

Super-Geometric Quantization

Izu Vaisman

Vienna, Preprint ESI 122 (1994)

August 16, 1994

Supported by Federal Ministry of Science and Research, Austria
Available via WWW.ESI.AC.AT

Super-Geometric Quantization

Izu Vaisman

August 1994

ABSTRACT. Let K be the complex line bundle where the Kostant-Souriau geometric quantization operators are defined. We discuss possible prolongations of these operators to the linear superspace of the K -valued differential forms, such that the Poisson bracket is represented by the supercommutator of the corresponding operators. We also discuss the possibility to obtain such super-geometric quantizations by (anti)Hermitian operators on a Hilbert superspace. We apply our general considerations to Kähler manifolds and to cotangent bundles of Riemannian manifolds.

1 Recalling Geometric Quantization

In differential geometry, the problem of geometric quantization is a two stage problem which can be stated in the following terms (e.g., [10], [9]).

Stage 1 - Prequantization. Let M be a Poisson manifold with the Poisson bracket

$$(1.1) \quad \{f, g\} = P(df, dg) \quad (f, g \in C^\infty(M)).$$

Find linear representations of the Lie algebra (1.1) on the space $\Gamma(K)$ of cross sections of a complex line bundle K over M by differential operators of order one and symbol equal to the Hamiltonian vector field X_f^P .

*This work was finalized during the author's stay at the Erwin Schrödinger International Institute for Mathematical Physics in Vienna, and the author wants to express here all his gratitude to the Institute, and to Professor Peter Michor, in particular, for their support and hospitality.

1991 Mathematics Subject Classification 58 F 06.

Key words and phrases: Geometric Quantization, Linear Superspace, Supercommutator.

Stage 2 - Quantization. Restrict prequantization in such a way as to obtain irreducible anti-Hermitian¹ representations of a subalgebra of $C^\infty(M)$ with bracket (1.1) on a Hilbert space derived from $\Gamma(K)$.

In this paper, we define the problem of *super-geometric quantization* as the problem of prolonging the representations mentioned above to linear and Hilbert superspaces.

Now, let us be more precise. While more general prequantization representations may exist [5], [9], we consider only the fundamental *Kostant-Souriau representation*. The latter is given by the operators

$$(1.2) \quad \hat{f}\sigma = \nabla_{X_f}\sigma + 2\pi\sqrt{-1}f\sigma \quad (f \in C^\infty(M), \sigma \in \Gamma(K)),$$

where ∇ is a connection on K which preserves a Hermitian metric h of K .

The condition that (1.2) is a representation means

$$(1.3) \quad \{\widehat{f}, \widehat{g}\} = \hat{f} \circ \hat{g} - \hat{g} \circ \hat{f},$$

and this condition is equivalent to

$$(1.4) \quad \Omega(X_f, X_g) = -2\pi\sqrt{-1}\{f, g\},$$

where Ω is the curvature of ∇ . In particular, (1.4) shows that K , h and ∇ exist iff P defines an integral Poisson cohomology class (namely, the image of the integral first Chern class of K) [9], and, then, we say that (M, P) is a *quantizable Poisson manifold*. In the symplectic case, the integrality condition is just that the symplectic form represents an integral cohomology class [10].

Furthermore, let \mathcal{D} be the bundle of complex valued *halfdensities* of M (e.g., [6], [7]). Then (1.2) extends to $\Gamma(K \otimes \mathcal{D})$ by

$$(1.5) \quad \hat{f}(\sigma \otimes \rho) = (\hat{f}\sigma) \otimes \rho + \sigma \otimes L_{X_f}\rho \quad (\sigma \in \Gamma(K), \rho \in \gamma(\mathcal{D}))$$

where L denotes the Lie derivative, and Stokes' theorem shows that the operators (1.5) are anti-Hermitian on $\Gamma_c(K \otimes \mathcal{D})$ (c means "with compact support") endowed with the scalar product

$$(1.6) \quad \langle \sigma_1 \otimes \rho_1, \sigma_2 \otimes \rho_2 \rangle = \int_M h(\sigma_1, \sigma_2)\rho_1\bar{\rho}_2$$

¹The fact that we use anti-Hermitian operators here is just a technicality. If these operators are multiplied by a purely imaginary constant they become Hermitian operators.

(the bar means complex conjugation) i.e., we have

$$(1.7) \quad \langle \hat{f}\alpha, \beta \rangle + \langle \alpha, \hat{f}\beta \rangle = 0 \quad (\alpha, \beta \in \Gamma_c(K \otimes \mathcal{D})).$$

Of course, we may complete $\Gamma_c(K \otimes \mathcal{D})$ to a Hilbert space but, we still remain in the prequantization stage since we do not have irreducibility.

Now, the stage of quantization is based on the notion of a *polarization*, for which we adopt here a new definition that includes the classical definition as a particular case. Let \mathcal{F} denote the sheaf of Poisson algebras of germs of complex valued C^∞ functions of M with the bracket (1.1). Then, a *polarization* \mathcal{P} of (M, P) is a subsheaf \mathcal{P} of \mathcal{F} whose stalks are abelian subalgebras of the stalks of \mathcal{F} .

If \mathcal{P} is given, we may look at the linear space

$$(1.8) \quad \Gamma_0(K) = \{\sigma \in \Gamma(K) / \nabla_{X_\varphi}\sigma = 0, \forall \varphi \in \mathcal{P}\},$$

and we may apply the operators (1.2) to $\Gamma_0(K)$ if $\Gamma_0(K) \neq \{0\}$. It is easy to see that, $\forall f \in C^\infty(M)$ such that $\{\varphi, f\} \in \mathcal{P}$ whenever $\varphi \in \mathcal{P}$, $\hat{f}(\Gamma_0(K)) \subseteq \Gamma_0(K)$. The set $\mathcal{Q}(M, \mathcal{P})$ of such functions f is a Lie subalgebra of $(C^\infty(M), \{, \})$ which includes all the real global sections ψ of \mathcal{P} , and for these ψ one has $\hat{\psi}\sigma = 2\pi\sqrt{-1}\psi\sigma$, $\forall \sigma \in \Gamma_0(K)$, as needed for irreducibility [10].

Furthermore, if $\Gamma_0(K)$ has nonzero elements with compact support, it may be possible to adapt conveniently the scalar product (1.6), and obtain a Hilbert space where (1.7) holds $\forall f \in \mathcal{Q}(M, \mathcal{P})$. Otherwise, the idea is to project the whole configuration onto a lower dimensional quotient manifold N , if possible, and get a similar scalar product by integration over N [10], [7], [9].

The basic types of polarizations encountered in applications are as follows (e.g., [10]).

1) Let (M^{2n}, ω) ($d\omega = 0$) be a quantizable symplectic manifold, with the Poisson brackets defined by ω , and assume that M has a real Lagrangian foliation \mathcal{L} . Then, the sheaf \mathcal{P} of germs of functions which are constant along the leaves of \mathcal{L} is a polarization called a *real Lagrangian polarization*. An important particular case is that of a cotangent bundle $M = T^*N$, where $\omega = d\theta$, $\theta :=$ the Liouville 1-form of T^*N , and \mathcal{L} is the foliation by the fibers of T^*N . In this case, if $\sigma \in \Gamma_0(K)$, $\text{supp } \sigma$ is noncompact (it is a union of

fibers), and the scalar product will be defined by integration over N and not over T^*N .

2) Let (M^{2n}, ω) be a quantizable symplectic manifold which admits compatible Kähler metrics. Then, if g is such a metric, the sheaf \mathcal{P} of germs of holomorphic functions with respect to the corresponding complex structure is a polarization called a *Kähler polarization*. In this case, K is a holomorphic line bundle (e.g., [7]), and

$$\mathcal{Q}(M, \mathcal{P}) = \{f \in C^\infty(M) / X_f = X_f^{1,0} + X_f^{0,1}, X_f^{1,0} \text{ holomorphic}\},$$

where the upper indices indicate the complex type. Equivalently, if J is the tensor of the complex structure, then $L_{X_f} J = 0$. We say that X is an analytic vector field, and we distinguish in this paper between the terms analytic and holomorphic for vector fields i.e., X is *analytic* and its component $X^{1,0}$ is *holomorphic*. Furthermore, we may forget about halfdensities, and make $\Gamma_{0c}(K)$ into a Hilbert space by the scalar product

$$(1.9) \quad \langle \sigma_1, \sigma_2 \rangle = \int_M h(\sigma_1, \sigma_2) d(\text{vol } g) \quad (\sigma_1, \sigma_2 \in \Gamma_{0c}(K)),$$

and the property (1.7) follows again from Stokes' theorem.

2 Super-Geometric Prequantization

Now, we proceed to the discussion of super-geometric quantization. We start with a quantizable Poisson manifold (M, P) and a quantization complex line bundle K . Let us emphasize that we do not intend to discuss geometric quantization of supermanifolds, as in [3]. Neither do we consider any kind of supermanifolds [1]. But, we shall use the terminology of superalgebra (e.g., [4]).

With (M, P, K) , we can associate a natural complex linear superspace

$$(2.1) \quad \mathcal{S}(K) = \mathcal{S}^+(K) \oplus \mathcal{S}^-(K).$$

where

$$\mathcal{S}^+ = \bigoplus_{i \geq 0} \wedge^{2i} (M, K), \quad \mathcal{S}^- = \bigoplus_{i \geq 0} \wedge^{2i+1} (M, K),$$

and $\wedge^h(M, K)$ are the spaces of K -valued forms on M , and it is possible to extend the Kostant-Souriau prequantization (1.2), (1.5) to $\mathcal{S}(K)$.

We did this in [8] as follows. Since $\wedge^h(M, K) = \Gamma((\wedge^h T^*M) \otimes K)$, one has the well-known *covariant exterior differential*

$$(2.2) \quad D(\alpha \otimes \sigma) = (d\alpha) \otimes \sigma + (-1)^{\deg \alpha} \alpha \wedge \nabla \sigma,$$

and the *covariant Lie derivative*

$$(2.3) \quad L_X^\nabla(\alpha \otimes \sigma) = (L_X \alpha) \otimes \sigma + \alpha \otimes \nabla_X \sigma,$$

where $\alpha \in \wedge^h M$, $\sigma \in \Gamma(K)$, and X is a vector field on M . These operators have the same global expressions as d and L_X , except for the fact that the action of X on functions is replaced by the action of ∇_X on sections of K . Notice also the formula

$$(2.4) \quad L_X^\nabla = Di(X) + i(X)D$$

which follows from (2.2) and (2.3).

Now, if (1.2) is extended to $\mathcal{S}(K)$ by

$$(2.5) \quad \hat{f}A = L_{X_f}^\nabla A + 2\pi\sqrt{-1}fA \quad (A \in \mathcal{S}(K)),$$

it follows from (1.4) that the commutator condition (1.3) is still valid. Indeed [8], using (2.3) we get

$$(2.6) \quad L_X^\nabla L_Y^\nabla A - L_Y^\nabla L_X^\nabla A - L_{[X,Y]}^\nabla A = \Omega(X, Y)A,$$

where Ω is the curvature of ∇ , and then, (1.4) is obtained by a straightforward computation.

The operators \hat{f} preserve the degree of a form. Thus, if we want to give a role to the structure (2.1), it is natural to define a *super-geometric prequantization* of M on K as a prolongation of (2.5) of the form

$$(2.7) \quad \tilde{f}A = \hat{f}A + 2\pi\sqrt{-1}l(f)(A) \quad (A \in \mathcal{S}(K)),$$

where $l(f)$ is an odd endomorphism of $\mathcal{S}(K)$, such that the following commutation condition holds

$$(2.8) \quad \{\widetilde{f}, \widetilde{g}\} = {}^s[\tilde{f}, \tilde{g}].$$

In the right hand side of (2.8), one has the supercommutator [4] of the operators \tilde{f}, \tilde{g} , and we denoted it by the index s . Brackets without this index will denote usual commutators.

Proposition 2.1. *The operation $*$ defined by*

$$(2.9) \quad f * \theta = [\hat{f}, \theta] = \hat{f}\theta - \theta\hat{f},$$

($f \in C^\infty(M)$, $\theta \in \text{End } \mathcal{S}(K)$) is a representation of the Lie algebra $(C^\infty(M), \{, \})$ on $\text{End } \mathcal{S}(K)$ which leaves $\text{End}_- \mathcal{S}(K)$ invariant, and (2.7) is a super-geometric prequantization iff l is a 1-cocycle with values in $\text{End}_- \mathcal{S}(K)$, and with respect to the representation (2.9), such that

$$(2.10) \quad l^2(f) = 0, \quad \forall f \in C^\infty(M).$$

Proof. The results are rather straightforward since, in view of (1.3), (2.8) is equivalent to

$$(2.11) \quad l(\{f, g\}) = [\hat{f}, l(g)] + [l(f), \hat{g}], \quad l(f)l(g) + l(g)l(f) = 0,$$

$\forall f, g \in C^\infty(M)$. Q.e.d.

Corollary 2.2. *$\forall c \in \text{End}_- \mathcal{S}(K)$ such that $[\hat{f}, c]^2 = 0$, $\forall f \in C^\infty(M)$, the operators*

$$(2.12) \quad \tilde{f}_c(A) = L_X^\nabla A + 2\pi\sqrt{-1}fA + 2\pi\sqrt{-1}[\hat{f}, c](A)$$

($A \in \mathcal{S}(K)$) define a super-geometric prequantization.

Proof. $[\hat{f}, c]$ is the coboundary of c in the Lie algebra cohomology mentioned in Proposition 2.1. Q.e.d.

We note some important particular cases in

Proposition 2.3. *Let θ be a complex valued 1-form, and V be a complex vector field on the Poisson manifold (M, P) . Then, (2.7) is a super-geometric prequantization for each of the following choices of l :*

$$(2.13) \quad l_1(f) = e(L_{X_f}\theta), \quad l_2(f) = i([X_f, V]),$$

$$l_3(f) = l_1(f) + l_2(f).$$

Proof. In (2.13), e means "exterior product by", and i means "interior product by". l_1 is obtained by using Corollary (2.2) for $c = e(\theta)$, and l_2 is obtained for $c = i(V)$. Q.e.d.

Remark 2.4. If we take $c = D$, then, using (2.4) and the well known fact that $D^2 = e(\Omega)$, we get

$$l(f) = e(i(X_f)\Omega - 2\pi\sqrt{-1}df),$$

which is 0 in the symplectic case because of (1.4).

Now, as in Section 1, we can relate super-geometric prequantization with a scalar product. Namely, we consider again the complex line bundle \mathcal{D} of halfdensities over M , and use the bundle $K \otimes \mathcal{D}$ instead of K . Then, instead of (2.1), we have

$$(2.14) \quad \tilde{\mathcal{S}}(K) := \mathcal{S}(K \otimes \mathcal{D}) := \tilde{\mathcal{S}}^+(K) \oplus \tilde{\mathcal{S}}^-(K),$$

which consists of forms with values in $K \otimes \mathcal{D}$ organized as those in (2.1).

Furthermore, we put on M a Riemannian metric g , and define a scalar product of $\wedge_c^p(M, K \otimes \mathcal{D})$ (i.e., forms with a compact support) by

$$(2.15) \quad \langle \alpha_1 \otimes \sigma_1 \otimes \rho_1, \alpha_2 \otimes \sigma_2 \otimes \rho_2 \rangle = \int_M g(\alpha_1, \alpha_2)h(\sigma_1, \sigma_2)\rho_1\bar{\rho}_2,$$

where $\alpha_i \in \wedge^p(M)$, $\sigma_i \in \Gamma(K)$, $\rho_i \in \Gamma(\mathcal{D})$ ($i = 1, 2$). Then, we get

Proposition 2.5. Assume that (2.7) is a super-geometric prequantization where the odd cocycle l is Hermitian with respect to gh . Then, the extension of (2.7) defined by

$$(2.16) \quad \tilde{f}(A \otimes \rho) = (\tilde{f}A) \otimes \rho + A \otimes L_{X_f}\rho$$

satisfies the commutator property (2.8), and, if X_f is a Killing vector field for g , \tilde{f} is anti-Hermitian with respect to (2.15).

Proof. That \tilde{f} of (2.16) also satisfies (2.8) follows by a straightforward calculation. (Notice that $l(f)$ extends to $\tilde{\mathcal{S}}(K)$ by $l(f)(A \otimes \rho) = (l(f)A) \otimes \rho$.) Furthermore, by the metric gh we mean

$$gh(\alpha_1 \otimes \sigma_1, \alpha_2 \otimes \sigma_2) = g(\alpha_1, \alpha_2)h(\sigma_1, \sigma_2),$$

and $l(f)$ are supposed to be gh -Hermitian. The anti-Hermitian character (1.7) of the present situation follows by using Stokes' theorem under the form (e.g., [6])

$$\int_M L_{X_f}(g(\alpha_1, \alpha_2)h(\sigma_1, \sigma_2)\rho_1\bar{\rho}_2) = 0.$$

Q.e.d.

3 Super-Geometric Quantization

Now, we combine super-geometric prequantization with a polarization and this process is *super-geometric quantization*.

Let (M, P) be a Poisson manifold endowed with the prequantization (2.7), (2.16), and the scalar product (2.15), and let \mathcal{P} be a polarization of M . Then, we shall define the linear superspace

$$(3.1) \quad \mathcal{S}_0(K) = \{A \in \mathcal{S}(K) / L_{X_\varphi}^\nabla A = 0, i(X_\varphi)A = 0, \forall \varphi \in \mathcal{P}\}.$$

Using (2.5), (2.6) and (1.4), we see easily that $\forall A \in \mathcal{S}_0(K), \forall f \in \mathcal{Q}(M, \mathcal{P})$, one has $\hat{f}A \in \mathcal{S}_0(K)$. We recall that (Section 1)

$$\mathcal{Q}(M, \mathcal{P}) = \{f \in C^\infty(M) / \{\varphi, f\} \in \mathcal{P}, \forall \varphi \in \mathcal{P}\}.$$

Furthermore, in order to deal with the odd part of (2.7), we restrict ourselves to $\mathcal{Q}'(M, \mathcal{P}) \subseteq \mathcal{Q}(M, \mathcal{P})$, where we define that $f \in \mathcal{Q}'(M, \mathcal{P})$ if it satisfies the following supplementary conditions

$$(3.2) \quad [L_{X_\varphi}^\nabla, l(f)] = 0, \quad {}^s[i(X_\varphi), l(f)] = 0, \quad \forall \varphi \in \mathcal{P}.$$

Then, we get

Proposition 3.1. $\forall f \in \mathcal{Q}'(M, \mathcal{P})$ and $\forall A \in \mathcal{S}_0(K)$, we have $\hat{f}A \in \mathcal{S}_0(K)$, for \hat{f} defined by (2.7). In particular, if the 1-form θ and the vector field V of M are such that $L_{X_\varphi}\theta = i(X_\varphi)\theta = 0, [X_\varphi, V] = 0, \forall \varphi \in \mathcal{P}$, the prequantizations of Proposition 2.3 induce quantization formulas on $\mathcal{S}_0(K), \forall f \in \mathcal{Q}(M, \mathcal{P})$.

Proof. The first assertion follows straightforwardly from the definitions. For the second assertion, we check that (3.2) holds for the cocycles l_1 and l_2 of (2.13), and $\forall B \in \mathcal{S}(K)$:

$$\begin{aligned} L_{X_\varphi}^\nabla((L_{X_f}\theta) \wedge B) - (L_{X_f}\theta) \wedge L_{X_\varphi}^\nabla B &= (L_{X_\varphi}L_{X_f}\theta) \wedge B = \\ &= (L_{X_f}L_{X_\varphi}\theta + L_{X_{\{\varphi, f\}}}\theta) \wedge B = 0, \\ i(X_\varphi)((L_{X_f}\theta) \wedge B) + (L_{X_f}\theta) \wedge (i(X_\varphi)B) &= (i(X_\varphi)L_{X_f}\theta)B = \\ &= (L_{X_f}i(X_\varphi)\theta + i(X_{\{\varphi, f\}})\theta)B = 0, \\ L_{X_\varphi}^\nabla i([X_f, V])B - i([X_f, V])L_{X_\varphi}^\nabla B &= i([X_\varphi, [X_f, V]])B = \end{aligned}$$

$$\begin{aligned}
&= i([X_{\{\varphi, f\}}, V])B + i([X_f, [X_\varphi, V]])B = 0, \\
&i(X_\varphi)i([X_f, V])B + i([X_f, V])i(X_\varphi)B = 0.
\end{aligned}$$

Q.e.d.

Furthermore, if we want a good scalar product, we may try to adapt conveniently formula (2.15), but, since we know from Proposition 2.5 that we shall need to ask X_f to be a Killing vector field for g , it is simpler to look at the subspace $\mathcal{S}_{0c}(K)$ of the elements of $\mathcal{S}_0(K)$ which have a compact support, and put

$$(3.3) \quad \langle \alpha_1 \otimes \sigma_1, \alpha_2 \otimes \sigma_2 \rangle = \int_M g(\alpha_1, \alpha_2)h(\sigma_1, \sigma_2)d(\text{vol } g)$$

(while, of course, M is assumed to be oriented). This scalar product vanishes on forms of different degrees. Hence, it makes $\mathcal{S}_{0c}(K)$ into a pre-Hilbert superspace, which, afterwards, will be completed to a Hilbert superspace. Then, just as for Proposition 2.5, we deduce

Proposition 3.2. *Assume that the cocycle l is Hermitian with respect to the metric gh , and put*

$$(3.4) \quad \mathcal{Q}''(M, \mathcal{P}) = \{f \in \mathcal{Q}'(M, \mathcal{P}) \mid L_{X_f}g = 0\}.$$

Then, the operator \tilde{f} of (2.7), associated with any $f \in \mathcal{Q}''(M, \mathcal{P})$ is anti-Hermitian with respect to the metric (3.3).

Proof. Make explicit the Lie derivative in the Stokes' formula

$$\int_M L_{X_f}(g(\alpha_1, \alpha_2)h(\sigma_1, \sigma_2)d(\text{vol } g)) = 0.$$

Q.e.d.

Corollary 3.3. *Assume that, $\forall \varphi \in \mathcal{P}$, $L_{X_\varphi}g = 0$. Assume that there exists a 1-form θ on M such that $\forall \varphi \in \mathcal{P}$ one has $L_{X_\varphi}\theta = 0$, $i(X_\varphi)\theta = 0$, and define $V = \sharp_g\theta$. Then, the cocycle l_3 of (2.13) is g -selfadjoint, and $\forall f \in \mathcal{Q}(M, \mathcal{P})$ such that X_f is a Killing vector field for g the superquantization \tilde{f} of (2.7) with $l = l_3$ is defined on $\mathcal{S}_0(K)$, and it is anti-Hermitian with respect to (3.3).*

Proof. By the definition of V , we have $g(V, Z) = \theta(Z)$ for any vector field Z of M , and the hypotheses of Proposition 3.1 are satisfied. Furthermore, we also see that $\sharp_g(L_{X_f}\theta) = L_{X_f}V = [X_f, V]$. Hence, the g -adjoint of $e(L_{X_f}\theta)$ is $i([X_f, V])$, and the result follows. Q.e.d.

4 Kähler and Lagrangian Polarizations

Now, we shall apply the general Propositions of Section 3 to the two basic examples mentioned in Section 1 i.e., where M is a symplectic manifold and \mathcal{P} is either a Kähler or a real Lagrangian polarization of M .

In the case of a Kähler polarization we get

Proposition 4.1. *Let (M, ω) be a quantizable symplectic manifold, and \mathcal{P} a Kähler polarization of M , with the corresponding complex structure J and metric g . Then K is a holomorphic line bundle, $\mathcal{S}_0(K)$ is the linear superspace of the K -valued holomorphic forms of (M, J) , and, $\forall f \in \mathcal{Q}(M, \mathcal{P})$, the Hamiltonian vector field X_f is Killing. Furthermore, if θ is a holomorphic 1-form on M , (2.7) with $l(f) = e(L_{X_f}\theta)$ is a super-geometric quantization on $\mathcal{S}_0(K)$, $\forall f \in \mathcal{Q}(M, \mathcal{P})$. Moreover, if*

$$\mathcal{Q}_0(M, \mathcal{P}) := \{f \in \mathcal{Q}(M, \mathcal{P}) \mid \sharp_g L_{X_f} \bar{\theta} \text{ is holomorphic}\},$$

then (2.7) with

$$(4.1) \quad l(f) = e(L_{X_f}\theta) + i([X_f, \sharp_g \bar{\theta}])$$

is an anti-Hermitian super-geometric quantization of $\mathcal{Q}_0(M, \mathcal{P})$ on $\mathcal{S}_0(K)$ seen as a Hilbert superspace with the scalar product (3.3).

Proof. We already recalled in Section 1 that K is holomorphic and that, $\forall f \in \mathcal{Q}(M, \mathcal{P})$, X_f is analytic ($L_{X_f}J = 0$). Since, of course, $L_{X_f}\omega = 0$, we also have $L_{X_f}g = 0$. The assertion about the super-geometric quantization with the odd cocycle $e(L_{X_f}\theta)$ follows from Proposition 3.1.

Finally, we claim that, $\forall f \in \mathcal{Q}_0(M, \mathcal{P})$, the conditions (3.2) are also satisfied for the cocycle $l(f) = i([X_f, \sharp_g \bar{\theta}])$. Indeed, the second condition (3.2) is well known, and, as shown during the proof of Proposition 3.1, the first condition (3.2) is satisfied if

$$(4.2) \quad [X_\varphi, [X_f, \sharp_g \bar{\theta}]] = 0, \quad \forall \varphi \in \mathcal{P}.$$

By taking φ equal to the local complex coordinates z^i of (M, J) , we see that the antiholomorphic tangent bundle $T_{0,1}M$ of (M, J) has local bases of the form $\{X_{\varphi_i}\}$, for some $\varphi_i \in \mathcal{P}$. Hence, (4.2) means that $[X_f, \sharp_g \bar{\theta}]$ preserves $T_{0,1}M$. On the other hand, since X_f is Killing, we have

$$(4.3) \quad [X_f, \sharp_g \bar{\theta}] = \sharp_g(L_{X_f}\bar{\theta}),$$

and this is a vector field of the complex type $(1, 0)$. Accordingly, (4.2) holds iff $[X_f, \sharp_g \bar{\theta}]$ is a holomorphic vector field, as claimed.

Now, if we use again Proposition 3.1, and the argument of Corollary 3.3, namely, that the adjoint of $e(\theta)$ is $i(\bar{\theta})$, we obtain the last assertion of Proposition 4.1. Q.e.d.

We shall also add a few more results about the space $\mathcal{Q}(M, \mathcal{P})$ of a Kähler polarization.

Proposition 4.2. *i) For a Kähler polarization \mathcal{P} , $f \in \mathcal{Q}(M, \mathcal{P})$ iff*

$$(4.4) \quad \nabla_{\bar{i}} \left(\frac{\partial f}{\partial \bar{z}^j} \right) = 0,$$

where (z^i) are complex coordinates and ∇ is the Riemannian connection of the Kähler manifold (M, g, J) . *ii) If the Kähler manifold M is compact, $f \in \mathcal{Q}(M, \mathcal{P})$ iff*

$$(4.5) \quad \Delta df - 2\sharp_r^{-1} \sharp_g df = 0,$$

where Δ is the Laplace operator and r is the Ricci tensor of g .

Proof. i) The condition (4.4) follows immediately from the local coordinate expression of a Hamiltonian vector field X_f .

ii) In (4.5) the definition of \sharp_r^{-1} is similar to that of \sharp_g^{-1} , but \sharp_r may not exist. It is well known that, if M is compact, X_f is analytic iff

$$\Delta(\sharp_g^{-1} X_f) - 2\sharp_r^{-1} X_f = 0$$

(e.g., see Proposition 2.140 in [2]). But, it follows easily that $\sharp_g^{-1} X_f = -df \circ J$, and, using the known properties of Δ in the Kähler case, the previous relation becomes

$$(\Delta df) \circ J + 2\sharp_r^{-1} J \sharp_g df = 0.$$

If this equality is composed by J , and if we remember that r is compatible with J , (4.5) follows. Q.e.d.

Remark 4.3. *1) If M is a compact connected Kähler-Einstein manifold, (4.5) becomes*

$$(4.6) \quad \Delta f - 2\kappa f = \text{const.},$$

where κ is the (constant) scalar curvature of g . 2) If the Ricci curvature of the compact Kähler manifold M is negative definite, $\mathcal{Q}(M, \mathcal{P}) = \mathbf{R}$. Indeed, in this case M has no non zero analytic vector fields (e.g., Proposition 2.138 in [2]).

In order to exemplify the case of a Lagrangian polarization, we consider the basic situation of a cotangent bundle $M = T^*N$ with the symplectic form

$$(4.7) \quad \omega = -d\theta + p^*F,$$

where θ is the Liouville form, $p : T^*N \rightarrow N$ is the natural projection, and F is an exact 2-form $F = d\lambda$ of N (the *electromagnetic term*). Thus, if q^i are local coordinates on N , and p_i are covector coordinates, we have (with the Einstein summation convention)

$$(4.8) \quad \theta = p_i dq^i, \quad \lambda = \lambda_i(q) dq^i.$$

Then, K may be taken trivial, the K -valued forms are just complex valued forms, the connection ∇ can be defined by the global, flat connection form $2\pi\sqrt{-1}(\theta - \lambda)$, and the prequantization formula (2.5) becomes

$$(4.9) \quad \hat{f}A = L_{X_f}A + 2\pi\sqrt{-1}(\theta(X_f) - \lambda(X_f) + f)A \quad (A \in \wedge M \otimes \mathbf{C}).$$

Furthermore, the polarization \mathcal{P} is defined as the sheaf of germs of lifts to T^*N of functions on N (i.e., functions of the (q^i) alone), and

$$(4.10) \quad \mathcal{Q}(M, \mathcal{P}) = \{f \in C^\infty(M) / f = \mu(Y) + \varphi\},$$

where Y is a tangent vector field of N , $\mu(Y)$ is its *momentum* $\mu(Y) = p_i Y^i$ ($Y = Y^i(\partial/\partial q^i)$), and $\varphi \in \mathcal{P}$ (e.g., [10]).

We shall use the notions of complete and vertical lift as defined, for instance, in [11]. Then, it is easy to obtain

$$(4.11) \quad X_\varphi = \text{vertical lift of } d\varphi = \frac{\partial\varphi}{\partial q^i} \frac{\partial}{\partial p_i}, \quad \forall \varphi \in \mathcal{P},$$

and, for a vector field Y of N

$$(4.12) \quad X_{\mu(Y)} = -\text{complete lift of } Y - \text{vertical lift of } i(Y)F =$$

$$= -Y^i \frac{\partial}{\partial q^i} + \text{vertical part}$$

(vertical means tangent to the fibers of T^*M).

From (4.11) we see easily that $\mathcal{S}_0(K)$ can be identified with the linear superspace of the complex valued differential forms of the base manifold N .

An odd cocycle l is provided by the Liouville form θ and, as we know, it is $l(f) = e(L_{X_f}\theta)$ ($f \in C^\infty(T^*N)$). In particular, using (4.11) and (4.12), we get for $f = \mu(Y) + \varphi \in \mathcal{Q}(M, \mathcal{P})$

$$(4.13) \quad l(\mu(Y) + \varphi) = e(-i(Y)F + d\varphi),$$

which is a 1-form on N . Hence, this cocycle l defines a super-geometric quantization of $\mathcal{Q}(M, \mathcal{P})$ on $\mathcal{S}_0(K)$. Moreover, we can prove

Proposition 4.4. *With the notation above, and with respect to a fixed Riemannian metric g on the base manifold N , the formula*

$$(4.14) \quad \hat{f}A = -L_Y A + 2\pi\sqrt{-1}(\varphi + \lambda(Y))A + 2\pi\sqrt{-1}(-i(Y)F + d\varphi) \wedge A + \\ + 2\pi\sqrt{-1}i(-i(Y)F + d\varphi)A \quad (A \in \wedge^* N \otimes \mathbf{C})$$

defines a super-geometric quantization of the observables $f = \mu(Y) + \varphi \in \mathcal{Q}(M, \mathcal{P})$, such that Y is a g -Killing vector field of N , on the linear superspace $\wedge_c^* N \otimes \mathbf{C}$ (c means "with compact support") with the odd-even grading. This quantization is by anti-Hermitian operators with respect to the scalar product defined by g on the forms of N .

Proof. In the right hand side of (4.14), the first two terms are $\hat{f}A$ (as one can see by using (4.9), (4.11), (4.12)), and the third term is the odd cocycle (4.13). Moreover, the operator of the fourth term is the g -adjoint of the operator of the third term. Therefore, we must only check that this fourth term behaves like a superquantization 1-cocycle i.e., it satisfies the conditions (2.11), $\forall f = \mu(Y) + \varphi, g = \mu(Z) + \psi$, where Y, Z are g -Killing vector fields of N , $\varphi, \psi \in C^\infty(N)$. The second condition (2.11) is obvious, and, for the first, we compute the corresponding expressions for $l(f) = e(-i(Y)F + d\varphi)$, and in the following cases.

a) $f = \varphi, g = \psi$. Then, with (4.11), $\{f, g\} = 0$, and $l(\{f, g\}) = 0$. Furthermore, $[\hat{\varphi}, l(\psi)] = 0, [l(\varphi), \hat{\psi}] = 0$.

b) $f = \varphi$, $g = \mu(Z)$. Then, $\{f, g\} = X_f g = Zf$, and $l(\{f, g\}) = i(dZ\varphi)$. Furthermore, we obtain

$$[\hat{\varphi}, l(g)] + [l(\varphi), \hat{g}] = i([Z, \sharp_g d\varphi]) = i(\sharp_g dZ\varphi) = i(dZ\varphi).$$

We used that $\forall \alpha \in \wedge^1(M)$, $i(\alpha) := i(\sharp_g \alpha)$, and that Z is Killing i.e., $L_Z \sharp_g = 0$.

c) $f = \mu(Y)$, $g = \mu(Z)$. Then

$$(4.15) \quad \{f, g\} = X_{\mu(Y)}\mu(Z) \stackrel{(4.12)}{=} -\mu([Y, Z]) - F(Y, Z),$$

and, since $dF = 0$,

$$(4.16) \quad \begin{aligned} l(\{f, g\}) &= i(i([Y, Z])F - d(F(Y, Z))) = \\ &= -i(i(Z)L_Y F - L_Y i(Z)F + d(F(Y, Z))) = \\ &= -i(i(Z)di(Y)F - i(Y)di(Z)F + 2d(F(Y, Z))). \end{aligned}$$

Furthermore, using again the general relations that exist among $L_X, i(X), d$ for any vector field X , we get

$$(4.17) \quad \begin{aligned} [\hat{f}, l(g)] + [l(f), \hat{g}] &= i([Z, \sharp_g i(Y)F] - \\ &- [Y, \sharp_g i(Z)F]) = i(\sharp_g L_Z i(Y)F - \sharp_g L_Y i(Z)F) \end{aligned}$$

(because Y, Z are Killing vector fields), and the final result will be the same as in (4.16). Q.e.d.

Remark 4.5. *If λ is used instead of θ , the same results as in Proposition 4.4, can be proven in the same way for*

$$(4.18) \quad \begin{aligned} \tilde{f}A &= -L_Y A + 2\pi\sqrt{-1}(\varphi + \lambda(Y))A - \\ &- 2\pi\sqrt{-1}(L_Y \lambda) \wedge A - 2\pi\sqrt{-1}i([Y, \sharp_g \lambda])A. \end{aligned}$$

In (4.18), the notation and the hypotheses are the same as in Proposition 4.4.

References

- [1] C. Bartocci, U. Bruzzo and D. Hernández-Ruipérez, The geometry of supermanifolds. Math. and Its Appl. 71, Kluwer, Dordrecht, 1991.
- [2] A. L. Besse, Einstein manifolds. Ergebnisse der Math. 10, Springer-Verlag, Berlin, 1987.
- [3] B. Kostant, Graded manifolds, graded Lie theory, and prequantization. In: Diff. Geom. Methods in Math. Physics (K. Bleuler and A. Reetz, eds.), Lecture Notes in Math. 570, Springer-Verlag, Berlin, 1977, pp.177-306.
- [4] Yu. I. Manin, Gauge field theory and complex geometry. Grundlehren Math. Wiss. 289, Springer-Verlag, Berlin, 1988.
- [5] R. W. Urwin, The prequantization representations of the Poisson-Lie algebra. Advances in Math., 50 (1983), 207-258.
- [6] I. Vaisman, Basic ideas of geometric quantization. Rend. Sem. Mat. Torino, 37 (1979), 31-41.
- [7] I. Vaisman, A coordinatewise formulation of geometric quantization. Ann. Inst. H. Poincaré, série A (Physique théorique), 31 (1979), 5-24.
- [8] I. Vaisman, Geometric quantization on spaces of differential forms. Rend. Sem. Mat. Torino, 39 (1981), 139-152.
- [9] I. Vaisman, On the geometric quantization of the Poisson manifolds. J. Math. Physics, 32 (1991), 3339- 3345.
- [10] N. Woodhouse, Geometric quantization. Clarendon Press, Oxford, 1980.
- [11] K. Yano and S. Ishihara, Tangent and cotangent bundles. M. Dekker, Inc., New York, 1973.

Department of Mathematics
and Computer Science
University of Haifa, Israel
E-mail: I.VAISMAN@UVM.HAIFA.AC.IL