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A Lecture on Poisson–Nijenhuis Structures

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ABSTRACT. This is an expository paper. In it, the Poisson-Nijenhuis structures are motivated and defined in the general algebraic framework of Gel'fand and Dorfman. Then, in the particular case of Lie algebroids and differentiable manifolds, the Poisson-Nijenhuis structures are related to the notion of a complementary 2-form, introduced and studied by the author in [10], and several examples of complementary forms and Poisson-Nijenhuis manifolds are given.

1 Motivation

It was established by several authors, and, in particular, by Magri (1978), Gel'fand-Dorfman (1979), Raţiu (1980), etc., that two Poisson structures (i.e., brackets which are "like" the classical Poisson brackets of mechanics), which are well correlated, lead to functions in involution, and these functions may provide the complete integrability of some important Hamiltonian systems, particularly, infinite-dimensional ones.

The basics beyond this procedure consists of the following general algebraic scheme of Gel'fand and Dorfman [1], where we recognize the fundamental ideas of Hamiltonian dynamics.

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Let χ be a real Lie algebra (not necessarily finite-dimensional) and

$$(1) \qquad \mathcal{C} = (\sum_{k=0}^{\infty} \Omega^k \, , \, d \, : \, \Omega^k \, \rightarrow \, \Omega^{k+1} \, , \, d^2 = 0)$$

a cochain complex of real linear spaces. A *representation* of χ on $\mathcal{C}$ is a mapping

$$X \mapsto i(X) \, \in \, L_{\mathbf{R}}(\Omega^k, \Omega^{k-1})$$

defined for all $X \, \in \, \chi$ and $k = 0, 1, 2, ...$, such that, $\forall X, Y \in \chi$ one has

$$(2) \qquad i(X)i(Y) + i(Y)i(X) = 0 \, , \, i([X,Y]) = L_X i(Y) - i(Y) L_X,$$

where

$$(3) \qquad L_X := di(X) + i(X)d \, .$$

If such a representation is defined, we say that $(\chi \, , \mathcal{C})$ is a *Gel'fand-Dorfman complex*. Of course, $i(X)f := 0$ for $f \in \Omega^0$. On the other hand, if we define $Xf := i(X)df$, we get a usual representation of the Lie algebra χ on Ω^0. We should also notice that (2) and (3) imply the usual formula

$$(4) \qquad i(X_k)...i(X_0)d\lambda = \sum_{i=0}^{k}(-1)^i L_{X_i}((i(X_k)...i(\widehat{X_i})...i(X_0)\lambda)+$$

$$+\sum_{i<j}(-1)^{(i+j)} i(X_k)...i(\widehat{X_j})...i(\widehat{X_i})...i(X_0)i([X_i,X_j])\lambda \;\; (\lambda \, \in \, \Omega^k).$$

The usual calculus of differential forms gives the basic example and the names for the operators. Another simple case is provided by any linear space $\mathcal{F}$ where χ has a representation, if we take $\Omega^k := L_{\mathbf{R}alt}(\underbrace{\chi \times \, ... \, \times \, \chi}_{k \; times}, \mathbf{R})$, and the usual formulas for d and $i(X)$.

Now, a *Hamiltonian structure* on $(\chi, \mathcal{C})$ is an element $H \in L_{\mathbf{R}}(\Omega^1, \chi)$ such that, if we define

$$(5) \qquad X_f := H(df) \qquad (f \in \Omega^0),$$

X_f behaves like a "Hamiltonian vector field". What we mean by this is that

$$(6) \qquad \{f,g\} :=< Hdf, dg >:= i(Hdf)(dg)$$

is a Lie bracket called the *Poisson bracket*. The skew-symmetry of (6) is ensured if we ask

$$< \lambda, H\mu >= - < H\lambda, \mu > \qquad (\lambda, \mu \in \Omega^1), \tag{7}$$

and a computation shows that the Jacobi identity holds iff

$$[H, H] = 0, \tag{8}$$

where this *Gel'fand-Dorfman bracket* is the operation defined for all H, K satisfying (7) by

$$[H, K](\alpha, \beta, \gamma) = \sum_{Cycl(\alpha,\beta,\gamma)} \{< KL_{H\alpha}\beta, \gamma > + < HL_{K\alpha}\beta, \gamma >\} \tag{9}$$

$(\alpha, \beta, \gamma) \in \Omega^1$.

Thus, *a Hamiltonian structure is H which satisfies* (7) *and* (8), and if the elements of χ represent a "time-evolution", the equation

$$\frac{du}{dt} = X_f \tag{10}$$

is a *Hamiltonian system*, with the *Hamiltonian* f.

Obviously, the scheme is applicable to time evolutions on finite dimensional or Banach manifolds, and it is also applicable in a *formal variational calculus* due to Gel'fand and Dikii [2]. This calculus is a general algebraic setting useful in the study of many partial differential equations.

The point is that, if a general Hamiltonian system (10) has *first integrals* in *involution* with f, and among themselves, these first integrals are helpful in the integration of (10), up to a possible *complete integration* à la Liouville, just as in classical mechanics. The "practical" conclusion is that one should look for methods of getting as many as possible elements of Ω^0 in involution. It turns out that this is possible if the *Hamiltonian complex* $(\chi, \mathcal{C}, H)$ also possesses a second Hamiltonian structure K which is *compatible* with H, in the sense that (see (9))

$$[H, K] = 0. \tag{11}$$

Accordingly, we call $(\chi, \mathcal{C}, H, K)$ a *bi-Hamiltonian complex*. In particular, we may also speak of *bi-Hamiltonian differentiable manifolds*. (Of course, a *Hamiltonian manifold* is just a *Poisson manifold*.)

Now, one way to obtain elements of Ω^0 ("functions") in involution is based on [1]:

Lemma 1 *In a bi-Hamiltonian complex, if for some* $\varphi, \psi, \chi \in \Omega^1$ *one has* $K\psi = H\varphi$, *and* $K\chi = H\psi$, *then* $\forall \xi, \eta \in \Omega^1$ *one has*

(12) $i(K\eta)i(K\xi)d\chi = i(H\eta)i(K\xi)d\psi + i(K\eta)i(H\xi)d\psi - i(H\eta)i(H\xi)d\varphi.$

The proof is by computations based on (7) (8), (9) and (11).

This lemma allows for an inductive proof of

Proposition 1 *With the notation above, assume that* $\mathcal{C}$ *is such that* $i(K\eta)i(K\xi)d\chi = 0$ *implies* $\chi = dh$ $(h \in \Omega^0)$, *and assume that* $\xi_i \in \Omega^1$ $(i = 0, 1, 2, ...)$ *are such that* $K\xi_{i+1} = H\xi_i$, *and* $\xi_0, \xi_1 \in d\Omega^0$. *Then, all* $\xi_i = dm_i$ $(m_i \in \Omega^0)$, *and* $\forall i, j \geq 0$, $\{m_i, m_j\}_H = 0$, $\{m_i, m_j\}_K = 0$.

Another possibility is used in more recent papers by Magri and collaborators [3]. Namely, let $\Omega^0\{\lambda\}$ be the space of formal power series in λ, over Ω^0. Then, the Poisson brackets associated with Hamiltonian structures on Ω^0 extend to $\Omega^0\{\lambda\}$, and it is easy to see that (H, K) is a bi-Hamiltonian structure iff $H - \lambda K := H_\lambda$ yields a Lie bracket on $\Omega^0\{\lambda\}$. Furthermore, let us agree to call any $h \in \Omega^0$ such that $\forall f \in \Omega^0$, $\{f, h\} = 0$ a *Casimir* of H, and let $h(\lambda) = \sum_{k\geq 0} h_k\lambda^k$ $(h_k \in \Omega^0)$ be a Casimir of H_λ. Then, it straightforwardly follows that we have

$$\{h_k, f\}_K = \{h_{k+1}, f\}_H \qquad (\forall f \in \Omega^0), \tag{13}$$

and, after several steps of this kind we get

$$\{h_k, h_{k+j}\}_K = \{h_k, h_{k+j}\}_H = 0. \tag{14}$$

Indeed

$$\{h_k, h_{k+1}\}_K = \{h_{k+1}, h_{k+1}\}_H = 0,$$

and then

$$\{h_k, h_{k+j}\}_K = \{h_{k+1}, h_{k+j}\}_H = -\{h_{k+j}, h_{k+1}\}_H =$$

$$= -\{h_{k+j-1}, h_{k+1}\}_K = \{h_{k+1}, h_{k+j-1}\}_K = ... = 0,$$

since we get either equal indices or indices which differ by 1, at the end. Therefore, Casimirs of H_λ yield systems of functions in involution. The KdV equation can be studied well by this method [3].

2 The Definition

Thus, if we have a Hamiltonian complex, it is of great interest to look for secondary, *compatible*, Hamiltonian structures of the complex. The theory of Poisson-Nijenhuis structures yields interesting pairs of compatible Hamiltonian structures, which are well suited for the application of the previous Proposition 1, since there exists a recursive construction of sequences $\{\xi_i\} \in \Omega^1$ as needed in this proposition, associated to a Poisson-Nijenhuis structure.

We shall give the definition of the Poisson-Nijenhuis structures following [4]. This definition is based on some general algebraic properties of Hamiltonian complexes.

Lemma 2 *Let χ be a Lie algebra represented on a complex $\mathcal{C}$, and $H \in L_{\mathbf{R}}(\Omega^1, \chi)$ be a skew- symmetric element (it satisfies (7)). $\forall \alpha, \beta \in \Omega^1$, define*

$$\{\alpha, \beta\} = L_{H\alpha}\beta - L_{H\beta}\alpha - d < H\alpha, \beta > . \tag{15}$$

Then, $\forall \gamma \in \Omega^1, \forall X \in \chi$ one has

$$< \gamma, H\{\alpha, \beta\} >=< \gamma, [H\alpha, H\beta] > +\frac{1}{2}[H, H](\alpha, \beta, \gamma), \tag{16}$$

$$\sum_{Cycl(\alpha,\beta,\gamma)} < \{\{\alpha, \beta\}, \gamma\}, X >= [H, L_X H](\alpha, \beta, \gamma)+ \tag{17}$$

$$+\frac{1}{2} \sum_{Cycl(\alpha,\beta,\gamma)} [H, H](\alpha, \beta, d < \gamma, X >).$$

In these formulas, we used the notation $< \gamma, X >:=< X, \gamma >:= i(X)\gamma$, and the "Lie derivative" is extended from "forms" to "tensors" as in the case of differentiable manifolds. The computations needed to prove these formulas are long, and we do not give them here. For (16) they are mainly technical, but for (17) they pass by the utilization of a "contravariant exterior differential" given by

$$(d_* X)(\alpha, \beta) := (H\alpha) < X, \beta > -(H\beta) < X, \alpha > - < X, \{\alpha, \beta\} > . \tag{18}$$

Then (17) follows by equating the results of two evaluations of $d_*^2 X$. (See [10].)

It follows from Lemma 2 that we have

Theorem 1 *i) Suppose that* $(\chi, \mathcal{C})$ *is such that* $< X, \gamma >= 0$ *for all* γ *implies* $X = 0$. *Then,* $[H, H] = 0$ *iff*

$$H\{\alpha, \beta\} = [H\alpha, H\beta]. \tag{19}$$

ii) Suppose also that $(\chi, \mathcal{C})$ *is such that* $< X, \gamma >= 0$ *for all* X *implies* $\gamma = 0$. *Then, if* $[H, H] = 0$, *the bracket (15) is a Lie bracket on* Ω^1, *and*

$$(\alpha \in \Omega^1, f \in \Omega^0) \longmapsto (H\alpha)f := i(H\alpha)df$$

is a representation of this Lie algebra Ω^1 *on* Ω^0.

i) follows from (16). ii) follows from (17), if we notice that $[H, H] = 0$ implies $L_X[H, H] = 2[H, L_X H] = 0$. A converse of ii) can be established under special hypotheses only [10].

Thus, under the mild supplementary hypotheses of Theorem 1, the space Ω^1 of a Hamiltonian complex is a Lie algebra with the bracket (15), which, hereafter, we call the *dual bracket.* Furthermore, if $\forall \gamma < X, \gamma >= 0$ implies $X = 0$, and $\forall X < X, \gamma >= 0$ implies $\gamma = 0$, we say that the Gel'fand-Dorfman complex $(\chi, \mathcal{C})$ is *regular.* (Sometimes, the second condition of this regularity is dropped [1].)

Now, with regard to the sequence (ξ_i) of Proposition 1 which satisfies $K\xi_{i+1} = H\xi_i$, it is natural to look for situations where such a sequence can be constructed by a recurrence formula $\xi_{i+1} = A\xi_i$, $A \in L_{\mathbf{R}}(\Omega^1, \Omega^1)$, such that $KA = H$. Such a situation might be obtained as the "dual" of the case where $K = BH$ for $B \in L_{\mathbf{R}}(\chi, \chi)$. (The skew-symmetry of H and K is essential.) It was discovered by Magri and Kosmann that the following definition leads to the situation desired. (As a matter of fact, we give here a more general definition following [9].)

Definition 1 *Let* $(\chi, \mathcal{C}, H)$ *be a regular Hamiltonian complex. A* generalized Poisson-Nijenhuis structure *of this complex is a triple* $([\ ,\ ]', d' : \Omega^k \longrightarrow \Omega^{k+1}\ (\forall k),\ B \in L_{\mathbf{R}}(\chi, \chi))$, *where* $[\ ,\ ]'$ *is a Lie algebra structure of* χ, $B : (\chi, [\ .\ ]') \longrightarrow (\chi, [\ ,\ ])$ *is a homomorphism of Lie algebras, and* d' *is a coboundary on* $\mathcal{C}$ $(d'^2 = 0)$, *such that:*

a) the composition by B *leaves* Ω^1 *invariant, and* $H_1 = BH$ *is skew-symmetric;*

b)i(X) is a representation of $(\chi, [\ ,\ ]')$ *on* $(\mathcal{C}, d')$, *and* $\forall f \in \Omega^0$ *one has* $i(X)d'f = i(BX)df$;

c) the dual brackets of $((\chi, [\ ,\ ]), \mathcal{C}, d, H_1)$ *and* $((\chi, [\ ,\ ]'), \mathcal{C}, d', H)$ *are equal;*

d) if $\forall X, Y \in \chi$ *we define*

$$S(X,Y) := [X,Y]' - [X,Y]_B, \tag{20}$$

where

$$[X,Y]_B := [BX,Y] + [X,BY] - B[X,Y],$$

then, $\forall \gamma \in \Omega^1$, $\forall Y \in \chi$ $S(H\gamma, Y) = 0$.

Furthermore, if we define the Nijenhuis torsion *of* B *by*

$$\mathcal{N}_B(X,Y) := [BX,BY] - B[X,Y]_B, \tag{21}$$

a Poisson-Nijenhuis structure *of* $(\chi, \mathcal{C}, H)$ *is an element* $B \in L_{\mathbf{R}}(\chi,\chi)$ *with* $\mathcal{N}_B = 0$ *(a "Nijenhuis tensor") such that the conditions a), b), c) above are satisfied for* $[\ ,\ ]' = [\ ,\ ]_B$*, and for some operator* d'*. (*$\mathcal{N}_B = 0$ *implies that* $[\ ,\ ]_B$ *is a Lie bracket, and* B *is a homomorphism of Lie algebras. Condition d) holds, since* $S = 0$*.)*

Remarks 1 *1) In the cases where a formula of the type* (4) *defines* $d\lambda$ *the coboundary* d' *is uniquely determined by* $([\ ,\ ], B)$*. This happens if* $\Omega^k = L_{\mathbf{R}alt}(\underbrace{\chi \times \ldots \times \chi}_{k\,times}, \mathbf{R})$*, and, in particular, for the differential forms of a Banach manifold.*

2) The regularity of the complex ensures that Ω^1 *may be seen as a subspace of* $L_{\mathbf{R}}(\chi, \mathbf{R})$*, and* B *leaves this subspace invariant.*

In the nongeneralized case, the following result is due to Magri [7].

Theorem 2 *Let* $(\chi, \mathcal{C}, H)$ *be a regular Hamiltonian complex,* $[\ ,\ ]'$ *a second Lie algebra structure of* χ*,* d' *a second coboundary of* $\mathcal{C}$*, and* $B : (\chi, [\ ,\ ]') \longrightarrow (\chi, [\ ,\ ])$ *a homomorphism of Lie algebras such that* $H_1 = BH$ *is skew-symmetric,* $i(X)d'f = i(BX)df, \forall f \in \Omega^0$*,* B *leaves* Ω^1 *invariant, and condition d) of 1 is satisfied. Then* $([\ ,\ ]', d', B)$ *is a generalized Poisson-Nijenhuis structure iff* $\forall \alpha, \beta \in \Omega^1$ *and* $\forall X \in \chi$ *one has*

$$C_{(H,B)}(\alpha, X, \beta) := i((L_{H\alpha}B)X)\beta - i((L_{H\beta}B)X)\alpha + \tag{22}$$

$$+ i(BX)d(i(H\alpha)\beta) - i(X)d(i(BH\alpha)\beta) = 0.$$

This result is a straightforward consequence of the equality of the corresponding dual brackets evaluated on $X \in \chi$. But, it is important since $\mathcal{C}_{(H,B)}$ is expressed only by means of $(\chi, \mathcal{C}, H, B)$. Moreover, it takes the same form for the generalized and the usual Poisson-Nijenhuis structures. For historical reasons we may call $C_{(H,B)}$ the *Schouten invariant* [7]. If we are not interested in the generalized case we may define a Poisson- Nijenhuis structure on $(\chi, \mathcal{C}, H)$ to be a Nijenhuis "tensor" B such that HB is skew-symmetric and $C_{(H,B)} = 0$ [7].

3 The Poisson Hierarchy

The main property of a generalized Poisson-Nijenhuis structure is that it admits a whole bunch of Poisson (i.e., Hamiltonian) structures. Namely, let $\{(\chi, [\ ,\ ]), (\mathcal{C}, d), H; [\ ,\ ]', d', B\}$ be a regular Hamiltonian complex with a generalized Poisson-Nijenhuis structure called Π_0. If in Π_0 we replace H by $H_k := B^k \circ H$ $(k = 0, 1, 2, \ldots)$, we obtain a *hierarchy of structures* Π_k, and the fundamental result of Magri [7], [4] is

Theorem 3 *All* Π_k $(k = 0, 1, 2, \ldots)$ *are generalized Poisson-Nijenhuis structures and all the corresponding Hamiltonian structures* H_k *are pairwise compatible.*

Proof. (Sketch). If we assume that all H_i $(i < k)$ are skew-symmetric, we easily get $< H_k\alpha, \beta >=< H_{k-2}(\alpha \circ B), \beta \circ B >$, and we see that the skew–symmetry of $H_0 = H, H_1 = BH$ implies the skew-symmetry of all the H_k.

Furthermore, if pertinent computations are made, one discovers the following formulas [7]

$$(23) \qquad [H_1, H_1](\alpha, \beta, \gamma) = [H, H](\alpha \circ B, \beta, \gamma \circ B) -$$

$$- C_{(H,B)}(\alpha, H\beta, \gamma \circ B) + < \gamma, \mathcal{N}_B(H\alpha, H\beta) >,$$

$$(24) \qquad C_{(H_1,B)}(\alpha, X, \beta) = C_{(H,B)}(\alpha, X, \beta \circ B) - < \beta, \mathcal{N}_B(H\alpha, X) > .$$

With (20) and (21), we have $\mathcal{N}_B(X, Y) = BS(X, Y)$ and, then, using d) of Definition 1, we see from (23) and (24) that H_1 is Hamiltonian and Π_1 is a generalized Poisson-Nijenhuis structure.

Recursively, the same result holds for all Π_k.

We still have to justify compatibility. This follows from one more bracket computation. Namely, if $Q \in \mathrm{L}_{\mathbf{R}}(\Omega^1, \chi)$ is skew-symmetric, the following formula holds [7]

$$(25) \qquad 2[BH,Q](\alpha,\beta,\gamma) = 2[H,Q](\alpha,\beta,\gamma \circ B) - C_{(Q,B)}(\alpha, H\beta, \gamma) +$$

$$+ C_{(Q,B)}(\beta, H\alpha, \gamma) + C_{(H,B)}(\alpha, Q\gamma, \beta).$$

Thus, the bracket $[H_{i+p}, H_i] = [B^p H_i, H_i]$ can be calculated by using (25) p times. At every step the right-hand side of the formula vanishes. Hence, we get $[H_{i+p}, H_i] = 0$. Q.e.d.

Now, it is clear that, under the "simply-connectedness" condition of Proposition 1, it is possible to obtain sequences of "first integrals in involution" as indicated by Proposition 1, for any regular Hamiltonian complex which has a Poisson-Nijenhuis structure.

Remark 3 *In the case of a true Poisson-Nijenhuis structure we have* $\mathcal{N}_B = 0$*, and it easily follows that one also has* $\mathcal{N}_{B^p} = 0 \quad (p = 0, 1, 2, \ldots)$*. Since, one also has* [9]*,*

$$(26) \qquad C_{(H,B^{p+1})}(\alpha, X, \beta) = C_{(H,B^p)}(\alpha, BX, \beta) +$$

$$+ C_{(H,B)}(\alpha \circ B^p, X, \beta) - \sum_{h=0}^{p-1} < \mathcal{N}_B(B^h H\alpha, X), \beta >,$$

we inductively get that all the structures $\Pi_{p,q} := (H_p, B^q)$ *are compatible Poisson-Nijenhuis structures. (In the generalized case* $\Pi_{p,q}$ *are not defined for* $q \neq 1$*.) In fact, the same is true for any structures* $\tilde{\Pi} := ((\sum_{i=0}^{\infty} a_i B^i) \circ H, \sum_{j=0}^{\infty} b_j B^j)$*, where the series involved are convergent power series with constant coefficients* [9]*.*

4 The Differential-Geometric Aspect

The usefulness of Poisson-Nijenhuis structures in the theory of dynamical Hamiltonian systems leads to the question of also studying these structures as objects of Differential Geometry. For this purpose, we will conveniently restrict our framework, and consider the Poisson-Nijenhuis structures of finite dimensional Lie algebroids only, and, in particular, of differentiable manifolds. All the manifolds and bundles are finite dimensional in what follows.

A *Lie algebroid* is a vector bundle $\pi : E \rightarrow M$ with a Lie bracket $[\ ,\ ]_E$ on the space ΓE of the global cross-sections of E and with a vector bundle morphism (*anchor*) $A : E \rightarrow TM$ which is Lie bracket preserving and satisfies

$$(27) \qquad [s_1, fs_2]_E = ((As_1)f)s_2 + f[s_1, s_2]_E \qquad (s_{1,2} \in \Gamma E).$$

For instance, this happens for $E = TM, A = Id$. Generally, if we use $E, E^*, \wedge E^*, \wedge E$ as $TM, T^*M, \wedge T^*M, \wedge TM$ are classically used, we obtain a differential calculus with operators such as d_E (exterior derivative), L^E_s (Lie derivative), $i(s)$ $(s \in \Gamma E)$, and these operators have all the usual properties. Indeed, since the linear spaces used now are reflexive, if we put $L^E_s f := (As)f \quad (s \in \Gamma E,\ f \in C^\infty(M))$, and take the usual algebraic $i(s)$, (4) defines d_E and $L^E_s := d_E i(s) + i(s) d_E$ defines the Lie derivative. d_E also acts on $f \in \wedge^0 E^* := C^\infty(M)$ by $(d_E f)(s) = (As)f$. We also have a Schouten-Nijenhuis bracket defined by the usual extension of the formula

$$(28) \qquad [s_1 \wedge \ldots \wedge s_k, s'_1 \wedge \ldots \wedge s'_h]_E =$$

$$= (-1)^{k+1} \sum_{i=1}^{k} \sum_{j=1}^{h} (-1)^{i+j} [s_i, s'_j] \wedge s_1 \wedge \ldots \wedge \hat{s}_i \wedge \ldots \wedge s_k \wedge s'_1 \wedge \ldots \wedge \hat{s}'_j \wedge \ldots \wedge s'_h.$$

A complete study of these operations can be found in [4], [6]. We will use the names E-tensors, E- forms, etc. for the analogs of tensors, forms, etc. of a differentiable manifold M.

Any Lie algebroid $(E, [\ ,\ ], A)$ has an associated cochain complex $(\mathcal{C}, d) := (\sum_{h=0}^{r} \wedge^h E^*, d_E)$ $(r = rankE,\ \wedge^0 E^* := C^\infty(M))$, and it easily follows that $i(s)$ is a representation of $\chi := \Gamma(E)$ on $\mathcal{C}$. We may define a *Poisson structure* of E to be a Hamiltonian structure of $(\chi, \mathcal{C})$, and, because of reflexivity, we may see H as an element $P \in \Gamma \wedge^2 E$, called the *Poisson* E- *bivector*. Namely, for the bivector P we get $H := \sharp_P$ by $< \sharp_P \alpha, \beta >= P(\alpha, \beta)$ $(\alpha, \beta \in \Gamma E^*)$. The condition $[H, H] = 0$ becomes $[P, P]_E = 0$, where the bracket is the Schouten-Nijenhuis bracket. For the particular case of (TM, Id) we regain the usual definition of a *Poisson manifold*. Similarly, the compatibility condition of two Poisson structures becomes $[P_1, P_2]_E = 0$.

Furthermore, if (E, P) is a *Poisson-Lie algebroid* (i.e., a Lie algebroid with a Poisson structure on it), we obtain the dual Lie bracket $\{\alpha, \beta\}$ $(\alpha, \beta \in \Gamma E^*)$ of Lemma 2. Together with the anchor map $A \circ \sharp_P$, this bracket makes E^* into the *dual Lie algebroid*, and $\{\alpha, \beta\}$ is the *dual bracket*. Moreover, since the Gel'fand-Dorfman complex $(\Gamma E, \mathcal{C}(E))$ is obviously regular, the results of Theorem 1 hold for any H associated to a bivector. (Notice that the Jacobi identity for $\{\alpha, \beta\}$ is not enough to ensure $[H, H] = 0$.) On the other hand, the dual Lie algebroid E^* also has the usual differential calculus including a Schouten-Nijenhuis bracket $\{\lambda, \mu\}_{E^*}$ for $\lambda, \mu \in \Gamma \wedge E^*$, and Theorem 1 holds again. In particular, we have

$$\sharp_P\{\lambda, \mu\}_{E^*} = [\sharp_P\lambda, \sharp_P\mu]_E, \tag{29}$$

where $\sharp_P$ is extended from 1-forms to k-forms.

Now, if we look at the definition of a generalized Poisson-Nijenhuis structure, we see that it makes sense to introduce some restrictions again. Namely, we will still ask for a second Lie algebra structure $[\ ,\ ]'$ of ΓE, but we will use $B \in Hom(E, E)$ rather than just $B \in L_{\mathbf{R}}(\Gamma E, \Gamma E)$, and $A \circ B$ will be a second anchor such that $(E, [\ ,\ ]', A \circ B)$ is a Lie algebroid structure. Then, we shall use the exterior differential of this Lie algebroid structure as d', and this is compatible with the general definition of generalized Poisson-Nijenhuis structures. If the conditions of this latter definition are satisfied (i.e., $\sharp_{P_1} := B \circ \sharp_P$ is skew-symmetric; the dual brackets of $([\ ,\ ]_E, P_1)$ and $([\ ,\ ]'_E, P)$ are equal; $i(\sharp_P\gamma)S_B = 0$), $([\ ,\ ]', B)$ is called a generalized Poisson-Nijenhuis structure of (E, P). Furthermore, if B is a "Nijenhuis tensor" (i.e., $\mathcal{N}_B = 0$), and $[\ ,\ ]' = [\ ,\ ]_B$, (P, B) is a Poisson-Nijenhuis structure of E, and (E, P, B) is a *Poisson-Nijenhuis Lie algebroid.* As a particular case, if $E = TM$, we shall speak of a *Poisson-Nijenhuis manifold* M.

Because of regularity, the Poisson-Nijenhuis algebroids and manifolds are characterized by the vanishing of the Schouten invariant (22), and they have a Poisson hierarchy.

Now, we will discuss a new notion which is useful in the geometry of Poisson-Nijenhuis algebroids.

Definition 2 *Let $(E, [\ ,\ ]_E, A; P)$ be a Poisson-Lie algebroid, and $\omega \in \Gamma \wedge^2 E^*$ a $2-E$-form. Then ω is called a* complementary 2-form*, and (E, P, ω) is a* complemented Lie-Poisson algebroid *(*complemented Poisson manifold*) if*

$$\{\omega, \omega\}_{E^*} = 0, \tag{30}$$

where the operation is the dual Schouten-Nijenhuis bracket of $[\ ,\ ]_E$, which satisfies (29).

The usefulness of complementary forms is due to

Theorem 4 *Let ω be a complementary 2-form of $(E, A; P)$, and put $B = \sharp_P \circ \flat_\omega$ ($\flat_\omega \in Hom(E, E^*)$, $\flat_\omega s := i(s)\omega$). Then, the bracket*

$$[s_1, s_2]'_E = [s_1, s_2]_B + \sharp_P i(s_1)i(s_2)d_E\omega \tag{31}$$

provides E with a Lie algebroid structure of anchor $A \circ B$.

Proof. (Sketch). If we look at ω as a skew-symmetric bivector of the complex $(\chi, \mathcal{C}) = ((\Gamma E^*, \{\ ,\ \}_{E^*}), (\Gamma \wedge E, d_{E^*}))$ of the dual Lie algebroid E^* of (E, P), Lemma 2 provides $\Gamma E = \Omega^1$ of our case with the bracket (15), now equal to

$$[s_1, s_2]^* := L^{E^*}_{\flat_\omega s_1} s_2 - L^{E^*}_{\flat_\omega s_2} s_1 - d_{E^*}(\omega(s_1, s_2)). \tag{32}$$

In view of Theorem 1, condition (30) ensures that (32) is a Lie bracket, and

$$\flat_\omega [s_1, s_2]^* = \{\flat_\omega s_1, \flat_\omega s_2\}_{E^*}. \tag{33}$$

Using also (29), we see that $[\ ,\ ]^*$ is a structure of a Lie algebroid with anchor $A \circ B$ on E.

The theorem is proven if we show that the brackets (31) and (32) are equal. This implies some lengthy calculations which, among other things, use the formulas

$$d_{E^*} s = -[P, s]_E = -L^E_s P, \tag{34}$$

(e.g., [4]). We skip over these calculations; they are given in our paper [10].

Now, the relations with Poisson-Nijenhuis structures are as follows:

Theorem 5 *With the notation of Theorem 4, if*

$$i(\sharp_P \alpha) i(\sharp_P \beta) d_E \omega = 0, \quad \forall \alpha, \beta \in \Gamma E^*, \tag{35}$$

$([\ ,\ ]', B)$ of (31) is a generalized Poisson-Nijenhuis structure. If the stronger condition

$$i(\sharp_P \alpha) d_E \omega = 0, \quad \forall \alpha \in \Gamma E^* \tag{36}$$

holds (in particular, if $d_E\omega = 0$), and if the anchor A of E is injective, (P, B) is a Poisson-Nijenhuis structure of E.

Proof. (Sketch). The "tensor" S of condition d) of the definition of a generalized Poisson-Nijenhuis structure is now $\sharp_P i(s_1)i(s_2)d_E\omega$, and (35) implies $i(\sharp_P\gamma)S = 0$ $(\gamma \in \Gamma E^*)$, as we needed it to be. And we have $S = 0$ if (36) holds. In the latter case we have

$$[ABs_1, ABs_2] = AB[s_1, s_2]'_E = AB[s_1, s_2]_B =$$

$$= A([Bs_1, Bs_2]_E - \mathcal{N}_B(s_1, s_2)) = [ABs_1, ABs_2] - A\mathcal{N}_B(s_1, s_2),$$

which implies $\mathcal{N}_B = 0$, if A is injective.

The rest of the proof consists in checking that the Schouten invariant vanishes. Again, the computation is too long to be given here, and we refer to [10] for this computation.

5 Poisson-Nijenhuis Manifolds

Now, we will use Theorem 4 and Theorem 5 in order to obtain several examples of Poisson-Nijenhuis manifolds.

Let M be a differentiable manifold endowed with a symplectic form σ. Let B be a $(1,1)$-tensor field of M. Then, the 2-form ω defined by

$$\flat_\omega = \flat_\sigma \circ B \tag{37}$$

will be called the *associated form* of (σ, B), and we have

Theorem 6 *If (M, σ, B) is a Poisson-Nijenhuis manifold, the associated 2-form ω is a closed complementary form of σ. Conversely, if ω is a closed complementary 2-form of (M, σ), and $B := \sharp_\sigma \circ \flat_\omega$, then (M, σ, B) is a Poisson-Nijenhuis manifold.*

Proof. The second assertion is directly implied by Theorems 4 and 5. For the first assertion, $\{\omega, \omega\}_\sigma = 0$ is equivalent to $[\sharp_\sigma\omega, \sharp_\sigma\omega] = 0$ i.e., $\sharp_\sigma\omega = P_1$ is a Poisson bivector. But, if (M, σ, B) is Poisson-Nijenhuis, P_1 belongs to the Poisson hierarchy, and it must be Poisson, indeed. Thus, ω is a complementary 2-form. It is also closed, since one may check that $\{\omega, \omega\}_\sigma = d\omega$.

Corollary 1 *Let M be a compact Hermitian symmetric space, with metric g and Kähler form σ. Then, any harmonic 2-form ω of M is associated with a Poisson-Nijenhuis structure $(M, \sigma, B = \sharp_\sigma \circ \flat_\omega)$.*

Proof. Since $d\omega = 0$, we must only check that $\{\omega, \omega\}_\sigma = 0$. By a formula of Koszul [5], we can get [8]

$$\{\omega, \omega\}_\sigma = 2(\delta\omega) \wedge \omega - \delta(\omega \wedge \omega), \tag{38}$$

where δ is the *symplectic codifferential* with respect to σ. In the Kähler case, $\delta = C\delta_g C$ where δ_g is the *Riemannian codifferential* and C is the transformation of the arguments of a form by the complex structure of M. Since ω is harmonic, $\delta\omega = 0$. But, on a compact Hermitian symmetric space $\omega \wedge \omega$ is harmonic as well, and $\delta(\omega \wedge \omega) = 0$. Therefore, $\{\omega, \omega\}_\sigma = 0$, and we are done.

Corollary 2 *Let (M, σ) be a symplectic manifold, and $\mathcal{F}$ be a foliation of M such that: i) the leaves of $\mathcal{F}$ are symplectic submanifolds; ii) for any pair of $\mathcal{F}$-projectable vector fields X, Y which are σ-orthogonal to $\mathcal{F}$, $\sigma(X, Y)$ is constant along the leaves. Let B be the projection of TM onto $T\mathcal{F}$ according to $TM = N\mathcal{F} \oplus T\mathcal{F}$, where $N\mathcal{F}$ is the σ-orthogonal bundle of $T\mathcal{F}$. Then (M, σ, B) is a Poisson-Nijenhuis manifold.*

Proof. In the given configuration, σ decomposes as $\sigma = \sigma_{T\mathcal{F}} + \sigma_{N\mathcal{F}}$, and condition ii) ensures that $\sigma_{T\mathcal{F}}$ is closed. On the other hand, $\sharp_\sigma \sigma_{T\mathcal{F}}$ is just the Poisson structure provided by the Poisson brackets along the symplectic leaves of $\mathcal{F}$ (see condition i)). (This is a so-called Dirac bracket, and the interest in Dirac brackets explains our interest in Corollary 2.) Therefore, $\sharp_\sigma \sigma_{T\mathcal{F}}$ is a Poisson bivector, hence $\sigma_{T\mathcal{F}}$ is a complementary form, and the result follows.

Another example is given by

Proposition 2 *Let (M, P) be a regular Poisson manifold, and ω be a 2-form of M. If there exists a Poisson connection ∇ (i.e., $\nabla P = 0$, and ∇ is torsionless) such that $\nabla\omega = 0$, ω is a complementary 2-form which provides (M, P) with a Poisson-Nijenhuis structure.*

Proof. The Koszul formula (38) extends to Poisson manifolds if we take $\delta := i(P)d - di(P)$. The corresponding local coordinate expression of $\{\omega, \omega\}_P$ contains only terms with ∇P and $\nabla\omega$. Hence, ω is a closed complementary 2-form. This proves Proposition 2.

Corollary 3 *A Riemannian manifold (M, g), which has a parallel 2-form ω, has a natural Poisson-Nijenhuis structure.*

Proof. The Poisson bivector will be $P := \sharp_g \omega$, and ω is complementary to P.

The number of examples can be increased. We give only one more example:

Proposition 3 *Any solution of the classical Yang-Baxter equation* $[\mathbf{r}, \mathbf{r}] = 0$ *of a finite dimensional Lie algebra* $\mathcal{G}$ *may be seen as a closed complementary* 2*-form of the Lie-Poisson structure* P *of the dual space* $\mathcal{G}^*$*.*

Proof. The bracket $[\mathbf{r}, \mathbf{r}]$ is defined as the extension of the Lie bracket of $\mathcal{G}$ to a Schouten-Nijenhuis bracket, and $\mathbf{r}$ may be seen as a "constant" 2-form on $\mathcal{G}$. An inspection of the definition of P immediately shows that $\{\ ,\ \}_P$ and $[\ ,\ ]$ are equal on constant forms. This implies the assertion made by Proposition 3.

Let us finish by the following results:

Proposition 4 *Let* $\varphi : (M_1, P_1) \longrightarrow (M_2, P_2)$ *be a Poisson mapping, and let* ω_2 *be a complementary* 2*-form of* P_2*. Then,* $\omega_1 := \varphi^* \omega_2$ *is a complementary* 2*-form of* P_1*. Furthermore, if* ω_2 *defines a (generalized) Poisson-Nijenhuis structure of* (M_2, P_2)*, and if*

$$\varphi_*(im\sharp_{P_1}) = im\sharp_{P_2} \tag{39}$$

(e.g., if (M_2, P_2) *is symplectic), then* ω_1 *also defines a (generalized) Poisson-Nijenhuis structure of* (M_1, P_1)*.*

Corollary 4 *1) The symplectic realizations of complemented Poisson manifolds are complemented symplectic manifolds. 2) If* (M, P, ω) *is a complemented Poisson manifold, the pullback of* ω *to either a symplectic leaf* S *of* P *or the local transversal germ* N *at* $x_0 \in S$ *is a complementary* 2*-form of* S *or* N*, respectively.*

Proposition 4 easily follows from the fact that, if φ is Poisson, φ^* is compatible with the bracket $\{\ ,\ \}$ of forms. Then, using Theorem 5, Corollary 4 follows from Proposition 4 since the inclusions of S and N in M are Poisson mappings.

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