Sketch of proof.

For $4I \leq k+1$ the similarity transformation

$$B_1 \to B_1^{(I)} = S B_1 S^{-1}, \tag{3.4}$$

where

$$S^{-1}_{\lambda\mu} = \begin{bmatrix} \mu \\ \lambda \end{bmatrix} \frac{[2I-\lambda]![\lambda+\mu]!}{[2I-\mu]![2\mu]!}, \tag{3.5}$$

diagonalizes B_1, the eigenvalues b_λ of B_1 appearing as

$$(B_1^{(I)})_{\lambda\mu} = \delta_{\lambda\mu} b_\lambda, \quad b_\lambda = (-1)^{2I-\lambda} q^{\lambda(\lambda+1)-2I(I+1)}. \tag{3.6}$$

For $4I \geq k+2$, B_i are not diagonalizable, but $B_1^{(I)}$ (3.4) (with S^{-1}, the inverse of S (3.3), only given by (3.5) for $\lambda, \mu \leq m$ and continued as in (3.3a) for $\mu > m$) still satisfies (3.6) for $\mu \leq m (= k - 2I)$. In both cases, we obtain from (3.1) and (3.2) the B_1 invariance condition for A:

$${}^tB_1 A = A B_1 \iff [B_1^{(I)}, D] = 0. \tag{3.7}$$

Verification of F or B_2 invariance requires more work and uses the explicit form of A:

$$A_{\lambda\mu} = \frac{(-1)^{\lambda+\mu}[\lambda]![\mu]!}{[2I-\lambda]![2I-\mu]!([2I+1]!)^2} \sum_{\nu=0}^{m} T_\nu(\lambda, \mu; I) \tag{3.8a}$$

where

$$T_\nu = \frac{[2I+\nu+1]!^2[2I-\nu]!^2[2\nu+1]}{[\lambda+\nu+1]![\mu+\nu+1]![\mu-\nu]![\lambda-\nu]!} (= 0 \text{ for } \nu > \min(\lambda, \mu)). \tag{3.8b}$$

As a consequence of Corollary 2.2 A also defines an invariant hermitean form.

Remark. An expression of the type (3.2) (3.8) was first derived in Section 3D of [7] using quantum group techniques. Apart from a misprint in (3.84c) of that reference the two formulas differ because of a different choice of normalization of the basis. The expression (3.8) is related to $Z_{\lambda\mu}$ (3.84) of [7] by

$$[2I+1]^2 A_{\lambda\mu} = \begin{bmatrix} 2I \\ \lambda \end{bmatrix} \begin{bmatrix} 2I \\ \mu \end{bmatrix} Z_{\lambda\mu}.$$

Clearly, $A_{\lambda\mu}$ are real for both q real ($q \neq 0$) and for q on the unit circle, $q\bar{q} = 1$. The cutoff in λ and μ in (3.2) and (3.3a) only occurs for q a root of unity.

3B. Galois automorphisms and positivity. We have not specified so far the choice of a primitive root q of -1 ((2.6)). Positivity of the invariant form (3.1) (3.2) requires setting

$$[2] = q + \bar{q} = 2\cos\frac{\pi}{h}. \tag{3.9}$$

All other properties of $\mathcal{B}_4^{(I)}$ remain valid for any primitive hth root of -1, and are, hence, invariant under Galois automorphisms* (1.3).

Lemma 3.2. *(i) If the form* (3.1) (3.2) *is totally positive, i.e. if for q satisfying* (3.9)

$$D_\lambda^{(k,I)}(q^n) > 0 \ \textit{for any}\ n, (2h,n) = 1, \ \textit{and}\ \lambda = 0, 1, \ldots, m, \tag{3.10}$$

then the $(m+1)$–dimensional representation $\mathcal{B}^{k,I}$ which leaves the non–degenerate form A invariant is a finite matrix group. (ii) Conversely, if the invariant hermitean form A is unique (or equivalently, if the representation $\mathcal{B}^{k,I}$ is irreducible), then the condition of total positivity of A is also necessary for the finiteness of $\mathcal{B}^{k,I}$.

This is a corollary of known results in algebraic number theory. The following sketch of a proof was communicated to the authors by Boris Venkov.

(i) The invariance group of a totally positive form A over a cyclotomic field is compact. Since $\mathcal{B}^{k,I}$ is discrete it follows that it is finite.

(ii) Since any (in our case, finite dimensional) representation of a compact (in particular, of a finite) group $\mathcal{B}^{k,I}$ is unitarizable, then the unique invariant form A should be positive together with all its Galois images.

This result gives a powerful criterion when is the group $\mathcal{B}^{k,I}$ finite. Before applying it we shall give a more explicit characterization of our group. In the so called "s–channel basis", in which the invariant form is diagonal we can write

$$\mathcal{B}^{k,I} = \{(B_{\lambda\mu}^{k,I}) = ((SBS^{-1})_{\lambda\mu}), B \in \mathcal{B}_4^{(I)}, \lambda,\mu = 0,1,\ldots,m\}. \tag{3.11}$$

Alternatively we can work in the original basis and factor the null subspace with respect to the form A (3.1) (3.8). An invariant condition for the $(m+1)\times(m+1)$ matrix generators of $\mathcal{B}^{k,I}$ is given by the characteristic equation

$$\prod_{\lambda=0}^{m} (B_i^{(k,I)} - b_\lambda) = 0, \quad i = 1, 2, \tag{3.12}$$

for b_λ given by (3.6).

*Galois automorphisms have been also used recently [14] for classifying modular invariant partition functions.

Theorem 3.3. *If the above described matrix group $\mathcal{B}^{k,I}$ acts irreducibly in $\mathcal{H}_{kI}$ then it is only finite in the following cases.*

(1) For the 1–dimensional representations corresponding to $m = 0$. These include the trivial representation for $I = 0$ and the "simple current" with $2I = k$. In the latter case the resulting group is $\mathbb{Z}_4$ for odd k, $\mathbb{Z}_2$ for $k = 4n + 2$ and again trivial for $k = 4n$.

(2) Three cases of 2–dimensional representations, $2I = 1$, corresponding to levels $k = 2$, 4 and 8. The commutator subgroup $C^{(k)}$, generated by the pair

$$b = B_1^{-1} B_2 = B_2 B_1 B_2^{-1} B_1^{-1} = \begin{pmatrix} 1 - q^2 & -\bar{q} \\ \bar{q} & -\bar{q}^2 \end{pmatrix}, \bar{b} = B_1 B_2^{-1} \tag{3.13}$$

is isomorphic to the 24 element double cover of the tetrahedron group for $k = 2$; to the 8 element group of quaternion units for $k = 4$; and to the 120 element double cover of the icosahedron group for $k = 8$. The latter two groups are also recovered for $2I = k - 1$, $k = 4, 8$.

(3) One 3–dimensional matrix group for $I = 1$, $k = 4$, that is a 27 element subgroup of SU_3.

Sketch of proof.

From the irreducibility assumption it follows that the invariant form A is unique. Therefore we can apply part (ii) of Lemma 3.2 and deduce that it is totally positive. This means according to (3.2) that all associated "quantum dimensions" should be positive

$$[2\lambda + 1] \geq 0, \quad \lambda = 0, 1, \ldots, m \tag{3.14}$$

for any primitive root of -1 (for a given $h = k + 2$). For $\lambda = 1$ this condition gives

$$[3] = [3]_{h,n} = 1 + 2 \cos \frac{2n\pi}{h} \geq 0 \text{ for } (2h, n) = 1. \tag{3.15}$$

A straightforward analysis shows that this is only possible for $h = 4, 6$ and 10. The inequality $[5]_{h,n} \geq 0$ (which arises for $2I \geq 2$) is only satisfied for $h = 6$ when $[5]_{6,n} = 1$ (for n=1,5,7).

The identification of various finite groups uses relations of the type (cf. [15])

$$b^3 = \bar{b}^3 = -1 = (b^{-1}\bar{b})^2 \text{ for } k = 2 \tag{3.16a}$$

$$b^2 = \bar{b}^2 = (b^{-1}\bar{b})^2 = -1 \text{ for } k = 4 \tag{3.16b}$$

$$(b^{-1}\bar{b}^2)^2 = (b^{-1}\bar{b})^3 = \bar{b}^5 = -1 \text{ for } k = 8 \tag{3.16c}$$

$$b^3 = \bar{b}^3 = (b\bar{b})^3 = (b^{-1}\bar{b})^3 = 1 \text{ for } k = 4, I = 1. \tag{3.16d}$$

Remark. The finite 2–dimensional braid groups (where the group algebra is a Hecke algebra) have been classified some 10 years ago by V. Jones [13] by a quite different method and the part (2) of the above theorem agrees with his result. Indeed, if $\hat{\mathcal{B}}_k$ is the subgroup of su_2 generated by the normalized 2×2 matrices $\hat{B}_i = \bar{q}^{\frac{k+1}{2}} B_i, i = 1, 2$, such that $\det \hat{B}_i = 1$, then $\hat{\mathcal{B}}_2$ is isomorphic to the 48 element binary octahedron group (the double cover of the symmetry group of the octahedron), $\hat{\mathcal{B}}_4$ is isomorphic to the 24 element binary tetrahedron group, while $C^{(k)}$ appear as their commutator subgroups. Only the binary icosahedron group $\hat{\mathcal{B}}_8$ coincides with its commutator subgroup $C^{(8)}$. The same finite braid groups appear in minimal conformal models. The corresponding algebraic 4-point functions have been computed in [16].

3C. Examples of reducible representations. The ADE classification of $\widehat{su}_2$ conformal theories [1] and the related classification of local extensions of chiral current algebras [8] imply the existence of reducible representations in the family $\mathcal{B}^{k,I}$ for some exceptional pairs (k, I). It turns out that the (one and) two dimensional irreducible components in the $k = 10$ E_6–theory is again a finite matrix group.

For $I = 2$ and $I = 3$ there exists a 1–dimensional $\mathcal{B}^{10,I}$ invariant subspace in each of the 5-dimensional spaces $\mathcal{H}_{10I}$ for $I = 2$ and 3. It corresponds to a second, degenerate invariant form whose "s–channel" expression $D^{(k,I)}$ is no longer diagonal but has the form

$$\tilde{D}^{(I)} = \begin{pmatrix} 1 & 0 & 0 & N_{I3} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ N_{I3} & 0 & 0 & N_{I3}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{aligned} N_{23} &= 2 - [3] (= 1 - \sqrt{3}) \\ N_{33} &= [3] - 3 (= \sqrt{3} - 2). \end{aligned} \tag{3.17}$$

We note that only the ratio

$$\frac{N_{23}^2}{D_3^{(10,2)}} = 2 = \frac{N_{33}^2}{D_3^{(10,3)}} \tag{3.18}$$

has an invariant (with respect to a rescaling of the type (2.12)) meaning. In both cases the vector $(1, 0, 0, N_{I3}, 0)$ spans a 1–dimensional representation of $\mathcal{B}^{10,I}$, that is $\mathbb{Z}_2$ for $I = 2$ and the trivial representation for $I = 3$.

For $2I = 3$ and $2I = 7$ there is a 2-dimensional $\mathcal{B}^{(10,I)}$ invariant subspace in each of the 4–dimensional spaces $\mathcal{H}_{10I}$ for $2I = 3$ and 7. The second invariant form $\tilde{D}^{(I)}$ is, in this case,

$$\tilde{D}^{(I)} = \begin{pmatrix} 1 & 0 & 0 & N_{I3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & N_{I2}^2 & 0 \\ N_{I3} & 0 & 0 & N_{I3}^2 \end{pmatrix} \tag{3.19a}$$

where

$$N_{\frac{3}{2}3} = -\frac{[2]}{2 + [3]} \left(= -\frac{1}{\sqrt{6}} \right), N_{\frac{3}{2}2}^2 = 1 \tag{3.19b}$$

$$N_{\frac{7}{2}3} = \frac{[3]-4}{6[2][3]} = \frac{2[3]-5}{6[2]}, N_{\frac{7}{2}2}^{2} = \frac{1}{2[2]^2} \left(= \frac{2-\sqrt{3}}{2} \right) \tag{3.19c}$$

The unitarized 2–dimensional sub-representation of $\mathcal{B}^{(10,I)}$ in the subspace spanned by $(1,0,0,N_{I3})$ ans $(0,0,1,0)$ is the same in the two cases and is generated by

$$\hat{B}_1 (= \hat{B}_1^{(I)}) = q^{3/2} \begin{pmatrix} q^3 & 0 \\ 0 & -\bar{q}^3 \end{pmatrix}, \quad \hat{F} = -\frac{[2]}{[3]} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{3.20}$$

where $[3]^2 = 2[2]^2$ for $q^{12} = -1, 2I = 3$ or 7. The generators $\hat{b}, \hat{\hat{b}}$ of the commutator subgroup,

$$\hat{b} = \hat{B}_1^{-1} \hat{F} \hat{B}_1 \hat{F} = \frac{1}{2} \begin{pmatrix} 1+q^6 & 1-q^6 \\ -1-q^6 & 1-q^6 \end{pmatrix} etc., \tag{3.21}$$

satisfy the relations (3.16a) for the double cover of the tetrahedron group. Furthermore, $\hat{B}_1$ satisfies the same equation, $\hat{B}_1^8 = -1$ as B_1 in the $k=2, 2I=1$ theory. There is, in fact, a series of rational conformal models of Virasaro central charge $c = n + 1/2$: the Ising model ($n=0$) and the level 1 $\widehat{so}(2n+1)$ model the $k=2$ $\widehat{su}_2$ and the E_6 model being the first examples. They exhibit striking similarities: all models have the same structure of superselection sectors, the same fusion rules, a $2c$ component Fermi field of conformal dimension $1/2$ and a primary chiral vertex operator of minimal positive dimension

$$\Delta(c) = \frac{c}{8}.$$

We would like to take this opportunity to thank P. Cohen, B. Dubrovin, L. Hadjiivanov, C. Itzykson, V. Pasquier, B. Venkov and J. B. Zuber for discussions at various stages of this work. The part of Boris Venkov in obtaining the results of Sec. 3B, that is central to the paper, was particularly important. The authors benefited from the hospitality and the stimulating atmosphere of the Erwin Schrödinger International Institute for Mathematical Physics where the major part of this work was done. We also acknowledge partial support by the Bulgarian Foundation for Scientific Research under contract F-11.

References

1. V. Pasquier, *Operator content of the A-D-E lattice model*, J. Phys. A. Math. Gren. **20** (1987), 5707-5717.

. A. Cappelli, C. Itzykson, J.B. Zuber, *ADE classification of minimal and $A_1^{(1)}$ conformal invariant theories*, Comm. Math. Phys. **113** (1987), 1-26.

. see also, Nucl. Phys. **B280** (1987), 445-465.

2. V.G. Knizhnik, A.B. Zamoldchikov, *Current algebra and Wess-Zumino model in two dimension*, Nucl. Physics **B247** (1984), 83-103.

3. H.A. Schwarz, *Über diejenigen Fälle in welhen die Gaußische hypergeometrische Reihe eine algebraische Funktion ihres vierten Elements darstellt*, Reine Angew.Math. (Crelle J.) **75** (1873), 292-335.

4. F. Beukers, G. Heckman, *Monodromy for the hypergeometric function* $_nF_{n-1}$, Invent. Math. **95** (1989), 325-354.
. F. Beukers, *Differential Galois theory, in: From Number Theory to Physics, M. Waldschmidt*, et.al.(eds.)(Springer), Berlin, 1992, 413-439.
5. P.B. Cohen, J. Wolfart, *Algebraic Appell–Lauricella functions*, Analysis **12** (1992), 359-376.
6. Ya.S.Stanev, I.T. Todorov, L.K. Hadjiivanov, *Braid invariant rational conformal models with a quantum group symmetry*, Phys. Lett. **B276** (1992), 87-94.
. *Braid invariant chiral conformal models with a quantum symmetry*, in Quantum Symmetries, Ed. by H.-D. Doebner, V.K. Dobrev, (World Scientific), Singapore, 1993, 24-40.
7. I.T. Todorov, Ya. S. Stanev, *Chiral current algebras and 2-dimensional conformal models*, Troisiène Cycle de la Physique en Suisse Romande **Lausanne lectures** (1992), 133.
8. L. Michel, Ya.S. Stanev, I.T. Todorov, *D-E classification of the local extensions of* su_2 *current algebras*, Theor. Mat. Fiz (Moscow) **92** (1992), 507-521.
. Ya.S. Stanev, I.T. Todorov, *Local 4-point functions and the Knizhnik-Zamoldchikov equation, Proceedings of the 1992 joint research conference on conformal field theory, and quantum groups. AMS book series. Contemporary mathematics (to be published)*, Ed.by P.J. Sally Jr.et al, AMS, USA.
9. A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nucl. Phys. **B241** (1984), 333-381.
10. A. Tsuchiya, Y. Kanie, *Vertex operators in the conformal field theory on* P^1 *and monodromy representations of the braid group*, Conformal field theory and solvable lattice models, Advanced studies in pure mathematics **16** (1988), 297-372;.
. -, Lett. Math. Phys. **13** (1987), 303-312.
11. V.G. Kac, *Infinite Dimensional Lie Algebras (Second edition)*, Cambridge University Press, Cambridge, 1985.
12. J. Howie, R.M. Thomas, *Proving certain groups infinite, Proceedings of the Symposium held in Sussex 1991*, London Mathematical Society Lecture Series, Vol. 1, ed. by G.A. Niblo, M.A. Roller, Cambridge University Press **181** (1993), 126-131.
13. V. Jones, *Braid groups, Hecke algebras and type* II_1 *factors*, Geometric methods in operator algebras (Kyoto 1983), Pitman Res. Notes in Math **123** (1986), 242-273.
14. A. Coste, T. Gannon, *Galois symmetry in RCFT*, Phys. Lett **B** (-), to appear.
-. Ph. Ruelle, E. Thirran, J. Weyers, *Implications of an arithmetical symmetry of the commutant for modular invariants*, Nuclear Physics **B 402** (1993), 693-708.
15. H.S.M. Coxeter, W.O.J. Moser, *Generators and Relations for Discrete Groups*, Springer Verlag, Berlin et. al, 1957.
16. K.-H. Rehren, B. Schroer, *Einstein causality and Artin braids*, Nucl. Phys. **B312** (1989), 715-750.

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