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SPECTRA OF BERNOULLI CONVOLUTIONS AS MULTIPLIERS IN L^p ON THE CIRCLE

NIKITA SIDOROV AND BORIS SOLOMYAK

Abstract. It is shown that the closure of the set of Fourier coefficients of the Bernoulli convolution μ_θ parameterized by a Pisot number θ, is countable. Combined with results of Salem and Sarnak, this proves that for every fixed $\theta > 1$ the spectrum of the convolution operator $f \mapsto \mu_\theta * f$ in $L^p(S^1)$ (where S^1 is the circle group) is countable and is the same for all $p \in (1, \infty)$, namely, $\overline{\{\widehat{\mu_\theta}(n) : n \in \mathbb{Z}\}}$. Our result answers the question raised by P. Sarnak in [8]. We also consider the sets $\overline{\{\widehat{\mu_\theta}(rn) : n \in \mathbb{Z}\}}$ for $r > 0$ which correspond to a linear change of variable for the measure. We show that such a set is still countable for all $r \in \mathbb{Q}(\theta)$ but uncountable (a non-empty interval) for Lebesgue-a.e. $r > 0$.

1. Introduction and main results

Let ν be a Borel probability measure on $S^1 = \mathbb{R}/\mathbb{Z}$, and let

$$T_\nu : L^p(S^1) \to L^p(S^1)$$

be the convolution operator, namely, $T_\nu f := \nu * f$. Put

$$\mathcal{F}_\nu := \{\widehat{\nu}(n) : n \in \mathbb{Z}\}.$$

It is shown by Sarnak [8] that if the closure $\overline{\mathcal{F}_\nu}$ has capacity zero, then the following identity relation for the spectra of T_ν in different $L^p(S^1)$ is satisfied:

$$\mathrm{sp}(T_\nu, L^p) \equiv \mathrm{sp}(T_\nu, L^2) = \overline{\mathcal{F}_\nu}, \quad 1 < p < +\infty. \tag{1.1}$$

The present paper deals with the case when ν is a Bernoulli convolution. Recall that for any $\theta > 1$ the *Bernoulli convolution* parameterized

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by θ is defined as follows:

$$\mu_\theta := * \prod_{k=0}^{\infty} \left(\frac{1}{2}\,\delta_{-\theta^{-k}} + \frac{1}{2}\,\delta_{\theta^{-k}} \right)$$

(where δ_x denotes the Dirac δ-measure at x). Thus,

$$\widehat{\mu_\theta}(t) = \prod_{k=0}^{\infty} \cos(2\pi\theta^{-k}t). \tag{1.2}$$

We can view μ_θ as a measure on the line, i.e., $t \in \mathbb{R}$ in (1.2). The induced measure on the circle has Fourier coefficients $\{\widehat{\mu_\theta}(n) : n \in \mathbb{Z}\}$.

As is well known, $\operatorname{supp}\mu_\theta \subset I_\theta = [-\frac{\theta}{\theta-1}, \frac{\theta}{\theta-1}]$ for any $\theta > 1$. Moreover, for $\theta > 2$ the measure μ_θ is usually called the *Cantor-Lebesgue measure* (parameterized by θ), and its support is the Cantor set with constant dissection ratio θ. On the other hand, $\operatorname{supp}\mu_\theta = I_\theta$ for $\theta \in (1, 2]$.

In [7, Th II, p. 40] it is shown that unless θ is a *Pisot number* (an algebraic integer greater than 1 whose conjugates are all less than 1 in modulus), $\widehat{\mu_\theta}(t) \to 0$ as $t \to +\infty$ along the reals, whence $\overline{\mathcal{F}_{\mu_\theta}}$ is countable and thus, (1.1) is satisfied.

Sarnak [8] considered the case of the classical Cantor-Lebesgue measure ($\theta = 3$), for which he proved that $\overline{\mathcal{F}_{\mu_\theta}}$ is countable and therefore (1.1) holds. As stated in [8], the same approach can be applied to the case of an arbitrary integer $\theta \geq 3$, and the only case left is the irrational Pisot numbers θ. The question about the limit points of the Fourier coefficients for this class of measures was raised in [8].

Theorem 1.1. *The set of limit points of the sequence $\{\widehat{\mu_\theta}(n) : n \in \mathbb{Z}\}$, with an irrational Pisot parameter θ, is countable, so (1.1) holds for $\nu = \mu_\theta$.*

Remark 1.2. There is apparently another way to obtain (1.1) for $\nu = \mu_\theta$, without getting countability of the spectrum. It is known that if a Borel probability measure ν on S^1 is L^p-improving[1], then, like in the case in question, (1.1) holds, see [5, Th. 4.1]. Christ [3] proved that μ_θ is L^p-improving for all $\theta > 2$, and he made a remark that the same argument works for $\theta \in (1, 2)$ as well.

Since we are considering measures on the circle, one may argue that it is in fact more natural to have the Bernoulli convolution measure supported on an interval of length 1, rather than on I_θ (whose length

[1]A measure ν is called *L^p-improving* for some $p \in (1, \infty)$, if there exists $q = q(p) > p$ such that $\nu * f \in L^q(S^1)$ for any $f \in L^p(S^1)$. If ν improves some L^p, then it improves all of them for $1 < p < +\infty$, see [5].

is $2\theta/(\theta-1) > 1$). This is achieved by a linear change of variable, resulting in the Fourier coefficients $\{\widehat{\mu}((\frac{\theta-1}{2\theta})n) : n \in \mathbb{Z}\}$, and we show that the analog of Theorem 1.1 is still valid (see below).

More generally, one may inquire what happens under an arbitrary scale change. It turns out that the situation is rather delicate.

For $r > 0$ put

$$\mathcal{F}_{\nu,r} := \{\widehat{\nu}(rn) : n \in \mathbb{Z}\}.$$

For a set $E \subset \mathbb{R}$ let E' denote the derived set of E, that is, the set of its limit points.

Theorem 1.3. *Let $\theta \neq 2$ be a Pisot number. Then*

(i) *for any positive $r \in \mathbb{Q}(\theta)$, the set $\mathcal{F}'_{\mu_\theta,r}$ is countable;*
(ii) *for Lebesgue a.e. $r > 0$, the set $\mathcal{F}'_{\mu_\theta,r}$ is a non-empty interval.*

Corollary 1.4. *For any $\theta > 2$, the spectrum in $L^p(S^1)$, $p > 1$, of the convolution operator corresponding to the Cantor-Lebesgue measure with the constant dissection ratio θ, constructed on $[0,1) \simeq S^1 = \mathbb{R}/\mathbb{Z}$, is countable.*

This follows from Theorem 1.3 (i), since $\frac{\theta-1}{2\theta} \in \mathbb{Q}(\theta)$ and translating the measure by $\frac{1}{2}$ results in multiplying the Fourier coefficients by $(-1)^n$.

Denote $E^{(1)} := E'$, $E^{(n+1)} := (E^{(n)})'$. For $r \in \mathbb{Q}(\theta)$, it is natural to ask what is the cardinality of the second, third, etc., derived sets for $\mathcal{F}_{\mu_\theta,r}$. The following theorem answers this question.

Theorem 1.5. *For any Pisot $\theta \neq 2$ and any positive $r \in \mathbb{Q}(\theta)$ the set $\mathcal{F}^{(n)}_{\mu_\theta,r}$ is countable for each $n \geq 1$.*

2. Proof of countability: the model case

The core of the paper is the proof of Theorem 1.3 (i) (which, of course, implies Theorem 1.1). Our proof is loosely based on the method used in [8] for $\theta = 3$. On the other hand, the case of irrational θ requires extra tools more common for the theory of Pisot numbers (in the spirit of the monograph [2, Chapter VIII]).

The structure of the rest of the paper is as follows: in this and the next sections we are going to show that

$$\operatorname{card} \mathcal{F}'_{\mu_\theta,r} \leq \aleph_0, \quad r \in \mathbb{Q}(\theta) \cap (0,\infty). \tag{2.1}$$

Note that our proof applies to the case $\theta \in \mathbb{N}$ as well. Combined with Theorem 1.5 (proved in Section 4) this yields Theorem 1.3 (i). Theorem 1.3 (ii) is also proved in Section 4.

The Pisot numbers $\theta \neq 2$ are distinguished by the fact that $\widehat{\mu}_\theta(t) \not\to 0$, as $t \to +\infty$ over the reals. This was proved by Erdős [4] (for integers $\theta > 2$ this had been known earlier). It is easy to see that also $\widehat{\mu}_\theta(n) \not\to 0$, as $n \to +\infty$ over the integers, see Section 4. If $\theta = 2$, then μ_θ is absolutely continuous, so there is nothing to prove.

For the rest of the paper, we fix a Pisot number $\theta \neq 2$ and denote by $\theta_2, \dots, \theta_m$ the conjugates of $\theta = \theta_1$. Since θ is Pisot,

$$\rho = \max_{i \geq 2} |\theta_i| \in [0, 1) \tag{2.2}$$

($\rho = 0$ if and only if $\theta \in \mathbb{N}$).

Let us begin the proof of the inequality (2.1). Denote by $\langle \cdot \rangle$, $\| \cdot \|$ the nearest integer and the distance to the nearest integer respectively.

To simplify notation, denote $\mu := \mu_\theta$. It suffices to prove that there are at most countable many limit points for the set $\{|\widehat{\mu}(rn)| : \ n \in \mathbb{N}\}$. Fix $\eta > 0$ and assume that integers $n_k \to +\infty$ are such that

$$|\widehat{\mu}(rn_k)| \to a, \quad a \geq \eta. \tag{2.3}$$

Similarly to [8], our goal is to show that there can be only a countable set of such a's for any fixed η; this will yield (2.1).

There exist $N_k \in \mathbb{Z}$ and $y_k \in [1, \theta)$ such that

$$y_k = 2rn_k \theta^{-N_k}. \tag{2.4}$$

Let

$$y_k \theta^j = K_j^{(k)} + \delta_j^{(k)}, \quad j = 1, \dots, N_k, \tag{2.5}$$

where $\delta_j^{(k)} \in (-\frac{1}{2}, \frac{1}{2}]$ and $K_j^{(k)} = \langle y_k \theta^j \rangle \in \mathbb{N}$. By (1.2) and (2.4),

$$|\widehat{\mu}(rn_k)| = \prod_{j=-\infty}^{N_k} |\cos(\pi y_k \theta^j)| = \prod_{j=-\infty}^{N_k} \cos\bigl(\pi \delta_j^{(k)}\bigr). \tag{2.6}$$

Let m be the degree of θ, with the minimal polynomial $x^m - d_1 x^{m-1} - \dots - d_m$. For the rest of the paper we fix a δ which satisfies

$$0 < \delta < (1 + |d_1| + \dots + |d_m|)^{-1}. \tag{2.7}$$

The reason for the choice of δ is the following

Lemma 2.1. *Suppose that $|\delta_j^{(k)}| \leq \delta$ for $j = A_k + 1, \dots, A_k + b$, where $0 \leq A_k \leq N_k - b$ and $b > m$. Then*

$$K_{j+m}^{(k)} = d_1 K_{j+m-1}^{(k)} + \dots + d_m K_j^{(k)}, \tag{2.8}$$

for $j = A_k + 1, \dots, A_k + b - m$.

Proof. By our condition and (2.7), for $j \geq A_k + 1$,

$$|K_{j+m}^{(k)} - d_1 K_{j+m-1}^{(k)} - \cdots - d_m K_j^{(k)}| \leq \delta(1 + |d_1| + \cdots + |d_m|) < 1.$$

As $K_j^{(k)}$'s and d_i's are integers, we are done. □

We want to estimate the number of $\delta_j^{(k)}$'s that are greater than δ in modulus. Let $L \in \mathbb{N}$ be such that $(\cos(\pi\delta))^L \leq \eta/2$. It follows from (2.3) and (2.6) that for k sufficiently large,

$$L_k := \#\{j \in [1, N_k] : \ |\delta_j^{(k)}| > \delta\} \leq L. \tag{2.9}$$

Since we only care about the limit, we can assume without loss of generality that (2.9) holds for all k.

The rest of the proof is somewhat technical, so we believe that it is helpful first to present a sketch in the special model case $L_k = 1$ and $r = 1$. This will be done in the rest of the section.

Thus, let us assume for the moment that $|\delta_j^{(k)}| \leq \delta$ for all $j = 1, \ldots, N_k$, except possibly $j = J_k$. There are three possibilities: (a) $\sup J_k < \infty$, (b) $\sup(N_k - J_k) < \infty$, and (c) $\sup J_k = \infty$ and $\sup(N_k - J_k) = \infty$. By passing to a subsequence, we can assume that we actually have one of the following cases:

Case 1: $J_k = R$ (independent of k);

Case 2: $J_k = N_k - R$;

Case 3: $J_k \to \infty$ and $N_k - J_k \to \infty$.

Case 1. By Lemma 2.1, the sequence $\{K_j^{(k)}\}$ satisfies the recurrence relation (2.8) for $j = R+1, \ldots, N_k - m$ (for k large enough to satisfy $N_k > R+m$). Then we can express $K_j^{(k)}$ in terms of θ and its conjugates $\theta_2, \ldots, \theta_m$ as follows:

$$K_j^{(k)} = c_1^{(k)}\theta^j + \sum_{i=2}^{m} c_i^{(k)}\theta_i^j, \quad j = R+1, \ldots, N_k. \tag{2.10}$$

Observe that the coefficients $c_i^{(k)}$ are completely determined by $K_j^{(k)}$ for $j = R+1, \ldots, R+m$. These $K_j^{(k)}$'s are integers bounded by $\theta^{R+m+1} + 1$ (as $y_k \leq \theta$ and $K_j^{(k)}$ is the nearest integer to $y_k\theta^j$). Thus, there are finitely many possibilities for $c_i^{(k)}$ and we can assume, passing to a subsequence, that $c_i^{(k)} = c_i$ do not depend on k.

Let $y := c_1$; the first important point is that $y \in \mathbb{Q}(\theta)$. This follows from the Cramer's Rule, solving the linear system (2.10), with $j = R+1, \ldots, R+m$, for c_i. Alternatively, note that $\|y\theta^j\| \to 0$ as $j \to +\infty$

by (2.10), and the fact that $y \in \mathbb{Q}(\theta)$ is a part of the well-known Pisot-Vijayaraghavan theorem (see [2]).

Now comes the crucial point—we have to use that n_k is an integer[2]. We assumed that $r = 1$, so $n_k = \frac{1}{2} y_k \theta^{N_k}$ from (2.4). We have $2n_k = K_{N_k}^{(k)}$, since both sides are integers. Hence by (2.5) and (2.10),

$$y_k \theta^{N_k} = K_{N_k}^{(k)} = y\theta^{N_k} + O(\rho^{N_k}),$$

where ρ is given by (2.2) and the implied constant in O is independent of k. Thus, $y_k = y + O(\theta^{-N_k}\rho^{N_k})$, and an elementary argument yields

$$|\widehat{\mu}(n_k)| \to \prod_{j=-\infty}^{\infty} |\cos(\pi y \theta^j)|.$$

(A more general statement is proved below, in Lemma 3.2.) Since the right-hand side depends only on $y \in \mathbb{Q}(\theta)$, the number of possible limit points in this case is at most countable.

Case 2. By Lemma 2.1, $K_j^{(k)}$'s satisfy the recurrence relation (2.8) for $j = 1, \dots, N_k - R - m - 1$, when k is sufficiently large. Passing to a subsequence, we can assume that $K_j^{(k)} = K_j$ do not depend on k for $j \le N_k - R - 1$ and

$$K_j = y\theta^j + \sum_{i=2}^{m} c_i \theta_i^j, \quad j = 1, \dots, N_k - R - 1. \tag{2.11}$$

Again we have $y \in \mathbb{Q}(\theta)$. Extend K_j by (2.11) to $j = N_k - R, \dots, N_k$; in other words, we extend K_j to satisfy the recurrence relation (2.8). We cannot claim that $K_j = K_j^{(k)}$ for $j = N_k - R, \dots, N_k$; however, it is easy to see from the recurrence that

$$\left|K_j^{(k)} - K_j\right| \le C_R, \quad j = N_k - R, \dots, N_k,$$

where C_R does not depend on k. (This is proved below, in Lemma 3.1.) Again, passing to a subsequence, we can assume that $K_{N_k}^{(k)} - K_{N_k} = A$ is a constant. Using that n_k is an integer, we obtain

$$y_k \theta^{N_k} = K_{N_k}^{(k)} = y\theta^{N_K} + A + O(\rho^{N_k}),$$

where the implied constant in O is independent of k. Thus $y_k = y + A\theta^{-N_k} + O(\theta^{-N_k}\rho^{N_k})$, and it is not hard to show that

$$|\widehat{\mu}(n_k)| \to \prod_{j=-\infty}^{\infty} |\cos(\pi y \theta^j)| \cdot \prod_{j=0}^{\infty} |\cos(\pi A \theta^{-j})|.$$

[2]Note that the set of limit points of $\widehat{\mu}(t)$, as $t \to \infty$ over the reals, is an interval—see Lemma 4.1 below.

(A more general statement will be proved below, in Lemma 3.2.) Since $y \in \mathbb{Q}(\theta)$ and $A \in \mathbb{Z}$, the number of possible limit points in this case is again at most countable.

Case 3. By Lemma 2.1, $\left\{K_j^{(k)}\right\}_{j=1}^{J_k-m-1}$ and $\left\{K_j^{(k)}\right\}_{j=J_k+1}^{N_k-m}$ satisfy the recurrence relation (2.8). As in Case 2, we can assume by passing to a subsequence that $K_j^{(k)} = K_j$ for $j = 1, \ldots, J_k - 1$, whence

$$K_j = y\theta^j + \sum_{i=2}^{m} c_i \theta_i^j, \quad j = 1, \ldots, J_k - 1,$$

for some $y \in \mathbb{Q}(\theta), c_2, \ldots, c_m$. Also, as in Case 2, we extend K_j to $j \geq J_k$ to satisfy the same recurrence relation and check that

$$\left|K_j^{(k)} - K_j\right| \leq C_R, \quad j = J_k + 1, \ldots, J_k + m.$$

Let $S_j^{(k)} = K_{J_k+j}^{(k)} - K_{J_k+j}$ for $j = 1, \ldots, N_k - J_k$. Hence there exist $b_i^{(k)} \in \mathbb{Q}(\theta_i)$, with $i = 1, \ldots, m$, such that

$$S_j^{(k)} = b_1^{(k)}\theta^j + \sum_{i=2}^{m} b_i^{(k)}\theta_i^j, \quad j = 1, \ldots, m.$$

Because of the bounds on $S_j^{(k)}$, there are finitely many possibilities for $b_i^{(k)}$, so we can assume that they do not depend on k, passing to a subsequence. Let $z = b_1 = b_1^{(k)}$. We have $S_j = S_j^{(k)} = z\theta^j + O(\rho^j)$ for $j \geq 1$. Now observe that

$$K_j^{(k)} = K_j + S_{j-J_k}, \quad j = J_k + 1, \ldots, N_k.$$

Thus, using that n_k is an integer, we obtain

$$y_k\theta^{N_k} = K_{N_k}^{(k)} = K_{N_k} + S_{N_k-J_k} = y\theta^{N_k} + z\theta^{N_k-J_k} + O(\rho^{N_k-J_k}),$$

where the implied constant in O is independent of k. Since $J_k \to \infty$ and $N_k - J_k \to \infty$, is not hard to show that

$$|\widehat{\mu}(n_k)| \to \prod_{j=-\infty}^{\infty} |\cos(\pi y\theta^j)| \cdot \prod_{j=-\infty}^{\infty} |\cos(\pi z\theta^j)|.$$

(We will prove a more general statement below, in Lemma 3.2.) As $y, z \in \mathbb{Q}(\theta)$, the number of possible limit points in this case is at most countable.

This concludes the sketch of the proof of (2.1) in the model case $L_k = 1$ and $r = 1$. The idea for the general case is as follows: we gather all indices j, for which $\left|\delta_j^{(k)}\right| > \delta$, in groups in such a way that the distance between any two adjacent groups goes to the infinity as

$k \to \infty$. Then we treat each group similarly to one of the three cases considered in this section, depending on the position of this group ("beginning", "middle" or "end") and finally, justify passing to the limit in the key Lemma 3.2.

3. Proof of countability: the general case

We continue with the proof of the general case where we left it, after the definition of L_k (2.9). Let $1 \le I_1^{(k)} < I_2^{(k)} < \ldots < I_{L_k}^{(k)} \le N_k$ be all the indices j for which $\left|\delta_j^{(k)}\right| > \delta$. Since $L_k \le L$, we can assume that $L_k = L'$ does not depend on k, passing to a subsequence. Further, passing to a subsequence, we can assume that for all $i = 1, \ldots, L'-1$, either $I_{i+1}^{(k)} - I_i^{(k)} = R_i$ (independent of k), or $I_{i+1}^{(k)} - I_i^{(k)} \to \infty$, as $k \to \infty$. Also, either $I_1^{(k)} = R_0$ or $I_1^{(k)} \to \infty$ and either $N_k - I_{L'}^{(k)} = R_{L'}$ or $N_k - I_{L'}^{(k)} \to \infty$. Let

$$R = \max\{R_i : \ i = 0, \ldots, L'\} + 1.$$

We can find $M \in \{1, \ldots, L'\}$ and integers

$$1 = J_0^{(k)} < J_1^{(k)} < \ldots < J_M^{(k)} < J_{M+1}^{(k)} = N_k$$

so that $J_{i+1}^{(k)} - J_i^{(k)} \to \infty$ for $i = 0, \ldots, M$, and

$$\left|\delta_j^{(k)}\right| \le \delta \quad \text{for all } j \in \{1, \ldots, N_k\} \quad \text{such that } \min_i \left|j - J_i^{(k)}\right| \ge R.$$

By Lemma 2.1, $\left\{K_j^{(k)}\right\}$ satisfy the recurrence relation (2.8) for $J_i^{(k)} + R \le j \le J_{i+1}^{(k)} - R - m$, with $i = 0, \ldots, M$ (for k large enough to satisfy $J_{i+1}^{(k)} - J_i^{(k)} > 2R + m$). In particular, this is true for $1 + R \le j \le J_1^{(k)} - R - m$. Thus we can write

$$K_j^{(k)} = c_1^{(k)} \theta^j + \sum_{i=2}^{m} c_i^{(k)} \theta_i^j, \quad j = R+1, \ldots, J_1^{(k)} - R$$

(for k sufficiently large). The coefficients $c_i^{(k)}$ are completely determined by $K_j^{(k)}$ for $j = R+1, \ldots, R+m$, which are integers bounded by $\theta^{R+m+1} + 1$. Thus, there are finitely many possibilities for $c_i^{(k)}$, $i = 1, \ldots, m$. Hence we can assume, passing to a subsequence, that $c_i^{(k)} = c_i$ do not depend on k. Thus, $K_j^{(k)} = K_j$ for $j \le J_1^{(k)} - R$. Denote $z_0 := c_1$, so that

$$K_j = z_0 \theta^j + O(\rho^j).$$

As in Case 1, we have $z_0 \in \mathbb{Q}(\theta)$ (z_0 is a natural analog of y from the previous section). Next we repeat the argument from Case 3. Extend

the sequence $\{K_j\}$ to $j > J_1^{(k)} - R$ so that it satisfies the recurrence relation for all j. We need the following lemma.

Lemma 3.1. *Suppose that $K_j^{(k)} = K_j$ for $j \le A_k$ and $\{K_j\}$ satisfies the recurrence relation (2.8) for all j. Then for any $p \in \mathbb{N}$ there exists $C_p = C_p(\theta) > 0$ such that*

$$|K_j^{(k)} - K_j| \le C_p, \quad j = A_k + 1, \dots, A_k + p. \tag{3.1}$$

Proof. This is proved by induction. Since $K_j^{(k)}$ is the nearest integer to $y_k\theta^j$, for any $j \ge 1$,

$$|K_{j+m}^{(k)} - d_1 K_{j+m-1}^{(k)} - \cdots - d_m K_j^{(k)}| \le \frac{1}{2}(1 + |d_1| + \cdots + |d_m|) =: \gamma.$$

It follows that we can take $C_1 = \gamma$ in (3.1). Suppose (3.1) is verified for some p. Then we have for $j = A_k + p + 1$,

$$\begin{aligned} |K_{j+1}^{(k)} - K_{j+1}| &= |K_{j+1}^{(k)} - d_1 K_j^{(k)} - \cdots - d_m K_{j-m+1}^{(k)} \\ &\quad + d_1 K_j^{(k)} + \cdots + d_m K_{j-m+1}^{(k)} - d_1 K_j - \cdots - d_m K_{j-m+1}| \\ &\le \gamma + (|d_1| + \cdots + |d_m|) C_p < \gamma(1 + 2C_p). \end{aligned}$$

Thus, we may put $C_{p+1} = \gamma(1 + 2C_p)$, and the lemma is proved. □

Let

$$S_j^{(k)} = K_{J_1^{(k)}+j}^{(k)} - K_{J_1^{(k)}+j}, \quad j = R + 1, \dots, R + m.$$

By Lemma 3.1,

$$\left|S_j^{(k)}\right| \le C_{2R+m}, \quad j = R + 1, \dots, R + m.$$

Therefore, we can assume (passing to a subsequence) that $S_j^{(k)}$'s do not depend on k for $j = R+1, \dots, R+m$. We can find $z_1 \in \mathbb{Q}(\theta), c_2', \dots, c_m'$ so that

$$S_j = S_j^{(k)} = z_1\theta^j + \sum_{i=2}^{m} c_i'\theta_i^j, \quad j = R + 1, \dots, R + m. \tag{3.2}$$

Extend S_j to $j > R + m$ by the formula (3.2), so that they satisfy the recurrence relation. Now observe that

$$K_j^{(k)} = K_j + S_{j-J_1^{(k)}}, \quad j = J_1^{(k)} + R + 1, \dots, J_2^{(k)} - R$$

(for k large enough to satisfy $J_2^{(k)} - J_1^{(k)} > 2R+m$), as both sides agree for $j = J_1^{(k)} + R+1, \dots, J_1^{(k)} + R+m$, and satisfy the same recurrence relation of length m. It follows that

$$K_j^{(k)} = z_0\theta^j + z_1\theta^{j-J_1^{(k)}} + O\big(\rho^{j-J_1^{(k)}}\big), \quad j = J_1^{(k)} + R + 1, \dots, J_2^{(k)} - R,$$

where the implied constant in O is independent of k.

Next we repeat the same argument and obtain, by induction, that for $i = 2, \dots, M$,

$$\begin{aligned} K_j^{(k)} =& z_0\theta^j + z_1\theta^{j-J_1^{(k)}} + \cdots + z_i\theta^{j-J_i^{(k)}} + O(\rho^{j-J_i^{(k)}}), \\ & j = J_i^{(k)} + R, \dots, J_{i+1}^{(k)} - R. \end{aligned} \tag{3.3}$$

Indeed, for each i extend $K_j^{(k)}$ from $j < J_i^{(k)} - R$ to larger j's by recurrence; denote them $Q_j^{(k)}$. Put

$$T_j^{(k)} := K^{(k)}_{J_i^{(k)}+j} - Q^{(k)}_{J_i^{(k)}+j}, \quad j = R+1, \dots, R+m.$$

Then $\left|T_j^{(k)}\right| \le C_{2R+m}$ for $j = R+1, \dots, R+m$. We can write $T_j^{(k)}$ as a linear combination of θ^j and the powers of its conjugates. As above, there are finitely many possibilities for the coefficients (as k varies), so we can assume without loss of generality that they do not depend on k. The coefficient at θ^j will be denoted by z_i, which yields (3.3).

For $i = M$ the formula (3.3) becomes

$$\begin{aligned} K^{(k)}_{N_k-R+j} =& z_0\theta^{N_k-R+j} + \sum_{i=1}^{M} z_i\theta^{N_k-J_i^{(k)}-R+j} + O\big(\rho^{N_k-J_M^{(k)}-R+j}\big), \\ & j = J_M^{(k)} + 2R - N_k, \dots, 0 \end{aligned} \tag{3.4}$$

(recall that $J_M^{(k)} - N_k \to -\infty$). As usual, the implied constant in O is independent of k. One last time extend $K_j^{(k)}$ by recurrence, to $j = N_k - R + 1, \dots, N_k, \dots$ Denote the resulting integer sequence by $\{L^{(k)}_{N_k-R+j}\}_{j=1}^{\infty}$. By Lemma 3.1,

$$|K_{N_k}^{(k)} - L_{N_k}^{(k)}| \le C_R. \tag{3.5}$$

We have

$$2rn_k = y_k\theta^{N_k} = K_{N_k}^{(k)} + \delta_{N_k}^{(k)}. \tag{3.6}$$

Now it's the time to use the fact that n_k is an integer; this is slightly more complicated than in Section 2, because the left-hand side of (3.6) need not be an integer. However, as $2r \in \mathbb{Q}(\theta) \setminus \{0\}$ by assumption, we can invert it in $\mathbb{Q}(\theta)$, i.e.,

$$(2r)^{-1} = a_0 + a_1\theta + \cdots + a_{m-1}\theta^{m-1}, \tag{3.7}$$

for some $a_i \in \mathbb{Q}$. Thus,

$$\begin{aligned} n_k &= (2r)^{-1}\big(K_{N_k}^{(k)} + \delta_{N_k}^{(k)}\big) \\ &= (a_0 + a_1\theta + \cdots + a_{m-1}\theta^{m-1})K_{N_k}^{(k)} + (2r)^{-1}\delta_{N_k}^{(k)} \\ &= a_0 L_{N_k}^{(k)} + a_1 L_{N_k+1}^{(k)} + \cdots + a_{m-1}L_{N_K+m-1}^{(k)} + A_k. \end{aligned} \tag{3.8}$$

Let us estimate the "error term" A_k. By the definition of the integers $L_{N_k-R+j}^{(k)}$, they satisfy (3.4), with K replaced by L, for $j = 1, 2, \ldots$ In particular,

$$L_{N_k+j}^{(k)} = z_0\theta^{N_k+j} + \sum_{i=1}^{M} z_i\theta^{N_k - J_i^{(k)}+j} + O(\rho^{N_k - J_M^{(k)}+j}), \quad j \geq 0. \tag{3.9}$$

Hence in view of $N_k - J_M^{(k)} \to +\infty$,

$$\lim_{k\to\infty}\big|L_{N_k+j+1}^{(k)} - \theta L_{N_k+j}^{(k)}\big| = 0, \quad 0 \leq j \leq m-1,$$

whence

$$\left| L_{N_k}^{(k)} \sum_{j=0}^{m-1} a_j\theta^j - \sum_{j=0}^{m-1} a_j L_{N_k+j}^{(k)} \right| \to 0, \quad k \to +\infty.$$

By (3.8), (3.5) and in view of $\big|\delta_{N_k}^{(k)}\big| \leq \frac{1}{2}$, we have $|A_k| \leq C'$ for some constant C' independent of k; namely, one may put for k large enough,

$$C' = 1 + (4r)^{-1}(2C_R + 1),$$

where C_R is as in (3.5).

On the other hand, it follows from (3.8) that $A_k \in s^{-1}\mathbb{Z}$ for some $s \in \mathbb{N}$ independent of k, because $a_i \in \mathbb{Q}$ and the n_k is an integer. Thus, there are finitely many possibilities for A_k, so, passing to a subsequence, we can assume that $A_k = A$ is a constant. Now we have by (3.9),

$$\begin{aligned} y_k = 2rn_k\theta^{-N_k} &= 2r\big(a_0 L_{N_k}^{(k)} + \cdots + a_{m-1}L_{N_K+m-1}^{(k)}\big)\theta^{-N_k} + 2rA\theta^{-N_k} \\ &= 2rz_0(a_0 + a_1\theta + \cdots + a_{m-1}\theta^{m-1}) \\ &+ 2r\sum_{i=1}^{M} z_i\big(a_0\theta^{-J_i^{(k)}} + \cdots + a_{m-1}\theta^{-J_i^{(k)}+m-1}\big) + 2rA\theta^{-N_k} \\ &+ O\big(\theta^{-N_k}\rho^{N_k - J_M^{(k)}}\big), \end{aligned}$$

and finally, by (3.7),

$$y_k = z_0 + \sum_{i=1}^{M} z_i\theta^{-J_i^{(k)}} + 2rA\theta^{-N_k} + O\big(\theta^{-N_k}\rho^{N_k - J_M^{(k)}}\big). \tag{3.10}$$

Lemma 3.2.

$$|\widehat{\mu}(n_k)| = \prod_{j=-\infty}^{N_k} |\cos(\pi y_k \theta^j)|$$
$$\to \prod_{i=0}^{M} \prod_{j=-\infty}^{\infty} |\cos(\pi z_i \theta^j)| \cdot \prod_{j=0}^{\infty} |\cos(2\pi r A \theta^{-j})|, \quad k \to +\infty.$$

Recall that by the construction of z_i we have for $0 \le i \le M$:

$$\|z_i \theta^j\| = O(\rho^j), \quad j \to \infty. \tag{3.11}$$

It follows that the bi-infinite products in the right-hand side converge. This lemma will clearly imply our theorem, as $z_i \in \mathbb{Q}(\theta)$ and $A \in \mathbb{Q}$, so there are countable many possible limits.

Proof of Lemma 3.2. We can find integers $E_i^{(k)}$, $i = 0, \dots, M$, so that

$$J_i^{(k)} < E_i^{(k)} < J_{i+1}^{(k)}, \quad \lim_{k\to\infty} \left(E_i^{(k)} - J_i^{(k)}\right) = \lim_{k\to\infty} \left(J_{i+1}^{(k)} - E_i^{(k)}\right) = +\infty.$$

We are going to show that

$$F_0(k) := \frac{\prod_{j=-\infty}^{E_0^{(k)}} |\cos(\pi y_k \theta^j)|}{\prod_{j=-\infty}^{\infty} |\cos(\pi z_0 \theta^j)|} \to 1, \quad k \to \infty; \tag{3.12}$$

$$F_i(k) := \frac{\prod_{j=E_{i-1}^{(k)}+1}^{E_i^{(k)}} |\cos(\pi y_k \theta^j)|}{\prod_{j=-\infty}^{\infty} |\cos(\pi z_i \theta^j)|} \to 1, \quad k \to \infty,\ i = 1, \dots, M; \tag{3.13}$$

$$F_{M+1}(k) := \frac{\prod_{j=E_M^{(k)}+1}^{N_k} |\cos(\pi y_k \theta^j)|}{\prod_{j=0}^{\infty} |\cos(2\pi r A \theta^{-j})|} \to 1, \quad k \to \infty. \tag{3.14}$$

These statements will imply the lemma.

First we verify (3.12). Observe that $\prod_{j=E_0^{(k)}}^{\infty} |\cos(\pi z_0 \theta^j)| \to 1$, since $E_0^{(k)} \to \infty$ and the denominator in $F_0(k)$ converges. Thus, it remains to show that

$$\prod_{j=-\infty}^{E_0^{(k)}} \frac{|\cos(\pi y_k \theta^j)|}{|\cos(\pi z_0 \theta^j)|} \to 1.$$

Note that $y_k = z_0 + O\left(\theta^{-J_1^{(k)}}\right)$ by (3.10). By assumption (2.3), $|\cos(\pi y_k \theta^j)| \ge \eta$ for $j \le N_k$. Since $|y_k \theta^j - z_0 \theta^j| = O\left(\theta^{j - J_1^{(k)}}\right)$, we

have $|\cos(\pi y'\theta^j)| \geq \eta/2$ for y' between z_0 and y_k, for all $j \leq E_0^{(k)}$, for k sufficiently large. Then we can take logarithm of each term and use the mean value theorem to get

$$\begin{aligned}\left|\log\left|\frac{\cos(\pi y_k\theta^j)}{\cos(\pi z_0\theta^j)}\right|\right| &\leq |\tan(\pi y'\theta^j)|\cdot|y_k - z_0|\theta^j \\ &\leq \frac{2}{\eta}|y_k - z_0|\theta^j = O\big(\theta^{j-J_1^{(k)}}\big),\end{aligned}$$

where y' is between z_0 and y_k. Summing over $j \leq E_0^{(k)}$ and letting $k \to \infty$ yields the desired claim, since $J_1^{(k)} - E_0^{(k)} \to +\infty$.

Now we verify (3.13). Since $E_{i-1}^{(k)} - J_i^{(k)} \to -\infty$, $E_i^{(k)} - J_i^{(k)} \to +\infty$, and the denominator in $F_i(k)$ converges, it suffices to show that

$$\prod_{j=E_{i-1}^{(k)}+1}^{E_i^{(k)}} \frac{|\cos(\pi y_k\theta^j)}{\left|\cos\big(\pi z_i\theta^{j-J_i^{(k)}}\big)\right|} \to 1. \tag{3.15}$$

In view of (3.10), we can write

$$y_k\theta^j = z_0\theta^j + \sum_{\ell=1}^{i-1} z_\ell\theta^{j-J_\ell^{(k)}} + z_i\theta^{j-J_i^{(k)}} + O\big(\theta^{j-J_{i+1}^{(k)}}\big),$$

for $j = E_{i-1}^{(k)}+1, \ldots, E_i^{(k)}$. By (3.11), for $j = E_{i-1}^{(k)}+1, \ldots, E_i^{(k)}$ we have

$$y_k\theta^j = O\big(\rho^{j-J_{i-1}^{(k)}}\big) + z_i\theta^{j-J_i^{(k)}} + O\big(\theta^{j-J_{i+1}^{(k)}}\big) \bmod \mathbb{Z}.$$

For k sufficiently large, the denominators in (3.15) are bounded away from 0, as above, when we checked (3.12), and we obtain

$$\left|\log\left|\frac{\cos(\pi y_k\theta^j)}{\cos\big(\pi z_i\theta^{j-J_i^{(k)}}\big)}\right|\right| \leq \text{const}\cdot\big(\rho^{j-J_{i-1}^{(k)}} + \theta^{j-J_{i+1}^{(k)}}\big).$$

Summing over $j = E_{i-1}^{(k)}+1, \ldots, E_i^{(k)}$, we obtain that the logarithm of the product in (3.15) is bounded in modulus by const $\cdot\big(\rho^{E_{i-1}^{(k)}-J_{i-1}^{(k)}} + \theta^{E_i^{(k)}-J_{i+1}^{(k)}}\big)$ which tends to 0, as $k \to \infty$.

It remains to check (3.14). Since $E_M^{(k)} - N_k \to -\infty$ and the denominator in $F_{M+1}(k)$ converges, it is sufficient to show that

$$\prod_{j=E_M^{(k)}+1}^{N_k} \frac{|\cos(\pi y_k\theta^j)|}{|\cos(2\pi r A\theta^{j-N_k})|} \to 1. \tag{3.16}$$

We have for $j = E_M^{(k)} + 1, \dots, N_k$, from (3.10), in view of (3.11):

$$y_k\theta^j = 2rA\theta^{j-N_k} + O\big(\rho^{j-J_M^{(k)}}\big) + O\big(\theta^{j-N_k}\rho^{N_k-J_M^{(k)}}\big) \bmod \mathbb{Z}.$$

For k sufficiently large, the denominators in (3.16) are bounded away from 0, as above, when we checked (3.12), and we can write

$$\left|\log\left|\frac{\cos(\pi y_k\theta^j)}{\cos(2\pi rA\theta^{j-N_k})}\right|\right| \le \text{const}\cdot(\rho^{j-J_M^{(k)}} + \theta^{j-N_k}\rho^{N_k-J_M^{(k)}}).$$

Summing over $j = E_M^{(k)}+1, \dots, N_k$, we obtain that the logarithm of the product in (3.16) is bounded above in modulus by $\text{const}\cdot(\rho^{E_M^{(k)}-J_M^{(k)}} + \rho^{N_k-J_M^{(k)}})$ which tends to 0, as $k \to \infty$. This concludes the proof of Lemma 3.2. Inequality (2.1) and thus, Theorem 1.3 and Theorem 1.1 are proved as well. □

4. Proofs of other results

4.1. **Proof of Theorem 1.3 (ii).** Let J_θ denote the set of limit points of $\{\widehat{\mu}_\theta(t) : t > 0\}$ as $t \to \infty$.

Lemma 4.1. *J_θ is a non-empty interval.*

Proof. By the theorem of Erdős [4], J_θ contains a non-zero point. On the other hand, $0 \in J_\theta$, since $\widehat{\mu}_\theta(\theta^n/4) = 0$, $n \ge 1$. Let $a = \inf J_\theta$, $b = \sup J_\theta$. Then there are sequences $u_i, v_i \to \infty$ such that $\widehat{\mu}_\theta(u_i) \to a$ and $\widehat{\mu}_\theta(v_i) \to b$. Without loss of generality, $u_i < v_i < u_{i+1}$ for all i. Since $t \mapsto \widehat{\mu}_\theta(t)$ is continuous, for any $\varepsilon > 0$, for all i sufficiently large, any value between $a+\varepsilon$ and $b-\varepsilon$ is assumed by $\widehat{\mu}_\theta(t)$ at least once in (u_i, v_i). Thus, $J_\theta = [a, b]$. □

Our goal is to prove that for a.e. $r > 0$, J_θ is in fact the set of limit points for the sequence $\{\widehat{\mu}_\theta(rn) : n \in \mathbb{Z}\}$ as well.

Let $\{y_k\}_{k\ge1}$ be a sequence dense in J_θ. By the definition of J_θ, for any $k \ge 1$, there is a sequence $t_i^{(k)} \to +\infty$ as $i \to \infty$, such that $\lim_{i\to\infty}\widehat{\mu}_\theta\big(t_i^{(k)}\big) = y_k$. Recall the following well-known fact.

Proposition 4.2. [6, Chap. 1, Sec. 4, Cor 4.3] *For any unbounded sequence $\{x_i\}_{i\ge1}$, the set $\{\alpha x_i\}_{i\ge1}$ is dense modulo 1 for Lebesgue-a.e. α.*

Thus, for a.e. $r > 0$, the sequence $\big\{r^{-1}t_i^{(k)}\big\}_{i\ge1}$ is dense modulo 1 for all $k \ge 1$. Fix such an r. Then for any $k \ge 1$, there is a subsequence $t_{i_j}^{(k)}$ such that $r^{-1}t_{i_j}^{(k)} \to 0 \bmod 1$ (of course, i_j may depend on k). Thus,

for any k, there exist $n_j \in \mathbb{N}$ such that $rn_j - t_{i_j}^{(k)} \to 0$, as $j \to \infty$. Note that $\left|\frac{d}{dt}\widehat{\mu_\theta}(t)\right| \le C$, since μ_θ has compact support on $\mathbb{R}$, whence

$$\{\widehat{\mu_\theta}(a_n) : n \in \mathbb{N}\}' = \{\widehat{\mu_\theta}(b_n) : n \in \mathbb{N}\}' \tag{4.1}$$

for any $a_n, b_n \to \infty$, with $a_n - b_n \to 0$. Therefore,

$$\lim_{j\to\infty} \widehat{\mu_\theta}(rn_j) = \lim_{j\to\infty} \widehat{\mu_\theta}\big(t_{i_j}^{(k)}\big) = y_k.$$

It follows that for a.e. $r > 0$, the set of limit points of $\{\widehat{\mu_\theta}(rn) : n \in \mathbb{N}\}$ contains all y_k, which are dense in J_θ, and hence it contains all of J_θ. □

4.2. **Proof of Theorem 1.5.** In view of Theorem 1.3 (i) and the fact that $\mathcal{F}_{\mu_\theta,r}^{(n+1)} \subset \mathcal{F}_{\mu_\theta,r}^{(n)}$, $n \ge 1$, it suffices to show that the n'th derived set of limit points of $\mathcal{F}_{\mu_\theta,r}$ is *at least* countable for $r \in \mathbb{Q}(\theta)$.

Since $r \in \mathbb{Q}(\theta)$, there exists $p \in \mathbb{Z}[x]$ such that $\Lambda = rp(\theta) \in \mathbb{Z}[\theta]$. Let $q \in \mathbb{Z}[\theta]$ be an arbitrary number. We have

$$\lim_{k\to\infty} \widehat{\mu_\theta}(r\langle p(\theta)q\theta^k\rangle) = \lim_{k\to\infty} \widehat{\mu_\theta}(\Lambda q\theta^k) = \prod_{j=-\infty}^{\infty} \cos(2\pi\Lambda q\theta^j).$$

The first equality holds by (4.1), as $\|h\theta^n\| = O(\rho^n) = o(1)$ for any $h \in \mathbb{Z}[\theta]$. The second equality follows from (1.2). Put

$$\varphi_\Lambda(q) = \prod_{j=-\infty}^{\infty} \cos(2\pi\Lambda q\theta^j)$$

and

$$\Omega_\Lambda := \{\varphi_\Lambda(q) : q \in \mathbb{Z}[\theta]\}.$$

We just proved that $\Omega_\Lambda \subset \mathcal{F}'_{\mu_\theta,r}$, so $\Omega_\Lambda^{(n)} \subset \mathcal{F}_{\mu_\theta,r}^{(n+1)}$, $n \ge 1$. Our next goal is to show first that Ω'_Λ (and hence $\mathcal{F}''_{\mu_\theta,r}$) is infinite for every $\Lambda \in \mathbb{Z}[\theta]$. Let $a, b \in \mathbb{Z}[\theta]$ and put $q_n(a,b) := a + b\theta^n$. Then

$$\varphi_\Lambda(q_n(a,b)) = \prod_{j=-\infty}^{\left[\frac{n}{2}\right]} \cos(2\pi\Lambda(a+b\theta^n)\theta^{-j}) \cdot \prod_{\left[\frac{n}{2}\right]}^{\left[\frac{3n}{2}\right]} \cos(2\pi\Lambda(a+b\theta^n)\theta^{-j})$$
$$\cdot \prod_{j=\left[\frac{3n}{2}\right]+1}^{\infty} \cos(2\pi\Lambda(a+b\theta^n)\theta^{-j}),$$

and similarly to Lemma 3.2, it is easy to see that the first product tends to $\varphi_\Lambda(a)$, the second one tends to $\varphi_\Lambda(b)$ and finally, the last one tends to 1. Hence

$$\varphi_\Lambda(q_n(a,b)) \to \varphi_\Lambda(a)\varphi_\Lambda(b), \quad n \to +\infty. \tag{4.2}$$

Recall that $\theta \neq 2$ (when $\varphi_\Lambda(q) \equiv 0$ for any Λ and q), so there always exists q such that $0 < |\varphi_\Lambda(q)| < 1$, whence Ω_Λ is infinite. Furthermore, since $q_n(a,b) \in \mathbb{Z}[\theta]$, (4.2) implies that $\Omega_\Lambda^2 \subset \Omega'_\Lambda$, where $\Omega_\Lambda^2 = \{\omega_1\omega_2 : \omega_i \in \Omega_\Lambda\}$. Therefore, Ω'_Λ is infinite as well. Now, $\Omega_\Lambda^4 \subset (\Omega'_\Lambda)^2 \subset (\Omega_\Lambda^2)' \subset \Omega''_\Lambda$, whence Ω''_Λ is also infinite, etc. By induction, $\mathcal{F}^{(n)}_{\mu_\theta,r}$ is countable for each $n \geq 1$. $\square$

5. Concluding remarks

1. Our first remark concerns the proof of Theorem 1.3 (ii) (see the beginning of the previous section). In fact, what we use is the following

Lemma 5.1. *Assume $f \in C(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, and J is the set of the limit points of f as $t \to +\infty$. Then the derived set for $\{f(rn) : n \in \mathbb{N}\}$ as $n \to +\infty$ is equal to J for a.e. $r > 0$.*

The proof of this lemma is exactly the same as above for $f(t) := \widehat{\mu_\theta}(t)$. This claim is probably known but we did not find it in the literature.

2. Our second remark consists in a simple observation that the expression for the limit points of $\mathcal{F}_{\mu_\theta,r}$ in Lemma 3.2 (without the moduli) is in fact a general formula for $x \in \mathcal{F}'_{\mu_\theta,r}$. More precisely, let

$$\mathcal{P}_\theta = \{\xi : \|\xi\theta^n\| \to 0,\ n \to +\infty\}.$$

As is well known, $\mathbb{Z}[\theta] \subsetneq \mathcal{P}_\theta \subsetneq \mathbb{Q}(\theta)$ (see, e.g., [2]), and $\mathcal{P}_\theta$ is obviously a group under addition. Then our claim is that for any $M \in \mathbb{Z}_+$, $(z_i)_{i=0}^M \in \mathcal{P}_\theta^{M+1}$ and $A \in \mathbb{Z}$,

$$x = \prod_{i=0}^{M} \prod_{j=-\infty}^{\infty} \cos(\pi z_i \theta^j) \cdot \prod_{j=0}^{\infty} \cos(2\pi r A \theta^{-j}) \in \mathcal{F}'_{\mu_\theta,r}.$$

Indeed, put

$$n_k = \langle (2r)^{-1}(z_0\theta^{(M+1)k} + z_1\theta^{Mk} + \cdots + z_M\theta^k)\rangle + A.$$

The proof is practically the same as for Lemma 3.2, and we leave it to the reader.

3. Our next remark concerns translations of Bernoulli convolutions. Let $\gamma \in \mathbb{R}$; then shifting the origin by γ results in multiplying $\widehat{\mu_\theta}(rn)$ by $e^{2\pi i\gamma n}$.

Proposition 5.2. *The closure of the set $\{\widehat{\mu_\theta}(nr)e^{2\pi i\gamma n} : n \in \mathbb{Z}\}$ is countable for $r \in \mathbb{Q}(\theta)$ and $\gamma \in \mathbb{Q}(\theta)$, but uncountable for $r \in \mathbb{Q}(\theta)$ and a.e. $\gamma \in \mathbb{R}$.*

Proof. First suppose that $r \in \mathbb{Q}(\theta)$ and $\gamma \in \mathbb{Q}(\theta)$. It follows from the proof of the main theorem that if $|\widehat{\mu}_\theta(rn_k)| \not\to 0$, then the formula (3.10) gives a general expression for n_k, with k sufficiently large. Now it is enough to note that for any $\xi \in \mathbb{Q}(\theta)$, the sequence $\{\|\xi\theta^n\| : n \in \mathbb{N}\}$ has finitely many limit points[3] and hence the sequence $\{\|\gamma n_k\| : \ k \in \mathbb{N}\}$ has finitely many limit points.

On the other hand, given $r \in \mathbb{Q}(\theta)$, we can fix n_k so that $\widehat{\mu}_\theta(rn_k) \to a \neq 0$. Using Proposition 4.2 again, we see that for a.e. $\gamma \in \mathbb{R}$, the sequence $\{\gamma n_k : k \in \mathbb{N}\}$ is dense modulo 1, and therefore, the set of limit points of the sequence $\{\widehat{\mu}_\theta(rn_k)e^{2\pi i\gamma n_k}\}$ is the circle of radius $|a|$. □

4. Denote by $\mu_\theta(r,\gamma)$ the measure on the circle whose Fourier coefficients are $\widehat{\mu}_\theta(rn)e^{2\pi in\gamma}$ (that is, the translation of a scaled copy of μ_θ). We have shown that for "most" (r,γ) the spectrum $\mathrm{sp}(T_{\mu_\theta(r)}, L^2)$ contains a continuum (an interval or a circle). In these cases we cannot use Sarnak's result [8] to claim that the spectra are the same in all L^p, for $p \in (1,\infty)$. However, the remarks in [3] indicate that $\mu_\theta(r,0)$ is L^p-improving for any $r > 1$ and $\theta > 1$, and any translation of the measure, obviously, preserves the property. Thus, by [5, Th. 4.1] we see that the claim on coincidence of spectra in all L^p is valid for all $\mu_\theta(r,\gamma)$.

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