ESI **The Erwin Schrodinger International**
Institute for Mathematical Physics

Boltzmanngasse 9
A-1090 Wien, Austria

Cyclic Homology and Pseudodifferential Operators, a Survey

Jacek Brodzki
Moulay-Tahar Benameur
Victor Nistor

Vienna, Preprint ESI 1249 (2002) December 5, 2002

CYCLIC HOMOLOGY AND PSEUDODIFFERENTIAL OPERATORS, A SURVEY

JACEK BRODZKI, MOULAY-TAHAR BENAMEUR, AND VICTOR NISTOR

ABSTRACT. We present a brief introduction to Hochschild and cyclic homology designed for researchers interested in pseudodifferential operators and their applications to index theory, spectral invariants, and asymptotics.

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INTRODUCTION

Singular cohomology is often used in Algebraic Topology to obtain invariants of topological spaces. In the same spirit, Hochschild and cyclic homology often provide interesting invariants of algebras. A possible important application of these algebra invariants is to the study of spaces with additional structures; these include, for instance, spaces with singularities or spaces endowed with group actions.

This paper is a rapid survey of Hochschild and cyclic homology, designed for mathematicians and physicists interested in pseudodifferential operators and their applications to index theory, spectral invariants, and asymptotics. We assume only very little familiarity with the subject, including homological algebra. Moreover, we have also included some short proofs when we felt that they are particularly helpful to the reader.

Hochschild homology is usually thought of as a generalization of the notion of differential forms and (periodic) cyclic homology as a generalization of (de Rham) cohomology, from smooth compact manifolds to essentially arbitrary algebras. The main guiding principle of this development is the correspondence

$$\text{“\textit{Space}”} \quad \leftrightarrow \quad \text{“\textit{Algebra of functions on that space}”},$$

which is at the heart of non-commutative geometry [5, 20, 21, 31, 34, 37, 38, 42, 43, 44]. This principle had been used before in Algebraic geometry where this becomes the correspondence (contravariant equivalence of categories) between affine algebraic varieties

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Benameur was partially supported by CNRS grant. Nistor was partially supported by NSF Grant DMS-9971951 and a “collaborative research” grant.

over a field $\mathfrak{k}$ and commutative, reduced, finitely generated algebras over $\mathfrak{k}$. In Functional analysis this principle is illustrated by the Gelfand–Naimark equivalence between the categories of compact topological spaces and of commutative, unital C^*-algebras. One of the main ideas of non-commutative geometry is that this correspondence somehow extends also to (classes of) non-commutative algebras. The theory of Quantum groups [39, 42] is a truly remarkable example of how this principle works in practice. Another remarkable example is that the singular cohomology of spaces and the long exact cohomology exact sequences of pairs of spaces extend to algebras – this is the Excision theorem in periodic cyclic homology [23].

Our main example of how this principle works in practice will be centered upon algebras like $\mathcal{C}^\infty(M)$ and $\Psi^\infty(M)$, where M is a smooth compact manifold and $\Psi^\infty(M)$ denotes the algebra of classical pseudodifferential operators of integral order on M. We shall see that the Hochschild homology of $\mathcal{C}^\infty(M)$ can be identified with the space of differential forms on M, whereas the periodic cyclic homology of $\mathcal{C}^\infty(M)$ is the same as the de Rham cohomology of M. The results on $\mathcal{C}^\infty(M)$ are interesting not only on their own, but also because they lead to a calculation of the Hochschild, cyclic, and periodic cyclic homology of algebras of pseudodifferential operators.

The contents of the paper are as follows. In Section 1, we define Hochschild homology and give some elementary properties and concrete computations. In Section 2, we introduce the cyclic homology and the Connes-Karoubi Chern character. We also recall the Cuntz-Quillen excision theorem together with an abstract cyclic cohomological index theorem. In Section 3, we extend our homologies and the complexes defining them to the setting of Frechet algebras and discover, in particular, that when dealing with topological algebras, choice of a suitable topological tensor product becomes important. Section 4 deals with algebras of complete pseudodifferential symbols on groupoids with corners. There, we give the computation of periodic cyclic homology and Hochschild homology for such algebras. Finally, Section 5 applies these results to some geometric examples in non-commutative geometry. In particular, it relates Hochschild homology computations with local residues. No results in this paper are new.

We have not attempted to provide a comprehensive list of references, restricting instead to the sources most relevant to the results presented here. The main sources of general information on cyclic type homology theories are the books [9, 21, 31, 37], which contain detailed bibliographic information.

We thank the referee for carefully reading our manuscript and for making several useful suggestions.

1. HOCHSCHILD HOMOLOGY

We begin by recalling the definitions of Hochschild homology groups of an algebra A, which we shall assume to be defined over the field of complex numbers. Later, we shall treat the case of algebras equipped with topology.

Let A be an algebra, not necessarily with unit. A *trace* on A is a linear map $\tau : A \to \mathbb{C}$ such that

$$\tau(a_0 a_1 - a_1 a_0) = 0$$

for all $a_0, a_1 \in A$. This definition captures an important property of the usual matrix trace Tr, defined on the algebra $A = M_n(\mathbb{C})$ by $Tr([a_{ij}]) = \sum a_{ii}$. The space of all traces on A is denoted $\mathrm{HH}^0(A)$, and is also called the 0*th Hochschild cohomology group*, for reasons that we explain now.

We shall need two differential complexes associated with the algebra A, denoted $\mathcal{H}(A)$ and $\mathcal{H}'(A)$. The space of n-chains for both will be the $n+1$-fold tensor product of A by itself. The two complexes are equipped with differentials b' and b defined as follows:

$$b'(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n, \tag{1}$$
$$b(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = b'(a_0 \otimes a_1 \otimes \ldots \otimes a_n) + (-1)^n a_n a_0 \otimes \ldots \otimes a_{n-1},$$

where $a_0, a_1, \ldots, a_n \in A$. Thus b and b' define linear maps

$$b,\, b' \,:\, A^{\otimes n+1} \to A^{\otimes n}.$$

By definition, *Hochschild homology groups* of the algebra A, denoted $\mathrm{HH}_n(A)$, are the homology groups of the complex

$$\mathcal{H}(A) := (A^{\otimes n+1}, b). \tag{2}$$

By contrast, the complex

$$\mathcal{H}'(A) := (A^{\otimes n+1}, b') \tag{3}$$

is often acyclic. This happens, for example when A has a unit, because

$$s(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = 1 \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_n$$

is a homotopy between 0 and the identity:

$$b's + sb' = 1.$$

In this case the complex $\mathcal{H}'(A)$ is a resolution of A by free A-bimodules. It has been noted by Wodzicki [60] that nonunital algebras whose $\mathcal{H}'(A)$ complex is acyclic have useful properties. Algebras with this property are now called *H-unital* (homologically unital).

The cohomology groups of the dual complex

$$\mathcal{H}^*(A) := (\mathrm{Hom}(A^{\otimes n+1}, \mathbb{C}), b^*)$$

are called the *Hochschild cohomology groups* of A and denoted $\mathrm{HH}^n(A)$. Since τ is a trace if, and only if, $\tau \circ b = 0$, we obtain that $\mathrm{HH}^0(A)$ is indeed the space of traces on A.

Clearly the groups $\mathrm{HH}_n(A)$ are *covariant* functors in A, in the sense that any algebra morphism $\phi : A \to B$ induces a morphism

$$\phi_* : \mathrm{HH}_n(A) \to \mathrm{HH}_n(B)$$

for any integer $n \geq 0$. Similarly, we also obtain a morphism $\phi^* : \mathrm{HH}^n(B) \to \mathrm{HH}^n(A)$. In other words, Hochschild cohomology is a *contravariant* functor. It is interesting to note that if Z is the center of A, then $\mathrm{HH}_n(A)$ is also a Z-module, where, at the level of complexes the action is given by

$$z(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = za_0 \otimes a_1 \otimes \ldots \otimes a_n. \tag{4}$$

for all $z \in Z$. As z is in the center of A, this action will commute with the Hochschild differential b.

When computing the Hochschild homology of an algebra A, it is often useful to replace the original complex $\mathcal{H}(A)$ with an equivalent complex. This can be achieved, for example, using the next Proposition, which is a standard result from homological algebra (see [40, 41], for instance). Before we can state that proposition, we need to recall the following terminology. An *A-bimodule* is an abelian group M which is at the same time a left- and right- A-module. A *free* A-bimodule is one of the form $A \otimes V \otimes A$, where V is a vector space and

$$a_0(a_0' \otimes v \otimes a_1')a_1 = (a_0a_0') \otimes v \otimes (a_1'a_1).$$

Let $f : A \otimes V_1 \otimes A \to A \otimes V_0 \otimes A$ be a morphism of free bimodules, and assume that $f(1 \otimes v \otimes 1) = \sum a_k \otimes w_k \otimes a_k'$. We shall denote by $f \otimes 1$ the map $A \otimes V_1 \to A \otimes V_0$ defined by

$$(f \otimes 1)(a \otimes v) = \sum a_k' a a_k \otimes w_k$$

Let A^{opp} be A as a vector space but with multiplication $\star$ given by $a \star a' := a'a$. Let $A^e := A \otimes A^{opp}$. The reader familiar with balanced tensor products will note that the above construction amount to taking the tensor product $- \otimes_{A^e} A$ of f and the respective free modules. (Note that a free A-bimodule is the same thing as a free A^e-module.)

Proposition 1. *Let A be a unital algebra and*

$$A \leftarrow A \otimes V_0 \otimes A \overset{d_1}{\leftarrow} A \otimes V_1 \otimes A \overset{d_2}{\leftarrow} \ldots$$

be an exact complex of A-bimodules. Then the homology groups of the complex

$$A \otimes V_0 \xleftarrow{d_1 \otimes 1} A \otimes V_1 \xleftarrow{d_2 \otimes 1} \ldots$$

are naturally isomorphic to the groups $\mathrm{HH}_n(A)$.

For example, when A is unital,[1] the complex $\mathcal{H}'(A)$ is exact, and when we apply to it the above proposition we obtain the Hochschild complex $\mathcal{H}(A)$. Another useful corollary of this proposition is derived when we use the exactness of the localization functor. The following result is due to Brylinski [14].

Proposition 2. *Let S be a multiplicative subset of the center Z of the algebra A. Then* $\mathrm{HH}_*(S^{-1}A) \cong S^{-1}\,\mathrm{HH}_*(A)$.

Let us use now Proposition 1 to determine the Hochschild homology of the algebra $\mathcal{O}[\mathbb{C}^n] := \mathbb{C}[X_1, \ldots, X_n]$ of polynomials in n-variables. We first need to recall the algebraic de Rham complex. Let W^* be the complex vector space generated by the 1-forms $dX_1, \ldots, dX_n$:

$$W^* = \mathbb{C}dX_1 \oplus \ldots \oplus \mathbb{C}dX_n$$

We define the space of *q-forms on $\mathbb{C}^n$* to be $\Omega^q(\mathbb{C}^n) = \mathcal{O}[\mathbb{C}^n] \otimes \Lambda^q W^*$. Here $\mathbb{C}^n$ is of course regarded as an affine algebraic variety. Then the de Rham differential

$$d : \Omega^q(\mathbb{C}^n) \to \Omega^{q+1}(\mathbb{C}^n)$$

is defined by

$$d\Big(PdX_{i_1} \ldots dX_{i_q}\Big) = \Big(\frac{\partial P}{\partial X_1}dX_1 + \ldots + \frac{\partial P}{\partial X_n}dX_n\Big)dX_{i_1} \ldots dX_{i_q}.$$

(We omited the $\wedge$ in the products above.) We can now define the *Hochschild-Kostant-Rosenberg-Connes map (HKRC-map)*

$$\chi : A^{\otimes q+1} \to \Omega^q(\mathbb{C}^n),$$

$A = \mathcal{O}[\mathbb{C}^n] = \mathbb{C}[X_1, \ldots, X_n]$, by the formula

$$\chi(a_0 \otimes \ldots \otimes a_q) = \frac{1}{q!}a_0 da_1 \ldots da_q. \tag{5}$$

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[1] we assumed A unital in the definition of $f \otimes 1$

Proposition 3. *We have* $\chi \circ b = 0$ *and hence* χ *defines an isomorphism* $\chi_* : \mathrm{HH}_q(A) \to \Omega^q(\mathbb{C}^n)$.

The proof of this proposition is a paradigm for the computation of the Hochschild homology of the commutative algebras, so we present it in detail.

Proof. Our main tool will be the Koszul complex, which provides a general recipe for building resolutions of $A = \mathcal{O}[\mathbb{C}^n]$. We construct it as follows. We fix the sequence $(v_1, v_2, \dots, v_n)$, $v_j = X_j \otimes 1 - 1 \otimes X_j$ of elements of $A \otimes A$. (This is a *regular sequence*, see [27] for the definition). Let V be the vector space with basis the elements v_j of this regular sequence. Also, let $\mathcal{K}_q = A \otimes \Lambda^q V \otimes A$. The spaces $\mathcal{K}_q$ will form the Koszul complex relative to the sequence $(v_1, \dots, v_n)$. The differentials of the Koszul complex are given by the following formula

$$\delta(a \otimes v_{i_1} \wedge \dots \wedge v_{i_q} \otimes a') := \sum_{j=1}^{q} (-1)^{j-1} \Big[aX_{i_j} \otimes v_{i_1} \wedge \dots \wedge \widehat{v_{i_j}} \wedge \dots \wedge v_{i_q} \otimes a' \\ - a \otimes v_{i_1} \wedge \dots \wedge \widehat{v_{i_j}} \wedge \dots \wedge v_{i_q} \otimes X_{i_j} a' \Big]$$

where $\widehat{v}$ means that the symbol v is to be omitted.

The Koszul complex $\mathcal{K}$ provides a resolution of A by free $A \otimes A$-modules (this follows from the fact that the sequence v_j is regular). In other words, the complex $\mathcal{K}$ has vanishing homology, except in dimension zero, where its homology is isomorphic to A via the multiplication map

$$\mathcal{K}_0 = A \otimes A \ni a \otimes a' \to aa' \in A\,.$$

Let $w = v_{i_1} \wedge \dots \wedge v_{i_q} \in \Lambda^q V$. Define $f : \mathcal{K} \to \mathcal{H}'(A) := (A^{\otimes q+2}, b')$ by

$$f(a \otimes w \otimes a') := \sum_{\sigma \in S_q} \operatorname{sign}(\sigma) a \otimes v_{i_{\sigma(1)}} \otimes v_{i_{\sigma(2)}} \otimes \dots \otimes v_{i_{\sigma(q)}} \otimes a',$$

where the sum is taken over the symmetric group on q letters and $\operatorname{sign}(\sigma)$ denotes the sign of the permutation σ. The map f is a chain map, which means that it commutes with the differentials:

$$f\delta = b'f,$$

and hence it induces a morphism of complexes (or, in this case, resolutions). The morphism f is the identity on $\mathcal{K}_0 = A \otimes A = \mathcal{H}'_0(A)$ and hence it induces the identity on the homology groups in dimension zero, that is, on the algebra A. (This is a standard homological fact that can easily be proved as an exercise.)

Using the remark just before Proposition 1 we see that the map

$$f \otimes 1 : \mathcal{K} \otimes_{A^e} A \to \mathcal{H}'(A) \otimes_{A^e} A \cong \mathcal{H}(A)$$

will also induce an isomorphism in homology.

The result then follows because $\delta \otimes 1 = 0$ and $\chi \circ f$ is the identity, if we identify V with W^* by $v_j \to dX_j$. ☐

The same argument can be used to prove the following result (see [29, 38]).

Theorem 1. *Let* Y *be a smooth, complex algebraic variety and* $\mathcal{O}[Y]$ *be the ring of regular functions on* Y *and* $\Omega^q(Y)$ *be the space of algebraic* q*-forms on* Y. *Then the HKRC map* χ *induces isomorphisms*

$$\chi : \mathrm{HH}_q(\mathcal{O}[Y]) \to \Omega^q(Y)$$

for all $q \geq 0$.

Another important class of examples deals with algebras associated with groups. The simplest situation is probably the following. Let Γ be a group (we do not assume that Γ has topology) and let $\mathbb{C}[\Gamma]$ be the group algebra of Γ. By definition, $\mathbb{C}[\Gamma]$ is the space of finite linear combinations $\sum_\gamma a_\gamma \gamma$, $\gamma \in \Gamma$, $a_\gamma \in \mathbb{C}$ equipped with the product $(a\gamma)(a'\gamma') := aa'(\gamma\gamma')$. For any $\gamma \in \Gamma$ we denote by

$$\Gamma_\gamma := \{g \in \Gamma, g\gamma = \gamma g\},$$

the *centralizer* of γ. Let $\mathrm{H}_q(\Gamma_\gamma)$ be the qth homology group of Γ_γ with complex coefficients (it is the homology with complex coefficients of any CW-complex whose fundamental group is Γ_γ and all other homotopy groups vanish). Let $\langle\Gamma\rangle$ denote the set of conjugacy classes of Γ and $\langle\gamma\rangle$ denote the conjugacy class of $\gamma \in \Gamma$. Then [17]:

Theorem 2. *Let Γ be a discrete group. Then*

$$\mathrm{HH}_q(\mathbb{C}[\Gamma]) \simeq \bigoplus_{\langle\gamma\rangle \in \langle\Gamma\rangle} \mathrm{H}_q(\Gamma_\gamma).$$

In the above sum, we take exactly one γ from each conjugacy class of Γ. Also, note that the groups $\mathrm{H}_q(\Gamma_x)$ and $\mathrm{H}_q(\Gamma_y)$ are *canonically* isomorphic if x and y are conjugate in Γ.

These results extend to crossed products. See [16, 18] for the calculation of the Hochschild, cyclic, and periodic cyclic homology crossed product algebras of the form $\mathcal{C}^\infty(M) \rtimes \Gamma$, where Γ is a discrete group acting on X by diffeomorphisms. Also, see [3] for some applications of these results to orbifold cohomology as well as [24, 51] for results on general crossed products $A \rtimes \Gamma$, with A not necessarily commutative. (When [51] was being written, the paper [24] was not available in Romanian libraries, which explains the overlap in results between the two papers.)

Hochschild homology is also compatible with completions, in the following sense. Let A be a finitely generated module over its center $\mathfrak{k}$ and let $\hat{M}$ denote the completion of a $\mathfrak{k}$–module M with respect to the topology defined by an ideal $I \subset \mathfrak{k}$, then we have [33]

Theorem 3. *Denote by $\mathrm{HH}_q^{top}(\hat{A})$ the homology of the completion of the Hochschild complex of A with respect to the natural filtration defined by $I^k A$. Then*

$$\mathrm{HH}_q^{top}(\hat{A}) \simeq \widehat{\mathrm{HH}}_q(A).$$

2. CYCLIC HOMOLOGY

We now define cyclic homology, which, as was the case with Hochschild homology, will appear as the homology of an explicitly defined differential complex. The cyclic complex will be introduced as the total complex of a double complex constructed using the Hochschild complex $\mathcal{H}(A)$, together with a new differential B whose definition uses in a crucial way the action of a cyclic group of a suitable size.

Let us assume first that A is a unital algebra. We shall denote by t the (signed) generator of cyclic permutations:

$$t(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1} \tag{6}$$

Using this operator and the contracting homotopy s of the complex $\mathcal{H}'(A)$ we construct the operator B of degree $+1$ in two steps. First define

$$B_0(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = s \sum_{k=0}^{n} t^k (a_0 \otimes a_1 \otimes \ldots \otimes a_n) \tag{7}$$

and then put $B = (1-t)B_0$. Then B is a differential, $B^2 = 0$ and it anticommutes with the Hochschild differential b: $[b, B]_+ := bB + Bb = 0$.

We have thus defined the following double complex, which fills up an octant in the plane:

$$\begin{tikzcd}
{} \arrow[d, "b"'] & {} \arrow[d, "b"] & {} \arrow[d, "b"] \\
A^{\otimes 3} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] \\
A^{\otimes 2} \arrow[d, "b"'] & A \arrow[l, "B"'] & \\
A & &
\end{tikzcd}$$

FIGURE 1. The cyclic bicomplex of the algebra A.

Note that columns in this complex are copies of the Hochschild complex $\mathcal{H}(A)$. The cyclic complex $\mathcal{C}(A)$ is by definition the total complex of this double complex. Thus the space of cyclic n-chains is defined by

$$\mathcal{C}(A)_n = \bigoplus_{k \geq 0} \mathcal{H}_{n-2k}(A), \tag{8}$$

we see that $(\mathcal{C}(A), b + B)$, is a complex, called *the cyclic complex* of A, whose homology is by definition the *cyclic homology* of A (see [20, 31, 59]).

There is a canonical operator $S : \mathcal{C}_n(A) \to \mathcal{C}_{n-2}(A)$, called the *Connes periodicity operator*, which shifts the cyclic complex down by two degrees, and which is defined by $S(\omega_n, \omega_{n-2}, \omega_{n-4}, \dots) \mapsto (\omega_{n-2}, \omega_{n-4}, \dots)$, where $\omega_i \in \mathcal{H}_i(A)$. This operator induces the short exact sequence of complexes

$$0 \to \mathcal{H}(A) \xrightarrow{I} \mathcal{C}(A) \xrightarrow{S} \mathcal{C}(A)[2] \to 0,$$

where the map I is the inclusion of the Hochschild complex as the first column of the cyclic complex and where on the right the cyclic complex has been shifted by two degrees. Applying the homology functor to this sequence we obtain the following exact sequence of Connes and Tsygan:

$$\dots \to \mathrm{HH}_n(A) \xrightarrow{I} \mathrm{HC}_n(A) \xrightarrow{S} \mathrm{HC}_{n-2}(A) \xrightarrow{B} \mathrm{HH}_{n-1}(A) \xrightarrow{I} \dots, \tag{9}$$

where B is the differential defined above, see [20, 38] for more details. Anticipating a little bit, this exact sequence exists whether or not A is endowed with a topology. The *periodic cyclic complex* of an algebra A is the complex

$$\mathcal{C}^{\mathrm{per}}(A) := \lim_{\leftarrow} \mathcal{C}(A),$$

the inverse limit being taken with respect to the periodicity morphism S. It is a $\mathbb{Z}/2\mathbb{Z}$-graded complex, whose chains are (possibly infinite) sequences of Hochschild chains with degrees of the same parity.

More generally, cyclic homology groups can be defined for “mixed complexes,” [32]. A *mixed complex* $(\mathcal{X}, b, B)$ is a $\mathbb{Z}_+$–graded complex vector space $\mathcal{X} = (\mathcal{X}_n)_{n \geq 0}$ endowed with two differentials b and B, $b : \mathcal{X}_n \to \mathcal{X}_{n-1}$ and $B : \mathcal{X}_n \to \mathcal{X}_{n+1}$, satisfying

$$b^2 = B^2 = bB + Bb = 0.$$

The *cyclic complex*, $\mathcal{C}(\mathcal{X})$, associated to a mixed complex $(\mathcal{X}, b, B)$ is the complex

$$\mathcal{C}_n(\mathcal{X}) = \mathcal{X}_n \oplus \mathcal{X}_{n-2} \oplus \mathcal{X}_{n-4} \ldots = \bigoplus_k \mathcal{X}_{n-2k}$$

with differential $b + B$. It is the total complex associated to the bicomplex shown in Figure 2. The cyclic homology groups of the mixed complex $\mathcal{X}$ are the homology groups of the cyclic complex of $\mathcal{X}$,

$$\mathrm{HC}_n(\mathcal{X}) = \mathrm{H}_n(\mathcal{C}(\mathcal{X}), b + B).$$

$$\begin{tikzcd}
{} \arrow[d, "b"'] & {} \arrow[d, "b"] & {} \arrow[d, "b"] \\
\mathcal{X}_2 \arrow[d, "b"'] & \mathcal{X}_1 \arrow[l, "B"'] \arrow[d, "b"] & \mathcal{X}_0 \arrow[l, "B"'] \\
\mathcal{X}_1 \arrow[d, "b"'] & \mathcal{X}_0 \arrow[l, "B"'] & \\
\mathcal{X}_0 & &
\end{tikzcd}$$

FIGURE 2. The bicomplex associated with a mixed complex.

We review below a few necessary results on mixed complexes. The interested reader can find more information in [30].

The homology of the complex $(\mathcal{X}_n, b)$ is called the Hochschild homology of the mixed complex $(\mathcal{X}, b, B)$ and is denoted $\mathrm{HH}_*(\mathcal{X})$. If A is an algebra with unit and if we let $\mathcal{X}_n = A^{\otimes n+1}$, with the differentials b and B defined above, then we recover the usual definitions of the Hochschild and cyclic homology groups of A. If A is an arbitrary algebra, denote by $A^+ = A + \mathbb{C}$ the algebra with an adjoined unit and let $\mathcal{X}_n(A) = \ker(\mathcal{X}_n(A^+) \to \mathcal{X}_n(\mathbb{C}))$. The cyclic and Hochschild homology groups of the resulting mixed complex $(\mathcal{X}_n(A), b, B)$ are, by definition, the cyclic and Hochschild homology groups of the not necessarily unital algebra A. When A is a unital algebra, the two ways of defining Hochschild and cyclic homology groups agree. Cyclic and Hochschild homology are covariant functors on the category of mixed complexes: a morphism $f : A \to B$ of algebras induces a morphism of the corresponding mixed complexes and hence it induces morphisms $f_* : \mathrm{HC}_*(A) \to \mathrm{HC}_*(B)$ and $f_* : \mathrm{HH}_*(A) \to \mathrm{HH}_*(B)$.

The cyclic complex of a mixed complex $(\mathcal{X}, b, B)$ has a periodicity endomorphism $S : \mathcal{C}_n(\mathcal{X}) \to \mathcal{C}_{n-2}(\mathcal{X})$ which maps $\mathcal{X}_n$ to 0 and is the identity on the other factors, as was seen in the case of the cyclic complex $\mathcal{C}(A)$. This defines a map $\mathrm{HC}_n(\mathcal{X}) \longrightarrow \mathrm{HC}_{n-2}(\mathcal{X})$, also denoted by S. The homology groups $\mathrm{HP}_i(\mathcal{X})$ of the projective limit complex

$$\mathcal{C}_n^{\mathrm{per}}(\mathcal{X}) = \lim_{\leftarrow} \mathcal{C}_{n+2k}(\mathcal{X}) \tag{10}$$

with respect to the periodicity operator S are called the *periodic cyclic homology* groups of $\mathcal{X}$:

$$\mathrm{HP}_i(\mathcal{X}) = \mathrm{H}_i(\mathcal{C}_n^{\mathrm{per}}(\mathcal{X}))\,. \tag{11}$$

Periodic cyclic homology is a $\mathbb{Z}/2\mathbb{Z}$–graded homology theory, $\mathrm{HP}_i = \mathrm{HP}_{i+2}$. Standard homological algebra shows that cyclic and periodic cyclic homology fit into the following $\lim^1$ exact sequence

$$0 \longrightarrow \underset{\leftarrow}{\lim}^1\ \mathrm{HC}_{m+1}(\mathcal{X}) \longrightarrow \mathrm{HP}_m(\mathcal{X}) \longrightarrow \underset{\leftarrow}{\lim}\ \mathrm{HC}_m(\mathcal{X}) \longrightarrow 0. \tag{12}$$

We shall write $\mathrm{HP}_m(A)$ for the periodic cyclic homology of the (mixed complex associated to the) algebra A. Again, functorial properties of mixed complexes show that the periodic cyclic homology of an algebra is a covariant functor.

Cyclic cohomology and *periodic cyclic cohomology* are defined by duality and, in case there is no topology, they are the duals of the corresponding homology groups.

The following standard lemma is useful for many calculations.

Lemma 1. *Let $f : X \to X'$ be a morphism of mixed complexes (*i.e. *commuting with b and B) that induces an isomorphism of the Hochschild homology groups. Then f induces an isomorphism of the cyclic homology, periodic cyclic homology, Hochschild cohomology, cyclic cohomology, and periodic cyclic cohomology groups.*

Proof. Consider filtrations of the cyclic complexes associated to X and X' by the columns of the corresponding double complexes. These filtrations define convergent spectral sequences and f is an isomorphism of these spectral sequences. The standard comparison theorem for spectral sequences is now enough to prove the isomorphisms of the cyclic homology groups of X and X'. (Alternatively, one can use here the SBI-exact sequence.) For periodic cyclic homology, use also the $\lim^1$ exact sequence above relating cyclic and periodic cyclic homologies.

The cohomology groups are dual (in this topology free setting) of the corresponding homology groups. □

Here is an application of this lemma. Consider

$$Tr_* : \mathcal{H}_q(M_N(A)) \to \mathcal{H}_q(A), \quad q \in \mathbb{Z}_+,$$

the map defined by $Tr_*(b_0 \otimes \ldots \otimes b_q) = Tr(m_0 m_1 \ldots m_q) a_0 \otimes \ldots a_q$, if $b_k = m_k \otimes a_k \in M_N(\mathbb{C}) \otimes A = M_N(A)$. Also consider the (unital) inclusion $\iota : A \to M_N(A)$ and ι_* be the morphism induced on the Hochschild complexes.

Proposition 4. *The map Tr_* is a morphism of mixed complexes. Both ι_* and Tr_* induce isomorphisms on Hochschild, cyclic, and periodic cyclic homologies and cohomologies such that $(\iota_*)^{-1} = N^{-1} Tr_*$.*

Proof. An easy calculation shows that Tr_* is a morphism of mixed complexes. The map ι_* induces an isomorphism in Hochschild homology, by the Künneth formula for Hochschild homology [37, 1.0.16, 1.2.4]. Thus ι_* induces an isomorphism of cyclic and periodic cyclic homology as well. The equation $Tr_* \iota_* = N$ is immediate and gives the rest of the proposition. □

For commutative algebras, cyclic homology can be calculated using the Hochschild homology and the equation $\chi \circ B = d_{DR} \circ \chi$, relating the de Rham differential d_{DR} and the differential B via the HKRC map χ. Let $A = \mathcal{O}[Y]$ be the algebra of regular functions on a smooth complex algebraic variety, as in the statement of Theorem 1. To compute $\mathrm{HC}_q(A)$ and the other groups associated to A, we first notice that χ defines a morphism

$$(A^{\otimes q+1}, b, B) \to (\Omega^q(Y), 0, d_{DR})$$

of mixed complexes, which is an isomorphism on Hochschild homology, by the same theorem of Loday and Quillen used above. This then gives the following.

Theorem 4. *Let Y be a smooth complex algebraic variety and $\mathrm{H}_q(Y)$ be its singular homology groups with complex coefficients. If $A = \mathcal{O}[Y]$, then*

$$\mathrm{HC}_q(A) \simeq \Omega^q(Y)/d\Omega^{q-1}(Y) \oplus \bigoplus_{j=1}^{[q/2]} \mathrm{H}_{q-2j}(Y),$$

with the periodicity morphism S identifying with the natural projection. In particular, $\mathrm{HP}_q(A) \simeq \bigoplus_{j\in\mathbb{Z}} \mathrm{H}_{q-2j}(Y)$.

Let us mention that the result on *periodic* cyclic homology of the above theorem extends to the case when Y is not necessarily smooth by a result of [25], but the proof is more difficult. (See [33] for a proof of this result in the spirit of this paper, using Theorem 3.)

We shall use these calculations to construct Chern characters. By taking Y to be a point, we obtain that $\mathrm{HC}_{2q}(\mathbb{C}) \simeq \mathbb{C}$ and $\mathrm{HC}_{2q+1}(\mathbb{C}) \simeq 0$. We can take these isomorphisms to be compatible with the periodicity operator S and such that for $q = 0$ it reduces to

$$\mathrm{HC}_0(\mathbb{C}) = \mathrm{HH}_0(\mathbb{C}) = \mathbb{C}/[\mathbb{C}, \mathbb{C}] = \mathbb{C}.$$

We shall denote by $\eta_q \in \mathrm{HC}_{2q}(\mathbb{C})$ the unique element such that

$$S^q \eta_q = 1 \in \mathbb{C} = \mathrm{HC}_0(\mathbb{C}).$$

If $e \in M_N(A)$ is a projection, it will induce a (non-unital) morphism $\psi : \mathbb{C} \to M_N(A)$ by $\lambda \mapsto \lambda e$. Then *Connes-Karoubi Chern character of e in cyclic homology* [20, 31] is defined by

$$Ch_q([e]) = Tr_*(\psi_*(\eta_q)) \in \mathrm{HC}_{2q}(A). \tag{13}$$

This map can be shown to depend only on the class of e in K-theory and to define a morphism

$$Ch_q : K_0(A) \to \mathrm{HC}_{2q}(A).$$

One can define similarly the Chern character in periodic cyclic homology and the Chern character on K_1 (algebraic K-theory). For the Connes-Karoubi Chern character on K_1, we use instead $Y = \mathbb{C}^*$, whose algebra of regular functions is $\mathcal{O}[Y] \simeq \mathbb{C}[z, z^{-1}]$, the algebra of Laurent polynomials in z and z^{-1} (this algebra, in turn, is isomorphic to the group algebra of $\mathbb{Z}$). Then $\mathrm{HC}_q(\mathcal{O}[Y]) \simeq \mathbb{C}$, for any $q \geq 1$. We are interested in the odd groups, which will be generated by elements $v_k \in \mathrm{HC}_{2k+1}(\mathcal{O}[Y])$, which can be chosen to satisfy $v_1 = z^{-1} \otimes z$ and $S^k v_{2k+1} = v_1$.

Then, if $u \in M_N(A)$ is an invertible element, it defines a morphism $\psi : \mathbb{C}[\mathbb{C}^*] \to M_N(A)$. The *Connes-Karoubi Chern character of u in cyclic homology* is thus defined by

$$Ch_q([u]) = Tr_*(\psi_*(v_q)) \in \mathrm{HC}_{2q+1}(A). \tag{14}$$

Again, this map can be shown to depend only on the class of u in K-theory and to define a morphism

$$Ch_q : K_1(A) \to \mathrm{HC}_{2q+1}(A).$$

Both the Chern character on K_0 and on K_1 are functorial, by construction.

Cyclic homology behaves to a large extent like Hochschild homology. Periodic cyclic homology however has a few additional properties that often make it easier to compute. The most important among them is the excision property, which we state here, although we shall not need it in this paper. The proof of excision in periodic cyclic homology relies in a crucial way on the following theorem of Goodwillie. (see [26]).

Theorem 5. *If $I \subset A$ is a nilpotent two-sided ideal, then the quotient morphism $A \to A/I$ induces an isomorphism $\mathrm{HP}_*(A) \to \mathrm{HP}_*(A/I)$.*

A deep and far reaching consequence of Goodwillie's theorem, and also of Wodzicki's Excision theorem in Hochschild homology [60], is the Excision Theorem in periodic cyclic homology established by Cuntz and Quillen [23].

Theorem 6. *Any two-sided ideal J of an algebra A over a characteristic 0 field gives rise to a periodic six-term exact sequence*

(15)
$$\begin{tikzcd}
\mathrm{HP}_0(J) \arrow[r] & \mathrm{HP}_0(A) \arrow[r] & \mathrm{HP}_0(A/J) \arrow[d, "\partial"] \\
\mathrm{HP}_1(A/J) \arrow[u, "\partial"] & \mathrm{HP}_1(A) \arrow[l] & \mathrm{HP}_1(J). \arrow[l]
\end{tikzcd}$$

It is interesting to note that this theorem holds in much greater generality in periodic cyclic homology than, for instance, in cyclic homology, where one needs to assume that the ideal J is an H-unital algebra.

Excision in periodic cyclic homology is compatible with excision in K-theory, which is seen from the following result [53].

Theorem 7. *Let $I \subset A$ the a two-sided ideal of a complex algebra A. Then the diagram*

$$\begin{tikzcd}
K_1(I) \arrow[r] \arrow[d] & K_1(A) \arrow[r] \arrow[d] & K_1(A/I) \arrow[r, "\partial"] \arrow[d] & K_0(I) \arrow[r] \arrow[d] & K_0(A) \arrow[r] \arrow[d] & K_0(A/I) \arrow[d] \\
\mathrm{HP}_1(I) \arrow[r] & \mathrm{HP}_1(A) \arrow[r] & \mathrm{HP}_1(A/I) \arrow[r, "\partial"] & \mathrm{HP}_0(I) \arrow[r] & \mathrm{HP}_0(A) \arrow[r] & \mathrm{HP}_0(A/I),
\end{tikzcd}$$

in which the vertical arrows are induced by the Chern characters $Ch : K_i \to \mathrm{HP}_i$, for $i = 0, 1$, commutes.

It is useful to mention here that periodic cyclic homology can be also be defined using a variant of the cyclic complex, the X-complex, which isobtained as follows. Let $TA \to A$ be the canonical surjection defined on the free algebra generated by A, regarded as a vector space. Denote by I the kernel of this morphism and complete the usual periodic cyclic complex of TA with respect to the powers of I. Since TA is free, a theorem of Loday and Quillen [38] allows us to replace its Hochschild complex with a canonical complex of length two. This descends to the completion. We obtain in this way a simpler complex for the periodic cyclic homology of A, which gives the same cohomology, by Goodwillie's result 5. See [22, 23, 46].

This result was used in [53, 54] to establish a link with index theory, which we now illustrate with the following simple example. Let M be a smooth, compact manifold. We shall denote by C_p the space of bounded linear maps T such that $Tr(|T|^p) < \infty$, $p \geq 1$. Also, we shall denote by $\Psi^m(M)$ the space of classical pseudodifferential operators on M of order m. It is known that $\Psi^{-1}(M) \subset C_p$, if $p > \dim M$. Since C_p is an ideal of the algebra $\mathcal{L}(\mathcal{H})$ of bounded operators on $L^2(M)$ and $\Psi^0(M)$ consists of bounded operators, we can define the algebra $\mathcal{E} := \Psi^0(M) + C_p$, which will contain C_p as a two-sided ideal. Denote by $S^*M = (T^*M \smallsetminus 0)/\mathbb{R}_+^*$, the cosphere bundle of M. Since $C_p \cap \Psi^0(M) = \Psi^{-1}(M)$ and the principal symbol $\sigma^{(0)} : \Psi^0(M)/\Psi^{-1}(M) \to \mathcal{C}^\infty(S^*M)$ is an isomorphism, we obtain the following exact sequence

$$0 \to C_p \longrightarrow \mathcal{E} \xrightarrow{\sigma^{(0)}} C(S^*M) \to 0.$$

The trace defines a cyclic cocycle $Tr_*(a_0, \dots, a_{2k+1}) = c_k \times Tr(a_0 a_1 \dots a_{2k+1})$ on C_p, if $2k+1 \geq p$, [20]. The connecting map ∂ in K-theory in the above diagram, when composed with Tr_* gives the analytic index

$$\text{Index}_a = Tr_* \circ \partial : K_1(C(S^*(M)) \longrightarrow K_0(C_p) \simeq \mathbb{Z}.$$

The Atiyah-Singer index theorem [1] can then be rephrased using Theorem 7 to say that

$$\partial[Tr_*] \simeq \pi^* \operatorname{Td}(M) \cap [S^*M] \in \bigoplus_k \mathrm{H}_{2k+1}(S^*M) = \mathrm{HP}^1(\mathcal{C}^\infty(S^*M)),$$

where $\partial : \mathrm{HP}^0(C^\infty(M \times M)) \to \mathrm{HP}^1(\mathcal{A}(M))$ is the boundary map is periodic cohomology, $\operatorname{Td}(M)$ is the Todd class of M and $[Tr_*]$ is the class of Tr_* in periodic cyclic cohomology. See also Theorem 8. The reader should be warned that there exists no proof of the Atiyah-Singer index theorem based on Theorem 7, although this would certainly be very interesting. This example can be generalized to give a proof of the Connes-Moscovici index theorem for coverings [54].

The last two theorems underscore strong similarities between periodic cyclic homology and K-theory. It is worth mentioning however that periodic cyclic homology is much easier to compute, and hence it is known for many more algebras, than K-theory.

3. INTRODUCING TOPOLOGY

In applications one is often interested in algebras with topology. Then in the definition of Hochschild and cyclic (co)homology one has to take the topology into account, and this means to complete $A^{\otimes q+1}$ in a suitable sense.

A locally convex algebra is a locally convex vector space A over $\mathbb{C}$ equipped with a separately continuous multiplication. In many examples the locally convex algebra A will be a Fréchet algebra, in which case the multiplication map will be jointly continuous. Given that cyclic type homology theories of an algebra are defined using complexes constructed from the tensor algebra of A it is clear that topology on A will force one to decide which topological tensor product to use.

The inductive tensor product $\otimes_i$ solves the universal problem for separately continuous bilinear maps in the sense that any such map $E \times F \to G$, where E, F, G are locally convex topological vector spaces, extends to a continuous linear map $E \otimes_i F \to G$. The projective tensor product $\otimes_\pi$ solves the same universal problem for jointly continuous bilinear maps. When E is a Fréchet space, then $E \otimes_\pi E \simeq E \otimes_\pi E$, as any separately continuous map is automatically jointly continuous, by [8, III.30, Corollary 1]. When E is also a *nuclear* space, a further simplification takes place as then there is a unique topological tensor product which is compatible with the algebraic tensor product.

A basic example of this situation is provided by the algebra A of smooth functions on a compact manifold. As a locally convex vector space, A is a complete, nuclear Fréchet space. In this case, a suitable topological tensor product to consider is the *completed projective tensor* product $\hat{\otimes}$ [28, Définition 2, p. 32]. In the remainder of this paper we shall restrict our attention to Fréchet algebras and so our topological tensor product of choice will be the completed projective tensor product. More details on the related issues can be found in [11, 22].

Let us assume that A is a Fréchet algebra. We can define then

$$\mathcal{H}_q(A) := A\hat{\otimes}A\hat{\otimes}\ldots\hat{\otimes}A, \qquad (q+1) \text{ factors },$$

where $\hat{\otimes}$ denotes as before the completed projective tensor product, which is the standard definition of the Hochschild complex for topological algebras. (The differentials b and b' then extend to the projective tensor product because of the continuity of the multiplication.) We shall denote the homology of the Hochschild complex $\mathcal{H}(A) := (\mathcal{H}_q(A), b)$ by $\mathrm{HH}_q(A)$, as before.

The following theorem from [20] generalizes Theorem 1 to the topological case. This theorem is one of the main reasons why Hochschild homology is regarded as a generalization of differentiable forms and why periodic cyclic homology is regarded as a generalization of de Rham cohomology.

Note that the HKRC-map extends to the projective tensor products.

Theorem 8. *Let M be a smooth compact manifold and let $\Omega^q(M)$ be the space of smooth q-forms on M. Then the HKRC map induces isomorphisms*

$$\chi : \mathrm{HH}_q(\mathcal{C}^\infty(M)) \to \Omega^q(M).$$

Consequently, $\mathrm{HP}_q(\mathcal{C}^\infty(M)) \simeq \bigoplus_{j\in\mathbb{Z}} \mathrm{H}^{q+2j}(M)$*. The dual identifications are valid for the corresponding cohomologies.*

Proof. Let $A = \mathcal{C}^\infty(M)$. We begin by noticing that [28, Ch. 2, Example 1, p. 80]

$$\mathcal{H}_q(A) := A^{\hat{\otimes} q+1} = \mathcal{C}^\infty(M^{q+1}). \tag{16}$$

Assume now that M is (diffeomorphic to) a compact ball in a Euclidean space. Then the proof of our theorem is exactly as that of Proposition 3, but only after replacing A^e by $\mathcal{C}^\infty(M \times M)$. (It is implicit here that the theory of resolutions and derived functors extends to topological algebras. Indeed, recall that this is achieved by requiring all maps to have closed, complemented images.)

Then we need a localization principle that will help us reduce the proof to the case of a closed ball in a Euclidean space. To this end, we shall adapt an argument from [16]. Let F_k be the set of functions $f : M^{q+1} \to \mathbb{C}$ that vanish on a neighborhood of the set $\{(x, x, \ldots, x, x_1, \ldots, x_{q-k})\} \subset M^{q+1}$, ($k+1$ repetitions of x), for any $x, x_1, \ldots, x_{q-k} \in M$.

Then $F_0 \subset F_1 \subset \ldots$ and $bF_k \subset F_k$. Let $F_{-1} = 0$. We claim that each complex F_k/F_{k-1} is acyclic. Indeed, fix a metric on M and choose for any $\epsilon > 0$ a smooth function $g_\epsilon : M^2 \to \mathbb{C}$ such that $g_\epsilon(x, y) = 0$ if the distance between x and y is less than $\epsilon/2$ and $g_\epsilon(x, y) = 1$ if the distance between x and y is greater than ϵ. Let

$$(L_\epsilon f)(x_0, x_1, \ldots, x_{q+1}) := g_\epsilon(x_{k-1}, x_k) f(x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{q+1}).$$

Then $(bL_\epsilon + L_\epsilon b)f - (-1)^k g(x_{k-1}, x_k) f \in F_{k-1}$ and $g(x_{k-1}, x_k) f = f$ for $\epsilon > 0$ small enough. This verifies our claim that F_k/F_{k-1} is acyclic.

Let $F_\infty = \bigcup F_n$. The spectral sequence associated to the resulting filtration of F_∞ is a first quadrant spectral sequence, and hence F_∞ is acyclic.

Denote by $\widetilde{\mathcal{H}}(M) := \mathcal{H}(M)/F_\infty$. Then the natural projection

$$\mathcal{H}(M) \to \mathcal{H}(M)/F_\infty$$

induces an isomorphism in homology (*i.e.* it is a quasi-isomorphism). We can thus replace $\mathcal{H}(M)$ with $\widetilde{\mathcal{H}}(M)$ in our calculation. This has the advantage that

$$\phi\widetilde{\mathcal{H}}(M) \simeq \phi\widetilde{\mathcal{H}}(B)$$

if $B \subset M$ is a closed manifold (possibly with boundary) such that ϕ has support in the interior of B. The action of ϕ was defined in Equation (4). Since the cohomology of $\widetilde{\mathcal{H}}(B)$ is known, determined by the HKRC-map χ and $\chi(\phi x) = \phi\chi(x)$ for any x in a Hochschild complex, a partition of unity argument can be used to complete the proof.

The corresponding result for cohomology follows by observing that the range of b in the Hochschild complex is closed, by the result we have just proved for homology. This is enough to conclude that the fact that χ is an isomorphism in homology implies that the dual map χ' is an isomorphism in cohomology. ☐

The resulting isomorphism helps identify the Chern characters on K_0 and K_1 with the classical Chern characters (up to normalization factors involving $2\pi\imath$), see [12, 52].

Our final remark in this section is that many general results proved for cyclic type homology theories for algebras without topology carry over to the topological case. The most important of these is excision. The method of Cuntz and Quillen has been extended by Cuntz to the case of topological locally multiplicatively convex algebras [22], and more recently, to the category of bornological algebras [46].

The periodic cyclic homology of algebras of this type satisfies the excision property with respect to extensions

$$0 \to J \to A \to A/J \to 0$$

of algebras from one of the categories mentioned above where the surjection on the right has a continuous (or bounded, in the bornological case) linear section. This implies that, as a topological vector space, the image of the ideal J is complemented in A. This requirement can be relaxed at the cost of requiring that J be H-unital [10].

4. ALGEBRAS OF PSEUDODIFFERENTIAL OPERATORS

The $K-$theory of many important algebras is difficult to compute [4]. Often a good substitute, especially when one is interested in index theorems, is provided by periodic cyclic homology.

If M is a smooth compact manifold, then an interesting algebra is the algebra $\mathcal{A}(M)$ of complete classical symbols on M. The computation of the homologies of this algebra was carried out in [15], and the result for the periodic cyclic homology is:

$$\mathrm{HP}_q(\mathcal{A}(M)) \simeq \bigoplus_{j\in\mathbb{Z}} \mathrm{H}^{q+2j}(S^*M \times S^1), \quad q = 0, 1,$$

where $\pi : S^*M \to M$ is the cosphere bundle of M.

Other interesting examples are provided by algebras of pseudodifferential operators on groupoids [47, 55]. Let $\mathcal{G}$ be a longitudinally smooth groupoid with corners with space of units a manifold with corners M. Denote by $d, r : \mathcal{G} \to M$ the domain and range map of $\mathcal{G}$. So $\mathcal{G}$ is itself a manifold with corners but we assume that the fibres of d are smooth manifolds (without boundary or corners). The algebra

$$\Psi^\infty(\mathcal{G}) = \bigcup_{m\in\mathbb{Z}} \Psi^m(\mathcal{G})$$

of (scalar) classical uniformly supported pseudodifferential operators on $\mathcal{G}$ is then defined, see [36, 47, 55]. For basic facts about pseudodifferential operators, see one of the many monographs available, for example [50], [56], or [57].

The algebra of smoothing operators is by definition

$$\Psi^{-\infty}(\mathcal{G}) := \bigcap_{m \in \mathbb{Z}} \Psi^m(\mathcal{G}).$$

The algebra

$$\mathcal{A}(\mathcal{G}) := \Psi^{\infty}(\mathcal{G})/\Psi^{-\infty}(\mathcal{G})$$

of complete symbols on $\mathcal{G}$ is endowed with an inductive limit topology. In order to provide a natural framework for a homological study of such algebras, one is led to a wider category of algebras with topology that we shall call *topologically filtered algebras* [5, 6].

The reader not familiar with groupoids can assume that our algebras of pseudodifferential operators are the algebras of classical pseudodifferential operators on a smooth, compact manifold. This corresponds to the case $\mathcal{G} = M \times M$.

We shall now study the homology of the algebras $\mathcal{A}(M)$, and, more generally, $\mathcal{A}(\mathcal{G})$. A convenient aproach is provided by topologically filtered algebras, which are algebras $\mathcal{A}$ endowed with bifiltrations $F_n^m\mathcal{A} \subset \mathcal{A}$ satisfying suitable conditions. Before formally formulating the definiton of a topologically filtered algebra, let us just say that in the case of $\mathcal{A} = \mathcal{A}(M)$, M compact, $F_n^m\mathcal{A}$ is independent of m. The second index, or filtration (with respect to m), is needed in order to treat algebras of compactly supported complete symbols on a non-compact manifold.

Recall that an algebra $\mathcal{A}$ with a given topology, is a *topologically filtered algebra* if there exists an increasing bifiltration $F_n^m\mathcal{A} \subset \mathcal{A}$,

$$F_n^m\mathcal{A} \subset F_{n'}^{m'}\mathcal{A}, \quad \text{if } n \leq n' \text{ and } m \leq m',$$

by closed, complemented subspaces, satisfying the following properties:

(1) $\mathcal{A} = \cup_{n,m} F_n^m\mathcal{A}$ and $F_n^{-1}\mathcal{A} = 0$;

(2) The union $\mathcal{A}_n := \cup_m F_n^m\mathcal{A}$ is a closed subspace such that

$$F_n^m\mathcal{A} = \mathcal{A}_n \cap \big(\cup_j F_j^m\mathcal{A}\big);$$

(3) Multiplication maps $F_n^m\mathcal{A} \otimes F_{n'}^{m'}\mathcal{A}$ to $F_{n+n'}^{m+m}\mathcal{A}$;

(4) The maps

$$F_n^m\mathcal{A}/F_{n-j}^{m'}\mathcal{A} \otimes F_{n'}^{m'}\mathcal{A}/F_{n'-j}^{m'}\mathcal{A} \longrightarrow F_{n+n'}^{m+m'}\mathcal{A}/F_{n+n'-j}^{m+m'}\mathcal{A}$$

induced by multiplication are continuous;

(5) The quotient $F_n^m\mathcal{A}/F_{n-j}^m\mathcal{A}$ is a nuclear Frechet space in the induced topology;

(6) The natural map

$$F_n^m\mathcal{A} \longrightarrow \lim_{\leftarrow} F_n^m/F_{n-j}^m\mathcal{A}, \quad j \to \infty$$

is a homeomorphism; and

(7) The topology on $\mathcal{A}$ is the strict inductive limit of the subspaces $F_n^n\mathcal{A}$, as $n \to \infty$ (recall that $F_n^n\mathcal{A}$ is assumed to be closed in $F_{n+1}^{n+1}\mathcal{A}$).

As the referee has pointed out, we can replace conditions (4), (5), and (7) with the condition that $\mathcal{A}$ be a nuclear LF-space with a separately continuous multiplication.

For topologically filtered algebras, the multiplication is not necessarily jointly continuous, and the definition of the Hochschild and cyclic homologies using the projective tensor product of the algebra $\mathcal{A}$ with itself, as in the previous section, is not very useful. For this reason, we change the definition of the spaces $\mathcal{H}_m(\mathcal{A})$ defining the Hochschild complexes as follows.

Consider

$$F'_p = \lim_{m\to\infty} \sum_{k_0+\ldots+k_q=p} \widehat{\otimes}_{j=0}^{q} F^m_{k_j}\mathcal{A}\,, \tag{17}$$

(projective tensor products) which defines an increasing sequence of subspaces (*i.e.* filtration) of $\mathcal{A}^{\widehat{\otimes} q+1}$. We use this filtration to define $\mathcal{H}_q(\mathcal{A})$ as a completion. Namely,

$$F_p\mathcal{H}_q(\mathcal{A}) := \lim_{\leftarrow} F'_p/F'_{p-j} \quad \text{and} \quad \mathcal{H}_q(\mathcal{A}) := \cup_p F_p\mathcal{H}_q(\mathcal{A}) \tag{18}$$

where $j \to \infty$ in the projective limit. The operators b and B extend to a well defined maps, still denoted b and B, defined on $\mathcal{H}_q(\mathcal{A})$, for any q, which allows us to define the cyclic complex and the cyclic homology of the algebra $\mathcal{A}$ as the homology of the complex $(\mathcal{C}_*(\mathcal{A}), b+B)$, with $\mathcal{C}_q(\mathcal{A}) := \oplus\mathcal{H}_{q-2k}(\mathcal{A})$, as for topological algebras.

For any topologically filtered algebra, we denote

$$Gr(\mathcal{A}) := \oplus_n\, \mathcal{A}_n/\mathcal{A}_{n-1}$$

the *graded algebra* associated to $\mathcal{A}$, where $\mathcal{A}_n$ is the union $\cup_m F^m_n\mathcal{A}$, as before. Its topology is that of an inductive limit of Frechet spaces:

$$Gr(\mathcal{A}) \simeq \lim_{N,m\to\infty} \oplus_{n=-N}^{N}\, F^m_n\mathcal{A}/F^m_{n-1}\mathcal{A},$$

which makes sense by (2) in the definition of the topologically filtered algebra $\mathcal{A}$.

For the algebras like $Gr(\mathcal{A})$, we need yet a third way of topologizing its iterated tensor products. For our purposes, the correct definition is then

$$\mathcal{H}_q(Gr(\mathcal{A})) := \lim_{N,m\to\infty} \big(\oplus_{n=-N}^{N}\, F^m_n\mathcal{A}/F^m_{n-1}\mathcal{A}\big)^{\widehat{\otimes} q+1}.$$

Note that this is not intended to be a "topological tensor product," but just a vector space, which happens to suit our purposes. This corrects the definition in [5], which is algebraically not so convenient as this one, although it does give a topological tensor product.

The Hochschild homology of $Gr(\mathcal{A})$ is the homology of the complex $(\mathcal{H}_*(Gr(\mathcal{A})),b)$. The operator B again extends to a map $B : \mathcal{H}_q(Gr(\mathcal{A})) \to \mathcal{H}_{q+1}(Gr(\mathcal{A}))$ and we can define the cyclic homology of $Gr(\mathcal{A})$ as above. The operators S, B and I associated to $\mathcal{H}_q(Gr(\mathcal{A}))$ are the graded operators associated with the corresponding operators (also denoted S, B and I) on $\mathcal{H}_q(\mathcal{A})$.

The Hochschild and cyclic complexes of the algebra $Gr(\mathcal{A})$ decompose naturally as direct sums of complexes indexed by $p \in \mathbb{Z}$. For example, $\mathcal{H}_q(Gr(\mathcal{A}))$ is the direct sum of the subspaces $\mathcal{H}_q(Gr(\mathcal{A}))_p$, where

$$\mathcal{H}_q(Gr(\mathcal{A}))_p = \lim_{m,N\to\infty} \bigoplus_{k_j} \left(\widehat{\otimes}_{j=0}^{q} F^m_{k_j}\mathcal{A}/F^m_{k_j-1}\mathcal{A}\right),$$
$$\text{where } k_0+k_1+\ldots+k_q = p \text{ and } -N \le k_j \le N.$$

The corresponding subcomplexes of the cyclic complex are defined similarly. We denote by $\mathrm{HH}_*(Gr(\mathcal{A}))_p$ and $\mathrm{HC}_*(Gr(\mathcal{A}))_p$ the homologies of the corresponding complexes (Hochschild and, respectively, cyclic).

The following two results are consequences of standard results in homological algebra (for topologically filtered algebras they were proved in [5]).

Lemma 2. *Let $\mathcal{A}$ be a topologically filtered algebra. Then the natural filtrations on the Hochschild and cyclic complexes of $\mathcal{A}$ define spectral sequences $\mathrm{EH}^r_{k,h}$ and $\mathrm{EC}^r_{k,h}$ such that*

$$\mathrm{EH}^1_{k,h} \simeq \mathrm{HH}_{k+h}(Gr(\mathcal{A}))_k \ \ \textit{and} \ \mathrm{EC}^1_{k,h} \simeq \mathrm{HC}_{k+h}(Gr(\mathcal{A}))_k .$$

Moreover, the periodicity morphism S induces a morphism $S' : \mathrm{EC}^r_{k,h} \to \mathrm{EC}^r_{k,h-2}$ of spectral sequences. For $r = 1$, the morphism S' is the graded map associated to the periodicity operator $S : \mathrm{HC}_n(Gr(\mathcal{A})) \to \mathrm{HC}_{n-2}(Gr(\mathcal{A}))$ and the natural filtration of the groups $\mathrm{HC}_n(Gr(\mathcal{A}))$.

Let us go back now to the algebra of complete symbols on our groupoid $\mathcal{G}$. We shall denote by $\mathcal{O}(M)$ the space of smooth functions on the interior of M that have only rational singularities at the boundary faces. If every hyperface H of M has a defining function x_H, then $\mathcal{O}(M)$ is the ring generated by $\mathcal{C}^\infty(M)$ and x_H^{-1}. Let then

$$\mathcal{A}_{\mathcal{L}}(\mathcal{G}) := \mathcal{O}(M)\mathcal{A}(\mathcal{G}).$$

Proposition 5. [5] *Assume that $\mathcal{G}$ and M are as above and that M is σ-compact. Then the quotients $\mathcal{A}(\mathcal{G})$ and $\mathcal{A}_{\mathcal{L}}(\mathcal{G})$ are topologically filtered algebras.*

The multifiltrations are given by the order of the symbols, an exhaustive sequence of compact subsets and the degree of the rational singularities. See [5, 6] for more precise constructions.

Let $A\mathcal{G}$ be the Lie algebroid of $\mathcal{G}$ and let $S^*\mathcal{G}$ be the sphere bundle of $A^*\mathcal{G}$, that is, the set of unit vectors in the dual of the Lie algebroid of $\mathcal{G}$. Denote $\mathrm{H}_c^{[q]} = \oplus_{k\in\mathbb{Z}} \mathrm{H}_c^{q+2k}$ the singular cohomology with compact support and coefficients in $\mathbb{C}$. The de Rham cohomology of compactly supported differential forms with only rational singularities at the corners will be denoted by $\mathrm{H}_{c,\mathcal{L}}^{[q]} = \oplus_{k\in\mathbb{Z}} \mathrm{H}_{c,\mathcal{L}}^{q+2k}$ and will be called *the Laurent-de Rham cohomology*.

The periodic cyclic homology of the algebras $\mathcal{A}(\mathcal{G})$ and $\mathcal{A}_{\mathcal{L}}(\mathcal{G})$ can be computed without any further assumption on the groupoid $\mathcal{G}$ [5].

Theorem 9. *Assume that the base M is σ-compact, then we have:*

$$\mathrm{HP}_m(\mathcal{A}(\mathcal{G})) \simeq \mathrm{H}_c^{[m]}(S^*\mathcal{G} \times S^1) \quad \textit{and} \quad \mathrm{HP}_m(\mathcal{A}_{\mathcal{L}}(\mathcal{G})) \simeq \mathrm{H}_{c,\mathcal{L}}^{[m]}(S^*\mathcal{G} \times S^1).$$

This theorem generalizes several earlier calculations. For instance, let $F \subset TM$ be an integrable subbundle of the tangent bundle TM to M and assume for simplicity that M is smooth. Then the algebra of complete symbols along the foliation defined by F coincides with Connes' algebra [19, 48], except that we require transverse smoothness [55]. We denote this algebra by $\mathcal{A}(M, F)$.

Theorem 10. *The periodic cyclic homology of the algebra $\mathcal{A}(M, F)$ is given by:*

$$\mathrm{HP}_q(\mathcal{A}(M, F)) \simeq \mathrm{H}_c^{[q]}(S^*F \times S^1).$$

Finer differential invariants can be captured by computing Hochschild homology. Recall that $A^*\mathcal{G} \to M$, the dual of the Lie algebroid of $\mathcal{G}$, is a Poisson manifold with corners. We shall then denote the Poisson differential associated with this structure by δ, see [13]. If $A^*\mathcal{G} \smallsetminus 0$ denotes the open submanifold of $A^*\mathcal{G}$ which is the complement of the zero section, then the radial action of $\mathbb{R}_+^*$ allows us to consider $l-$homogeneous $k-$forms $\Omega^k(A^*\mathcal{G} \smallsetminus 0)_l$ and Laurent type homogeneous forms $\Omega^k_{\mathcal{L}}(A^*\mathcal{G} \smallsetminus 0)_l$. It is then easy to check that the Poisson differential sends $\Omega^k(A^*\mathcal{G} \smallsetminus 0)_l$ (resp. $\Omega^k_{\mathcal{L}}(A^*\mathcal{G} \smallsetminus 0)_l$) to $\Omega^{k-1}(A^*\mathcal{G} \smallsetminus 0)_{l-1}$ (resp. $\Omega^{k-1}_{\mathcal{L}}(A^*\mathcal{G} \smallsetminus 0)_{l-1}$). We shall denote the resulting homology spaces by $\mathrm{H}^\delta_k(A^*\mathcal{G} \smallsetminus 0)_l$ and $\mathrm{H}^\delta_{k,\mathcal{L}}(A^*\mathcal{G} \smallsetminus 0)_l$.

Proposition 6. *The algebra $\mathcal{A}_{\mathcal{L}}(\mathcal{G})$ is H-unital. Let*

$$\chi : \mathrm{HH}_l(Gr(\mathcal{A}))_d \to \Omega^l(A^*(\mathcal{G}) \smallsetminus 0)_d$$

be the HKRC-isomorphism, and let $d_1 : \mathrm{EH}^1_{k,h} \to \mathrm{EH}^1_{k-1,h}$ be the first differential of the spectral sequence associated to $\mathcal{A}$ by Lemma 2. Then $\chi \circ d_1 \circ \chi^{-1} = -\sqrt{-1}\delta$, and hence $\mathrm{EH}^2_{k,h} \simeq \mathrm{H}^{\delta}_{k+h}(A^*(\mathcal{G}) \smallsetminus 0)_k$.

Proof. As we have already observed the term $\mathrm{EH}^1_{k,h}$ of the above spectral sequence coincides with the Hochschild homology space $\mathrm{HH}_{k+h}(Gr(\mathcal{A}_{\mathcal{L}}(\mathcal{G})))_k$. Since the graded algebra $Gr(\mathcal{A}_{\mathcal{L}}(\mathcal{G}))$ is commutative, we can use the HKRC-map to identify $\mathrm{EH}^1_{k,h}$ with the space of $(k+h)-$forms on $A^*\mathcal{G} \smallsetminus 0$ that are $k-$homogeneous with respect to the radial action of $\mathbb{R}^*_+$.

The rest of the proof consists in identifying the differential

$$d^1 : \mathrm{EH}^1_{k,h} \longrightarrow \mathrm{EH}^1_{k-1,h}$$

under the HKRC–map.

Let q be a quantization map $q : S^m(A^*\mathcal{G}) \to \Psi^m(\mathcal{G})$ (this map is a local inverse for the principal symbol). Choose an anti-symmetric tensor in the last m-variables

$$\eta = \sum \mathrm{sign}(\sigma) f_0 \otimes f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(m)},$$

with $f_j \in S^\infty(A^*\mathcal{G})$. We denote by

$$q(\eta) = \sum \mathrm{sign}(\sigma) q(f_0) \otimes q(f_{\sigma(1)}) \otimes \ldots \otimes q(f_{\sigma(m)})$$

the *quantization* of η. Let $k = \deg f_0 + \ldots + \deg f_m$ be the total degree. Because

$$[q(a), q(b)] = -\sqrt{-1} q(\{a, b\}) + \ldots,$$

where the dots represent terms of order at most $\deg a + \deg b - 2$, the quantity $b \circ q(\eta)$ is of total order at most $k - 1$ and hence, modulo terms of order $k - 2$, $\chi \circ b \circ q(\eta))$ is easily checked to be exactly $\delta(\eta)$. $\square$

5. APPLICATIONS AND EXAMPLES

5.1. **Manifolds with corners.** When the groupoid $\mathcal{G}$ is the groupoid describing the b-calculus (*i.e.* the “stretched product,” M^2_b, in Melrose’s terminology) calculus on the manifold with corners M, the spectral sequence associated with Hochschild homology satisfies

$$E^2_{k,h} = 0 \quad h \neq n,$$

and hence it collapses at EH^2. The asymptotic completeness of the algebras of complete symbols shows that this spectral sequence also converges (this is part of a more general result on topologically filtered algebras, [6]). Therefore, for these algebras, the computation of Hochschild homology is complete.

It turns out that many algebras of complete symbols on manifolds with corners become isomorphic when introducing Laurent coefficients [6, 45, 49, 35]. Because of this, we shall denote the algebra of complete symbols on the groupoid M^2_b simply by $\mathcal{A}_{\mathcal{L}}(M)$.

Denote by $H^q_{\mathcal{L}}(X)$ the homology of the complex $(\mathcal{O}(X)\Omega^*(X), d)$ of de Rham differential forms with Laurent singularities at the boundary.

Theorem 11. *Let M be a manifold with corners. We have*

$$\mathrm{HH}_q(\mathcal{A}_{\mathcal{L}}(M)) \simeq \mathrm{H}^{2n-q}_{c,\mathcal{L}}(S^*M \times S^1).$$

As an easy consequence of this theorem, we obtain the dimension of the space of residue traces on manifolds with corners.

Corollary 1. *The dimension of the space of traces of $\mathcal{A}_{\mathcal{L}}(M)$ is the number of minimal faces of M.*

We now turn to the computation of cyclic homology. A direct inspection shows that the operator B is trivial in this case. This allows us to deduce also the cyclic homology of $\mathcal{A}_{\mathcal{L}}(M)$.

Theorem 12. *Let M be a smooth, compact manifold with corners. We have*

$$\mathrm{HC}_q(\mathcal{A}_{\mathcal{L}}(M)) \simeq \bigoplus_{k\geq 0} \mathrm{HH}_{q-2k}(\mathcal{A}_{\mathcal{L}}(M)).$$

5.2. **Fibrations by manifolds with corners.** Assume now that the groupoid $\mathcal{G}$ describes the vertical pseudodifferential calculus on a connected fibration with corners $\pi : M \to B$ over a smooth manifold B. Then we shall denote the algebra $\mathcal{A}_{\mathcal{L}}(\mathcal{G})$ by $\mathcal{A}_{\mathcal{L}}(M|B)$, because these algebras turn out to be isomorphic under some pretty general conditions.

When the manifold M has no corners and $\mathcal{G} = M \times_B M$ is the space of pairs of points with the same projection on B, we recover the Atiyah-Singer algebra of families of smooth complete symbols along the fibers of the fibration $M \to B$, described for example in [2, 5]. Denote by n the dimension of M, by p the dimension of the fibres and by q the dimension of B.

Denote by $\mathcal{F}^j$ the local coefficient system over B defined by

$$\mathcal{F}^j(b) := \mathrm{HH}_{2p-j}(\mathcal{A}_{\mathcal{L}}(\pi^{-1}(b))) \simeq \mathrm{H}^j_{c,\mathcal{L}}(\pi_0^{-1}(b)),$$

where $\pi_0 : S^*_{vert}M \times S^1 \to B$ is the natural projection and $S^*_{vert}M$ is the vertical cosphere bundle.

Theorem 13. *Let $M \to B$ be a fibration of manifolds with corners, with B smooth, then*

$$\mathrm{HH}_m(\mathcal{A}_{\mathcal{L}}(M|B)) \simeq \bigoplus_{k+h=m} \Omega^h_c(B, \mathcal{F}^{2p-k}).$$

As an easy consequence and in the case without corners for simplicity, we obtain that (when $p \geq 2$), residue traces are in one-one correspondence with distributions on the base manifold B.

5.3. **Longitudinal symbols on foliations.** Foliations provide several examples in noncommutative geometry. We shall look hence at the case of complete symbols on the holonomy groupoid of a foliation, that is the case of the algebra of complete symbols along the leaves of a foliation.

We first need some definitions and notations. Let (X, F) be a smooth manifold X of dimension n_0 equipped with a regular smooth foliation F. The transverse bundle to the foliation (X, F) is the quotient vector bundle $\nu = TX/F$. We denote by p_0 the dimension of F. The codimension will be denoted by q_0 so that $n_0 = p_0 + q_0$.

The space $\Omega^{k,h}(X, F)$ denotes the space of differential forms of bidegree (k, h), *i.e.* of smooth sections of the bundle $\Lambda^k F^* \otimes \Lambda^h \nu^*$. A choice of a supplementary subbundle H to F in TX induces the splittings

$$\Theta_H : T^*X \cong F^* \bigoplus \nu^* \quad \text{and} \quad \Omega^d(X) \cong \bigoplus_{k+h=d} \Omega^{k,h}(X, F). \tag{19}$$

The splitting (19) endows $\Omega^*(X)$ with a bigrading so that the de Rham differential decomposes as a sum of three bihomogeneous components

$$d = d_F + d_{\dashv} + \partial,$$

where d_F is the $(1,0)$ component called the longitudinal differential, $d_{\dashv}$ is the $(0,1)$ component and ∂ is an extra component which can be shown to have bidegree $(-1,2)$ [58].

For $s \in \{0, \cdots, q_0\}$, we get longitudinal de Rham complexes $(\Omega^{*,s}(X,F), d_F)$

$$0 \to \Omega^{0,s}(X,F) \xrightarrow{d_F} \Omega^{1,s}(X,F) \xrightarrow{d_F} \dots \xrightarrow{d_F} \Omega^{p_0,s}(X,F) \to 0.$$

The r^{th} homology space of this longitudinal de Rham complex will be denoted by $\mathrm{H}^{r,s}(X,F)$. The de Rham cohomology spaces of the smooth manifold X will be denoted by $\mathrm{H}^k(X)$, as it is customary.

Dual to (k,h)-differential forms, we define a (k,h)-current as a generalized section of the bundle $\Lambda^{p_0-k}(F^*) \otimes \Lambda^{q_0-h}(\nu^*) \otimes \mathbb{C}_\nu$, where $\mathbb{C}_\nu$ is the orientation line bundle of ν. We denote the space of (k,h)-currents by $A_{k,h}(X,F)$. By choosing a transverse distribution H, we can view any (k,h)-current as a continuous linear form on the space of compactly supported differential (k,h)-forms with respect to H. By duality, we define a longitudinal differential on (k,h)-currents, still denoted d_F, and get in this way longitudinal complexes $(A_{*,h}(X,F), d_F)_{0 \le h \le q_0}$:

$$0 \to A_{p_0,h} \xrightarrow{d_F} A_{p_0-1,h} \xrightarrow{d_F} \dots \xrightarrow{d_F} A_{0,h} \to 0.$$

The homology of this complex is denoted $\mathrm{H}_{*,h}(X,F)$.

Let now $(M,\mathcal{F})$ be a new smooth connected foliated manifold with $\dim(M) = n$ and $\dim(\mathcal{F}) = p$. We assume that the bundle $\mathcal{F}$ is oriented and we denote by q the codimension of the foliation, so that $n = p + q$. The above manifold X will be a total space of some fibration over M with an induced foliation F, as we shall see.

If we denote by $\Psi^m(M,\mathcal{F})$ the set of (compactly supported) pseudodifferential operators of integer order $\le m$ along the leaves of $\mathcal{F}$, then $\Psi^{-\infty}(M,\mathcal{F}) = \cap_{m\in\mathbb{Z}} \Psi^m(M,\mathcal{F})$ is isomorphic to the smooth convolution algebra $\mathcal{C}_c^\infty(\mathcal{G})$ of the holonomy groupoid $\mathcal{G}$ associated to $\mathcal{F}$, [55]. We shall denote as usual by $\Psi^\infty(M,\mathcal{F}) = \cup_{m\in\mathbb{Z}} \Psi^m(M,\mathcal{F})$ the set of all classical pseudodifferential operators of integer order along the leaves of $\mathcal{F}$. Then we obtain the usual exact sequence

$$0 \to \mathcal{C}_c^\infty(\mathcal{G}) \longrightarrow \Psi^\infty(M,\mathcal{F}) \longrightarrow \mathcal{A}(M,\mathcal{F}) \to 0,$$

with the quotient $\mathcal{A}(M,\mathcal{F})$ being the algebra of *complete symbols along the leaves of* $\mathcal{F}$.

We endow $X = S^*\mathcal{F}^* \times S^1$ with the foliation F whose leaves are the total spaces of the restriction of $X \to M$ to the leaves of $(M,\mathcal{F})$. In particular, (X,F) has the same codimension as $(M,\mathcal{F})$.

Theorem 14. [7] *Let* $(M,\mathcal{F})$ *be a foliated manifold, and let* $(\mathrm{EH}^r, d^r)_{r\ge1}$ *be the spectral sequence associated with the Hochschild homology of the algebra* $\mathcal{A}(M,\mathcal{F})$ *of complete symbols along the leaves of* $\mathcal{F}$*, as before. Then the spectral sequence* $(\mathrm{EH}^r, d^r)_{r\ge1}$ *converges to the Hochschild homology of* $\mathcal{A}(M,\mathcal{F})$ *and we have:*

$$\mathrm{EH}^2_{k,h} \simeq \mathrm{H}^{p-k,h-p}(S^*F \times S^1, F).$$

The space of residue traces along the leaves of $(M,\mathcal{F})$ can then be deduced.

Corollary 2. *The space* $\mathrm{HH}_0(\mathcal{A}(M,\mathcal{F}))$ *is isomorphic to the space* $\mathrm{H}^{2p,0}(S^*\mathcal{F} \times S^1, F)$*. Moreover when the dimension of the foliation* $\mathcal{F}$ *is* ≥ 2*,*

$$\mathrm{HH}_0(\mathcal{A}(M,\mathcal{F})) \simeq \mathrm{H}^{p,0}(M,\mathcal{F}).$$

Thus, the space of residue traces on the foliation $(M, \mathcal{F})$ is isomorphic to the space of $(2p, 0)$-invariant currents on $(S^*\mathcal{F} \times S^1, F)$. More precisely and when $p \geq 2$, we get

$$\mathrm{HH}^0(\mathcal{A}(M, \mathcal{F})) \simeq \mathrm{H}_{p,0}(M, \mathcal{F}).$$

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UNIVERSITY OF SOUTHAMPTON, U. K.
E-mail address: `j.brodzki@maths.soton.ac.uk`

PENNSYLVANIA STATE UNIVERSITY AND INST. DESARGUES, UNIV. LYON 1, FRANCE
E-mail address: `benameur@igd.univ-lyon1.fr`

INST. DESARGUES AND PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802
E-mail address: `nistor@math.psu.edu`