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A universal symmetry structure in open string theory

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Abstract

In this paper, we arrive from different starting points at the conclusion that the symmetry given by an action of the Grothendieck-Teichmüller group GT on the so called extended moduli space of string theory can not be physical - in the sense that it does not survive the inclusion of general nonperturbative vacua given by boundary condition on the level of two dimensional conformal field theory - but has to be extended to a quantum symmetry given by a self-dual, noncommutative, and noncocommutative Hopf algebra $\mathcal{H}_{GT}$. First, we show that a class of two dimensional boundary conformal field theories always uniquely defines a trialgebra and find $\mathcal{H}_{GT}$ as the universal symmetry of such trialgebras (in analogy to the definition of GT as the universal symmetry of quasi-triangular quasi-Hopf algebras). Second, we argue in a more heuristic approach that the $\mathcal{H}_{GT}$ symmetry can also be found in a more geometric picture using the language of gerbes. Finally, we will see that the fact that the GT symmetry can not be physical in the above sense can also be seen by trying to understand why the action of GT on the Duflo-Kirillov isomorphism which is considered in [Kon 1999] trivializes.

1 Introduction

For the case of perturbative vacua in string theory, i.e. two dimensional superconformal field theory, there is a large body of results on certain symmetry structures which are linked to these. First of all, there are a number of reconstruction theorems (see [CP], [FK]) which allow to view the fusion structure of these superconformal theories as given by the representation category of a quasitriangular quasi-Hopf algebra. Beyond this, one can find a universal symmetry of all quasitriangular quasi-Hopf algebras in the form of the so called Grothendieck-Teichmüller group GT (see [Dri]). The basic idea is the following: Ask for the possibility to transform the R-matrix and the Drinfeld associator α of a quasi-triangular quasi-Hopf algebra H while keeping the rest of the structure of H completely fixed. In order to define a nontrivial group from these transformations, one has to take a certain closure by including formal deformations of R and α in the sense of a class of formal power series. Doing this, one arrives at GT (for the full technical details of the definition of GT we refer the reader to [Dri], for a convenient short description see also [CP]). As a consequence of these two levels of symmetry structure - the Hopf algebra structures of two dimensional superconformal field theory and the universal symmetry of GT - there exists an action of GT on the so called extended moduli space of two dimensional superconformal field theories (see [Kon 1994], [Wit 1991] for the introduction and general structure of this space and [Kon 1999], [KoSo] for the GT-action on it). Mathematically, this space is expected to be describable as the moduli space of a triangulated version of A_∞-categories (see [Kon 1994]). On the one hand, this extended moduli space has been related to D-branes, i.e. to boundary conditions for open strings (see e.g. [GZ], [Laz]). On the other hand, even this extended moduli space (it is extended in comparison to the usual moduli space of a two dimensional superconformal field theory which is locally generated by the truely marginal operators) is restricted to see finitely many degrees of freedom, only (see [Kon 1994], [Wit 1991]), i.e. it basically describes a topological string theory.

An understanding of the structure of the relevant moduli space of vacua for string theory is considered to be of outermost relevance for the long term goal of a full fledged nonperturbative and background independent formulation of the theory. Finding a symmetry like the GT action on extended moduli space is of deep principal interest. The decisive question is, then, if this symmetry is of physical relevance. If it would be, one could hope that it tells an important lesson about the ultimate nature of a complete formulation of string theory. After all, we know from nearly all of the examples of theories in physics that these are to a high degree determined by knowing the fundamental symmetry principle. The examples range from the Galilei invariance of classical mechanics and the Lorentz invariance of the Maxwell equations up to the gauge symmetries in elementary particle physics and the diffeomorphism invariance of general relativity.

A first step toward the question of a physical relevance of the GT-symmetry is taken by asking the following more concrete question, instead: Does the GT-action extend to a physically relevant full moduli space of open string theory, i.e. to general boundary conditions in the non-topological setting (formulated more concisely: to a general moduli space of two dimensional boundary conformal field theories)? We will sometimes call this space simply the full moduli space, in the sequel.

We will in this paper show that the GT-symmetry can not extend to the full moduli space, i.e. in this sense the GT-symmetry can not be physically relevant (one can not take the GT-invariance as a candidate for a physical principle in full non-topological string theory). But we find that there is a self-dual, noncommutative, and noncocommutative Hopf algebra $\mathcal{H}_{GT}$ which extends the GT-symmetry to the full moduli space. In more physical terms, we can view this as saying that the GT-symmetry does not survive the inclusion of general nonperturbative vacua but that in the full setting a quantum analog of this classical symmetry is seen.

We will first arrive at this result by starting from the question of what takes the role of the Hopf algebra symmetries behind two dimensional superconformal field theories if we pass to the setting of boundary conformal field theories. We then ask for the universal symmetry of these algebraic structures, replacing the GT-symmetry of quasi-triangular quasi-Hopf algebras. This will be the content of section 2.

In section 3, we approach the same problem from a more geometrical perspective: Boundary conditions in the two dimensional conformal field theories geometrically are linked to the appearance of a 2-form field B with 3-form field strength H. Compared to closed 2-forms (which define connections on vector bundles), closed 3-forms - like H - are linked to connections on gerbes which, roughly speaking, can be seen as principal bundles with categories as their fibers. Paralleling the way in which vector bundles define by Serre duality finite-dimensional projective modules over the algebra $\mathcal{C}^{\infty}(M)$ of smooth functions on the base manifold M, a class of gerbes induces $\mathcal{C}^{\infty}(M)$-linear ring categories. We will see in a simple but instructive example of such a category $\mathcal{C}$ that the analog of the Hochschild cohomology of a projective module for $\mathcal{C}$ is, again, related to an action of the Hopf algebra $\mathcal{H}_{GT}$. So, we can argue that the more geometrical picture of gerbes leads to the same conclusion concerning a universal $\mathcal{H}_{GT}$-symmetry.

Section 4 is devoted to the Duflo-Kirillov isomorphism. In [Kon 1999] the Duflo-Kirillov isomorphism is approached with the aim of showing an action of GT on it as a concrete ramification of the GT-action on the moduli space of deformation quantizations (which is closely related to moduli spaces in conformal field theory by the results of [CaFe]). But it turns out, finally, that the transformations which satisfy conditions, necessary for the GT-action, all vanish. We will present an argument which leads to the view that the fact that the GT-action on the Duflo-Kirillov isomorphism is trivial has to be attributed to quantization, i.e. the GT-symmetry does not survive the process of quantization. This is completely in accordance with the results of the two foregoing sections which also show that the GT-symmetry can not survive the inclusion of general nonperturbative vacua. Finally, we discuss in this section the conjecture that the $\mathcal{H}_{GT}$-symmetry should not only be stable against quantization but that in this case the classical and the quantum description should even be equivalent.

Section 5 contains some concluding remarks.

2 Trialgebras and boundary conformal field theory

Two well known results are the starting point for the considerations in this section: First, starting from a three dimensional topological quantum field theory, one can get a two dimensional conformal field theory as living on the boundary of the 3-manifold ([FFFS], [Wit 1989]). By the fact that the boundary of a boundary is empty, one can get only conformal field theories without boundary in this way. Second, three dimensional topological field theories can be formulated in a purely algebraic way as vector space valued functors on the category of three dimensional cobordisms ([Ati]) and these functors can in turn be shown to be constructible from modular categories (see [Tur]). So, two dimensional conformal field theories can be constructed starting from certain modular categories (see [Seg] for a different approach leading to this result). In [KL] an algebraic framework is presented in order to extend this approach to the case of two dimensional boundary conformal field theories. The algebraic notion of topological quantum field theory of [Ati] is extended (algebraically by using double functors, i.e. morphisms between double categories) to include 3-manifolds which do not only have boundaries but where also corners on the boundaries are allowed. This introduces the necessary freedom for two dimensional boundary conformal field theories to appear on the boundary of the 3-manifold (see [KL] for the details). The central result of [KL] is the following:

The extended topological quantum field theories in the above sense are in one to one correspondence with $\mathbb{C}$-linear, bounded (i.e. equivalent to a category of finite dimensional modules over a finite dimensional algebra), balanced, abelian, rigid, braided, monoidal categories $\mathcal{C}$ together with a Hopf algebra object H in them (see, again, [KL] for the detailed definitions).

So, the algebraic framework introduced in [KL] leads to the conclusion that at least a class of two dimensional boundary conformal field theories can be defined starting from such categories $\mathcal{C}$ together with a Hopf algebra object H. It is shown in [KL] that the input data of [Tur] give a special case of the data $(\mathcal{C}, H)$. This means that the data, given by the introduction of a Hopf algebra object H, are directly linked to boundary conditions for the conformal field theory.

In order to proceed, we now introduce a new algebraic concept, called a trialgebra:

Definition 1 *A trialgebra $(A, *, \Delta, \cdot)$ with $*$ and $\cdot$ associative products on a vector space A (where $*$ may be partially defined, only) and Δ a coassociative coproduct on A is given if both $(A, *, \Delta)$ and $(A, \cdot, \Delta)$ are bialgebras and the following compatibility condition between the products is satisfied for arbitrary elements $a, b, c, d \in A$:*

$$(a * b) \cdot (c * d) = (a \cdot c) * (b \cdot d)$$

whenever both sides are defined.

Trialgebras were first suggested in [CF] as an algebraic means for the construction of four dimensional topological quantum field theories. It was observed there that the representation categories of trialgebras have the structure of so called Hopf algebra categories (see [CF]) and it was later shown explicitly in [CKS] that from the data of a Hopf category one can, indeed, construct a four dimensional topological quantum field theory. The first explicit examples of trialgebras were constructed in [GS 2000a] and [GS 2000b] by applying deformation theory, once again, to the function algebra on the Manin plane and some of the classical examples of quantum algebras and function algebras on quantum groups. In [GS2001] it was shown that one of the trialgebras constructed in this way appears as a symmetry of a two dimensional spin system. Besides this, the same trialgebra can also be found as a symmetry of a certain system of infinitely many coupled q-deformed harmonic oscillators.

We then have the following lemma:

Lemma 1 *The data $(\mathcal{C}, H)$ of [KL] define for each choice of $\mathcal{C}$ and H a trialgebra.*

Proof. By definition of $\mathcal{C}$, $\mathcal{C}$ is equivalent to a category of representations of an algebra A (see [KL]). So, on each object of $\mathcal{C}$ there is a representation $\cdot$ of the product of A. Now, the Hopf algebra object H is an object in $\mathcal{C}$ together with a product $*$ and a coproduct Δ. So, we have two associative products and a coassociative coproduct given. It remains to check compatibilities. The compatibility of Δ and $*$ is given by the definition of a Hopf algebra object. The compatibility of Δ with $\cdot$ follows from the fact that Δ is a morphism in $\mathcal{C}$. Finally, the compatibility of $\cdot$ and $*$ follows also from the fact that $*$ is a morphism in $\mathcal{C}$ together with the canonical extension of $\cdot$ to the tensor product of a representation of A with itself. ■

Remark 1 *It was already observed in [CF] that trialgebras define certain monoidal bicategories. While two dimensional conformal field theories without boundary arise from modular categories, the above lemma shows that the two dimensional boundary conformal field theories which can be defined through the construction of [KL] correspond to a special class of monoidal bicategories.*

Remark 2 *One might ask which physically interesting conformal field theories can arise from the [KL] approach. Since the data of [Tur] give a special case, it is clear that at least the type of conformal field theory induced by Chern-Simons theory is contained in this class, i.e. the WZW model.*

As the next step, we can - in analogy to [Dri] - ask for the universal symmetry of (quasi-) trialgebras (where we allow for the coproduct and one of the products to be quasi-associative, only) which are quasi-triangular (respectively, coquasi-triangular) with respect to Δ and one of the products. This question was considered in [Sch] and it was found that instead of GT we get a Hopf algebra $\mathcal{H}_{GT}$ which was shown to be self-dual, noncommutative, and noncocommutative. In addition, $\mathcal{H}_{GT}$ can be shown to be a sub-Hopf algebra of the Drinfeld double of GT.

Together with the above lemma, this shows that the action of the classical group GT is too restrictive to hold for general two dimensional boundary conformal field theories. Instead, we have a quantum symmetry in the form of an action of the Hopf algebra $\mathcal{H}_{GT}$, here. In this sense, the GT-symmetry on extended moduli space can not be physical since it does not extend to general nonperturbative vacua. The GT-symmetry is extended in this case to a quantum symmetry $\mathcal{H}_{GT}$.

Remark 3 *Since the associator can be seen to basically determine the structure of GT already (see [Dri], [Kon 1999]), the assumption of quasi-triangularity and coquasi-triangularity for the trialgebra which we made above is not a decisive restriction.*

We conclude this section by a remark on some additional structural properties of trialgebras: Though examples of trialgebras can be constructed by further deformation quantization of quantum groups and the concept carries some analogy to Hopf algebras (e.g dual pairings of trialgebras can be introduced and a system of coupled matrix equations can be given, replacing the RTT-relations in the trialgebraic case, see [GS 2000b]), trialgebras are in several respects quite different objects.

To give a simple example of this kind, we mention that nontrivial trialgebras are never unital. By a unital trialgebra one would mean one with a unit element 1 which is compatible with both products where the algebraic notion of compatibility, usually used in the case of two products, is the requirement that one and the same element 1 should act as the unit for both products. Such unital trialgebras are always trivial in the sense that they are commutative bialgebras, i.e. the two products necessarily agree and the product is commutative. The proof consists of a simple Eckmann-Hilton type argument, for if such an element 1 would exist, we would have

$$\begin{aligned} a \cdot b &= (a * 1) \cdot (1 * b) \\ &= (a \cdot 1) * (1 \cdot b) \\ &= a * b \\ &= (1 * a) \cdot (b * 1) \\ &= b * a \end{aligned}$$

A much more involved result, showing that trialgebras are objects with new and interesting algebraic properties, is given by the fact that - loosely speaking - trialgebras can not be further deformed to algebraic structures with e.g. two asscociative products and two coassociative coproducts, all linked in a compatible way (see [Sch]). In this sense, trialgebras are the end of the story in the deformation process leading from groups to Hopf algebras to trialgebras. We call this property of trialgebras *ultrarigidity*.

3 Cohomology of gerbes

Geometrically, the introduction of boundary conditions in two dimensional conformal field theory is related to the appearance of a 2-form field B with a closed 3-form H as its field strength. While closed 2-forms lead to connections on vector bundles, closed 3-forms are interpreted as a kind of connection on a gerbe (i.e. a kind of principal bundle with categories as fibers). In [BM] a detailed development of the necessary geometrical theory of gerbes is started.

We will in this section be rather brief. Our aim is not to introduce the heavy machinery of gerbes but mainly to show that the results, which have been given in a completely rigorous way from an algebraic approach in the foregoing section, can also be seen in this more geometrical picture.

Following [BM], we consider a gerbe $\mathcal{P}$ over a base manifold M with cover $(U_i)_{i \in I}$. Denote by $\mathcal{P}_{U_i}$ the fiber categories of $\mathcal{P}$ and let

$$U_{ij} = U_i \cap U_j$$

Then sections of $\mathcal{P}$ can be written as pairs of data (x_i, ϕ_{ij}) with x_i an object in $\mathcal{P}_{U_i}$ and

$$\phi_{ij} : x_j|_{U_{ij}} \to x_i|_{U_{ij}}$$

a morphism in $\mathcal{P}_{U_{ij}}$.

Let us now assume that $\mathcal{P}$ is a gerbe with gauge group (see [BM] for this concept) $GL(n,\mathbb{C})$ (or $U(n)$ for the unitary case). Then we have the following result:

Lemma 2 *For $\mathcal{P}$ a gerbe with gauge group $GL(n,\mathbb{C})$ (or $U(n)$) over a smooth base manifold M, the sections of $\mathcal{P}$ define the structure of a $\mathcal{C}^{\infty}(M)$-linear ring category.*

Proof. $\mathcal{P}$ leads to a vector bundle analog $\widetilde{\mathcal{P}}$ of a gerbe with sections $\left(\widetilde{x}_i, \widetilde{\phi}_{ij}\right)$ where the pointwise restrictions of x_i and ϕ_{ij} lead to vector spaces and linear maps, respectively. For a ring category we need a tensor product $\otimes$ and a direct sum $\oplus$ with the obvious properties. But these are induced from $\oplus$ and $\otimes$ of the category of $\mathbb{C}$-vector spaces. So, the category $\mathcal{S}$ of sections is a ring category. $\mathbb{C}$-linearity of $\mathcal{S}$ follows from the fact that pointwise we have $\mathbb{C}$-vector spaces and $\mathbb{C}$-linear maps. The stronger result of $\mathcal{C}^{\infty}(M)$-linearity of $\mathcal{S}$ is a consequence of the fact that we have smooth data over M. ■

Remark 4 *This result is the analog of the fact that - by Serre duality - vector bundles induce finite dimensional projective modules over $\mathcal{C}^{\infty}(M)$.*

Since there is a highly developed theory of Hochschild cohomology of projective modules (see [GeSch]), we can now ask for the analog of Hochschild cohomology for such $\mathcal{C}^{\infty}(M)$-linear ring categories $\mathcal{S}$. We will restrict to a simple but instructive example, here (and make a remark about the more general case, below). Namely, consider the case of such a category, consisting only of one object. Since the morphism classes of $\mathcal{S}$ are $\mathbb{C}$-linear, especially, it is immediately clear that $\mathcal{S}$ can be reinterpreted as a $\mathbb{C}$-algebra A where the product $\cdot$ of A is just the composition of $\mathcal{S}$. The tensor product gives a second product $*$ on A and, using well known coherence theorems, we can assume without loss of generality that $*$ is associative, too. We will call such a structure $(A,\cdot,*)$ a double algebra, in the sequel. Observe that the compatibility of $\cdot$ and $*$ is, once again,

$$(a\cdot b)*(c\cdot d)=(a*c)\cdot(b*d) \tag{1}$$

for $a,b,c,d\in A$.

Remark 5 *Since our aim is to consider deformation theory, in the sequel, we forget about the additional sum* $\oplus$ *for the moment because the additive structures remain fixed, anyway.*

Let us now come to the question of the cohomology of such a double algebra. It is clear that $(A, \cdot, *)$ defines two Hochschild complexes - one for each product - but these are not independent but linked by compatibility conditions for the cohomology groups induced from condition (1). E.g. for the second cohomology of $(A, \cdot, *)$ we do not have arbitrary pairs (B_1, B_2) where B_1 is a Hochschild 2-cocycle for $\cdot$ and B_2 a Hochschild 2-cocycle for $*$ but only pairs where $B_1$and B_2 satisfy the constraint

$$\begin{aligned} & (a \cdot b) * B_1(c, d) + B_1(a, b) * (c \cdot d) - B_1(a * c, b * d) \\ = \; & (a * c) \cdot B_2(b, d) + B_2(a, c) \cdot (b * d) - B_2(a \cdot b, c \cdot d) \end{aligned}$$

calculated from the first order perturbation theory of condition (1).

Remark 6 *Actually, the structure of* $\mathcal{S}$ *induces for* A *even the structure of a double algebra over* $\mathcal{C}^\infty(M)$ *(instead of simply a* $\mathbb{C}$*-linear double algebra). One could therefore suppose at first sight that one can actually introduce, in addition, deformations of the product in* $\mathcal{C}^\infty(M)$ *and that therefore three different Hochschild complexes are involved. But observe that in the undeformed case - by the composition of the linear maps* $\widetilde{\phi}_{ij}$ *- the composition in* $\mathcal{S}$ *and the product on* $\mathcal{C}^\infty(M)$ *are not independent but the product of* $\mathcal{C}^\infty(M)$ *can be seen as induced from the composition which we would have for the case of* 1×1 *matrices. In this sense, we do not consider the product on* $\mathcal{C}^\infty(M)$ *as another possibility for deformations and therefore restrict in the cohomology theory to cohomology of a* $\mathbb{C}$*-linear double algebra.*

Next, remember that on Hochschild cohomology of associative algebras there is - by the Deligne conjecture - a hidden action of the Grothendieck-Teichmüller group GT (see [Kon 1999] where a proof of the Deligne conjecture is announced). Basically, the action of GT can be imagined as deriving from the possibility to weaken the associative product to a quasi-associative one and from the transformation possibilities for such an associator. We can therefore give the following heuristic argument for the case of two associative products $\cdot$ and $*$: Here, we have the possibility to introduce two different associators α, β and the compatibility condition for α and β is

$$[\alpha, \beta] = 0 \tag{2}$$

(observe that α and β operate on the same space, i.e. we can introduce a commutator for the successive application of the two maps). The constraint resulting from condition (2) for transformations of α and β was calculated in [Sch] and it was derived there that, again, instead of GT the Hopf algebra $\mathcal{H}_{GT}$ is the correct algebraic object to describe the common transformations of α and β respecting condition (2).

Our heuristic argument therefore leads to the view that on the total cohomology of a $\mathbb{C}$-linear double algebra $(A, \cdot, *)$ we have to expect a hidden action of the Hopf algebra $\mathcal{H}_{GT}$.

Remark 7 *We expect that the $\mathcal{H}_{GT}$-action should also hold for the case of a general category with the structure of $\mathcal{S}$ because in the case of the GT-action this is also not affected by the passage from associative algebras to A_∞-categories.*

The main conclusion of this section is therefore that also in the more geometrical picture of gerbes we find an argument that the GT-symmetry can not be physical in the sense that it does not hold for general nonperturbative backgrounds but that it has to be extended to the quantum symmetry $\mathcal{H}_{GT}$ in this case. The arguments of this section are of a more heuristic nature but one should keep in mind that we have given a completely rigorous approach in the algebraic framework in the previous section. The aim of this one was mainly to convey the general view which presents itself from the geometrical side, in this question.

4 The Duflo-Kirillov isomorphism

In this section, we will see from a quite different perspective - than the one taken in the two previous ones where we started basically from two dimensional boundary conformal field theory - that the GT-symmetry can not be physical. To be more precise, we will see that it is unstable against quantization (which is, of course, a similar statement as found in the form of an instability against the inclusion of general nonperturbative backgrounds) but this time our starting point will be the consideration of the Duflo-Kirillov isomorphism (see [Kir]) for finite-dimensional Lie algebras.

In [Kon 1999] the Duflo-Kirillov isomorphism is considered with the aim of showing an action of the Grothendieck-Teichmüller group GT on it. But it turns out that the requirements an action of GT has to satisfy lead to trivial data, only (at least in the category of vector spaces). We will try to understand the reason why the GT-action trivializes on the Duflo-Kirillov isomorphism, now.

There are basically two possibilities: As is well known (see [Dri]) GT is closely linked to the Ihara algebra Ih (Ih is closely related to the Lie algebra of GT, see [Dri], [Iha 1987], [Iha 1989] for the technical definition of the Ihara algebra), i.e. on Ih there has to be a kind of adjoint action of GT. On the other hand, evaluation of Ih on finite-dimensional metrized Lie algebras g (i.e. endowed with an invariant inner product) leads to an algebra of complex valued functions on $g \times g$ and the Lie bracket of Ih turns into the Kirillov bracket on this algebra of functions. Let us call this algebra $\mathcal{F}_g$. The action of GT on Ih canonically induces an action of GT on $\mathcal{F}_g$, then. Now, the first possibility to explain the fact that there is no visible action of GT on the Duflo-Kirillov isomorphism is that the action of GT on $\mathcal{F}_g$ is always trivial. This would mean that the GT action is an effect which we can not see upon evaluation of Ih on finite-dimensional Lie algebras (i.e. an effect which can only be seen "globally"). In this case, the fact that the Duflo-Kirillov example does not work in [Kon 1999] would simply tell us this nature of the GT-action and would beyond this not be too interesting.

To understand the second possibility, recall that canonical quantization of the function algebra on the dual g^* of g (endowed with the Kirillov bracket) leads to the universal enveloping algebra $U(g)$ of g (see e.g. [Kon 1997]). The Duflo-Kirillov isomorphism is then an isomorphism between the center of $U(g)$ and the algebra $Sym(g)^g$ of g-invariant polynomials on g^*, i.e. the Duflo-Kirillov isomorphism appears only after quantization of the Kirillov bracket. The second possibility is, then, that the GT-action does not survive the quantization process. In this case, the fact that we do not see the GT-action on the Duflo-Kirillov isomorphism would carry an instructive lesson for an understanding of the physical relevance of the GT-action on the extended moduli space in string theory.

We therefore give an argument, now, which decides between the two possibilities. In fact, we will find that the - more interesting - second possibility is the correct one.

Let $w \in GT$, $\varphi \in Ih$, $ad(.)$ the representation of GT on Ih discussed above, $ad_g(.)$ the induced representation on $\mathcal{F}_g$, i.e.

$$ad_g(w).\varphi_g = (ad(w).\varphi)_g$$

where φ_g denotes the evaluation of φ at g.

Since $ad(.)$ is basically the adjoint representation of GT, we have

$$ad(w) \neq 0 \tag{3}$$

for an appropriate choice of w and, consequently,

$$ad(w).\varphi \neq 0 \tag{4}$$

for an appropriate choice of φ. Let w, φ be chosen in accordance with (3) and (4).

Since for $\varphi \neq 0$ there exists always a finite-dimensional Lie algebra g with $\varphi_g \neq 0$ (see [Dri]), there exists a finite-dimensional Lie algebra g with

$$(ad(w).\varphi)_g \neq 0$$

i.e. we have

$$ad_g(w) \neq 0$$

But this means we have a non-trivial action of GT on $\mathcal{F}_g$ for this g.

In conclusion, on the classical level (i.e. before the quantization of the Kirillov bracket) we can not trivialize the GT-action simply by evaluation at finite-dimensional Lie algebras g.

Next, consider the question of quantization of Ih. Denote by $*_g$ the Kontsevich product for $\mathcal{F}_g$ using the Kirillov bracket as the Poisson bracket. For $\varphi, \psi \in Ih$ we have

$$\varphi_g *_g \psi_g = \chi_g$$

and by the universality of the Kirillov bracket together with the fact that the Kontsevich formula is universal, only depending in turn on the data of the Poisson bracket, χ is independent of g. Hence, we can define $\varphi * \psi$ by

$$(\varphi * \psi)_g = \varphi_g *_g \psi_g$$

and $*$ gives a noncommutative deformation of the Ihara algebra.

Here, we use, in addition, the fact that the S_3 action, involved in the definition of *Ih*, goes over to the quantized case because it is defined there in complete analogy to the classical case.

Now, suppose the GT-action would also hold after deformation quantization, i.e. on the deformed Ihara algebra we would also have a non-trivial action $\widehat{ad}(.)$. Choose, again, w, φ with

$$\widehat{ad}(w).\varphi \neq 0$$

Define

$$\widehat{ad}_g(w).\varphi_g = \left(\widehat{ad}(w).\varphi\right)_g$$

We have by the above result of Drinfeld, again, that there exists g with

$$\widehat{ad}_g(w).\varphi_g \neq 0$$

So, $\widehat{ad}_g$ gives a non-trivial representation of GT on $U(g)$.

Let $\widehat{ad}_g^c$ be the induced representation on the center of $U(g)$. Since $\widehat{ad}_g(w)$ is an automorphism of $U(g)$, especially, it has trivial kernel, $\widehat{ad}_g^c$ is also a non-trivial representation of GT. But $\widehat{ad}_g^c$ induces a non-trivial action of GT on isomorphisms

$$Center(U(g)) \to Sym(g)^g$$

In conclusion, the argument that the triviality of the GT action on the Duflo-Kirillov isomorphism can not derive from evaluation on finite-dimensional Lie algebras g does also hold in the quantized setting. Consequently, the triviality of the GT-action on the Duflo-Kirillov isomorphism can only mean that the GT-action on *Ih* does not survive the quantization process.

Let us formulate this result in more physical terms: That the GT-symmetry does not hold after quantization means that this symmetry does not survive the inclusion of off shell states. But this is basically the same conclusion we arrived at when noting in the previous sections that upon extension of the moduli space to general nonperturbative vacua the GT-symmetry does not survive but has to be extended to a quantum symmetry given by the Hopf algebra $\mathcal{H}_{GT}$.

Let us therefore conclude this section by passing to the question if the $\mathcal{H}_{GT}$-symmetry is stable against quantization. In this part, we can only give a heuristic argument at this time. We will argue that for the $\mathcal{H}_{GT}$-symmetry we should not only have stability against quantization but that in this case the quantum and the classical description should even be equivalent.

Observe, first, that the canonical quantization of the Kirillov bracket can be described as a deformation quantization given by the Kontsevich formula (see [Kon 1997]). We suppose that a further deformation quantization of $\mathcal{H}_{GT}$ is not possible as a consequence of ultrarigidity and that in this sense in the presence of the $\mathcal{H}_{GT}$-symmetry the classical and the quantum descriptions are already equivalent, i.e. there is no analog of the quantization of the Kirillov bracket upon evaluation of the elements of $\mathcal{H}_{GT}$ on finite-dimensional Lie algebras but the involved brackets remain classical.

If there would exist a further deformation quantization of $\mathcal{H}_{GT}$, we would have a class of deformations of representations of $\mathcal{H}_{GT}$. But by the considerations in the previous section, this would mean that there exists a further deformation of the cohomology of double algebras which we have briefly mentioned, there. But this should be excluded by the same line of argument used in [Sch] to establish ultrarigidity because on such a deformation a trialgebra would have to act instead of $\mathcal{H}_{GT}$ but this trialgebra would necessarily trivialize (see [Sch]).

In conclusion, there should be no analog of the Duflo-Kirillov isomorphism for the $\mathcal{H}_{GT}$-symmetry case and we expect the $\mathcal{H}_{GT}$-symmetry to be stable which gives a first hint at the possibility that it might be a symmetry of physical relevance.

5 Conclusion

We have in this paper given arguments from three different perspectives - the algebraic formulation of boundary conformal field theories of [KL], the cohomology of gerbes, and the Duflo-Kirillov isomorphism - that the GT-symmetry on extended moduli space can not represent a physically relevant symmetry but that upon inclusion of general nonperturbative vacua it has to be extended to a quantum symmetry represented by a self-dual, noncommutative, and noncocommutative Hopf algebra $\mathcal{H}_{GT}$.

A more detailed investigation of the implications of a universal $\mathcal{H}_{GT}$-symmetry in open string theory will follow in subsequent work.

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