

**Deligne Cohomology for Orbifolds,  
Discrete Torsion and B-Fields**

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Vienna, Preprint ESI 1229 (2002)

October 31, 2002

Supported by the Austrian Federal Ministry of Education, Science and Culture  
Available via <http://www.esi.ac.at>

# DELIGNE COHOMOLOGY FOR ORBIFOLDS, DISCRETE TORSION AND B-FIELDS.

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In this paper we introduce the concept of Deligne cohomology of an orbifold. We prove that the third Deligne cohomology group  $H^3(G, \mathbb{Z}(3)_{\mathbb{D}}^{\infty})$  of a smooth étale groupoid classify gerbes with connection over the groupoid. We argue that the  $B$ -field and the discrete torsion in type II superstring theories are special kinds of gerbes with connection, and finally, for each one of them, using Deligne cohomology we construct a flat line bundle over the inertia groupoid, namely a Ruan inner local system<sup>11</sup> in the case of an orbifold.

## 1 Introduction

D-brane fields in type II superstring theory can be interpreted as a global object in  $K$ -theory, as has been argued by Witten<sup>14</sup>. In his work he studies the proper quantization condition for the D-brane fields. He shows that the fields have a charge in integral  $K$ -theory in much the same way in which the electromagnetic charge of a  $U(1)$ -field is the first Chern class, namely an element in integral second cohomology.

When we incorporate the Neveu-Schwarz  $B$ -field with 3-form field strength  $H$  and characteristic class  $[H] \in H^3(M, \mathbb{Z})$ , the corresponding statement is that the recipient of the charge is a twisted  $K$ -theory group  $K_{[H]}(M)$ . Witten has posed the question of generalizing this framework to the orbifold case. We have constructed the appropriate recipient of the charge for types IIA and IIB superstring theories in a previous paper<sup>6</sup>. In that paper we argue that the corresponding version of  $K$ -theory is a twisting via a *gerbe* of the  $K$ -theory of the *stack* associated to the orbifold. We have done this for a general orbifold not only for orbifolds known as *global quotients*, that is to say orbifolds of the form  $X = M/G$  with  $G$  a finite group. In order to do so we have used extensively the notion of *étale groupoid*. This will be explained in more detail below. The appearance of gerbes is natural even in the smooth case.

Sharpe has argued<sup>13</sup> that orbifold string theories can detect the stacky

nature of the orbifold. This is consistent with the fact that the orbifold Euler characteristic<sup>3</sup> is a property of the stack associated to the orbifold (actually of its inertia stack<sup>7</sup>). This point of view also fits nicely with our construction of the orbifold  $K$ -theory twisted by a gerbe.

In this paper we argue that both discrete torsion and the  $B$ -field can be interpreted as classes in the Deligne cohomology group  $H^3(X, \mathbb{Z}(3)_D^\infty)$ . For this we first introduce Deligne cohomology for orbifolds (in fact we do this for foliations). Then we show that the Deligne cohomology group  $H^3(X, \mathbb{Z}(3)_D^\infty)$  classifies gerbes with connection over the orbifold (we have introduced the concept of gerbe with connection over an orbifold in a previous paper<sup>6</sup>).

The basic idea of interpreting the  $B$ -field and the discrete torsion as Deligne cohomology classes is related to the  $K$ -theory twisting in the following way. We have showed that every twisting can be interpreted as a gerbe over the orbifold, and that gerbes are classified by the third integral cohomology of the classifying space  $H^3(BX, \mathbb{Z})$ . In particular we showed how to produce such a gerbe for the case of discrete torsion. In fact in the orbifold case the cohomology class  $[H]$  corresponding to the one used by Witten in his twisting is now a class  $[H] \in H^3(BX, \mathbb{Z})$ . But if we want to actually consider the forms themselves up to isomorphisms, then both the  $B$ -field and the discrete torsion will then be classes in the enhanced cohomology group  $H^3(X, \mathbb{Z}(3)_D^\infty)$ . The mathematical expression of the exact relation between the actual  $B$ -field  $B \in H^3(X, \mathbb{Z}(3)_D^\infty)$  and its class  $[H]$  is given by a forgetting map  $H^3(X, \mathbb{Z}(3)_D^\infty) \rightarrow H^3(BX, \mathbb{Z})$ .

Dixon, Harvey, Vafa and Witten<sup>3</sup> have shown that from the point of view of string theory the right notion of the loop space for the global quotient orbifold  $X = M/G$  should include the twisted sectors, namely  $LX = \coprod_{(g)} L_g X / C(g)$  where the disjoint union runs over all the conjugacy classes  $(g)$  of elements  $g \in G$ , and  $C(g)$  is the centralizer of  $g$  in  $G$ . We have generalized this argument to the case of an arbitrary orbifold<sup>7</sup> (even one that is not a global quotient) by means of the concept of the *loop groupoid*  $LX$ . Even in the case of global quotients the loop groupoid has the virtue that it is invariant of the representation chosen for the orbifold. For example if  $X = M_1/G_1 = M_2/G_2$  as orbifolds then the loop groupoid will be the same for both representations. We have also shown that the loop groupoid  $LX$  has a circle action, and that the  $S^1$  invariant loops in  $LX$  correspond exactly with the twisted sectors of the orbifold that we will write as  $\wedge X$  (we call these the *inertia groupoid*<sup>7</sup> of the orbifold  $X$ ).

In the case of a global quotient  $X = M/G$  and a discrete torsion  $\alpha \in H^3(G, \mathbb{Z})$  we constructed a gerbe  $L_\alpha$  over  $X$  (corresponding to the cohomology map  $H^3(G, \mathbb{Z}) \rightarrow H^3(BX, \mathbb{Z})$ ), and then from here by using the holonomy of

this gerbe we produced a line bundle over  $\wedge X$  (corresponding to the restriction of the holonomy line bundle over  $LX$  of the gerbe  $L_\alpha$  to  $\wedge X$ ).

In this paper we prove that such line bundle admits a natural *flat* connection. In fact we prove something more interesting and general. For *any* gerbe with a connection over  $X$  we will construct a *flat* line bundle over the twisted sectors  $\wedge X$ . We will do this using Deligne cohomology in the following way. We will prove that  $H^3(X, \mathbb{Z}(3)_D^\infty)$  classifies *gerbes with connection* over  $X$ , that  $H^2(Y, \mathbb{Z}(2)_D^\infty)$  classifies *line bundles with connection* over  $Y$ , and finally that there exists a natural holonomy map

$$H^3(X, \mathbb{Z}(3)_D^\infty) \longrightarrow H^2(\wedge X, \mathbb{Z}(2)_D^\infty) \quad (1)$$

whose image lands in the family of *flat* line bundles over  $\wedge X$ .

To finish this introduction let us mention that the mathematical results of this paper can be approached from a more general point of view that includes all the Deligne cohomology groups (not only the second and the third) and their geometric interpretations<sup>8</sup>.

## 2 Preliminaries

In this section we will review the basic concepts of groupoids, sheaves over groupoids and the cohomologies associated to them. For a more detailed description we recommend to see Haefliger<sup>4</sup>, Crainic and Moerdijk<sup>2</sup> and Lupercio and Uribe<sup>7</sup>.

### 2.1 Topological Groupoids

The groupoids we will consider are small categories  $G$  in which every morphism is invertible. By  $G_1$  and  $G_0$  we will denote the space of morphisms (arrows) and of objects respectively, and the structure maps

$$G_1 \times_s G_1 \xrightarrow{m} G_1 \xrightarrow{i} G_1 \xrightleftharpoons[t]{s} G_0 \xrightarrow{e} G_1$$

where  $s$  and  $t$  are the source and the target maps,  $m$  is the composition of two arrows,  $i$  is the inverse and  $e$  gives the identity arrow over every object.

The groupoid will be called *topological (smooth)* if the sets  $G_1$  and  $G_0$  and the structure maps belong to the category of topological spaces (smooth manifolds). In the case of a smooth groupoid we will also require that the maps  $s$  and  $t$  must be submersions, so that  $G_1 \times_s G_1$  is also a manifold.

A topological (smooth) groupoid is called *étale* if all the structure maps are local homeomorphisms (local diffeomorphisms). For an étale groupoid we

will mean a topological étale groupoid. In what follows, sometimes the kind of groupoid will not be specified, but it will be clear from the context to which one we are referring to. We will fix notation for the groupoids and they will be denoted only by letters of the type  $G, H, S$ .

Orbifolds are a special kind of étale groupoids, they have the peculiarity that the map  $(s, t) : G_1 \rightarrow G_0 \times G_0$  is proper, groupoids with this property are called *proper*. Whenever we write orbifold, a proper étale smooth groupoid will be understood.

A morphism of groupoids  $\Psi : H \rightarrow G$  is a pair of maps  $\Psi_i : H_i \rightarrow G_i$   $i = 1, 2$  such that they commute with the structure maps. The maps  $\Psi_i$  will be continuous (smooth) depending on which category we are working on.

The morphism  $\Psi$  is called *Morita* if the following square is a cartesian square

$$\begin{array}{ccc} H_1 & \xrightarrow{\Psi_1} & G_1 \\ (s, t) \downarrow & & \downarrow (s, t) \\ H_0 \times H_0 & \xrightarrow{\Psi_0 \times \Psi_0} & G_0 \times G_0. \end{array}$$

If the Morita morphism is between étale groupoids and the map  $s\pi_2 : H_0 \times_{\Psi_0} G_1 \rightarrow G_0$  is an étale surjection (local homeomorphism, diffeomorphism) then the morphism is called *étale Morita*; for orbifolds we will require the Morita morphisms to be étale.

Two groupoids  $G$  and  $H$  are Morita equivalent if there exist another groupoid  $K$  with Morita morphisms  $G \xrightarrow{\sim} K \xrightarrow{\sim} H$ .

For an étale groupoid  $G$ , we denote by  $G_n$  the space of  $n$ -arrows

$$x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} \dots \xrightarrow{g_n} x_n$$

The spaces  $G_n$  ( $n \geq 0$ ) form a simplicial space:

$$\dots \rightrightarrows G_2 \rightrightarrows G_1 \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} G_0$$

that together with the face maps  $d_i : G_n \rightarrow G_{n-1}$

$$d_i(g_1, \dots, g_n) = \begin{cases} (g_2, \dots, g_n) & \text{if } i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & \text{if } 0 < i < n \\ (g_1, \dots, g_{n-1}) & \text{if } i = n \end{cases}$$

form what is called the *nerve of the groupoid*. Its geometric realization<sup>12</sup> is the classifying space of  $G$ , denoted  $BG$ . A Morita equivalence  $H \xrightarrow{\sim} G$  induces a weak homotopy equivalence  $BH \xrightarrow{\sim} BG$ .

## 2.2 Sheaves and cohomology

From now on, we will restrict our attention to the case where  $G$  is an étale groupoid and smooth when required.

A  $G$ -sheaf  $\mathcal{F}$  is a sheaf over  $G_0$ , namely a topological space with a projection  $p : \mathcal{F} \rightarrow G_0$  which is a local homeomorphism on which  $G_1$  acts continuously. This means that for  $a \in \mathcal{F}_x = p^{-1}(x)$  and  $g \in G_1$  with  $s(g) = x$ , there is an element  $ag$  in  $\mathcal{F}_{t(g)}$  depending continuously on  $g$  and  $a$ . The action is a map  $\mathcal{F}_p \times_s G_1 \rightarrow \mathcal{F}$ .

All the properties of sheaves and its cohomologies of topological spaces can be extended for the case of étale groupoids as is done in Haefliger<sup>4</sup> and Crainic and Moerdijk<sup>2</sup>.

For  $\mathcal{F}$  a  $G$ -sheaf, a section  $\sigma : G_0 \rightarrow \mathcal{F}$  is called invariant if  $\sigma(x)g = \sigma(y)$  for any arrow  $x \xrightarrow{g} y$ .  $\Gamma_{inv}(G, \mathcal{F})$  is the set of invariant sections and it will be an abelian group if  $\mathcal{F}$  is an abelian sheaf.

For an abelian  $G$  sheaf  $\mathcal{F}$ , the *cohomology* groups  $H^n(G, \mathcal{F})$  are defined as the cohomology groups of the complex:

$$\Gamma_{inv}(G, \mathcal{T}^0) \rightarrow \Gamma_{inv}(G, \mathcal{T}^1) \rightarrow \dots$$

where  $\mathcal{F} \rightarrow \mathcal{T}^0 \rightarrow \mathcal{T}^1 \rightarrow \dots$  is a resolution of  $\mathcal{F}$  by injective  $G$ -sheaves. When the abelian sheaf  $\mathcal{F}$  is locally constant (for example  $\mathcal{F} = \mathbb{Z}$ ) is a result of Moerdijk<sup>9</sup> that

$$H^*(G, \mathcal{F}) \cong H^*(BG, \mathcal{F})$$

where the left hand side is sheaf cohomology and the right hand side is simplicial cohomology.

There is a *basic spectral sequence* associated to this cohomology. Pulling back  $\mathcal{F}$  along

$$\epsilon_n : G_n \rightarrow G_0 \tag{2}$$

$$\epsilon_n(g_1, \dots, g_n) = t(g_n)$$

it induces a sheaf  $\epsilon_n^* \mathcal{F}$  on  $G_n$  (where the  $G$  action on  $G_n$  is the natural one, i.e.  $(g_1, \dots, g_n)h = (g_1, \dots, g_n h)$ ;  $G_n$  becomes in this way a  $G$ -sheaf) such that for fixed  $q$  the groups  $H^q(G_p, \epsilon_p^* \mathcal{F})$  form a cosimplicial abelian group, inducing a spectral sequence:

$$H^p H^q(G_\bullet, \mathcal{F}) \Rightarrow H^{p+q}(G, \mathcal{F})$$

So if  $0 \rightarrow \mathcal{F} \rightarrow \mathcal{F}^0 \rightarrow \mathcal{F}^1 \rightarrow \dots$  is a resolution of  $G$ -sheaves with the property that  $\epsilon_p^* \mathcal{F}^q$  is an acyclic sheaf on  $G_p$ , then  $H^*(G, \mathcal{F})$  can be computed from the double complex  $\Gamma(G_p, \epsilon_p^* \mathcal{F}^q)$ .

We conclude this section by introducing the algebraic gadget that will allow us to define Deligne cohomology. Let  $\mathcal{F}^\bullet$  be a cochain complex of abelian sheaves, then the *hypercohomology groups*  $\mathbb{H}^n(\mathbb{G}, \mathcal{F})$  are defined as the cohomology groups of the double complex  $\Gamma_{inv}(\mathbb{G}, \mathcal{T}^\bullet)$  where  $\mathcal{F}^\bullet \rightarrow \mathcal{T}^\bullet$  is a quasi-isomorphism into a cochain complex of injectives.

### 3 Deligne Cohomology

In what follows we will define the smooth Deligne cohomology of a smooth étale groupoid; we will extend the results of Brylinski<sup>1</sup> to groupoids and will follow very closely the description given in there.

We will assume all through out this paper that the set of objects  $G_0$  of our groupoid  $\mathbb{G}$  has an open cover by subsets which are each paracompact, Hausdorff, locally compact and of bounded cohomological dimension depending on  $G_0$ . If not specified explicitly, when working on a groupoid we will have always in mind its description given by this cover; in other words, we will think of  $G_0$  as the disjoint union of this cover and  $G_1$  its respective space of arrows that will keep the groupoid in the same Morita class.

Deligne cohomology is related to the De Rham cohomology. We will consider the De Rham complex of sheaves and we will truncate it at level  $p$ ; what interests us is the degree  $p$  hypercohomology classes of this complex. To be more specific, let  $\mathbb{Z}(p) := (2\pi\sqrt{-1})^p \cdot \mathbb{Z}$  be the cyclic subgroup of  $\mathbb{C}$ ,  $A^p(G_0)_{\mathbb{C}} := A^p(G_0) \otimes \mathbb{C}$  the space of complex-valued  $p$  forms on  $G_0$  and  $\mathcal{A}_{\mathbb{G}, \mathbb{C}}^p$  the  $\mathbb{G}$ -sheaf of complex-valued differential  $p$ -forms; as  $\mathbb{G}$  is a smooth étale groupoid the maps  $s$  and  $t$  are local diffeomorphisms, then the action of  $\mathbb{G}$  into the sheaf over  $G_0$  of complex-valued differential  $p$ -forms is the natural one given by the pull back of the corresponding diffeo.

Let  $\mathbb{Z}(p)_{\mathbb{G}}$  be the constant  $\mathbb{Z}(p)$ -valued  $\mathbb{G}$ -sheaf, and  $i : \mathbb{Z}(p)_{\mathbb{G}} \rightarrow \mathcal{A}_{\mathbb{G}, \mathbb{C}}^0$  the inclusion of constant into smooth functions.

**Definition 3.0.1** *Let  $\mathbb{G}$  be a smooth étale groupoid. The smooth Deligne complex  $\mathbb{Z}(p)_{\mathbb{D}}^\infty$  is the complex of  $\mathbb{G}$ -sheaves:*

$$\mathbb{Z}(p)_{\mathbb{G}} \xrightarrow{i} \mathcal{A}_{\mathbb{G}, \mathbb{C}}^0 \xrightarrow{d} \mathcal{A}_{\mathbb{G}, \mathbb{C}}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}_{\mathbb{G}, \mathbb{C}}^{p-1}$$

*The hypercohomology groups  $\mathbb{H}^q(\mathbb{G}, \mathbb{Z}(p)_{\mathbb{D}}^\infty)$  are called the smooth Deligne cohomology of  $\mathbb{G}$ .*

Formally, this description will do the job, but we would like to have a more concrete definition of this cohomology. This is done using a *Leray* description of the groupoid and then calculating the cohomology of the respective Čech double complex. For a manifold  $M$ , a *Leray* cover  $\mathcal{U}$  is one on which all

the open sets and its intersections are contractible. With this cover we can calculate the De Rham cohomology of  $M$  by calculating the cohomology of the Čech complex with real coefficients; for the De Rham cohomology of the open sets in the cover is trivial. The same idea can be applied to the hypercohomology of the Deligne complex on a manifold, but in this case we obtain a Čech double complex. This will be explained in more detail in the next section.

**Definition 3.0.2** *Let  $H$  be a smooth étale groupoid. A Leray description of  $H$  is an étale Morita equivalent groupoid  $G$ , provided with an étale Morita morphism  $G \rightarrow H$ , on which all the sets  $G_i$ , for  $i \geq 0$ , are diffeomorphic to a disjoint union of contractible sets.*

The existence of such a groupoid for orbifolds (namely proper étale smooth groupoids) can be proved using the results of Moerdijk and Pronk<sup>10</sup>. In any case, this is clear for most relevant examples. In order to make the calculations clearer, where are going to work with a quasi-isomorphic complex of sheaves to the Deligne one, which is a bit simpler.

**Definition 3.0.3** *Let  $\mathbb{C}^\times(p)_G$  be the following complex of sheaves:*

$$\mathbb{C}_G^\times \xrightarrow{d \log} \mathcal{A}_{G, \mathbb{C}}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}_{G, \mathbb{C}}^{p-1}$$

It's easy to see that there is a quasi-isomorphism between the complexes  $(2\pi\sqrt{-1})^{-p+1} \cdot \mathbb{Z}(p)_D^\infty$  and  $\mathbb{C}^\times(p)_G[-1]$  (this fact is explained in Brylinski<sup>1</sup> page 216)

$$\begin{array}{ccccccc} (2\pi\sqrt{-1})^{-p+1} \cdot \mathbb{Z}(p)_G & \longrightarrow & \mathbb{C}_G & \xrightarrow{d} & \mathcal{A}_{G, \mathbb{C}}^1 & \xrightarrow{d} & \cdots \xrightarrow{d} & \mathcal{A}_{G, \mathbb{C}}^{p-1} \\ & & \downarrow \text{exp} & & \downarrow & & & \downarrow \\ & & \mathbb{C}_G^\times & \xrightarrow{d \log} & \mathcal{A}_{G, \mathbb{C}}^1 & \xrightarrow{d} & \cdots \xrightarrow{d} & \mathcal{A}_{G, \mathbb{C}}^{p-1} \end{array}$$

hence there is an isomorphism of hypercohomologies:

$$\mathbb{H}^{q-1}(G, \mathbb{C}^\times(p)_G) \cong (2\pi\sqrt{-1})^{-p+1} \cdot \mathbb{H}^q(G, \mathbb{Z}(p)_D^\infty) \quad (3)$$

Now let  $G$  be such Leray description of the groupoid. We are going to define the Čech double complex associated to the  $G$ -sheaf complex  $\mathbb{C}^\times(p)_G$ . Consider the space

$$C^{k,l} = \check{C}(G_k, \mathcal{A}_{G, \mathbb{C}}^l) := \Gamma(G_k, \epsilon_k^* \mathcal{A}_{G, \mathbb{C}}^l)$$

of global sections of the sheaf  $\epsilon_k^* \mathcal{A}_{G, \mathbb{C}}^l$  over  $G_k$  as in (2). The vertical differential  $C^{k,l} \rightarrow C^{k,l+1}$  is given by the maps of the complex  $\mathbb{C}^\times(p)_G$  and the

horizontal differential  $C^{k,l} \rightarrow C^{k+1,l}$  is obtained by  $\delta = \sum (-1)^i \delta_i$  where for  $\sigma \in \Gamma(\mathbf{G}_k, \epsilon_k^* \mathcal{A}_{\mathbf{G},\mathbb{C}}^l)$

$$(\delta_i \sigma)(g_1, \dots, g_{k+1}) = \begin{cases} \sigma(g_1, \dots, g_k) \cdot g_{k+1} & \text{for } i = k + 1 \\ \sigma(g_1, \dots, g_i g_{i+1}, \dots, g_{k+1}) & \text{for } 0 < i < k + 1 \\ \sigma(g_2, \dots, g_{k+1}) & \text{for } i = 0 \end{cases}$$

**Definition 3.0.4** For  $\mathbf{G}$  a Leray description of a smooth étale groupoid, let's denote by  $\check{C}(\mathbf{G}, \mathbb{C}^\times(p)_{\mathbf{G}})$  the total complex induced by the double complex

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & & \vdots \\ & \delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \delta \uparrow \\ \check{C}(\mathbf{G}_2, \mathbb{C}_{\mathbf{G}}^\times) & \xrightarrow{d \log} & \check{C}(\mathbf{G}_2, \mathcal{A}_{\mathbf{G},\mathbb{C}}^1) & \xrightarrow{d} & \check{C}(\mathbf{G}_2, \mathcal{A}_{\mathbf{G},\mathbb{C}}^2) & \xrightarrow{d} & \dots & \xrightarrow{d} & \check{C}(\mathbf{G}_2, \mathcal{A}_{\mathbf{G},\mathbb{C}}^{p-1}) \\ & \delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \delta \uparrow \\ \check{C}(\mathbf{G}_1, \mathbb{C}_{\mathbf{G}}^\times) & \xrightarrow{d \log} & \check{C}(\mathbf{G}_1, \mathcal{A}_{\mathbf{G},\mathbb{C}}^1) & \xrightarrow{d} & \check{C}(\mathbf{G}_1, \mathcal{A}_{\mathbf{G},\mathbb{C}}^2) & \xrightarrow{d} & \dots & \xrightarrow{d} & \check{C}(\mathbf{G}_1, \mathcal{A}_{\mathbf{G},\mathbb{C}}^{p-1}) \\ & \delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \delta \uparrow \\ \check{C}(\mathbf{G}_0, \mathbb{C}_{\mathbf{G}}^\times) & \xrightarrow{d \log} & \check{C}(\mathbf{G}_0, \mathcal{A}_{\mathbf{G},\mathbb{C}}^1) & \xrightarrow{d} & \check{C}(\mathbf{G}_0, \mathcal{A}_{\mathbf{G},\mathbb{C}}^2) & \xrightarrow{d} & \dots & \xrightarrow{d} & \check{C}(\mathbf{G}_0, \mathcal{A}_{\mathbf{G},\mathbb{C}}^{p-1}) \end{array}$$

The Čech hypercohomology of the complex of sheaves  $\mathbb{C}^\times(p)_{\mathbf{G}}$  is defined as the cohomology of the Čech complex  $\check{C}(\mathbf{G}, \mathbb{C}^\times(p)_{\mathbf{G}})$ :

$$\check{H}^*(\mathbf{G}, \mathbb{C}^\times(p)_{\mathbf{G}}) := H^* \check{C}(\mathbf{G}, \mathbb{C}^\times(p)_{\mathbf{G}}).$$

Due to all the conditions imposed to the Leray description, the previous cohomology actually matches the hypercohomology of the complex  $\mathbb{C}^\times(p)_{\mathbf{G}}$ , so we get

**Lemma 3.0.5** Let  $\mathbf{H}$  be a smooth étale groupoid and  $\mathbf{G}$  a Leray description of it. Then the cohomology of the Čech complex  $\check{C}(\mathbf{G}, \mathbb{C}^\times(p)_{\mathbf{G}})$  is isomorphic to the hypercohomology of  $\mathbb{C}^\times(p)_{\mathbf{H}}$

$$\check{H}^*(\mathbf{G}, \mathbb{C}^\times(p)_{\mathbf{G}}) \xrightarrow{\cong} \mathbb{H}^*(\mathbf{G}, \mathbb{C}^\times(p)_{\mathbf{G}}) \xleftarrow{\cong} \mathbb{H}^*(\mathbf{H}, \mathbb{C}^\times(p)_{\mathbf{H}})$$

where the second isomorphism is induced by the map  $\mathbf{G} \xrightarrow{M} \mathbf{H}$ .

We will postpone the proof of this result to a forthcoming paper<sup>8</sup>. This Čech description will be the one that will allow us understand the relationship between the  $B$ -field and the discrete torsion and also gives us an inside view of what the hypercohomology calculates.

We conclude this section by observing that with the definition of *gerbe with connection* over an orbifold given in our previous work<sup>6</sup> we can easily prove the following

**Proposition 3.0.6** *For  $G$  a Leray description of a smooth étale groupoid, a gerbe with connection is a 2-cocycle of the complex  $\check{C}(G, \mathbb{C}^\times(3)_G)$ , that is, a triple  $(h, A, B)$  with  $B \in \check{C}(G_0, \mathcal{A}_{G, \mathbb{C}}^2)$ ,  $A \in \check{C}(G_1, \mathcal{A}_{G, \mathbb{C}}^1)$  and  $h \in \check{C}(G_2, \mathbb{C}_G^\times)$  that satisfies  $\delta B = dA$ ,  $\delta A = d \log h$  and  $\delta h = 1$ . Two such gerbes with connection are isomorphic if they lie in the same cohomology class, hence they are classified by  $\mathbb{H}^3(G, \mathbb{Z}(3)_D^\infty)$ .*

## 4 B-field and Discrete Torsion

### 4.1 Manifolds

A  $B$ -field over a manifold  $M$  (see Hitchin<sup>5</sup>) is a choice of gerbe with connection which in terms of a Leray cover  $\{U_\alpha\}$  of  $M$  is described by a collection of 2-forms  $B_\alpha$  over each  $U_\alpha$ , 1-forms  $A_{\alpha\beta}$  over the double intersections  $U_{\alpha\beta} := U_\alpha \cap U_\beta$  and  $\mathbb{C}^\times$ -valued functions  $h_{\alpha\beta\gamma}$  over triple overlaps  $U_{\alpha\beta\gamma}$  satisfying

$$\begin{aligned} B_\alpha - B_\beta &= dA_{\alpha\beta} \\ A_{\alpha\beta} + A_{\beta\gamma} - A_{\alpha\gamma} &= d \log h_{\alpha\beta\gamma} \\ h_{\alpha\beta\gamma}^{-1} h_{\alpha\beta\nu} h_{\alpha\gamma\nu}^{-1} h_{\beta\gamma\nu} &= 1. \end{aligned}$$

If we consider the description as a groupoid of  $M$  given by the Leray cover  $\{U_\alpha\}$ :

$$\mathbf{M}_0 := \bigsqcup_{\alpha} U_\alpha \quad \text{and} \quad \mathbf{M}_1 := \bigsqcup_{\alpha\beta} U_{\alpha\beta}$$

and we collect the information of these functions as sections of the sheaves defined in the previous section, in other words  $B \in \check{C}(\mathbf{M}_0, \mathcal{A}_{\mathbf{M}, \mathbb{C}}^2)$ ,  $A \in \check{C}(\mathbf{M}_1, \mathcal{A}_{\mathbf{M}, \mathbb{C}}^1)$  and  $h \in \check{C}(\mathbf{M}_2, \mathbb{C}_{\mathbf{M}}^\times)$ , it's easy to see that those equations become:

$$\begin{aligned} \delta B &= dA \\ \delta A &= d \log h \\ \delta h &= 1 \end{aligned}$$

then the triple  $(h, A, B)$  determines a cocycle in the Čech complex  $\check{C}(\mathbf{M}, \mathbb{C}^\times(3)_{\mathbf{M}})$  (see theorem 5.3.11<sup>5</sup>); hence, by lemma 3.0.5 we obtain

**Theorem 4.1.1** *A choice of B-field in a manifold  $M$  determines a cocycle  $(h, A, B)$  of the complex  $\check{C}(M, \mathbb{C}^\times(3)_M)$  and vice versa. Moreover, the isomorphism classes of choices of B-field are classified by the cohomology class of  $(h, A, B)$  in the hypercohomology group  $\mathbb{H}^2(M, \mathbb{C}_M^\times \xrightarrow{d \log} \mathcal{A}_{M, \mathbb{C}}^1 \xrightarrow{d} \mathcal{A}_{M, \mathbb{C}}^2)$ .*

Then, in view of the isomorphism 3 we conclude

**Corollary 4.1.2** *The choice of B-field over a manifold  $M$  is classified by the third Deligne cohomology group  $\mathbb{H}^3(M, \mathbb{Z}(3)_D^\infty)$  of the manifold.*

Here we should point out that we<sup>6</sup> have generalized the picture described by Hitchin<sup>5</sup> to the case of orbifolds, and hence the previous statements remain true for the orbifold case.

#### 4.2 Discrete torsion

In what follows we will argue that over an orbifold the discrete torsion is just another choice of gerbe with connection, as is the  $B$ -field. In other words, the discrete torsion and the  $B$ -field are both extreme cases of the same picture, namely gerbes with connection; while the  $B$ -field only takes into account the differentiable structure of the orbifold, the discrete torsion one only considers the extra structure added by the action of the groups.

In the case of a global quotient  $M/G$  with  $G$  a finite group acting via diffeomorphisms over  $M$ , we can take one étale groupoid that models it<sup>6</sup>, i.e.  $X_0 := M$  and  $X_1 := M \times G$  with the natural source and target maps:  $s(z, g) = z$  and  $t(z, g) = zg$ ; denoting by  $\bar{G} := * \times G \rightrightarrows *$  the groupoid associated to  $G$  we have a natural morphism

$$X \rightarrow \bar{G}. \quad (4)$$

As  $\bar{G}$  is formed by a discrete set of points we have that

$$H^2(G, \mathbb{C}^\times) \cong \mathbb{H}^2(\bar{G}, \mathbb{C}_{\bar{G}}^\times \xrightarrow{d \log} \mathcal{A}_{\bar{G}, \mathbb{C}}^1 \xrightarrow{d} \mathcal{A}_{\bar{G}, \mathbb{C}}^2),$$

as there is a natural monomorphism induced by (4)

$$\mathbb{H}^2(\bar{G}, \mathbb{C}_{\bar{G}}^\times \xrightarrow{d \log} \mathcal{A}_{\bar{G}, \mathbb{C}}^1 \xrightarrow{d} \mathcal{A}_{\bar{G}, \mathbb{C}}^2) \rightarrow \mathbb{H}^2(X, \mathbb{C}_X^\times \xrightarrow{d \log} \mathcal{A}_{X, \mathbb{C}}^1 \xrightarrow{d} \mathcal{A}_{X, \mathbb{C}}^2)$$

we get a map

$$H^2(G, \mathbb{C}^\times) \rightarrow \mathbb{H}^2(X, \mathbb{C}_X^\times \xrightarrow{d \log} \mathcal{A}_{X, \mathbb{C}}^1 \xrightarrow{d} \mathcal{A}_{X, \mathbb{C}}^2).$$

**Theorem 4.2.1** *For the orbifold  $X = [M/G]$  the homomorphism*

$$H^2(G, \mathbb{C}^\times) \rightarrow \mathbb{H}^2(X, \mathbb{C}_X^\times \xrightarrow{d \log} \mathcal{A}_{X, \mathbb{C}}^1 \xrightarrow{d} \mathcal{A}_{X, \mathbb{C}}^2)$$

is injective. Therefore the choice of discrete torsion is a subgroup of the equivalence classes of gerbes with connection over the groupoid, which is classified by the third Deligne cohomology of the orbifold, namely

$$(2 \in \sqrt{-1})^{-2} \cdot \mathbb{H}^3(\mathcal{X}, \mathbb{Z}(3)_D^\infty).$$

Let  $c : G \times G \rightarrow \mathbb{C}^\times$  be a 2-cocycle and  $\bar{c} : M \times G \times G \rightarrow \mathbb{C}^\times$ ,  $\bar{c}(x, g, h) := c(g, h)$  its image under the morphism. If  $(\bar{c}, 0, 0) = 0$  in  $\mathbb{H}^2(\mathcal{X}, \mathbb{C}^\times(3)_\mathcal{X})$  then there exist a map  $f : M \times G \rightarrow \mathbb{C}^\times$  such that  $\delta f = \bar{c}$  and  $df = 0$ . As

$$(\delta f)(x, g, h) = f(x, g)f(x, gh)^{-1}f(x, h) = \bar{c}(x, g, h) = c(g, h)$$

we get that the cocycle  $c$  is also exact; take  $\sigma : G \rightarrow \mathbb{C}^\times$  with  $\sigma(g) := f(x, g)$  for any fixed  $x$ , then  $\delta\sigma = c$ .

## 5 Gerbes with connection and the Inertia groupoid

In this section we are going to construct the holonomy bundle of a gerbe with connection, which in the case of a groupoid will be a flat line bundle over the inertia groupoid. To do this we need to recall some definitions.

### 5.1 Line bundles with connection

From theorem 2.2.12 of Brylinski<sup>1</sup> we know that the group of isomorphism classes of line bundles with connection over a manifold  $M$  is canonically isomorphic to its second Deligne cohomology, namely

$$(2\pi\sqrt{-1})^{-1} \cdot \mathbb{H}^2(M, \mathbb{Z}(2)_D^\infty) \cong \mathbb{H}^1(M, \mathbb{C}_M^\times \xrightarrow{d \log} \mathcal{A}_{M, \mathbb{C}}^1).$$

The same result can be extended to cover smooth étale groupoids, let's explain the idea. For  $\mathbf{G}$  a Leray description of a groupoid  $\mathbf{G}$ , a line bundle with connection over it is a morphism of groupoids  $\rho : \mathbf{G} \rightarrow \mathbb{C}^\times$  and a 1-form  $A$  over  $\mathbf{G}_0$  such that

$$s^*A - t^*A = d \log \rho_1.$$

But  $\rho$  is a morphism of groupoids if and only if  $\delta\rho_1 = 1$ ; considering  $\rho_1$  also as an element of  $\check{C}(\mathbf{G}_1, \mathbb{C}_\mathbf{G}^\times)$  (recall that  $\rho_1 : \mathbf{G}_1 \rightarrow \mathbb{C}^\times$ ); i.e.

$$(\delta\rho_1)(g_1, g_2) = \rho_1(g_2)\rho_1(g_1g_2)^{-1}\rho_1(g_1) = 1.$$

**Proposition 5.1.1** *The line bundle with connection  $(\rho, A)$  over  $\mathbf{G}$  represents a 1-cocycle in the complex  $\check{C}(\mathbf{G}, \mathbb{C}^\times(2)_\mathbf{G})$  and its isomorphism class is classified by the respective element in  $\mathbb{H}^1(\mathbf{G}, \mathbb{C}_\mathbf{G}^\times \xrightarrow{d \log} \mathcal{A}_\mathbf{G}^1)$ .*

## 5.2 The inertia groupoid

The inertia groupoid  $\wedge G$  is defined in the following way:

$$\begin{aligned}\wedge G_0 &= \{a \in G_1 \mid s(a) = t(a)\} \\ \wedge G_1 &= \{(a, b) \in G_2 \mid a \in \wedge G_0\}\end{aligned}$$

with  $s(a, b) = a$  and  $t(a, b) = b^{-1}ab$ . Here we consider the case in which this groupoid is smooth. We know (see theorem 6.4.2<sup>7</sup>) that a gerbe over an étale groupoid  $\theta : G_2 \rightarrow \mathbb{C}^\times$  with  $\delta\theta = 1$  determines a line bundle over the inertia groupoid  $\rho : \wedge G \rightarrow \mathbb{C}^\times$  with

$$\rho_1(a, b) = \frac{\theta(a, b)}{\theta(b, b^{-1}ab)}$$

(we recommend the reader to see our previous paper<sup>7</sup> to get acquainted with the terminology).

Now we want to extend the previous result to define a line bundle with connection over the inertia groupoid from a gerbe with connection.

**Lemma 5.2.1** *Let  $G$  be a Leray description of a smooth étale groupoid and  $(h, A, B)$  a 2-cocycle of  $\check{C}(G, \mathbb{C}^\times(3)_G)$  (a gerbe with connection). Then the pair  $(\rho, \nabla)$  where*

$$\rho(a, b) := \frac{h(a, b)}{h(b, b^{-1}ab)} \quad \text{and} \quad \nabla := A|_{\wedge G_0}$$

is a 1-cocycle of the induced complex  $\check{C}(\wedge G, \mathbb{C}^\times(2)_{\wedge G})$ .

We just need to prove that  $\delta\nabla = d \log \rho$ . We have that in  $\check{C}(\wedge G, \mathbb{C}^\times(2)_{\wedge G})$ :

$$(\delta\nabla)(a, b) = \nabla(s(a, b)) \cdot b - \nabla(t(a, b)) = \nabla(a) \cdot b - \nabla(b^{-1}ab)$$

and in  $\check{C}(G, \mathbb{C}^\times(3)_G)$ :

$$(\delta A)(a, b) = A(a) \cdot b - A(ab) + A(b) = d \log h(a, b)$$

$$(\delta A)(b, b^{-1}ab) = A(b) - A(ab) + A(b^{-1}ab) = d \log h(b, b^{-1}ab).$$

By definition  $\nabla(a) = A(a)$  and  $\nabla(b^{-1}ab) = A(b^{-1}ab)$ , so we get

$$(\delta\nabla)(a, b) = (\delta A)(a, b) - (\delta A)(b, b^{-1}ab) = d \log \frac{h(a, b)}{h(b, b^{-1}ab)} = d \log \rho(a, b).$$

□

In the same way as before, 1-cocycles in  $\check{C}(G, \mathbb{C}^\times(3)_G)$  induce 0-cocycles in  $\check{C}(\wedge G, \mathbb{C}^\times(2)_{\wedge G})$ , so we get that there is a morphism

$$H^2 \check{C}(G, \mathbb{C}^\times(3)_G) \longrightarrow H^1 \check{C}(\wedge G, \mathbb{C}^\times(2)_{\wedge G}) \quad (5)$$

that extends to a morphisms in hypercohomology:

**Theorem 5.2.2** *Let  $G$  be a smooth étale groupoid, then there exists a holonomy homomorphism*

$$\begin{array}{ccc} \mathbb{H}^2(G, \mathbb{C}_G^\times \xrightarrow{d \log} \mathcal{A}_{G, \mathbb{C}}^1 \xrightarrow{d} \mathcal{A}_{G, \mathbb{C}}^2) & \longrightarrow & \mathbb{H}^1(\wedge G, \mathbb{C}_{\wedge G}^\times \xrightarrow{d \log} \mathcal{A}_{\wedge G, \mathbb{C}}^1) \\ \downarrow \cong & & \downarrow \cong \\ (2\pi\sqrt{-1})^{-2} \cdot \mathbb{H}^3(G, \mathbb{Z}(3)_D^\infty) & \xrightarrow{\text{holo}} & (2\pi\sqrt{-1})^{-1} \cdot \mathbb{H}^2(\wedge G, \mathbb{Z}(2)_D^\infty) \end{array}$$

which to every gerbe with connection assigns a line bundle with connection over the inertia groupoid  $\wedge G$ . Moreover, this line bundle is flat.

Let's assume  $G$  is the Leray description of such a groupoid, from lemma 3.0.5

$$H^2 \check{C}(G, \mathbb{C}^\times(3)_G) \cong \mathbb{H}^2(G, \mathbb{C}_G^\times \xrightarrow{d \log} \mathcal{A}_{G, \mathbb{C}}^1 \xrightarrow{d} \mathcal{A}_{G, \mathbb{C}}^2)$$

and in the same manner as proposition 1.3.4 of Brylinski<sup>1</sup>, there is a canonical homomorphism

$$H^1 \check{C}(\wedge G, \mathbb{C}^\times(2)_{\wedge G}) \rightarrow \mathbb{H}^1(\wedge G, \mathbb{C}_{\wedge G}^\times \xrightarrow{d \log} \mathcal{A}_{\wedge G, \mathbb{C}}^1)$$

which together with the morphism (5) implies (5.2.2).

Using the notation of lemma 5.2.1 we know that  $\delta B = dA$ , or in other words,  $B(s(g)) - B(t(g)) = dA(g)$  for  $g \in G_1$ . Then is clear that for  $a \in \wedge G_0$   $dA(a) = 0$ , hence  $d\nabla = 0$ . The induced line bundle with connection over the groupoid is flat.  $\square$

In the case that  $G$  is an orbifold, these induced flat line bundles over the inertia groupoid are precisely what Ruan<sup>11</sup> denoted by *inner local systems* (see the last section of our previous paper<sup>7</sup>), then we can conclude

**Proposition 5.2.3** *A gerbe with connection over an orbifold  $G$  determines an inner local system<sup>11</sup> over the twisted sectors  $\widetilde{\Sigma}_1 G$ .*

The homomorphism of theorem 5.2.2 can be generalized to all the Deligne cohomology groups. So we get the following result which will be proved in a forthcoming paper<sup>8</sup>

**Theorem 5.2.4** *Let  $G$  be an smooth étale groupoid, then there exist a natural morphism of complexes*

$$\check{C}(G, \mathbb{C}^\times(p)_G) \rightarrow \check{C}(\wedge G, \mathbb{C}^\times(p-1)_{\wedge G})$$

which induces for all  $p > 0$  a morphism of cohomologies

$$\begin{array}{ccc} \mathbb{H}^q(G, \mathbb{C}^\times(p)_G) & \longrightarrow & \mathbb{H}^{q-1}(\wedge G, \mathbb{C}^\times(p-1)_{\wedge G}) \\ \downarrow \cong & & \downarrow \cong \\ (2\pi\sqrt{-1})^{-p+1} \cdot \mathbb{H}^{q+1}(G, \mathbb{Z}(p)_D^\infty) & \longrightarrow & (2\pi\sqrt{-1})^{-p+2} \cdot \mathbb{H}^q(\wedge G, \mathbb{Z}(p-1)_D^\infty). \end{array}$$

Observe that these facts provide a generalization of the concept of inner local system for the twisted multisectors of Ruan. It is reasonable to predict that this will be relevant in the full theory of Gromov-Witten invariants on orbifolds.

### Acknowledgments

We would like to thank conversations with M. Crainic, I. Moerdijk and Y. Ruan regarding several aspects of this work.

The first author would like to thank Paula Lima-Filho for some conversations that motivated his interest in Deligne Cohomology.

The second author would like to express its gratitude to the organizers of the Summer School on Geometric and Topological Methods for Quantum Field Theory, and specially to its sponsors the Centre International de Mathématiques Pures et Appliquées (CIMPA) and the Universidad de los Andes, for the invitation to take part in the school where some of the previous results were presented.

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