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ESTIMATES OF SEMIINVARIANTS FOR THE ISING MODEL AT LOW TEMPERATURES

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§1. Formulation of the result

Let $V \subset \mathbb{Z}^d$ be a finite subset of d-dimensional integer lattice $\mathbb{Z}^d$ and $X(V) = \{-1,1\}^V$ be the set of configurations $x = (x_t, t \in V)$, where $x_t = \pm 1$. The (symmetric ferromagnet) *Ising distribution in the volume V with the plus-boundary condition and a converse temperature $\beta > 0$* is the probability distribution on the set $X(V)$ such that the probability

$$p_V(x) = (Z(V))^{-1} \exp\{-\beta U_V(x)\}, \quad x \in X(V), \tag{1.1}$$

where the *partition function*

$$Z(V) = \sum_{x \in X(V)} \exp\{-\beta U_V(x)\} \tag{1.2}$$

and the *energy*

$$U_V(x) = -\left(\sum_{\{s,t\} \subset V : |s-t|=1} x_s x_t + \sum_{\{s,t\} : t \in V, s \in V^c, |s-t|=1} x_s x_t \right), \quad x \in X(V). \tag{1.3}$$

Let $T \subset V$ be a finite set. The *generating function of the values of the field on the set T* is defined by the relation

$$F_T(V; z_t, t \in T) = \sum_{x \in X(V)} p_V(x) \exp\left\{ \sum_{t \in T} z_t x_t \right\}, \tag{1.4}$$

where $z_t, t \in T$, are complex numbers. Let $r = (r_t, t \in T)$ be a function with integer values $r_t \geq 1, t \in T$, and $|r| = \sum_{t \in T} r_t$. The *semiinvariant* (in other terminologies *cumulant, truncated correlation function*) *of order r of the values of the field on the set T* is defined as the value of the partial derivate

$$s_V(T, r) = \left. \frac{\partial^{|r|} \ln F_T(V; z_t, t \in T)}{\prod_{t \in T} \partial z_t^{r_t}} \right|_{z_t \equiv 0, t \in T,}. \tag{1.5}$$

We introduce a geometrical characteristic of the set T

$$q(T) = \max_{S \subseteq T} \frac{|S|}{(\text{diam } S)^{d-1}} \tag{1.6}$$

which we call the *concentration coefficient of the set T*.[1] This coefficient do not exceed $T^{\frac{1}{d}}$ and takes values of this order for sets like cubes or spheres but can be arbitrary small for sets evenly spread on the lattice.

The following estimate is the main result of the paper.

[1] Here and in the following $|A|$ is the number of elements in a finite set A.

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Theorem 1.1. *For each $d \geq 2$ there exist constants $\tilde{\beta} = \tilde{\beta}_d > 0$ and $K = K_d < \infty$ such that for all $\beta > \tilde{\beta}$, all finite sets $T \subset V \subset \mathbb{Z}^d$, where $|T| > 1$, and all $r = (r_t, t \in T)$ the following estimates for the semiinvariants hold*

$$|s_V(T, r)| \leq (K \max(q(T), 1))^{|r|} \exp\{-4(\beta - \tilde{\beta})\ diam\ T\}. \tag{1.7}$$

Note 1.1. The estimate (1.7) is essentially stronger than the estimates which were obtained earlier ([Ma], [MM], §6.5). The multiplier in (1.7) which exponentially depend on diam T seems natural since it estimates the probability that the set T lies inside a contour arounding a piece of another phase. Strong treelike estimates of semiinvariants (see [MM], [DS]) which are natural for the high-temperature case are not possible in the low-temperature situation. The improvement in the comparison with the previous estimates is connected with the preexponential multiplier which is larger than $2^{2^{|r|}}$ in [Ma], [MM].

Note 1.2. In the derivation of the formulated theorem we use a new variant of the cluster expansion method and instead of a direct application of these expansions to contours we apply them to some more complex objects adequate to the considered problem. It gives an improvement of the estimates. This new variant of the cluster expansion method is described in §2 of this paper. There are a lot of publications devoted to development and applications of the powerful cluster expansion method. (See books and review paper [Br], [GJ], [Ma], [MM], [Se], [Sim] and references there). A general model used in §2 is taken from the Kotecky and Preiss paper [KP] (see also [HKZ], Appendix B, and [MS]). In a difference with the other approaches we apply elementary facts of the theory of analytical functions instead of involved combinatorial estimates for terms of cluster expansion and it simplifies the construction and gives estimates convenient for applications. This approach was used earlier by the author and his coauthors ([D1], [D2], [DM], [DW]) but for special situations and in a more complex variant.

Note 1.3. Without essential changes in constructions the main result can be generalized in some directions. So instead semiinvariants of values of the field it is possible to obtain similar estimates for semiinvariants of functions of finite collections of values of the field. Also instead of the ferromagnetic Ising model it is possible to consider other spin models which can be studied by the contour method (see [Sin], for example).

§2. Cluster expansions in animal models

Let Θ be a countable or a finite set. Its elements will be called *animals*.[2] Assume that a structure of an undirected graph without loops and multiple edges with the set Θ of its vertices is fixed. We say that the elements $\theta_1, \theta_2 \in \Theta$ connected by an edge of the graph are *incompatible* and will write $\theta_1 \nsim \theta_2$. If the animals $\theta_1, \theta_2 \in \Theta$ are not connected by an edge, we say that they are *compatible* and shall

[2] In the previous papers the elements of this set were called polymers or alternatively contours. But in the terminology of statistical mechanics these words are attached to some concrete objects, even in applications (including §4 of this papers) these elements can have various and sometimes exotic nature. So we propose the term animal and even develop further this animal terminology.

write $\theta_1 \sim \theta_2$. A finite subset $\pi \subseteq \Theta$ is called *a herd*, if any two distinct animals $\theta_1, \theta_2 \in \pi$ are compatible. For any finite $\Lambda \subseteq \Theta$ the set of all herds π such that $\pi \subseteq \Lambda$ will be denoted $H(\Lambda)$ and will be called the set of all *herds in* Λ. (The set $H(\Lambda)$ includes the empty herd which is denoted by $\emptyset$.)

Assume that a complex-valued function $w(\theta), \theta \in \Theta$, is given. We call the number $w(\theta)$ *the weight of the animal* θ. Let a finite set $\Lambda \subseteq \Theta$. The number

$$Z_w(\Lambda) = \sum_{\pi \in H(\Lambda)} \prod_{\theta \in \pi} w(\theta) \tag{2.1}$$

is called *the partition function in* Λ *defined by the weights* $w = w(\theta)$. (In the case $\pi = \emptyset$ the product in (2.1) is interpreted as the number 1. So if Λ is an empty set the partition function $Z_w(\Lambda) = 1$.) Assume that a positive-valued function $b(\theta), \theta \in \Theta$, is given. The value $b(\theta)$ will be called *the might of the animal* θ.

Theorem 2.1. *Assume that a positive weight function $w_0 = w_0(\theta) > 0$ is fixed such that for any animal $\theta \in \Theta$*

$$1 - w_0(\theta) \exp\left\{ \sum_{\tilde\theta \in \Theta : \tilde\theta \not\sim \theta} w_0(\tilde\theta) b(\tilde\theta) \right\} \geq \exp\{-w_0(\theta) b(\theta)\}. \tag{2.2}$$

In particular, the condition (2.2) implies that

$$w_0(\theta) \exp\left\{ \sum_{\tilde\theta \in \Theta : \tilde\theta \not\sim \theta} w_0(\tilde\theta) b(\tilde\theta) \right\} < 1. \tag{2.3}$$

Let W_0 be the set of all weight functions $w = w(\theta)$ of $\theta \in \Theta$ such that

$$|w(\theta)| \leq w_0(\theta), \theta \in \Theta, \tag{2.4}$$

and a weight function $w \in W_0$. Then for any finite set $\Lambda \subseteq \Theta$ the partition function $Z_w(\Lambda) \neq 0$ and for any finite $\Lambda' \subseteq \Lambda$

$$\left| \ln \left| \frac{Z_w(\Lambda)}{Z_w(\Lambda')} \right| \right| \leq \sum_{\theta \in \Lambda \setminus \Lambda'} w_0(\theta) b(\theta). \tag{2.5}$$

Proof. We shall use an induction in the number of elements $|\Lambda|$ in the set Λ. In the case $|\Lambda| = 0$, i.e. $\Lambda = \emptyset$, the statement of the Theorem is evident. So we fix a set Λ and suppose that the estimate (2.3) is valid for any sets with numbers of elements smaller than $|\Lambda|$. We fix also a subset $\Lambda' \subset \Lambda$ and an animal $\theta_0 \in \Lambda \setminus \Lambda'$. (The case of $\Lambda = \Lambda'$ is trivial.) Consider also the set $\widehat\Lambda = \Lambda \setminus \{\theta_0\}$. Then

$$\frac{Z_w(\Lambda)}{Z_w(\Lambda')} = \frac{Z_w(\Lambda)}{Z_w(\widehat\Lambda)} \frac{Z_w(\widehat\Lambda)}{Z_w(\Lambda')}. \tag{2.6}$$

It follows from the induction hypothesis that

$$\left|\ln\left|\frac{Z_w(\widehat{\Lambda})}{Z_w(\Lambda')}\right|\right| \leq \sum_{\theta\in\widehat{\Lambda}\setminus\Lambda'} w_0(\theta)b(\theta). \tag{2.7}$$

So we shall prove the statement of the Theorem, if we show that

$$\left|\ln\left|\frac{Z_w(\Lambda)}{Z_w(\widehat{\Lambda})}\right|\right| \leq w_0(\theta_0)b(\theta_0). \tag{2.8}$$

Let

$$\Lambda_0 = \{\theta \in \widehat{\Lambda} : \theta \sim \theta_0\}. \tag{2.9}$$

It is clear from the definition (2.1) that

$$Z_w(\Lambda) = Z_w(\widehat{\Lambda}) + w(\theta_0)Z_w(\Lambda_0). \tag{2.10}$$

Since $\Lambda_0 \subseteq \widehat{\Lambda}$ and $|\Lambda_0| < |\Lambda|$ we find using the induction conjecture and the definition (2.4) that

$$\left|\frac{w(\theta_0)Z_w(\Lambda_0)}{Z_w(\widehat{\Lambda})}\right| \leq w_0(\theta_0)\exp\left\{\sum_{\tilde\theta:\tilde\theta\in\Lambda,\tilde\theta\nsim\theta_0} w_0(\tilde\theta)b(\tilde\theta)\right\}. \tag{2.11}$$

Observe now that for any complex z such that $|z|<1$ we have $1-|z| \leq |1+z| \leq 1+|z|$ and so

$$|\ln|1+z|| \leq \max(\ln(1+|z|), -\ln(1-|z|)) \leq -\ln(1-|z|). \tag{2.12}$$

It follows from the condition (2.3) that the absolute value of the right part in (2.11) is smaller than 1. So applying the relations (2.10), (2.11) and (2.12) we find that

$$\begin{aligned}\left|\ln\left|\frac{Z_w(\Lambda)}{Z_w(\widehat{\Lambda})}\right|\right| &= \left|\ln\left|1+\frac{w(\theta_0)Z_w(\Lambda_0)}{Z_w(\widehat{\Lambda})}\right|\right| \\ &\leq -\ln\left(1 - w_0(\theta_0)\exp\left\{\sum_{\tilde\theta:\tilde\theta\in\Lambda:\tilde\theta\nsim\theta_0} w_0(\tilde\theta)b(\tilde\theta)\right\}\right).\end{aligned} \tag{2.13}$$

The estimate (2.13) together with the main condition (2.2) implies the desired estimate (2.8). □

Note 2.1. Instead of the condition (2.2) which appeared in a natural way in the proof of Theorem 2.1 it is convenient to use the following a little more strong condition: for any $\theta \in \Theta$

$$\exp\left\{\sum_{\tilde\theta\in\Theta:\tilde\theta\nsim\theta} w_0(\tilde\theta)b(\tilde\theta) + w_0(\theta)b(\theta)\right\} \leq b(\theta). \tag{2.14}$$

Check that the condition (2.2) is really weaker then the condition (2.14). It follows from (2.14) that

$$w_0(\theta)\exp\left\{\sum_{\tilde\theta\in\Theta:\tilde\theta\nsim\theta} w_0(\tilde\theta)b(\tilde\theta)\right\} \le w_0(\theta)b(\theta)\exp\{-w_0(\theta)b(\theta)\}. \tag{2.15}$$

We let $x = w_0(\theta)b(\theta)$ and see that the condition (2.2) follows from (2.15) and the general inequality

$$1 - e^{-x} \ge xe^{-x} \quad \text{for } x \ge 0. \tag{2.16}$$

To derive this inequality it is enough to check that for $x \ge 0$ the function $f(x) = 1-(x+1)e^{-x}$ has an nonnegative derivate for $x \ge 0$. The condition (2.14) was used in the Kotecky and Preiss paper [KP].

Now we want construct a cluster expansion for logarithms of partition functions. For any finite $\Lambda \subseteq \Theta$ the set of all pairs $\rho = (\bar\rho, \alpha)$ such that $\bar\rho \subseteq \Lambda$ is a set and $\alpha = \alpha(\theta) \ge 1, \theta \in \bar\rho$, is an integer-valued function of $\theta \in \bar\rho$ will be denoted by $D(\Lambda)$ and will be called a *group of animals in* Λ. The set $\bar\rho$ will be called the *support of the group* and the value $\alpha(\theta)$ will be interpreted as the multiplicity of animals of the kind θ in the group ρ. We say that a group $\rho = (\bar\rho, \alpha)$ is a *sum of groups* $\rho_i = (\bar\rho_i, \alpha_i), i = 1, 2, \dots, k$, if $\bar\rho_i \subseteq \bar\rho, i = 1, 2, \dots, k$ and

$$\alpha(\theta) = \sum_{i=1,2,\dots,k:\theta\in\bar\rho_i} \alpha_i(\theta), \quad \theta \in \bar\rho. \tag{2.17}$$

A *gang of animals in* Λ is an non-empty group of animals $\rho = (\bar\rho, \alpha) \in D(\Lambda)$ such that for any two animals $\theta, \theta' \in \bar\rho$ there is a sequence $\theta = \theta_1, \theta_2, \dots, \theta_n = \theta'$ of animals in $\bar\rho$ such that the animals θ_i and θ_{i+1} are incompatible for all $i = 1, 2 \dots, n-1$, i.e. $\bar\rho$ is a connected subset of the graph Θ. The set of all gangs in Λ will be denoted by $G(\Lambda)$.

Theorem 2.2. *Let the conditions of Theorem 2.1 are fulfilled and a finite set* $\Lambda \subseteq \Theta$ *be fixed. Consider a polydisk* $W_0(\Lambda) = \{w = (w(\theta), \theta \in \Lambda) : |w(\theta)| \le w_0(\theta), \theta \in \Lambda\} \subset \mathbb{C}^\Lambda$ *and the set* $W_0^{\mathrm{in}}(\Lambda)$ *of all inner points of the polydisk* $W_0(\Lambda)$. *The partition function* $Z_w(\Lambda)$ *will be treated as a function of* $w \in \mathbb{C}^\Lambda$. *For any* $w \in W_0^{\mathrm{in}}(\Lambda)$ *a convergent expansion*

$$\ln Z_w(\Lambda) = \sum_{\rho\in G(\Lambda)} q_w(\rho) = \sum_{\rho\in G(\Lambda)} r(\rho)\prod_{\theta\in\bar\rho} w(\theta)^{\alpha(\theta)} \tag{2.18}$$

holds. The coefficients $r(\rho)$ *are real numbers depending on the restriction of the graph structure on* Θ *to* $\bar\rho$ *only. For any gang* $\rho = (\bar\rho, \alpha)$

$$|q_w(\rho)| = |r(\rho)\prod_{\theta\in\bar\rho} w(\theta)^{\alpha(\theta)}| \le \left(\sum_{\theta\in\bar\rho} w_0(\theta)b(\theta)\right)\left(\prod_{\theta\in\bar\rho}\left(\frac{|w(\theta)|}{w_0(\theta)}\right)^{\alpha(\theta)}\right). \tag{2.19}$$

Proof. For any $w \in W_0(\Lambda)$ let

$$F_\Lambda(w) = \ln Z_w(\Lambda). \tag{2.20}$$

It follows from Theorem 2.1 that $Z_w(\Lambda)$ do not vanish and so the function $F_\Lambda(w)$ is a holomorphic function of $w \in W_0^{\text{in}}(\Lambda)$. The Tailor expansion of this function at the point $w = 0$ can be written as

$$F_\Lambda(w) = \sum_{\rho \in D(\Lambda)} r_\Lambda(\rho) \prod_{\theta \in \bar\rho} w(\theta)^{\alpha(\theta)}, \tag{2.21}$$

For $\bar\rho = (\theta_1, \theta_2, \ldots, \theta_n)$ the coefficients

$$r_\Lambda(\rho) = (\alpha(\theta_1)!\alpha(\theta_2)! \ldots \alpha(\theta_n)!)^{-1} \left. \frac{\partial^{\alpha(\theta_1)+\alpha(\theta_2)+\ldots+\alpha(\theta_n)} F_\Lambda(w)}{\partial^{\alpha(\theta_1)} w(\theta_1) \partial^{\alpha(\theta_2)} w(\theta_2) \ldots \partial^{\alpha(\theta_n)} w(\theta_n)} \right|_{w=0}. \tag{2.22}$$

For any set $\bar\rho \subseteq \Lambda$ and any function $w = (w(\theta), \theta \in \bar\rho) \in W_0(\bar\rho)$ consider its continuation $w_\Lambda = (w_\Lambda(\theta), \theta \in \Lambda)$ to Λ such that $w_\Lambda(\theta) = 0$ for $\theta \in \Lambda \setminus \bar\rho$. It follows from the definitions (2.1) and (2.20) that

$$F_\Lambda(w_\Lambda) = F_{\bar\rho}(w) \tag{2.23}$$

and so we see from the relation (2.22) that

$$r_\Lambda(\rho) = r_{\bar\rho}(\rho). \tag{2.24}$$

Now letting $r(\rho) = r_{\bar\rho}(\rho)$ we can rewrite the expansion (2.21) as

$$F_\Lambda(w) = \sum_{\rho \in D(\Lambda)} r(\rho) \prod_{\theta \in \bar\rho} w(\theta)^{\alpha(\theta)}. \tag{2.25}$$

The coefficients $r(\rho)$ are real, since the function $F_{\bar\rho}(w)$ takes real values for real w.

Since $Z_w(\Lambda) = 1$ for $w = 0$ the coefficient $r(\rho) = 0$ for the empty group of animals $\rho = \emptyset$. Let now an non-empty group of animals $\rho \in D(\Lambda) \setminus G(\Lambda)$. There exists a representation $\bar\rho = \bar\rho_1 \cup \bar\rho_2$, where the sets $\bar\rho_1, \bar\rho_2$ are non-empty, their intersection $\bar\rho_1 \cap \bar\rho_2 = \emptyset$ and for any $\theta_1 \in \bar\rho_1, \theta_2 \in \bar\rho_2$ the animals θ_1, θ_2 are compatible. Then it follows from the definitions (2.1) and (2.20) that

$$F_{\bar\rho}(w) = F_{\bar\rho_1}(w) + F_{\bar\rho_2}(w). \tag{2.26}$$

Differentiating we obtain from (2.22) and (2.26) that

$$r(\rho) = r_{\bar\rho}(\rho) = 0, \quad \text{if } \rho \in D(\Lambda) \setminus G(\Lambda), \tag{2.27}$$

and so the expansion (2.25) is reduced to the desired expansion (2.18).

It follows from the relation (2.22) and the Cauchy formula that

$$r(\rho) = (2\pi i)^{-|\bar\rho|} \oint_{w:|w(\theta)|=w_0(\theta), \theta \in \bar\rho} \frac{F_{\bar\rho}(w)}{\prod_{\theta \in \bar\rho} (w(\theta))^{\alpha(\theta)+1}} \, dw. \tag{2.28}$$

The inequality (2.5) applied for $\Lambda' = \emptyset$ gives a priori estimate

$$|F_{\bar\rho}(w)| \leq \sum_{\theta \in \bar\rho} w_0(\theta) b(\theta), \quad w \in W_0(\Lambda), \tag{2.29}$$

The desired estimate (2.19) follows from the estimates (2.28) and (2.29). □

§4. Contour representation

The proof of the main theorem will use the well-known contour description of the Ising ferromagnetic model for law temperatures. Following a tradition of the literature on application of contour method we restrict its exposition by the simplest case, when the dimension $d = 2$. The generalization to the case $d > 2$ is really direct and even sometimes discussed in details (for example in [Sim], §V.8 for the case $d = 3$.)

Let $\mathbb{Z}^2$ be the two-dimensional integer lattice. Assume that $\mathbb{Z}^2 \subset \mathbb{R}^2$. Let $\mathbb{Z}^{2\star}$ be the *conjugate lattice* with vertices $(n_1+1/2, n_2+1/2)$, $n_1, n_2 \in \mathbb{Z}^1$. Let $\mathbb{E}$ be the set of *edges of the congugate lattice*, i.e. the set of all closed intervals of the length 1 connecting the adjacent points of this lattice. For the each edge $e \in \mathbb{E}$ there are two vertices of the original lattice $\mathbb{Z}^2$ on the distance $1/2$ from e. We say that they are *vertices adjacent to the edge* e. Let $V \subset \mathbb{Z}^2$ be a finite set. The set of edges $e \in \mathbb{E}$ such that at least one of two points to which the edge e is adjacent belongs to V is called the set of *edges in the volume* V and is denoted by $\mathbb{E}(V)$.

Return now to the Ising distribution with the plus-boundary condition described in §1. For each configuration $x \in X(V)$ we define its *boundary* $B(x)$ as the set of all edges $e \in \mathbb{E}(V)$ of the conjugate lattice such that if t, t' are the points of the original lattice adjacent to e the values $x_t \neq x_{t'}$. (We assume here that $x_t = 1$ for $t \in V^c = \mathbb{Z}^2 \setminus V$ and it means that we introduce plus-boundary conditions.) The contour method of a study of the Ising model is founded on a possibility to represent the boundary $B(x)$ in a unique way as a sum of contours, i.e. non-selfintersecting closed lines consisting of edges $e \in E(V)$, but it is necessary to be careful in the definition of contours. (What is a unique natural way to divide to contours the boundary of the configuration x such that $x_t = (-1)^{t_1+t_2}, t = (t_1, t_2) \in V$? See the answer at the right part of Fig. 1.)

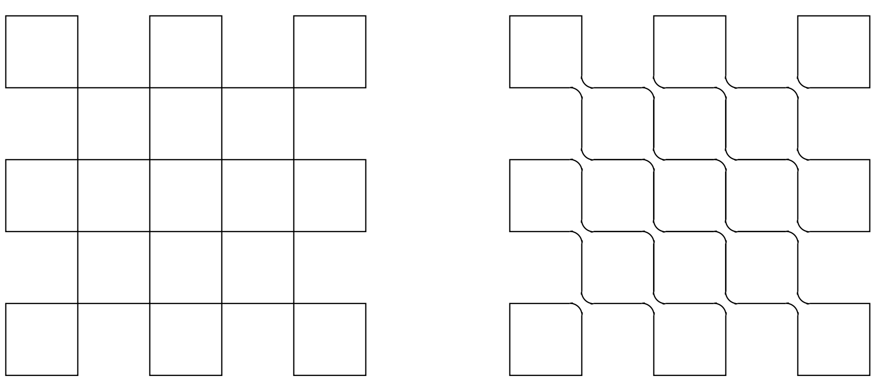

Figure 1 Contour representation of the boundary of a configuration

So we introduce the following convention. Let $e_1, e_2 \in \mathbb{E}$ be two different edges containing a general vertex $t = (t_1, t_2) \in \mathbb{Z}^{2\star}$. We say that these vertices make a *legitimate turn*, if either one of these edges connects the vertex t with the vertex (t_1+1, t_2) and the other vertex connects it with the vertex (t_1, t_2+1) or if one of these edges connects the vertex t with the vertex (t_1-1, t_2) and the other vertex connects it with the vertex $(t_1, t_2 - 1)$. A contour is defined as a sequence $e_1, e_2, \dots, e_k$ of mutually distinct edges such that the edges $e_i, e_{i+1}, i = 1, 2, \dots, k$ (here $k + 1 = 1$) have a general vertex and if there is another pair of edges $e_{i'}, e_{i'+1}$ of this contour having the same general vertex the vertices e_i, e_{i+1} make a legitimate turn. In other words a contour is a closed broken line consisting of mutually different edges such that a small deformation conserving the legitimate turns transforms it to a non-selfintersecting closed curve. The set of all contours is denoted by $\mathbb{G}$ We say that a contour $? \in \mathbb{G}$is a *contour in the volume* V, if all its edges belong to $\mathbb{E}(V)$. The set of all such contours is denoted by $\mathbb{G}(V)$. The number of edges in a contour $? \in \mathbb{G}$ is denoted $|?|$ and is called the *length* of this contour. The set of all points $t \in \mathbb{Z}^2$ such that there are no continuous curve in $\mathbb{R}^2$ which do not intersect a contour ? and connect the point $t \in \mathbb{Z}^2 \subset \mathbb{R}^2$ with "infinity" will be called the *interior of the contour* ? and will be denoted Int ?. We say that the contours $?_1$ and $?_2$ are *compatible* if they have no general edges and at any vertex which is contained in both of contours these contours make legitimate turns.

Let $H(V)$ be the set of all sets $\pi \subseteq \mathbb{G}(V)$ of contours in the volume V such that any two different contours in π are compatible ($H(V)$ includes the empty set of contours). It is easy to understand that for any finite volume V and any configuration $x \in X(V)$ there exists a unique system of contours $\pi(x) \in H(V)$ such that

$$B(x) = \cup_{\Gamma \in \pi(x)} ?. \tag{3.1}$$

Further for any system of contours $\pi \in H(V)$ there is a unique configuration $x(\pi) = (x_t(\pi), t \in V)$ such that

$$\pi(x(\pi)) = \pi. \tag{3.2}$$

Really this configuration can be defined by the following construction. For any point $t \in \mathbb{Z}^2$ denote by $O(t)$ the set of all contours $? \in \mathbb{G}$such that $t \in$ Int ?. Then

$$x_t(\pi) = (-1)^{|\pi \cap O(t)|}, \quad t \in V, \pi \in H(V). \tag{3.3}$$

The definition (1.3) implies that the energy

$$U_V(x) = 2|\pi(x)| - K_V, \quad x \in X(V), \tag{3.4}$$

where we let

$$|\pi| = \sum_{\Gamma \in \pi} |?|, \quad \pi \in H(V), \tag{3.5}$$

and where the constant K_V equals the number of terms in the sums in (1.3) and does not depend on x. So recalling the definition (1.1) we find that

$$p_V(x) = \tilde{p}_V(\pi(x)), \quad x \in X(V), \tag{3.6}$$

where *the contour probability distribution*

$$\tilde{p}_V(\pi) = (\tilde{Z}(V))^{-1} \exp\{-2\beta|\pi|\}, \quad \pi \in H(V), \tag{3.7}$$

and *the contour partition function*

$$\tilde{Z}(V) = \sum_{\pi \in H(V)} \exp\left\{-2\beta \sum_{\Gamma \in \pi} |?|\right\}. \tag{3.8}$$

It is the well-known reduction of the ferromagnetic Ising model to the contour Ising model used for its study in the case of large β.

Comparing the definitions (1.4) and the relation (3.6) we find that the generating function

$$F_T(V; z_t, t \in T) = \frac{Z_T(V; z_t, t \in V)}{\tilde{Z}(V)}, \tag{3.9}$$

where the partition function

$$Z_T(V; z_t, t \in V) = \sum_{\pi \in H(V)} \exp\left\{-2\beta \sum_{\Gamma \in \pi} |?| + \sum_{t \in T} z_t x_t(\pi)\right\}. \tag{3.10}$$

So we can rewrite the definition (1.5) as

$$s_V(T, r) = \left.\frac{\partial^{|r|} \ln Z_T(V; z_t, t \in T)}{\prod_{t \in T} \partial z_t^{r_t}}\right|_{z_t \equiv 0, t \in T,}. \tag{3.11}$$

This contour representation of semiinvariants is the starting point of the following constructions.

The following simple statement about contours will be used below. For any set $V_0 \subset \mathbb{Z}^2$ we denote by $O(V_0)$ the system of all contours which contain at least one of points of the set V_0 inside of them:

$$O(V_0) = \sum_{t \in V_0} O(t). \tag{3.12}$$

Lemma 3.1. *There exists a constant $\beta_1 > 0$ such that for any finite $V_0 \subset \mathbb{Z}^2$*

$$\sum_{\Gamma \in O(V_0)} \exp\{-2\beta_1 |?|\} \leq \frac{1}{2}|V_0|. \tag{3.13}$$

Proof. Let $t_0 = (t_0^1, t_0^2) \in V_0$ and e_{a,t_0}, where $a = 0, 1, \ldots,$ be the edge of the lattice connecting the vertices $(t_0^1 + a + 1/2, t_0^2 + 1/2)$ and $(t_1^0 + a + 1/2, t_0^2 - 1/2)$ of the conjugate lattice. It is clear that any contour $? \in O(t_0)$ contains at least one of the edges e_{a,t_0} with $a \leq |?|$. The number of different contours of a length d containing a fixed edge do not exceed 3^d and so

$$\sum_{\Gamma \in O(t_0)} \exp\{-2\beta|?|\} \leq \sum_{a=0}^{\infty} \sum_{d=a}^{\infty} \exp\{-(2\beta - \ln 3)d\}. \tag{3.14}$$

The right part of (3.14) tends to 0 as $\beta \to \infty$, and now it follows from the definition (3.12) that the desired estimate (3.13) holds for large enough β_1. $\square$

§4. Animal model representation

The derivation of the main estimate (1.7) for semiinvariants uses a special representation of the partition function (3.10) as the partition function of a special animal model (see §2). Fixing a finite set $T \subset \mathbb{Z}^2$ we describe this animal model. The set of animals $\Theta = \Theta(T)$ consists of elements of two types. First all contours $\mathbb{G} \subset \Theta(T)$. Besides it the set of animals $\Theta(T)$ contains some additional elements which we call *amoebas*. The amoebas are pairs $A = (S, \pi_S)$, where $S \subseteq T$ are finite nonempty sets, the elements of which will be called *nuclei* of the amoeba A, and π_S are finite systems of pairwise compatible contours such that $\pi_S \subset O(S)$ (see (3.12)) and if $|S| > 1$ there is a contour $? \in \pi_S$ which contains all the set S inside of it: $S \subseteq \mathrm{Int}\ ?$. The largest (in the sense of the number of $|\mathrm{Int}\ ?)$ of contours ? containing S will be called the *membrane* of the amoebas A and will be denoted $M(A)$ (In the case $|S| = 1$ the system π_S can be the empty system and the membrane be absent). (See Fig. 2.) The set of all amoebas will be denoted $\mathbb{A}(T)$.

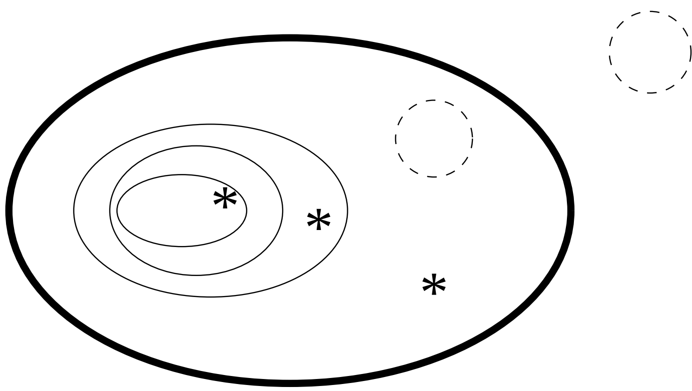

Figure 2 An amoeba with three nuclei and two (dashed) contours compatible with it.

We need to describe the incompatibility relation ($\nsim$) in $\Theta(T)$. Contours $?_1, ?_2 \in \mathbb{G} \subset \Theta(T)$ are incompatible, if they are incompatible in the sense described in §3. A contour $? \in \mathbb{G}$ and an amoeba $A = (S, \pi_S) \in \mathbb{A}(T)$ are incompatible if either one of contours $?' \in \pi_S$ is incompatible with the contour ? or the contour $? \in O(S)$. Two amoebas $A_1 = (S_1, \pi^1_{S_1}), A_2 = (S_2, \pi^2_{S_2}) \in \mathbb{A}(T)$ are incompatible if either there is a pair of incompatible contours $?^1 \in \pi^1_{S_1}$ and $?^2 \in \pi^2_{S_2}$ or the intersection $S_1 \cap S_2$ is not empty or at last the set of contours $(\pi^1_{S_1} \cap O(S_2)) \cup (\pi^2_{S_2} \cap O(S_1))$ is not empty.

At last fixing $\beta > 0$ and complex numbers $z_t, t \in T$ we define the weight function (4.1)

$$w(\theta) = \begin{cases} \exp\{-2\beta|?|\}, & \text{if } \theta = ? \in \mathbb{G}, \\ \exp\{-2\beta|\pi_S|\} \prod_{t \in S}(\exp\{z_t x_t(\pi_S)\} - 1), & \text{if } \theta = (S, \pi_S) \in \mathbb{A}(T). \end{cases}$$

Proposition 4.1. *Let $\Lambda(T, V)$, where $T \subseteq V \subset \mathbb{Z}^2$ are finite sets, be the set of animals $\theta \in \Theta(T)$ such that either $\theta = ? \in \mathbb{G}(V)$ or $\theta = (S, \pi_S) \in \mathbb{A}(T)$, where*

$\pi_S \in H(V)$ (see §3). Then (see the definition (2.1)) the partition function

$$Z_T(V; z_t, t \in T) = Z_w(\Lambda(T, V)). \tag{4.2}$$

Let

$$w_0(\theta) = \begin{cases} \exp\{-2\beta_0 |?|\}, & \text{if } \theta = ? \in \mathbb{G}, \\ (v(T))^{|S|} \exp\{-2\beta_0 |\pi_S|\}, & \text{if } \theta = (S, \pi_S) \in \mathbb{A}(T), \end{cases} \tag{4.3}$$

where (recall (1.6))

$$v(T) = \frac{1}{2} e^{-\frac{3}{2}} \min((q(T))^{-1}, 1) \tag{4.4}$$

and

$$b(\theta) = \begin{cases} \exp\{2|?|\}, & \text{if } \theta = ? \in \mathbb{G}, \\ \exp\{2|\pi_S| + |S|\}, & \text{if } \theta = (S, \pi_S) \in \mathbb{A}(T). \end{cases} \tag{4.5}$$

Then there exists a value $\beta_0 > 0$ such that the main condition (2.2) of Theorem 2.1 is fulfilled.

Proof. We can rewrite the partition function (3.10) as

$$\begin{aligned} Z_T(V; z_t, t \in V) = \sum_{\pi \in H(V)} \exp\left\{-2\beta \sum_{\Gamma \in \pi} |?|\right\} \prod_{t \in T} (\exp\{z_t x_t(\pi)\} - 1 + 1) = \\ \sum_{S \subseteq T, \pi \in H(V)} \exp\left\{-2\beta \sum_{\Gamma \in \pi} |?|\right\} \prod_{t \in S} (\exp\{z_t x_t(\pi)\} - 1). \end{aligned} \tag{4.6}$$

Show that for any fixed pair (S, π) where $S \subseteq T$ and $\pi \in H(V)$ there exists a unique representation

$$\begin{aligned} S = S_1 \cup S_2 \cup \ldots \cup S_k, \\ \pi \cap O(S) = \pi_{S_1} \cup \pi_{S_2} \ldots \cup \pi_{S_k}, \end{aligned} \tag{4.7}$$

such that $k, 1 \le k \le |S|$ is some integer and $A_1 = (S_1, \pi_{S_1}), A_2 = (S_2, \pi_{S_2}), \ldots, A_k = (S_k, \pi_{S_k})$ are mutually compatible amoebas. Really the partition $S = S_1 \cup S_2 \cup \ldots \cup S_k$ is uniquely generated by the following construction. We say that points $s_1, s_2 \in S$ are connected if there is a contour $? \in \pi$ such that both points $s_1, s_2 \in \text{Int } ?$ are situated inside of this contour. Observe now that if two compatible contours $?_1$ and $?_2$ and two sets $S_1, S_2 \subset \mathbb{Z}^2$ are such that $S_1 \subseteq \text{Int } ?_1$, $S_2 \subseteq \text{Int } ?_2$ and the intersection $S_1 \cap S_2$ is not empty, then at least one of these two contours contains the sum $S_1 \cup S_2$ in its interior. So if we choose the sets $S_i, i = 1, 2, \ldots, k$, as maximal components of mutually connected points of S, we find that there are contours $?_i \in \pi, i = 1, 2, \ldots, k$, such that $S_i \subseteq \text{Int } ?_i$, if $|S_i| > 1$, and no contour $? \in \pi$ can contain in its interior some points of two different sets $?_i, ?_j, i \neq j, i, j = 1, 2, \ldots, k$. So we obtain the representation (4.7) if we let

$$\pi_{S_i} = \pi \cap O(S_i), \quad i = 1, 2, \ldots, k. \tag{4.8}$$

From the other side it is easy to check that the existence the representation (4.7) implies that the sets S_i and π_{S_i} can be obtained by the construction described above. To the each pair (S,π) we can correspond a herd of animals $\rho(\pi)=\{? \in \pi\setminus O(S), A_1=(S_1,\pi_{S_1}), A_2=(S_2,\pi_{S_2}),\dots, A_k=(S_k,\pi_{S_k})\}\in H(\Lambda(T,V))$ and it will be an one-to-one correspondence between the set of all pairs (S,π), where $S\subseteq T$ and $\pi\in H(V)$, and the set of all herds $\rho\in H(\Lambda(T,V))$. The weights (4.3) were chosen in such a way that for any pair (S,π)

$$\exp\left\{-2\beta\sum_{\Gamma\in\pi}|?|\right\}\prod_{t\in S}(\exp\{z_t x_t(\pi)\}-1)=\prod_{\theta\in\rho(\pi)} w(\theta). \tag{4.9}$$

So the desired identity (4.2) follows from the equality (4.6).

Instead of the condition (2.2) we check a more strong condition (2.14) which states that for any animal $\theta\in\Theta(T)$

$$\sum_{\tilde\theta\in\Theta:\tilde\theta\nsim\theta} w_0(\tilde\theta)b(\tilde\theta)+w_0(\theta)b(\theta)\le \ln b(\theta). \tag{4.10}$$

With this aim we need some estimates of sums of terms $w_0(A)b(A)$ where $A\in\mathbb{A}(T)$. Let $\mathbb{A}_S(T)$ be the set of all amoebas with a fixed set of nuclei $S\subseteq T$. In the first place we observe using the definitions (4.3), (4.4) and (4.5) and Lemma 3.1 that for any finite set $S\subseteq T$ and any number $a\ge 0$

$$\begin{aligned}
&\sum_{A=(S,\pi_S)\in\mathbb{A}_S(T):|\pi_S|\ge a} w_0(A)b(A)\\
&\quad=(v(T))^{|S|}\sum_{A=(S,\pi_S)\in\mathbb{A}_S(T):|\pi_S|\ge a}\exp\{-2(\beta_0-1)|\pi_S|+|S|\}\\
&\quad\le (v(T))^{|S|}e^{|S|}\exp\{-2(\beta_0-\beta_1-1)a\}\sum_{A=(S,\pi_S)\in\mathbb{A}_S(T)}\exp\{-2\beta_1|\pi_S|\}\\
&\quad\le (v(T))^{|S|}e^{|S|}\exp\{-2(\beta_0-\beta_1-1)a\}\prod_{\Gamma\in O(S)}(1+\exp\{-2\beta_1|?|\})\\
&\quad\le (v(T))^{|S|}e^{|S|}\exp\{-2(\beta_0-\beta_1-1)a\}\exp\left\{\sum_{\Gamma\in O(S)}\exp\{-2\beta_1|?|\}\right\}\\
&\quad\le\left(\frac{1}{2}\right)^{|S|}\min((q(T))^{-|S|},1)\exp\{-2(\beta_0-\beta_1-1)a\}.
\end{aligned} \tag{4.11}$$

Observe also using the estimate (4.11) and the definition (1.6) that for any fixed contour $?^0 \in \mathbb{G}$ and number $a \ge 0$

(4.12)
$$\begin{aligned}
&\sum_{A=(S,\pi_S)\in\mathbb{A}_S(T):S\subseteq \mathrm{Int}\ \Gamma^0, |\pi_S|\ge a} w_0(A)b(A) \\
&\le \exp\{-2(\beta_0-\beta_1-1)a\} \sum_{S:S\subseteq \mathrm{Int}\ \Gamma^0\cap T} (2q(T))^{-|S|} \\
&\le \exp\{-2(\beta_0-\beta_1-1)a\}(1+(2q(T))^{-1})^{|\mathrm{Int}\ \Gamma^0\cap T|} \\
&\le \exp\{-2(\beta_0-\beta_1-1)a+(2q(T))^{-1}|\mathrm{Int}\ ?^0\cap T|\} \\
&\le \exp\left\{-2(\beta_0-\beta_1-1)a+\frac{1}{2}\mathrm{diam}\ (\mathrm{Int}\ ?^0\cap T)\right\} \\
&\le \exp\left\{-2(\beta_0-\beta_1-1)a+\frac{1}{4}|?^0|\right\}.
\end{aligned}$$

Similarly fixing a point $t \in \mathrm{Int}\ ?^0 \cap T$ and observing that $\mathrm{diam}\ (\mathrm{Int}\ ?^0 \setminus \{t\}) \le \frac{1}{2}|?^0|$ we find that

(4.13)
$$\begin{aligned}
&\sum_{A=(S,\pi_S)\in\mathbb{A}_S(T):S\subseteq \mathrm{Int}\ \Gamma^0, t\in S, |\pi_S|\ge a} w_0(A)b(A) \\
&\le \frac{1}{2}\min((q(T))^{-1},1)\exp\{-2(\beta_0-\beta_1-1)a\} \sum_{S'\subseteq(\mathrm{Int}\ \Gamma^0\cap T)\setminus\{t\}} (2q(T))^{-|S'|} \\
&\le \frac{1}{2}\min((q(T))^{-1},1)\exp\{-2(\beta_0-\beta_1-1)a\}(1+(2q(T))^{-1})^{|\mathrm{Int}\ \Gamma^0\cap T|\setminus\{t\}|} \\
&\le \frac{1}{2}\min((q(T))^{-1},1)\exp\left\{-2(\beta_0-\beta_1-1)a+\frac{1}{4}|?^0|\right\}.
\end{aligned}$$

Fix a point $t \in \mathbb{Z}^2$ and a contour $?^0 \in O(t)$ and assume that $\mathbb{A}_{t,\Gamma^0}(T)$ is the set of all amoebas $A = (S, \pi_S) \in \mathbb{A}(T)$ such that $t \in S$ and the membrane $M(A) = ?^0$. Since $|\pi(S)| \ge |?^0|$ for $A = (S, \pi_S) \in \mathbb{A}_{t,\Gamma^0}(T)$, it follows from the estimate (4.13) that for $\beta_2 = 2\beta_1 + \frac{9}{8}$

(4.14)
$$\begin{aligned}
&\sum_{A=(S,\pi_S)\in\mathbb{A}_{t,\Gamma^0}(T):|\pi_S|\ge a} w_0(A)b(A) \\
&\le \frac{1}{2}\min((q(T))^{-1},1)\exp\left\{-2\left(\beta_0-\beta_1-1\right)\max(a,|?^0|)+\frac{1}{4}|?^0|\right\} \\
&= \frac{1}{2}\min((q(T))^{-1},1)\exp\{-2(\beta_0-\beta_2)\max(a,|?^0|)\} \\
&\times\ \exp\left\{-2(\beta_1+\frac{1}{8})\max(a,|?^0|)+\frac{1}{4}|?^0|\right\} \\
&\le \frac{1}{2}\min((q(T))^{-1},1)\exp\{-2(\beta_0-\beta_2)a\}\exp\{-2\beta_1|?^0|\}.
\end{aligned}$$

Using the estimate (4.11) for $S = \{t\}$, the estimate (4.14) and Lemma 3.1 and recalling that $\beta_2 \geq \beta_1 + 1$ we find that for any fixed point $t \in \mathbb{Z}^2$

(4.15)

$$\begin{aligned}\sum_{A=(S,\pi_S)\in\mathbb{A}(T):t\in S,|\pi_S|\geq a} w_0(A)b(A) &\leq \sum_{A=(\{t\},\pi_{\{t\}})\in\mathbb{A}(T):|\pi_{\{t\}}|\geq a} w_0(A)b(A)\\ &+ \sum_{\Gamma^0\in O(t)}\ \sum_{A=(S,\pi_S)\in\mathbb{A}(T):|S|>1,t\in S,|\pi_S|\geq a,M(A)=\Gamma^0} w_0(A)b(A)\\ &\leq \frac{1}{2}\min((q(T))^{-1},1)\\ &\times\left(\exp\{-2(\beta_0-\beta_1-1)a\}+\exp\{-2(\beta_0-\beta_2)a\}\sum_{\Gamma^0\in O(t)}\exp\{-\beta_1|?^0|\}\right)\\ &\leq \frac{3}{4}\min((q(T))^{-1},1)\exp\{-2(\beta_0-\beta_2)a\}.\end{aligned}$$

Now we obtain from (4.15) and (1.6) that for any contour $?_0 \in \mathbb{G}$ and $\beta_3 = \beta_2 + \frac{1}{2}$

(4.16)

$$\begin{aligned}&\sum_{A=(S,\pi_S)\in\mathbb{A}(T):\Gamma_0\in\pi_S} w_0(A)b(A)\\ &\quad\leq \sum_{t\in\mathrm{Int}\ \Gamma^0\cap T}\ \sum_{A=(S,\pi_S)\in\mathbb{A}(T):t\in S,|\pi_S|\geq|\Gamma^0|} w_0(A)b(A)\\ &\quad\leq \exp\{-2(\beta_0-\beta_2)|?^0|\}(q(T))^{-1}|\mathrm{Int}\ ?^0\cap T|\\ &\quad\leq \exp\{-2(\beta_0-\beta_2)|?^0|\}\mathrm{diam}\ (\mathrm{Int}\ ?^0\cap T) \leq \exp\{-2(\beta_0-\beta_2)|?^0|\}|?^0|\\ &\quad\leq \exp\{-2(\beta_0-\beta_3)|?^0|\}.\end{aligned}$$

In a similar way applying (4.15) for $a = 0$ we find that for any contour $?_0 \in \mathbb{G}$

(4.17)

$$\begin{aligned}\sum_{A=(S,\pi_S)\in\mathbb{A}(T),\Gamma_0\in O(S)} w_0(A)b(A) &\leq \sum_{t\in\mathrm{Int}\ \Gamma^0\cap T}\ \sum_{A=(S,\pi_S)\in\mathbb{A}(T):t\in S} w_0(A)b(A)\\ &\leq (q(T))^{-1}|\mathrm{Int}\ ?^0\cap T| \leq \mathrm{diam}\ (\mathrm{Int}\ ?^0\cap T) \leq \frac{1}{2}|?^0|.\end{aligned}$$

Now we can begin to prove the desired estimate (4.10). In the first place we assume that $\theta = ? \in \mathbb{G}$ Let $R(G)$ be the set consisting of all contours $\tilde{?} \in \mathbb{G}$ which are incompatible with the contour ? and the contour ? itself. It follows from the definition of incompatibility that for $\theta = ?$

$$\begin{aligned}&\sum_{\tilde\theta\in\Theta:\tilde\theta\nsim\theta} w_0(\tilde\theta)b(\tilde\theta)+w_0(\theta)b(\theta)\\ (4.18)\qquad &\leq \sum_{\tilde\Gamma\in R(\Gamma)}\left(w_0(\tilde{?})b(\tilde{?})+\sum_{A=(S,\pi_S)\in\mathbb{A}(T):\tilde\Gamma\in\pi_S} w_0(A)b(A)\right)\\ &+\sum_{A=(S,\pi_S)\in\mathbb{A}(T):\Gamma\in O(S)} w_0(A)b(A).\end{aligned}$$

It follows from the estimate (4.16) applied for $?^0 = \tilde{?}$ that

$$\sum_{A=(S,\pi_S)\in\mathbb{A}(T):\tilde{\Gamma}\in\pi_S} w_0(A)b(A) \le \exp\left\{-2(\beta_0-\beta_3)|\tilde{?}|\right\}. \tag{4.19}$$

So recalling also the definitions (4.3) and (4.5) we find that

$$\begin{aligned}\sum_{\tilde{\Gamma}\in R(\Gamma)} &\left(w_0(\tilde{?})b(\tilde{?}) + \sum_{A=(S,\pi_S)\in\mathbb{A}(T):\tilde{\Gamma}\in\pi_S} w_0(A)b(A)\right)\\ &\le \sum_{\tilde{\Gamma}\in R(\Gamma)} (\exp\{-2(\beta_0-1)|\tilde{?}|\} + \exp\{-2(\beta_0-\beta_3)|\tilde{?}|\}).\end{aligned} \tag{4.20}$$

Since each of contours $\tilde{?} \in R(?)$ contains a vertex of the conjugate lattice belonging also to $?$ and the number of different contours $\tilde{?}$ of a length d containing a fixed vertex do not exceed 3^d, it follows from the estimate (4.20) that for some large enough $\beta_0 > 0$

$$\sum_{\tilde{\Gamma}\in R(\Gamma)} \left(w_0(\tilde{?})b(\tilde{?}) + \sum_{A=(S,\pi_S)\in\mathbb{A}(T):\tilde{\Gamma}\in\pi_S} w_0(A)b(A)\right) \le |?|. \tag{4.21}$$

The estimate (4.17) shows that

$$\sum_{A=(S,\pi_S)\in\mathbb{A}(T):\Gamma\in O(S)} w_0(A)b(A) \le \frac{1}{2}|?|. \tag{4.22}$$

The estimates (4.18), (4.21) and (4.22) implies that for $\theta = ?$

$$\sum_{\tilde{\theta}\in\Theta:\tilde{\theta}\nsim\theta} w_0(\tilde{\theta})b(\tilde{\theta}) + w_0(\theta)b(\theta) \le 2|?| \tag{4.23}$$

and it is the desired estimate (4.10) for the considered case.

At last we assume that $\theta = A = (S,\pi_S) \in \mathbb{A}(T)$. Using the definition of incompatibility we find that

$$\begin{aligned}&\sum_{\tilde{\theta}\in\Theta:\tilde{\theta}\nsim\theta} w_0(\tilde{\theta})b(\tilde{\theta}) + w_0(\theta)b(\theta)\\ &\quad\le \sum_{\Gamma\in\pi_S}\sum_{\tilde{\Gamma}\in R(\Gamma)} \left(w_0(\tilde{?})b(\tilde{?}) + \sum_{\tilde{A}=(\tilde{S},\tilde{\pi}_{\tilde{S}})\in\mathbb{A}(T):\tilde{\Gamma}\in\tilde{\pi}_{\tilde{S}}} w_0(\tilde{A})b(\tilde{A})\right)\\ &\quad+ \sum_{\tilde{A}=(\tilde{S},\tilde{\pi}_{\tilde{S}})\in\mathbb{A}(T):\tilde{S}\cap S\neq\emptyset} w_0(\tilde{A})b(\tilde{A}) + \sum_{\Gamma\in\pi_S}\sum_{\tilde{A}=(\tilde{S},\tilde{\pi}_{\tilde{S}})\in\mathbb{A}(T):\Gamma\in O(\tilde{S})} w_0(\tilde{A})b(\tilde{A})\\ &\quad+ \sum_{\tilde{\Gamma}\in O(S)}\sum_{\tilde{A}=(\tilde{S},\tilde{\pi}_{\tilde{S}})\in\mathbb{A}(T):\tilde{\Gamma}\in\tilde{\pi}_{\tilde{S}}} w_0(\tilde{A})b(\tilde{A}).\end{aligned} \tag{4.24}$$

We shall estimate the separate terms in the right part of the estimate (4.24). In the first place we observe using the estimate (4.21) that

$$\sum_{\Gamma\in\pi_S}\sum_{\bar{\Gamma}\in R(\Gamma)}\left(w_0(\tilde{?})b(\tilde{?})+\sum_{\tilde{A}=(\tilde{S},\tilde{\pi}_{\tilde{S}})\in\mathbb{A}(T):\bar{\Gamma}\in\tilde{\pi}_{\tilde{S}}}w_0(\tilde{A})b(\tilde{A})\right)\leq|\pi_S|. \tag{4.25}$$

Further using the estimate (4.15) for $a=0$ we find that

$$\sum_{\tilde{A}=(\tilde{S},\tilde{\pi}_{\tilde{S}})\in\mathbb{A}(T):\tilde{S}\cap S\neq\emptyset}w_0(\tilde{A})b(\tilde{A})\leq\sum_{t\in S}\sum_{\tilde{A}=(\tilde{S},\tilde{\pi}_{\tilde{S}})\in\mathbb{A}(T):t\in\tilde{S}}w_0(\tilde{A})b(\tilde{A})\leq\frac{3}{4}|S|. \tag{4.26}$$

It follows from the estimate (4.17) that

$$\sum_{\Gamma\in\pi_S}\sum_{\tilde{A}=(\tilde{S},\tilde{\pi}_{\tilde{S}})\in\mathbb{A}(T):\Gamma\in O(\tilde{S})}w_0(\tilde{A})b(\tilde{A})\leq\frac{1}{2}|\pi_S|. \tag{4.27}$$

At last using the estimate (4.16) and then Lemma 3.1 and the estimate $\min|\bar{?}|\geq 4$ we obtain that for some large enough β_0

(4.28)

$$\begin{aligned}\sum_{\bar{\Gamma}\in O(S)}\sum_{\tilde{A}=(\tilde{S},\tilde{\pi}_{\tilde{S}})\in\mathbb{A}(T):\bar{\Gamma}\in\tilde{\pi}_{\tilde{S}}}w_0(\tilde{A})b(\tilde{A})&\leq\sum_{\bar{\Gamma}\in O(S)}\exp\{-2(\beta_0-\beta_3)|\bar{?}|\}\\&\leq\exp\{-4(\beta_0-\beta_3-\beta_1)\}\sum_{\bar{\Gamma}\in O(S)}\exp\{-\beta_1|\bar{?}|\}\leq\frac{1}{4}|S|.\end{aligned}$$

Gathering the estimates (4.24) – (4.28) and recalling the definition (4.5) we find that for any $\theta=A=(S,\pi_S)\in\mathbb{A}(T)$

$$\sum_{\tilde{\theta}\in\Theta:\tilde{\theta}\nsim\theta}w_0(\tilde{\theta})b(\tilde{\theta})+w_0(\theta)b(\theta)\leq 2|\pi_S|+|S|=\ln b(A). \tag{4.29}$$

We have checked the condition (4.10) in all the cases. □

§5. Estimate of semiinvariants

Now using the animal representation constructed in §4 we can prove the main Theorem 1.1. For it we need the following lemma.

Lemma 5.1. *Let $G(\mathbb{G})$ be the set of all gangs of animals consisting of contours $? \in\mathbb{G}$ only. There exists a constant $\hat{\beta}>0$ such that for any edge $e\in\mathbb{E}$ the sum*

$$\sum_{\rho\in G(\mathbb{G}):e\in\cup_{\Gamma\in\bar{\rho}}\Gamma}\exp\left\{-2\hat{\beta}\sum_{\Gamma\in\bar{\rho}}\alpha(?)|?|\right\}\leq 1. \tag{5.1}$$

Proof. Fix an edge $e \in \mathbb{E}$. By $R(e)$ we denote the set of all gangs $\rho \in G(\mathbb{G})$ such that for all $? \in \bar{\rho}$ the edge $e \in ?$. Observe that for any $\beta > 0$

$$\begin{aligned} \sum_{\rho \in R(e)} \exp\left\{-2\beta \sum_{\Gamma \in \bar{\rho}} \alpha(?)|?|\right\} &= \prod_{\Gamma \in \mathbb{G} e \in \Gamma} \left(1 + \sum_{a=1}^{\infty} \exp\{-2\beta a|?|\}\right) - 1 \\ &= \prod_{\Gamma \in \mathbb{G} e \in \Gamma} \left(1 + \exp\{-2\beta|?|\}(1 - \exp\{-2\beta|?|\})^{-1}\right) - 1 \\ &\leq \exp\left\{\sum_{\Gamma \in \mathbb{G} e \in \Gamma} \exp\{-2\beta|?|\}\left(1 - \exp\{-2\beta|?|\}\right)^{-1}\right\} - 1. \end{aligned} \tag{5.2}$$

(Really, calculating the product in second part of this relation we obtain sum of terms corresponding to any subset of the set of all contours containing the edge e. The term -1 arises since we need to exclude the empty subset.) Since the number of different contours $?$ such that $|?| = d$ and $e \in ?$ do not exceed 3^{d-1} the series in the exponent in the right part of (5.2) converges for $\beta > \ln 3$ and its sum goes to 0, as $\beta \to \infty$. So there exists a value $\hat{\beta}$ so large that

$$\sum_{\rho \in R(e)} \exp\left\{-(2\hat{\beta} - \ln 2) \sum_{\Gamma \in \bar{\rho}} \alpha(?)|?|\right\} \leq 1. \tag{5.3}$$

We prove below that for this value of $\bar{\beta}$ the statement of the Lemma holds.

Consider a finite set $C \subset \mathbb{E}$ and an edge $e \in C$. The set of all gangs $\rho \in G(\mathbb{G})$ such that $e \in \cup_{\Gamma \in \bar{\rho}} ? \subseteq C$ will be denoted by $R(e, C)$. Instead of (5.1) it is enough to prove that for any finite $C \subset \mathbb{E}$ and any edge $e \in C$

$$\sum_{\rho \in R(e,C)} \exp\left\{-2\hat{\beta} \sum_{\Gamma \in \bar{\rho}} \alpha(?)|?|\right\} \leq 1. \tag{5.4}$$

We prove this estimate by induction in the number of elements $|C|$. In the case $|C| = 1$ the set of gangs $R(e, C)$ is empty and so the estimate (5.4) is evident. We use it as the initial step of induction. So we suppose that for any edge $\tilde{e} \in C, \tilde{e} \neq e$

$$\sum_{\rho \in R(\tilde{e}, C \setminus \{e\})} \exp\left\{-2\hat{\beta} \sum_{\Gamma \in \bar{\rho}} \alpha(?)|?|\right\} \leq 1. \tag{5.5}$$

Any gang of contours $\rho \in R(e, C)$ can be represented (may be in a nonunique way) as a sum (see the definition of sums of groups of animals in §2)

$$\rho = \rho_0 + \sum_{(e_1, e_2, \dots, e_k)} \rho_{e_i}, \tag{5.6}$$

where a gang $\rho_0 \in R(e)$, the gangs $\rho_{e_i} \in R(e_i, C \setminus \{e\})$ and the summation is taken over all subsets of edges $(e_1, e_2, \dots, e_k) \subseteq \cup_{\Gamma \in \bar{\rho}_0} ? \setminus \{e\}$ (including the empty subset with $k = 0$). Let $R(e, C, \rho_0)$ be the set of gangs $\rho \in R(e, C)$ which can be represented in the form (5.6) with a fixed gang ρ_0. Then using the induction conjecture (5.5) we find that

(5.7)
$$\begin{aligned}
&\sum_{\rho \in R(e,C,\rho_0)} \exp\left\{-2\hat\beta \sum_{\Gamma\in\bar\rho} \alpha(?)|?|\right\} \\
&\le \exp\left\{-2\hat\beta \sum_{\Gamma\in\bar\rho_0} \alpha(?)|?|\right\} \\
&\quad\times\left(\sum_{(e_1,e_2,\dots,e_k)} \prod_{i=1}^{k}\left(\sum_{\rho\in R(e_i,C\setminus\{e\})} \exp\left\{-2\hat\beta\sum_{\Gamma\in\bar\rho}\alpha(?)|?|\right\}\right)\right) \\
&\le \exp\left\{-2\hat\beta \sum_{\Gamma\in\bar\rho_0} \alpha(?)|?|\right\} \prod_{\tilde e\in\cup_{\Gamma\in\bar\rho_0}\Gamma}\left(1+\sum_{\rho\in R(\tilde e,C\setminus\{e\})}\exp\left\{-2\hat\beta\sum_{\Gamma\in\bar\rho}\alpha(?)|?|\right\}\right) \\
&\le \exp\left\{-2\hat\beta \sum_{\Gamma\in\bar\rho_0} \alpha(?)|?|\right\} 2^{\sum_{\Gamma\in\bar\rho_0}|\Gamma|} \le \exp\left\{-(2\hat\beta-\ln 2)\sum_{\Gamma\in\bar\rho_0}\alpha(?)|?|\right\}.
\end{aligned}$$

Returning to the representation (5.6) we see that

$$(5.8)\quad \sum_{\rho\in R(e,C)}\exp\left\{-2\hat\beta\sum_{\Gamma\in\bar\rho}\alpha(?)|?|\right\} \le \sum_{\rho_0\in R(e)}\exp\left\{-(2\hat\beta-\ln 2)\sum_{\Gamma\in\bar\rho_0}\alpha(?)|?|\right\}$$

and the desired induction estimate (5.4) follows from the estimate (5.3) proved above. □

Proof of Theorem 1.1. Proposition 4.1 gives a possibility to apply the cluster expansion (2.18) to the logarithm of partition functions $Z_T(V; z_t, t \in T) = Z_w(\Lambda(T, V))$ (see (4.2)). So if (see (4.1) and (4.3))

$$(5.9)\qquad |w(\theta)| < w_0(\theta), \theta \in \Theta(T),$$

we have the expansion

$$(5.10)\qquad \ln Z_T(V; z_t, t \in T) = \sum_{\rho\in G(\Lambda(T,V))} q_w(\rho).$$

The condition (5.9) is fulfilled if $\beta > \beta_0$ and the values of variables $(z_t, t \in T)$ belong to the polydisk

$$(5.11)\qquad P_T = \{(z_t, t\in T) : |z_t| < \ln(1 + v(T))\} \subset \mathbb{C}^T.$$

So $\ln Z_T(V; z_t, t \in T)$ and the terms $q_w(\rho)$ of the series (5.10) are functions of $z_t, t \in T$, holomorphic in this polydisk and continuous in its closure and we can differentiate it in z_t term wise. Let $G_T(V)$ be the set of all gangs $\rho \in G(\Lambda(T,V))$ consisting of animals $\theta \in \Lambda(T,V)$ such that the sum of nuclei of amoebas included in ρ is equal T. The derivates of $q_w(\rho)$ with respect of all the variables $z_t, t \in T$, vanish if $\rho \notin G_T(V)$. So we obtain from the relations (3.11) and (5.10) that

$$s_V(T,r) = \sum_{\rho \in G_T(V)} \frac{\partial^{|r|} q_w(\rho)}{\prod_{t\in T} \partial z_t^{r_t}}\bigg|_{z_t \equiv 0, t \in T}. \tag{5.12}$$

The derivates included in (5.12) will be estimated by the Cauchy formula with integrating over the boundary of the polydisk (5.11). It follows from the estimate (2.19) and the definitions (4.1) and (4.3) that for $|z_t| = \ln(1+(v(t))), t \in T$,

$$\begin{aligned} |q_w(\rho)| \le & \left(\sum_{\theta\in\bar\rho} w_0(\theta) b(\theta)\right) \\ & \times \exp\left\{-2(\beta-\beta_0)\left(\sum_{\Gamma\in\mathbb{G}(\bar\rho)} \alpha(?)|?| + \sum_{A=(S,\pi_S)\in\mathbb{A}(\bar\rho)} \alpha(A)|\pi_S|\right)\right\}, \end{aligned} \tag{5.13}$$

where we denote the set of all contours $? \in \mathbb{G}$ such that $? \in \bar\rho$ by $\mathbb{G}(\bar\rho)$ and the set of all amoebas $A \in \mathbb{A}$ such that $A \in \bar\rho$ by $\mathbb{A}(\bar\rho)$. It follows from (5.12) and (5.13), the Cauchy formula and the estimate $\ln(1+x) \le x$ for $x \ge 0$ that
(5.14)

$$\begin{aligned} |s_V(T,r)| \le & (\ln(1+(v(t)))^{-|r|} \sum_{\rho\in G_T(V)} \left(\sum_{\theta\in\bar\rho} w_0(\theta) b(\theta)\right) \\ & \times \exp\left\{-2(\beta-\beta_0)\left(\sum_{\Gamma\in\mathbb{G}(\bar\rho)} \alpha(?)|?| + \sum_{A=(S,\pi_S)\in\mathbb{A}(\bar\rho)} \alpha(A)|\pi_S|\right)\right\}. \end{aligned}$$

The definitions (4.3), (4.4), (4.5) implies (if we assume $\beta_0 > 1$) that for any gang ρ

$$\sum_{\theta\in\bar\rho} w_0(\theta) b(\theta) \le |\bar\rho| \le \exp\left\{\sum_{\Gamma\in\mathbb{G}(\bar\rho)} \alpha(?)|?| + \sum_{A=(S,\pi_S)\in\mathbb{A}(\bar\rho)} \alpha(A)|\pi_S|\right\}. \tag{5.15}$$

So we can rewrite the estimate (5.14) as
(5.16)

$$\begin{aligned} & |s_V(T,r)| \le (\ln(1+(v(t)))^{-|r|} \\ & \times \sum_{\rho\in G_T(V)} \exp\left\{-2(\beta-\beta_0-1)\left(\sum_{\Gamma\in\mathbb{G}(\bar\rho)} \alpha(?)|?| + \sum_{A=(S,\pi_S)\in\mathbb{A}(\bar\rho)} \alpha(A)|\pi_S|\right)\right\}. \end{aligned}$$

This formula is the starting point for the following estimates.

Fix a gang of animals $\rho \in G_T(V)$. We need to describe the structure of its support $\bar\rho$. It follows from the definition of the set $G_T(V)$ that the set $\mathbb{A}(\bar\rho)$ is not empty and

$$\bigcup_{A=(S,\pi_S)\in\mathbb{A}(\bar\rho)} S = T. \tag{5.17}$$

Consider the set of contours

$$K(\bar\rho) = \mathbb{G}(\bar\rho) \bigcup_{A=(S,\pi_S):A\in\mathbb{A}(\bar\rho)} \pi_S. \tag{5.18}$$

Treating $K(\bar\rho)$ as a subgraph of the graph of all contours $\mathbb{G}$ with the vertices connecting incompatible contours we consider the system of all connected components of this subgraph: $K_1(\bar\rho), K_2(\bar\rho), \dots, K_N(\bar\rho)$, where an integer $N = N(K(\bar\rho))$. Observe that each components $K_i(\bar\rho)$ contains a contour $?_i \in O(T)$. Indeed in the opposite case $K_i(\bar\rho) \subseteq \mathbb{G}(\bar\rho)$ and there are no path along the incompatible animals belonging to $\bar\rho$ which connect $K_i(\bar\rho)$ with the amoebas from $\mathbb{A}(\bar\rho)$ what contradicts to the definition of a gang. Denote by $Q_i(\bar\rho)$ the set of all $t \in T$ such that the intersection $K_i(\bar\rho) \cap O(t) \neq \emptyset$. Since we assumed that $|T| > 1$ the definition of compatibility of animals and the condition (5.17) exclude the situations when $O(t) \cap K(\bar\rho) = \emptyset$ for some $t \in T$. So

$$\bigcup_{i=1}^{N} Q_i(\bar\rho) = T. \tag{5.19}$$

Observe also that if $Q_i(\bar\rho) \cap Q_j(\bar\rho) \neq \emptyset$ then either $Q_i(\bar\rho) \subseteq Q_j(\bar\rho)$ or $Q_j(\bar\rho) \subseteq Q_i(\bar\rho)$. Indeed if $t \in Q_i(\bar\rho) \cap Q_j(\bar\rho)$ and $?_i \in K_i(\bar\rho) \cap O(t)$ and $?_j \in K_j(\bar\rho) \cap O(t)$, then either $\mathrm{Int}\ ?_i \supseteq \cup_{\Gamma\in K_j(\bar\rho)} \mathrm{Int}\ ?$ or $\mathrm{Int}\ ?_j \supseteq \cup_{\Gamma\in K_i(\bar\rho)} \mathrm{Int}\ ?$. At last the situation when there are two nonempty disjoint sets $\widehat{T}_1$ and $\widehat{T}_2$ such that each of the sets $Q_i(\bar\rho)$ either embedded in $\widehat{T}_1$ or in $\widehat{T}_2$ is also impossible. Indeed in the opposite case there are no path along the incompatible animals belonging to $\bar\rho$ which connect amoebas with the nuclei belonging to $\widehat{T}_1$ with amoebas with the nuclei belonging to $\widehat{T}_2$ what again contradicts to the definition of a gang. So the only possibility remains for some i_0 the set $Q_{i_0}(\bar\rho) = T$. But then

$$\sum_{\Gamma\in K(\bar\rho)} |?| \geq \sum_{\Gamma\in K_{i_0}(\bar\rho)} |?| \geq 2 \ \mathrm{diam}\ T \tag{5.20}$$

and this estimate is the main conclusion from the previous consideration.

Fix a sufficiently large constant $\bar\beta$ which will be specified below. Observe using the estimates (5.16) and (5.20) that

(5.21)

$$|s_V(T,r)| \leq \exp\{-4(\beta - \bar\beta - \beta_0 - 1)\ \mathrm{diam}\ T\}(\ln(1 + (v(t)))^{-|r|}$$
$$\times \sum_{\rho\in G_T(V)} \exp\left\{-4\bar\beta\left(\sum_{\Gamma\in\mathbb{G}(\bar\rho)} \alpha(?)|?| + \sum_{A=(S,\pi_S)\in\mathbb{A}(\bar\rho)} \alpha(A)|\pi_S|\right)\right\}.$$

We need to estimate the sum over $\rho \in G_T(V)$ in (5.21). Let $\mathbb{A}(\rho) = (\mathbb{A}(\bar\rho), \alpha)$ be the group of animals consisting of all amoebas A included in the gang ρ with the same multiplicity $\alpha(A)$ which they have in ρ. It follows from the description of connected components of the set $K(\bar\rho)$ given above that the gang ρ can be represented (may be in a nonunique way) as a sum (see (2.17))

$$\rho = \mathbb{A}(\rho) + \sum_{e\in E(\mathbb{A}(\bar\rho))} \rho_e, \tag{5.22}$$

where $E(\mathbb{A}(\rho))$ is the set of all edges of the conjugate lattice having nonempty intersection with contours included in $\mathbb{A}(\bar{\rho})$ and ρ_e is either a gang of contours such that some of its contours contain e or the empty group of contours. Assuming that $\bar{\beta} \geq \hat{\beta}$ we can estimate the contributions of all possible gangs ρ_e with a fixed e with a help of Lemma 5.1. So we find that

$$\begin{aligned} &\sum_{\rho \in G_T(V)} \exp\left\{-4\bar{\beta}\left(\sum_{\Gamma \in \mathbb{G}(\bar{\rho})} \alpha(?)|?| + \sum_{A=(S,\pi_S)\in\mathbb{A}(\bar{\rho})} \alpha(A)|\pi_S|\right)\right\} \\ &\qquad \leq \sum_{\rho \in \mathcal{A}_T} \exp\left\{-(4\bar{\beta} - \ln 6)\left(\sum_{A=(S,\pi_S)\in\bar{\rho}} \alpha(A)|\pi_S|\right)\right\}, \end{aligned} \tag{5.23}$$

where $\mathcal{A}_T$ is the set of all groups of amoebas $\rho = (\bar{\rho}, \alpha), \bar{\rho} \in \mathbb{A}(T)$. (The term $\ln 6$ in the right part of (5.23) arises in the following way. In the representation $6 = 3(1+1)$ one of 1 originates in the estimate (5.1) , another 1 originates from a possibility that ρ_e is the empty set and the multiplier 3 originates in the following evident estimate: the number of edges e having a nonempty intersection with a contour ? do not exceed $3|?|$.) Let $\mathcal{A}_t, t \in T$, be the set of all groups of amoebas $\rho = (\bar{\rho}, \alpha) \in \mathcal{A}_T$ such that $t \in S$ for all $A = (S, \pi_S) \in \bar{\rho}$. It is clear that

$$\begin{aligned} &\sum_{\rho \in \mathcal{A}_T} \exp\left\{-(4\bar{\beta} - \ln 6)\left(\sum_{A=(S,\pi_S)\in\bar{\rho}} \alpha(A)|\pi_S|\right)\right\} \\ &\leq \prod_{t \in T}\left(1 + \sum_{\rho \in \mathcal{A}_t} \exp\left\{-(4\bar{\beta} - \ln 6)\left(\sum_{A=(S,\pi_S)\in\bar{\rho}} \alpha(A)|\pi_S|\right)\right\}\right). \end{aligned} \tag{5.24}$$

Further for any $t \in T$

$$\begin{aligned} &\sum_{\rho \in \mathcal{A}_t} \exp\left\{-(4\bar{\beta} - \ln 6)\left(\sum_{A=(S,\pi_S)\in\bar{\rho}} \alpha(A)|\pi_S|\right)\right\} \\ &\qquad \leq \prod_{A=(S,\pi_S)\in\mathbb{A}(T): t \in S, \alpha=1,2,\ldots} \left(1 + \exp\{-(4\bar{\beta} - \ln 6)\alpha|\pi_S|\}\right) \\ &\qquad \leq \exp\left\{\sum_{A=(S,\pi_S)\in\mathbb{A}(T): t \in S, \alpha=1,2,\ldots} \exp\{-(4\bar{\beta} - \ln 6)\alpha|\pi_S|\}\right\}. \end{aligned} \tag{5.25}$$

At last we need the following statement: for a sufficiently large $\bar{\beta}$ and all T and $t \in T$

$$\sum_{A=(S,\pi_S)\in\mathbb{A}(T): t \in S, \alpha=1,2,\ldots} \exp\{-(4\bar{\beta} - \ln 6)\alpha|\pi_S|\} \leq \ln 2. \tag{5.26}$$

Really first we find summing geometrical progressions with ratios $\exp\{-(4\bar\beta - \ln 6)|\pi_S|\}$ that for large enough $\bar\beta$

$$\sum_{A=(S,\pi_S)\in\mathbb{A}(T):t\in S,\alpha=1,2,\dots} \exp\{-(4\bar\beta-\ln 6)\alpha|\pi_S|\} \leq 2 \sum_{A=(S,\pi_S)\in\mathbb{A}(T):t\in S} \exp\{-(4\bar\beta-\ln 6)|\pi_S|\}. \tag{5.27}$$

After it we can almost literally repeat the derivation of the relation (4.15) for $a = 0$ and find that the right part of (5.27) goes to 0, as $\bar\beta \to \infty$. Gathering the estimates (5.21), (5.23), (5.24), (5.25) and (5.26) we obtain that

$$|s_V(T,R)| \leq 3^{|T|}\exp\{-4(\beta-\bar\beta-\beta_0-1)\ \mathrm{diam}\ T\}(\ln(1+(v(t)))^{-|r|}. \tag{5.28}$$

Observing that

$$\ln(1+x) \geq x - \frac{x^2}{2} \geq \frac{1}{2}x \quad \text{for } 0 \leq x \leq 1 \tag{5.29}$$

and that $|T| \leq |r|$ we obtain from (5.28) and the definition (4.4) the desired estimate (1.7). □

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