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Stable twisted trace formula: elliptic terms

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Abstract: This paper deals with the stabilization of the contribution of elliptic elements to the geometric side of the general twisted trace formula. We extend the results of Langlands, Kottwitz and Kottwitz-Shelstad to all elliptic elements for the general twisted trace formula.

Introduction

1 – Outline

Langlands has shown, in [Lan], how to stabilize the contribution of regular elliptic elements to the ordinary trace formula under the transfer assumption, for arbitrary connected reductive groups. This was extended by Kottwitz to all elliptic elements [Ko4]. Langlands result has been extended in an other direction: in [KS], Kottwitz and Shelstad stabilize the contribution of strongly regular elliptic elements in the general twisted trace formula. In [LBC] we have treated the case of all elliptic elements for the twisted trace formula but only in the base change situation. Here we stabilize the contribution of all elliptic elements for the general twisted trace formula. As in [Lan],[Ko4] and [KS], the last step of the stabilization uses the transfer assumption.

Together with the cohomological abstract nonsense developped in the first chapter of [LBC] and the results on Tamagawa measures proved in [Lab], the present paper gives an essentially self-contained proof of the stabilization of the elliptic trace for the general twisted case, under the transfer assumption.

2 – About the method

Before describing the contents of the paper let us make some technical comments for readers familiar with the subject. The proof given in [Lan] uses lengthy cocycle computations and permanently appeals to Poitou-Tate duality. Using the results of [Ko3] some simplifications were brought in by Kottwitz in [Ko4], but in that paper and again in [KS], the stabilization is somewhat complicated, if not very difficult. The origin of the complications is easy to track: in these papers the cohomological objects necessary for the stabilization are often introduced via the dual picture using Poitou-Tate-Nakayama-Langlands duality.

As was shown in [LBC], some new, but elementary, nonabelian cohomological construction leads to quite simple and direct "geometric" definitions for the objects that show up in the stabilization process of the geometric side of the trace formula (except that here we do not discuss transfer factors), and to very simple proofs of their properties making little, if any, use of duality even in the definition of endoscopic groups. Although it was clear that the framework developped in [LBC] for the geometric stabilization should work in general, we could not deal with the most general case for a technical reason: the notion of norm used in [LBC] made sense only for base change. What we provide here is a definition valid in the most general case together with an existence proof. Using this, we reprove and generalize most of the results of [Lan], [Ko4] and [KS] (except chapters 4 and 5 of [KS] that deal with transfer factors).

It turns out that, besides a few class field theory arguments and the transfer assumption – which is used in all quoted papers and only occurs in the last step – the key results needed for the stabilization, are the theorems of Kneser ([Kn1], [Kn2]) and Kneser-Harder-Chernousov ([Kn3], [Har1], [Har2] and [Ch]) on Hasse principle for simply-connected semisimple groups used here via references to [Lab].

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3 – The contents

The twisted trace formula arises when considering a connected reductive group G over a global field F (we shall consider here only number fields) and an automorphism θ of G. The way G and θ occur is better understood by introducing the category of twisted space. Their definition and basic properties occupy section I. A twisted space is a pair (G, L) where G is a group and L is a G-principal homogenous space with a G-equivariant map from L to the group of automorphisms of G. There is a left and a right action of G on L. Any twisted G-space L is isomorphic to some $G\rtimes\theta$, but such an isomorphism is not canonical. Then we introduce the group $\mathrm{Int}\,(L)$ of inner automorphisms i.e. those induced by elements of G acting by conjugacy and by elements of Z_G, the center of G, acting by left translations.

Section II deals with centralizers and endoscopic groups. Consider $\delta \in L$, let G^δ be its centralizer and G_δ the connected component of G^δ. For the study of endoscopy it is necessary to use a group I_δ in between:

$$G_\delta \subset I_\delta \subset G^\delta$$

we call the stable centralizer of δ. Section II continues with group theoretic results about roots, automorphisms and centralizers, mainly due to Steinberg, using the notion of norm $N\alpha$ and the sign ϵ_α attached to the θ-orbit of a root α. We give in II.3 a classification of stable centralizers of semisimple elements $\delta \in L$, when G is a simple group, in order to compute the group of outer automorphisms of I_δ induced by conjugation under an auxilliary subgoup $J_\delta \subset G$. In II.4, we introduce endoscopic groups, but without rational structure at this stage. Consider a torus T in a θ-stable Borel pair (B, T) and let T_{sc} be the corresponding maximal torus in the simply connected cover G_{SC} of the derived group of G. Denote by T_θ the group of θ-coinvariants in T. Let $\overline{\kappa}$ be a character of the $\mathbb{Z}$-module $\check{X}^\theta_{sc}$ of θ-invariant cocharacters of T_{sc}. We define the endoscopic root system $R_{\overline{\kappa}}$ for T_θ as dual to the coroot system $\check{R}_{\overline{\kappa}}$ defined as follows: elements in $\check{R}_{\overline{\kappa}}$ are the cocharacters of T_θ induced by coroots $\check{\alpha}$ for T such that $\overline{\kappa}(N\check{\alpha}) = \epsilon_\alpha$. The endoscopic group attached to $(T, \overline{\kappa})$ is the reductive group, well defined up to inner automorphisms, with a maximal torus T_H isomorphic to T_{θ^*}, with root system and coroot system isomorphic to $R_{\overline{\kappa}}$ and $\check{R}_{\overline{\kappa}}$.

Section III deals with rational structures and norms. One defines inner forms of twisted spaces using classes in $\mathbf{H}^1(F, \mathrm{Int}\,(L))$. Any twisted G-space L is an inner form of a quasi-split twisted space $L^* = G^* \rtimes \theta^*$ where G^* is the quasi-split inner form of G and θ^* is an automorphism that preserves a splitting in G^*. Let φ denote the isomorphism over the algebraic closure $\varphi : L \to L^*$. The main result of this section is the existence of special prenorms (called abstract norms in [KS]). Their definition is pretty technical. We say that $\delta^* \in L(\overline{F})$ is a special prenorm of an almost semisimple element $\delta \in L(F)$ if δ^* is a $G^*(\overline{F})$-conjugate of $\varphi(\delta)$ such that there is a torus T defined over F in G^* that belongs to a θ^*-stable Borel pair (B, T) with

$$\delta^* \in T\rtimes\theta^* \subset L^*$$

and such that the image of δ^* in T_{θ^*} is rational, up to some central twist that depends on L; moreover we ask that the stable centralizer I_{δ^*} of δ^* is defined over F and is an inner form of I_δ. The existence of special prenorms for strongly regular elements elements is an easy consequence of a theorem of Steinberg, but for the singular ones the proof is more involved and uses the case by case study of centralizers in II.2 to show one can fulfill the inner form condition.

Section IV deals with endoscopy. We first recall the non-abelian hypercohomological objects introduced in [LBC] that describes stable conjugacy. Then we discuss the Langlands criterion IV.2.2 that allows to test when an element which is locally everywhere a prenorm is a global prenorm. Then we discuss endoscopic spaces. Let δ be an elliptic element in $L(F)$ with special prenorm δ^* and let κ be an endoscopic character i.e. a character of

$$\mathbf{H}^0_{ab}(\mathbb{A}_F/F, I_\delta\backslash G) \simeq \mathbf{H}^0_{ab}(\mathbb{A}_F/F, I_{\delta^*}\backslash G^*)\ .$$

We introduce the endoscopic group H^* over F attached to a pair (T, κ). The space H is isomorphic to $H^* \rtimes 1$ over $\overline{F}$ but the rational structure differs by the central twist alluded to above. The conorm map $T \to T_{\theta^*}$

induces a bijection from T-conjugacy classes in $T\rtimes\theta^*$ to $T_{\theta^*}\subset H^*$. Given $\delta\in L(F)$ with prenorm δ^*, the conorm map yields a $\gamma\in H(F)$ we call the norm of δ. The section ends with the comparison of $\tilde\imath_{G^*}(\delta^*)$, the number of points in $\mathbf{H}^0(F,G^{\delta^*}/I_{\delta^*})$, and of $\tilde\imath_{H^*}(\gamma)$.

The trace formula appears in section V. The twisted trace formula is the renormalized trace of an operator $\rho(f,\omega)$defined by a smooth compactly supported function f on $L(\mathbb{A}_F)$ and a character ω of $G(\mathbb{A}_F)$ trivial on $G(F)$. The operator has a kernel

$$K(x,y)=\sum_{\delta\in L(F)}\omega(x)\,f^1(x^{-1}\delta\,y)$$

and the renormalized trace is obtained from a truncated integration over the diagonal. The contribution of elliptic elements need not be regularized: it is convergent and will be called the elliptic trace. It will be denoted $\mathbf{T}_e(f,\omega)$. Elementary manipulations show it can be expressed as a linear combination, indexed by a set of representatives of conjugacy classes, of orbital integrals $\mathrm{O}_{\delta,\omega}(f^0)$:

$$\mathbf{T}_e(f,\omega)=J(L)\sum_{\delta\in\{L_e\}}a^L(\delta)\ \mathrm{O}_{\delta,\omega}(f^0)\ .$$

There are three main steps in its stabilization. A first step V.1.3, called pre-stabilization, expresses the elliptic trace as a linear combination, indexed by representatives of stable elliptic conjugacy classes, of sums of κ-orbital integrals $\mathrm{O}^\kappa_\delta(f)$ where κ runs over the group $\mathfrak{K}(I_\delta,G;F)$ of endoscopic characters for δ that induce the character ω:

$$\mathbf{T}_e(f,\omega)=\tau(L)\sum_{\delta\in\{\{L_e\}\}}\frac{1}{\tilde\imath_G(\delta)}\sum_{\{\kappa\in\mathfrak{K}(I_\delta,G;F)\ |\ [\kappa]=\omega\}}\mathrm{O}^\kappa_\delta(f^0)\ .$$

The pre-stabilization is immediate using the results of [Lab]. In a second step V.2.2, using the existence of special prenorms III.4.7 and the Langlands criterion IV.2.2, the elliptic trace is expanded as a linear combination indexed by equivalence classes of triples (T,δ^*,κ) of objects for L^*, of a variant of the κ-orbital integrals:

$$\mathbf{T}_e(f,\omega)=\tau(L)\sum_{\{(T,\delta^*,\kappa)\ |\ [\kappa]=\omega\}}\frac{N(T,\delta^*,\kappa)}{\tilde\imath_{G^*}(\delta^*)}\ \mathrm{O}^\kappa_{\delta^*}(f^0)\ .$$

In the third and last step we write the elliptic trace as a sum of stable elliptic traces for endoscopic spaces. There we use the transfer assumption V.3.1 to express κ-orbital integrals as stable orbital integrals for the endocopic space H attached to $\mathcal{E}=(T,\kappa)$. The assumption is that given $f\in\mathcal{C}^\infty_c(L(\mathbb{A}_F))$ there exists $f_{\mathcal{E}}\in\mathcal{C}^\infty_c(H(\mathbb{A}_F))$ such that, for $\gamma\in H(F)$ conorm of δ^* we have:

$$\mathrm{O}^1_\gamma(f_{\mathcal{E}})=\mathrm{O}^\kappa_{\delta^*}(f)\ .$$

We shall take for granted the transfer assumption, without discussing transfer factors and the – still conjectural – Fundamental Lemma. Our main result, Theorem V.3.2, gives the expansion of the elliptic trace as a linear combination of stable elliptic traces (or rather a slight variant of it) for endoscopic spaces:

$$\mathbf{T}_e(f,\omega)=\sum_{\mathcal{E}\in\mathcal{E}(\omega)}a(\mathcal{E})\ \mathbf{ST}^{\mathcal{E}}_e(f_{\mathcal{E}})$$

where

$$\mathbf{ST}^{\mathcal{E}}_e(f_{\mathcal{E}})=\tau(H)\sum_{\gamma\in\{\{H_e\}\}_L}\frac{1}{\tilde\imath_{H^*}(\gamma)}\ \mathrm{O}^1_\gamma(f^0_{\mathcal{E}})\ .$$

Thus we have extended to all elliptic elements the main result of [KS].

4 – Perspectives

In [LBC] we have been able to prove the transfer assumption in the base change situation, for the main endoscopic group, for a large class of functions. Unfortunately, we are not able to do anything similar, in the more general setting treated here, since the twisted fundamental lemma is not known in any generality besides base change. Hence, as regards applications to transfer of automorphic representations we can but refer to [LBC].

We have not tried here to reformulate the definition of the transfer factors in a more geometric way. We hope to return to this in a future paper.

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I – The setting

I.1 – Basic notation

We shall try to use a set of notation as close as possible from what is used in [KS] and in [LBC]. In particular, F is a field of characteristic zero, $\overline{F}$ an algebraic closure and we denote by

$$\Gamma := \mathrm{Gal}(\overline{F}/F)$$

the Galois group. A typical element in Γ will be denoted σ (the letter γ being used for elements in endoscopic groups). If V is an algebraic variety defined over F, we shall often, by abuse of notation, write V for $V(\overline{F})$.

Let G be a group, we denote by Z_G its center. Let $\mathrm{Aut}\,(G)$ be the group of its automorphisms; we denote by $\mathrm{Int}\,(G)$ the subgroup of inner automorphisms and by $\mathrm{Out}\,(G)$ the group of outer automorphisms: the quotient of $\mathrm{Aut}\,(G)$ by $\mathrm{Int}\,(G)$. For $\theta \in \mathrm{Aut}\,(G)$ we denote by G^θ the subgroup of θ-fixed points.

Recall that the exact sequence

$$1 \to \mathrm{Int}\,(G) \to \mathrm{Aut}\,(G) \to \mathrm{Out}\,(G) \to 1\ .$$

is split when G is a linear algebraic connected reductive group. A splitting can be defined by the choice a triple $(B, T, \{X_\alpha\})$ where B is a Borel subgroup, T a maximal torus in B and $\{X_\alpha\}$ a collection of root vectors indexed by simple roots. In fact an inner automorphism that preserves such a triple is trivial while any two such triples are conjugate. Such a triple is called a splitting for G.

If G is a reductive group, we denote by G^0 its neutral connected component, by G_{der} the derived subgroup of G^0, by G_{SC} the simply connected covering of G_{der} and by G_{ad} the adjoint quotient of G^0. If T is a torus in G^0 we shall denote by T_{sc} its preimage in G_{SC}. We denote by Z_{sc} the center of G_{SC}. By abuse of notation we shall often use the same letter to denote an element in G_{SC} and its image in G. If G is a connected reductive group there is a surjective map

$$G_{SC} \times Z_G \to G\ .$$

Assume that G is reductive and let T be a maximal torus in G^0. Let S be the centralizer of T in G. We define the Weyl group of G to be the quotient of the normalizer of S in G, modulo S. It will be denoted $\Omega(G, T)$ or $\Omega(G, S)$.

For the study of twisted endoscopy we cannot work only with connected reductive groups; we need a slightly larger category that contains also diagonalizable groups. We say that a reductive group G is quasi-connected if G is the kernel of a surjective map

$$G = \ker[G_1 \to T_0]$$

where G_1 is a connected reductive group and T_1 a torus. Quasi-connected group have been studied in [LBC] and [Lab]. Let G be a quasi-connected group, Z_G its center and G_{der} the derived group of G^0, then $G = Z_G.G_{der}$. Observe that the category of quasi-connected group contains the groups $O(2n+1)$ but not the groups $O(2n)$.

I.2 – Non abelian hyper-cohomology

The notion of crossed sets and the definition of their Galois cohomology have been introduced in [LBC]. We shall use them freely. When dealing with Galois cohomology of crossed sets we write as usual $\mathbf{H}^*(F,\mathfrak{X})$ for the inductive limits of cohomology sets over finite Galois extensions E/F:

$$\mathbf{H}^*(F,\mathfrak{X}) = \mathbf{H}^*(\mathrm{Gal}(\overline{F}/F),\mathfrak{X}(\overline{F})) = \lim_{\rightarrow}\mathbf{H}^*(\mathrm{Gal}(E/F),\mathfrak{X}(E))$$

(in degrees where it is defined). We shall not distinguish between cohomology and hyper-cohomology and, unless otherwise stated, complexes will be in positive homological degrees:

$$\cdots X_2 \to X_1 \to X_0 \ .$$

For the definition and the properties of adelic cohomology we refer the reader to [LBC].

For a diagonalizable group S we denote by $X(S)$ the finitely generated $\mathbb{Z}$-module $\mathrm{Hom}(S,\mathbb{G}_m)$ of its characters. If T is a torus we denote by $\check{X}(T)$ the finitely generated free $\mathbb{Z}$-module $\mathrm{Hom}(\mathbb{G}_m,T)$ of cocharacters of T:

$$\check{X}(T) = \mathrm{Hom}(\mathbb{G}_m,T) = \mathrm{Hom}(X(T),\mathbb{Z}) \ .$$

For a diagonalizable group $\check{X}(S)$ must be defined as an element in the derived category $\mathcal{DX}$ of complexes of finitely generated $\mathbb{Z}$-module, of finite length: $\check{X}(S) = \mathbf{RHom}(X(S),\mathbb{Z})$. Let $\mathcal{DT}$ be the derived category of complexes of diagonalizable groups, of finite length. The map $S \mapsto X(S)$ defines a contravariant equivalence between $\mathcal{DT}$ and $\mathcal{DX}$. We also have an equivalence induced by the derived tensor product

$$\check{X} \mapsto \check{X} \overset{\mathbf{L}}{\otimes} \mathbb{G}_m \ .$$

Consider a connected reductive group G. The small complex

$$[G_{SC} \to G] \, ,$$

with G in degree zero is a "stable crossed module" (cf. [LBC, appendice B]). The abelianized cohomology in the sense of Borovoi [Bo] can be defined for $-1 \le i \le 1$ as

$$\mathbf{H}^i_{ab}(F,G) := \mathbf{H}^i(F, G_{SC} \to G) \ .$$

There is a functorial map

$$\mathbf{H}^i(F,G) \to \mathbf{H}^i_{ab}(F,G)$$

for $i = 0$ or 1. Let T be a maximal torus and T_{sc} its preimage in G_{SC}; the complex $[T_{sc} \to T]$ is a sub-complex quasi-isomorphic to $[G_{SC} \to G]$ and hence the natural map

$$\mathbf{H}^i(F, T_{sc} \to T) \to \mathbf{H}^i(F, G_{SC} \to G)$$

is an isomorphism. Such considerations easily extend to quasi-connected groups but in this case T has to replaced by the centralizer T^+ of a maximal torus T of G^0. Since T^+ is a diagonalizable group, $[T_{sc} \to T^+]$ defines an object in $\mathcal{DT}$, denoted G_{ab}. It is independent of the choice of T; in fact G_{ab} can also be represented by $[Z_{sc} \to Z_G]$ which is a subcomplex quasi isomorphic to $[T_{sc} \to T^+]$. This also shows that if G' is an inner form of a quasi-connected reductive group G, they define the same object in $\mathcal{DT}$: $G'_{ab} = G_{ab}$. There is a canonical isomorphism

$$\mathbf{H}^i(F,G_{ab}) \to \mathbf{H}^i(F, G_{SC} \to G)$$

for $-1 \le i \le 1$. But $\mathbf{H}^i(F,G_{ab})$ makes sense for all i and this allows to define $\mathbf{H}^i_{ab}(F,G)$ for all i by

$$\mathbf{H}^i_{ab}(F,G) := \mathbf{H}^i(F,G_{ab}) \ .$$

I.3 – Twisted spaces

A twisted space is a pair (G,L) where G is a group and L is a G-principal homogenous space with a G-equivariant map from L to the group of automorphisms of G:

$$\mathrm{Ad}_L : L \to \mathrm{Aut}\,(G)\ .$$

We shall also say that L is a twisted G-space. The action of G on L: $G \times L \to L$ will be denoted by left translations $(x,\delta) \mapsto x\delta$. The equivariance of the map Ad_L is the following compatibility condition: for $x \in G$ and $\delta \in L$ we have

$$\mathrm{Ad}_L(x\delta) = \mathrm{Ad}_G(x) \circ \mathrm{Ad}_L(\delta)$$

where $\mathrm{Ad}_G(x)$ is the inner automorphism defined by $x \in G$. One defines right translations by

$$(y,\delta) \mapsto \delta y^{-1} := (\mathrm{Ad}_L(\delta) y^{-1})\,\delta$$

and conjugacy by

$$(x,\delta) \mapsto x\delta x^{-1} := x(\mathrm{Ad}_L(\delta) x^{-1})\,\delta\ .$$

Observe that since L is homogenous the image of the composed map

$$L \to \mathrm{Aut}\,(G) \to \mathrm{Out}\,(G)$$

is reduced to a single point in $\mathrm{Out}\,(G)$. Consider $\theta \in \mathrm{Aut}\,(G)$ in the image of Ad_L. For $z \in Z_G$ the map $z \mapsto z^\theta$ is independent of the choice of θ. We denote by Z_L the subgroup of θ-fixed points in Z_G: this is the centralizer of L in G:

$$Z_L = \{x \in G \mid \mathrm{Ad}_L(\delta)\, x = x \quad \text{for all} \quad \delta \in L\} = \{x \in G \mid x\delta\, x^{-1} = \delta \quad \text{for all} \quad \delta \in L\}\ .$$

A morphism of twisted spaces (G,L) and (H,M) is a map $\varphi : L \to M$ such that there exists a group homomorphism $\psi : G \to H$ with

$$\varphi(x\delta y) = \psi(x)\varphi(\delta)\psi(y) \qquad \text{for} \qquad \delta \in L\ ,\quad x \in G \quad \text{and} \quad y \in G\ .$$

Observe that ψ is uniquely determined by φ.

We denote by $\mathrm{Aut}\,(L)$ the group of automorphisms of (G,L). The map $\varphi \mapsto \psi$ yields a homomorphism $\mathrm{Aut}\,(L) \to \mathrm{Aut}\,(G)$. The center Z_G acting by left translations on L defines a subgroup of $\mathrm{Aut}\,(L)$, isomorphic to Z_G, called the subgroup of central automorphisms. This is the kernel of the map $\varphi \mapsto \psi$. An other subgroup is defined by conjugacy by elements of G. The kernel of the map $G \to \mathrm{Aut}\,(L)$ is Z_L. The group of automorphisms generated by the image of Z_G acting by translation and of G acting by conjugacy, will be called the group of inner automorphisms and will be denoted $\mathrm{Int}\,(L)$. The above discussion shows that

$$1 \to Z_G \to \mathrm{Int}\,(L) \to \mathrm{Int}\,(G) \to 1$$

is an exact sequence. Consider the map

$$\upsilon : Z_G \to G \times Z_G$$

defined by $\upsilon : z \mapsto z \times z^{1-\theta}$ then

$$1 \to Z_G \xrightarrow{\upsilon} G \times Z_G \to \mathrm{Int}\,(L) \to 1$$

is also an exact sequence.

Assume that G is reductive; let G_{SC} be the simply connected cover of the derived group of G and Z_{sc} the center of G_{SC}. Consider the complex

$$[Z_{sc}\xrightarrow{v}G_{SC}\times Z_G]$$

where, as above, the map from Z_{sc} to $G_{SC}\times Z_G$ is $v: z\mapsto (z\times z^{1-\theta})$ and by abuse of notation we use the same letter for elements in Z_{sc} and their projection in Z_G.

I.3.1. Lemma. – *If G is reductive, the morphism of complexes*

$$[Z_{sc}\xrightarrow{v}G_{SC}\times Z_G]\to[1\to \mathrm{Int}\,(L)]$$

is a quasi-isomorphism.

Proof: It suffices to observe that G is the quotient of $G_{SC}\times Z_G$ by the image of Z_{sc} via the map $z\mapsto(z,z^{-1})$ and hence the map

$$[Z_{sc}\xrightarrow{v}G_{SC}\times Z_G]\to[Z_G\xrightarrow{v}G\times Z_G]$$

is a quasi-isomorphism.

□

Consider $\theta\in\mathrm{Aut}\,(G)$, one has a natural structure of twisted G-space on the subset

$$G\rtimes\theta\subset G\rtimes\mathrm{Aut}\,(G)\ .$$

Now, given a twisted G-space L, consider $\delta_0\in L$ and let $\theta=\mathrm{Ad}_L(\delta_0)$. Since L is a principal homogenous space, the map $x\mapsto x\delta_0$ is a bijection from G onto L and it induces a bijection between θ-conjugacy classes in G and conjugacy classes in L. The map

$$x\delta_0\mapsto x\rtimes\theta$$

is an isomorphism

$$L\to G\rtimes\theta$$

of pointed twisted G-space. But this isomorphism is not canonical due to the existence of non trivial central automorphisms, unless $Z_G=1$.

Let T be an abelian group and θ an automorphism. We denote by T_θ the group of θ-coinvariants in T; this is the quotient of T by the subgroup, usually denoted $T^{1-\theta}$, of elements of the form $t\theta(t)^{-1}$. We have an exact sequence

$$1\to T^\theta\to T\xrightarrow{1-\theta}T\to T_\theta\to 1\ .$$

This can be reinterpreted as follows. Consider $\delta=t\rtimes\theta\in T\rtimes\theta$ and denote by $[\delta]$ the image of t in T_θ. The conorm map

$$T\rtimes\theta\to T_\theta$$

defined by $\delta\mapsto[\delta]$ induces a bijection between the set of T-conjugacy classes in $T\rtimes\theta$ and the group T_θ.

Let G_0 be a group and θ_0 an automorphism of it. We shall say that the twisted space $L=G\rtimes\theta$ is induced from $L_0=G_0\rtimes\theta_0$ if

$$G=G_0\times\cdots\times G_0$$

and

$$\theta(g_1,\cdots,g_m)=(g_2,\cdots,g_m,\theta_0(g_1))\ .$$

I.3.2. Lemma. – *Assume that L is induced from L_0. The map*

$$g=(g_1,\cdots,g_m)\mapsto Ng=g_1g_2\cdots g_m$$

induces a bijection between G-conjugacy classes in L and G_0-conjugacy classes in L_0.

Proof: It suffices to observe that if $g=(g_1,\cdots,g_m)$ and $u=(1,g_1,\cdots,g_1g_2\cdots g_{m-1})$, then

$$u\,g\,\theta(u)^{-1}=(1,\cdots,1,Ng)$$

and that if $v=(v_1,\cdots,v_m)$, then

$$N(vg\theta(v)^{-1})=v_1\,.\,Ng\,.\,\theta_0(v_1)^{-1}\ .$$

□

We shall say that a twisted G-space L is simple if G is a simple group. We shall say L is irreducible if G is a product of m copies of a simple group G_0 and any θ in the image of Ad_L permutes transitively the factors.

I.3.3. Lemma. – *An irreducible twisted space L is isomorphic to a twisted space $G\rtimes\theta$ induced from a simple twisted space $L_0=G_0\rtimes\theta_0$. Moreover, if σ is an automorphism of G that commutes with θ, then there is an integer $n(\sigma)$ and an automorphism σ_0 of G_0, that commutes with θ_0 such that*

$$\sigma(g_1,\cdots,g_m)=\theta^{n(\sigma)}(\sigma_0(g_1),\cdots,\sigma_0(g_m))\ .$$

Proof: By assumption $L\simeq(G_0\times\cdots\times G_0)\rtimes\theta$ with G_0 simple. Since θ permutes transitively the m factors there are automorphisms θ_i of G_0 such that

$$\theta(g_1,\cdots,g_m)=(\theta_1(g_2),\cdots,\theta_m(g_1))\ .$$

The diagonal automorphism of G:

$$\alpha(g_1,\cdots,g_m):=(\alpha_1(g_1),\cdots,\alpha_m(g_m))$$

with $\alpha_1=1$, $\alpha_2=\theta_1$ and $\alpha_m=\theta_1\cdots\theta_{m-1}$ is such that $\alpha\theta\alpha^{-1}$ is of the desired form with $\theta_0=\theta_1\cdots\theta_m$. Now assume $L=G\rtimes\theta$ is induced and let σ be an automorphism of G that commutes with θ. Since G_0 is simple σ must also permute the factors and hence there are automorphisms σ_i of G_0 and a permutation s of $\{1,\cdots,m\}$ such that

$$\sigma(g_1,\cdots,g_m)=(\sigma_1(g_{s(1)}),\cdots,\sigma_m(g_{s(m)}))\ .$$

Since $\theta\sigma=\sigma\theta$ the permutation s is a power $n(\sigma)$ of the cyclic permutation. Then $\sigma=\theta^{n(\sigma)}\sigma'$ where σ' fixes each factor and commute with θ. It must be diagonal of the form $\sigma'=(\sigma_0,\cdots,\sigma_0)$ where σ_0 commutes with θ_0.

□

II – Centralizers and endoscopic groups

II.1 – Centralizer and stable centralizer

Let G be a connected algebraic group and let L be a twisted G-space. For $\delta\in L$ we denote by G^δ the group of fixed points under the automorphism $\mathrm{Ad}_L(\delta)$. This is the centralizer of δ. In particular G^δ contains Z_L. If G is a connected reductive there is a map

$$\mathrm{Aut}\,(G)\rightarrow\mathrm{Aut}\,(G_{SC})$$

and hence $\delta \in L$ defines an automorphism of G_{SC}. We denote by G_{SC}^{δ} the subgroup of fixed points.

Assume from now on that G is a connected reductive group. Let B be a Borel subgroup and T a maximal torus in B; we say that the pair (B,T) is a Borel pair. Let L be a twisted G-space. We say that $\delta \in L$ is almost semisimple if $\mathrm{Ad}_L(\delta)$ preserves a Borel pair; this is equivalent to say that the automorphism $\mathrm{Ad}_L(\delta)$ is almost semisimple in the terminology of [KS]. In particular $\mathrm{Ad}_L(\delta)$ defines a semisimple element in the group of automorphism of G_{der}. The centralizer of an almost semisimple element is reductive. But the action on the Lie algebra of the center of G need not be semisimple. We shall say that δ is semisimple if $\mathrm{Ad}_L(\delta)$ is semisimple.

II.1.1. Definition. – *Let G be a connected reductive group and L a twisted G-space. We call stable centralizer of $\delta \in L$ the group denoted I_δ image in G of $G_{SC}^{\delta} \times Z_L$ (i.e. of the centralizer of δ in $G_{SC} \times Z_G$).*

Following Jim Arthur's notation G_δ will denote the connected component of G^δ. In particular

$$G_\delta \subset I_\delta \subset G^\delta$$

and I_δ is a normal subgroup of G^δ.

Remark – The above definition for the stable centralizer is different from the one given in [LBC]. For semisimple elements in the base change situation, which is the case studied in [LBC], and also in general for regular semisimple elements, the two definitions yield the same object. Nevertheless they do not coincide in general. We hope that the present definition will turn out to be the right one, at least for semisimple elements (see below).

II.1.2. Lemma. – *Let δ be an almost semisimple element. Then I_δ is the group generated by Z_L and G_δ.*

Proof: Observe that if $\delta \in L$ is almost semisimple, it induces a semisimple automorphism of G_{SC}. Hence, the centralizer G_{SC}^{δ} of δ in G_{SC} is connected [St2, Theorem 8.2] and the image in G of $G_{SC}^{\delta} \times Z_L^0$ is G_δ.

□

II.1.3. Lemma. – *Let δ be an almost semisimple element and let (B,T) be a δ-stable Borel pair. Then $S = T \cap G_\delta$ is a maximal torus in G_δ. Conversely, if S is a maximal torus in G_δ, its centralizer in G is a maximal torus T that belongs to a δ-stable Borel pair.*

Proof: This is a part of Theorem 1.1.1 of [KS], for the proof of which they simply refer to [St2]. For the convenience of the reader we sketch an argument. Since δ preserves the positive Weyl chamber defined by B there is a strongly regular semisimple element $t \in G_\delta \cap T$ whose centralizer in G is T. The centralizer $S = T \cap G_\delta$ of t in G_δ is a maximal torus in G_δ. Now since all maximal tori in a connected reductive group are conjugate we see that conversely, if S is a maximal torus in G_δ, its centralizer in G is a maximal torus T that belongs to a δ-stable Borel pair.

□

II.1.4. Lemma. – *Let δ be an almost semisimple element in L and let (B,T) be a Borel pair fixed by δ. Then I_δ is a quasi-connected reductive group that contains T^δ. Moreover there is a group isomorphism*

$$\Omega(G^\delta,T^\delta)/\Omega(I_\delta,T^\delta) \xrightarrow{\sim} G^\delta/I_\delta\ .$$

Proof: Since Z_L is diagonalizable and centralizes G_δ the group I_δ is quasi-connected. Choose an automorphism θ in the image of Ad_L that preserves a splitting $(B,T,\{X_\alpha\})$. Let T^1 be the connected component of

T^θ and let G^1 be the connected component of G^θ. Then, as seen in II.1.3, the torus T^1 is a maximal torus in G^1 and $T^1 = G^1 \cap T^\theta$. Since θ preserves a splitting we have $G^\theta = G^1 Z_L$ (see [KS, section 1.1]) but since $Z_L \subset T$ we see that $Z_L T^1 = T^\theta$. Now $\mathrm{Ad}_L(\delta)$ and θ differ by an inner automorphism that fixes the Borel pair (B,T) hence we have $\mathrm{Ad}_L(\delta) = \mathrm{Ad}_G(t) \circ \theta$ for some $t \in T$ and hence $T^\delta = T^\theta$. This implies $T^1 \subset G_\delta \subset I_\delta$ and this shows that $T^\delta \subset I_\delta$. Recall that we denote by $\Omega(G^\delta, T^\delta)$ the quotient of the normalizer of T^δ in G^δ by T^δ. Given $g \in G^\delta$ the torus $gT^\delta g^{-1}$ is also a conjugate of T^δ in I_δ. This shows that the normalizer of T^δ in G^δ maps surjectively on G^δ/I_δ and hence

$$\Omega(G^\delta, T^\delta)/\Omega(I_\delta, T^\delta) \xrightarrow{\sim} G^\delta/I_\delta \ .$$

□

Let δ be an almost semisimple element in L and let (B,T) be a Borel pair fixed by δ. We shall denote by J_δ the group generated by G_δ and T. It is normalized by δ. Moreover, II.1.3 shows that J_δ is independent of the choice of the Borel pair. Clearly I_δ is contained in the centralizer of δ in J_δ; but they may not coincide (see II.3.2).

II.2 – Centralizers and θ-orbits of roots

We shall recall some classical results mainly due to Steinberg [St2] on automorphisms of reductive groups, in particular on θ-orbits of roots (see also [KS, section 1.3]). Let G be a connected reductive group with a splitting $(B, T, \{X_\alpha\})$. Let θ be an automorphism of G that preserves the splitting and let $L = G \rtimes \theta$. Denote by G^1 (instead of G_θ) the connected component of G^θ and let $T^1 = T \cap G^1$.

II.2.1. Lemma. – *The centralizer of T^1 in G is T. There are canonical isomorphism*

$$\Omega(G^1, T^1) \xrightarrow{\sim} \Omega(G^\theta, T^\theta) \xrightarrow{\sim} \Omega(G,T)^\theta \ .$$

A Weyl group element $w \in \Omega(G,T)$ preserves T^θ if and only if $\theta(w) = w$.

Proof: The first assertion holds since θ preserves a Borel pair containing T (II.1.3). The second assertion holds since θ preserves a splitting containing T ([KS, section 1.3]). Let n be an element in the normalizer of T that induces $w \in \Omega(G,T)$. If w preserves T^θ, n normalizes T^θ and then $\theta(n)n^{-1}$ fixes T^θ pointwise. But the centralizer of T^θ is T and hence $\theta(n)n^{-1} \in T$. This is equivalent to $\theta(w) = w$.

□

Assume now that G is adjoint. Then the automorphism θ is of finite order ℓ. We denote by $R = R(G,T)$ the set of roots of T in G. A root $\alpha \in R$ defines, by restriction to T^θ, the sub-torus of θ-invariants, a restricted root α_{res}. Let us denote by $N\alpha$ the sum of the roots in the orbit of α under θ:

$$N\alpha = \sum_{i=0}^{\ell_\alpha - 1} \theta^i(\alpha)$$

the number of roots in the orbit being ℓ_α. This is a character of T that factors through T_θ the torus of θ-coinvariants:

$$N\alpha : T \to T_\theta \to \mathbb{G}_m \ .$$

The composition of maps

$$T^\theta \to T \to T_\theta$$

allows to see $N\alpha$ and α_{res} as characters of the same torus. We observe that under this identification $N\alpha$ and α_{res} are proportional:

$$N\alpha = \ell_\alpha \ \alpha_{res} \ .$$

Using the classification of root systems one checks that the map $\alpha \mapsto \alpha_{res}$ induces a bijection between the set of θ-orbits of roots and the set of restricted roots. The set of restricted roots forms a non-necessarily reduced root system R_{res} for T^θ. The set of characters $N\alpha$ defines a reduced root system NR for T_θ. Assume $L = G\rtimes\theta$ is simple and θ of order ℓ. We have the following cases:

1 - If $\theta = 1$ then $R = NR = R_{res}$.

2 even - If R is of type A_{2n} and $\ell = 2$ then NR is of type B_n and R_{res} is non reduced of type BC_n.

2 odd - If R is of type A_{2n-1} with $n \geq 2$ and $\ell = 2$ then NR is of type B_n and R_{res} is of type C_n.

3 - If R is of type D_n with $n \geq 4$ and $\ell = 2$ then NR is of type C_n and R_{res} is of type B_n.

4 - If R is of type D_4 and $\ell = 3$ then NR and R_{res} are of type G_2.

5 - If R is of type E_6 and $\ell = 2$ then then NR and R_{res} are of type F_4.

One defines a number $\epsilon_\alpha \in \{\pm 1\}$ by

$$\theta^{\ell_\alpha}(X_\alpha) = \epsilon_\alpha X_\alpha$$

where X_α is a root vector for α. The roots in R are of three types:

- α is of type 1 if $2\alpha_{res}, \frac{1}{2}\alpha_{res} \notin R_{res}$, $\epsilon_\alpha = 1$
- α is of type 2 if $2\alpha_{res} \in R_{res}$, $\epsilon_\alpha = 1$
- α is of type 3 if $\frac{1}{2}\alpha_{res} \in R_{res}$, $\epsilon_\alpha = -1$.

It follows from the classification that, if G is simple, roots of type 2 and 3 occur only when G is of type A_{2n} and $\ell = 2$.

Let $\delta = t\rtimes\theta \in T\rtimes\theta$ we denote by P_δ the set of roots α such that $t^{N\alpha} = \epsilon_\alpha$. This is equivalent to

$$\mathrm{Ad}_L(\delta)^{\ell_\alpha} X_\alpha = X_\alpha \ .$$

We observe that roots in a θ-orbit are linearly independent. This implies that the subgroup J_δ introduced at the end of II.1 is the subgroup of G generated by T and the one parameter subgroups defined by the X_α with $\alpha \in P_\delta$. The root system of J_δ is set of roots in $R(G,T)$ that belong to the $\mathbb{Z}$-linear span of P_δ.

The set P_δ and the group J_δ are θ-stable. They are also invariant under T-conjugacy: they only depend on the image of δ in T_θ the group of θ-coinvariants in T. We emphasize the following obvious remark: consider $s \in T$ then, $s^\alpha = 1$ for all $\alpha \in P_\delta$ if and only if s belongs to the center of J_δ. The subgroup J_δ is of interest to us because of the next two lemmas.

II.2.2. Lemma. – *Assume that θ preserves a splitting $(B, T, \{X_\alpha\})$ in G. For $t \in T$, let $\delta = t \rtimes \theta$. The map $\alpha \mapsto \alpha_{res}$ induces a bijection from the set of θ-orbits of roots in P_δ onto the set of roots of T^θ in I_δ. If $s \in J_\delta$ centralizes I_δ then s belongs to the center of J_δ.*

Proof: Let X be a vector in the Lie algebra which is a linear combination of root vectors

$$X = \sum c_\alpha X_\alpha \ .$$

Then, X is a fixed vector under $\mathrm{Ad}_L(\delta) = \mathrm{Ad}_G(t) \circ \theta$, if and only if

$$t^{\theta(\alpha)} c_\alpha \theta(X_\alpha) = c_{\theta(\alpha)} X_{\theta(\alpha)}$$

which implies that c_α can be non zero if and only if $t^{N\alpha} = \epsilon_\alpha$. This shows that the θ-orbits of roots in P_δ map onto roots of T^θ in I_δ. Now if s centralizes I_δ it centralizes T^θ and hence it belongs to T. But an $s \in T$ that centralizes a linear combination of root vectors

$$X = \sum c_\alpha X_\alpha$$

is such that $s^\alpha = 1$ whenever $c_\alpha \neq 1$. Hence if $s \in T$ centralizes I_δ one has $s^\alpha = 1$ for all $\alpha \in P_\delta$ and s is in the center of J_δ.

□

II.2.3. Corollary. – *The group J_δ is the connected centralizer of the center of I_δ.*

Proof: It follows from II.2.2 that the center of I_δ is the subgroup of T^θ intersection of the kernels of the restricted roots α_{res} with $\alpha \in P_\delta$. Its connected centralizer is generated by T and the one parameter subgroups defined by the X_α with $\alpha \in P_\delta$. We have already seen that this group is J_δ.

□

II.2.4. Lemma. – *Assume that $a \in J_\delta$ normalizes I_δ. If the automorphism induced by a is inner then $a = cb$ where c takes its values in I_δ and b takes its values in the center of J_δ.*

Proof: Assume that the automorphism induced by a is inner, then a is a product $a = cb$ where $c \in I_{\delta^*}$ induces this inner automorphism and $b \in J$ centralize I_{δ^*}. It follows from II.2.2 that b takes its values in the center of J_δ.

□

II.3 – Outer automorphisms of stable centralizers

Let $\delta \in L$. We need to compute the group of outer automorphisms of I_δ that are induced by elements in J_δ. We shall rely on the classification of pairs (J_δ, I_δ) modulo the center of J_δ. Since the center does not matter we may assume Z_G trivial. The classification is immediately reduced to the case where L is irreducible. Assume now this is the case. Then I.3.3 shows that $L \simeq G \rtimes \theta$ is induced from $G_0 \rtimes \theta_0$. A θ-stable torus T in G is of the form

$$T = T_0 \times \cdots \times T_0$$

where T_0 is θ_0-stable. Lemma I.3.2, or rather its proof, shows that, up to θ-conjugacy, we may take $t \in T$ of the form

$$t = (1, \cdots, 1, t_0) \ .$$

If $c \in G$ centralizes $\delta = t \rtimes \theta$ then $c = (c_0, \cdots, c_0)$ where c_0 centralizes $\delta_0 = t_0 \rtimes \theta_0$ in G_0. This shows that $I_\delta \simeq I_{\delta_0}$ embedded diagonally in

$$J_\delta \simeq J_{\delta_0} \times \cdots \times J_{\delta_0} \ .$$

Hence we are now reduced to the case G simple.

Observe that θ defines an automorphism of the twisted space $G \rtimes \theta$ and it makes sense to consider $\theta(\delta)$. Assume that moreover G is reductive and θ preserves a splitting; one can view $G \rtimes \theta$ as a subset of the group $G \rtimes \mathrm{Out}\,(G)$. This allows to consider powers δ^n of elements $\delta \in G \rtimes \theta$.

II.3.1. Lemma. – *Let ℓ be the order of the automorphism of the root system induced by θ. Then δ^ℓ is central in J_δ.*

Proof: For $\delta \in G \rtimes \theta$ we have $\delta^\ell \in G \rtimes 1 \simeq G$. But δ^ℓ belongs to $T \subset J_\delta$ and centralizes I_δ; now II.2.2 shows that it must be central in J_δ.

□

Consider $\delta \in L$; we denote by J_{ad} the adjoint quotient of J_δ. Since δ^ℓ is central in J_δ its image in the adjoint quotient is trivial and $\mathrm{Ad}_L(\delta)$ induces an automorphism θ' of order ℓ of J_{ad}. Now split J_{ad} into its simple factors and consider a simple factor J_0. Assume G simple, then $\ell \leq 3$. Since ℓ is prime, either θ' preserves J_0 and δ defines an element in $J_0 \rtimes \theta'$ or θ' induces the cyclic permutation of ℓ copies of J_0. It suffices to compute I_δ when $J_\delta = G$ is a simple group. In such a case II.2.2 shows that I_δ is adjoint.

II.3.2. Proposition. – *Consider $L = G \rtimes \theta$ where θ fixes a splitting $(B, T, \{X_\alpha\})$. Assume G is simple. Let ℓ be the order of the automorphism of the root system of G induced by θ. Consider $\delta \in T \rtimes \theta$ such that $J_\delta = G$. Then up to conjugacy by some element of T, we may choose δ so that $\delta^\ell = 1$ and $\delta = \theta(\delta)$. A case by case description is then as follows:*

1 - $\ell = 1$ then $\delta = 1 \rtimes 1$, $G = J_\delta = I_\delta$.

2a - the root system is of type A_{2n} with $n \geq 1$, $\ell = 2$ and $\delta = 1 \rtimes \theta$ then I_δ is of type B_n.

2b - the root system is of type A_{2n-1} with $n \geq 2$, $\ell = 2$ and $\delta = 1 \rtimes \theta$ then I_δ is of type C_n.

2c - the root system is of type A_{2n-1} with $n \geq 2$, $\ell = 2$ and $\delta \neq 1 \rtimes \theta$ then I_δ is of type D_n.

3 - the root system is of type D_n with $n \geq 4$, and $\ell = 2$ then I_δ is of type $B_{n_+} \cup B_{n_-}$ with $n_+ + n_- = n - 1$.

4a - the root system is of type D_4, $\ell = 3$ and $\delta = 1 \rtimes \theta$ then I_δ is of type G_2.

4b - the root system is of type D_4, $\ell = 3$ and $\delta \neq 1 \rtimes \theta$ then I_δ is of type A_2.

5 - the root system is of type E_6, $\ell = 2$ then I_δ is of type F_4, C_4 or $B_3 \cup A_1$.

Proof: We have seen II.3.1 in that δ^ℓ is central in J_δ. But, since $J_\delta = G$ is simple $\delta^\ell = 1$. We have $\delta = t \rtimes \theta$ then $\delta^\ell = 1$ is equivalent to $t\theta(t)\cdots\theta^{r-1}(t) = 1$. In particular, if α is a root such that $\ell_\alpha = \ell$ we have $t^{N\alpha} = 1$. A variant of lemma I.3.2 shows that there exist $a \in T$ such that

$$(a\,t\,\theta(a)^{-1})^\alpha = 1$$

for each simple root with $\ell_\alpha = \ell$ while $(a\,t\,\theta(a)^{-1})^\alpha = t^\alpha$ (and $(t^\alpha)^\ell = 1$) if $\ell_\alpha = 1$ i.e. if α is a fixed root. Hence

$$\theta(a\,t\,\theta(a)^{-1}) = (a\,t\,\theta(a)^{-1}) \ .$$

This yields the first assertion. To prove the other assertions we shall compute J_δ when $\delta^\ell = 1$ and $\delta = \theta(\delta)$ for the various triples (R, Δ, θ) of an irreducible root system, a set of simple roots, and an automorphism. When the automorphism is non trivial it will be defined via an isometry, again denoted θ, of an euclidean vector space $\tilde{V}$ with an orthonormal basis $\{e_i\}_{i \in I}$ such that θ preserve the basis up to signs. The $\{e_i\}_{i \in I}$ belong to the lattice of characters of a torus $\tilde{T}$. The root system generates a subspace $V \subset \tilde{V}$. Instead of looking to simple groups it is equivalent, but may be simpler for some computations, to consider an auxilliary group $\tilde{G}$ whose adjoint quotient is the simple group G and to prove in $\tilde{G}$ the assertions up to the center. We now study the various cases.

1 – The automorphism θ is trivial i.e. $\ell = 1$ then $I_\delta = G$ since $\delta = 1$.

2 – The root system is of type A_{m-1} with $m \geq 3$, and $\ell = 2$. Here we take $\tilde{G} = GL(m)$ and $\tilde{T}$ will be the group of diagonal matrices and t^{e_i} is the i-th entry of the diagonal matrix $t \in \tilde{T}$. The roots of $\tilde{T}$ in $\tilde{G}$ can be written $(e_i - e_j)$ with $i \neq j$. The automorphism is defined by $\theta(e_i) = -e_{m+1-i}$. The characters $\beta_i = (e_i - e_{m+1-i})$ are θ-invariant; they are roots unless $2i = m + 1$. Let $\lambda_i = t^{\beta_i}$. Since $\delta = t \rtimes \theta$ is such that $\delta^2 = 1$ in G it suffices to study the t such that $t^{\beta_i} = \pm 1$. Let $n = [\frac{m}{2}]$; define $\tau \in \tilde{T}$ by

$$\tau^{e_i} = \lambda_i \quad \text{if } \lambda_i = -1 \text{ and } 1 \leq i \leq n, \qquad \tau^{e_i} = 1 \quad \text{otherwise} \ .$$

Observe that for $a \in \tilde{T}$

$$(a\,t\,\theta(a)^{-1})^{e_i} = t^{e_i} a^{e_i + e_{m+1-i}} \ .$$

Hence, one can find $a \in \tilde{T}$ such that $a\,t\,\theta(a)^{-1} = \tau$. Then $(\tau\theta(\tau)^{-1})^{e_i} = -1$ if $\lambda_i = -1$ and equals 1 otherwise. Let $\alpha = e_i - e_j$ be a root with a non trivial θ-orbit i.e. $j \neq m+1-i$; then $\alpha \in P_\delta$ if and only if $\lambda_i = \lambda_j$ and we have $(\tau\theta(\tau)^{-1})^\alpha = 1$. Now if α is a θ-fixed root, i.e. $\alpha = \beta_i$ for some i, and belongs to P_δ then $\lambda_i = t^\alpha = \epsilon_\alpha = \pm 1$ and we have $(\tau\theta(\tau)^{-1})^\alpha = \lambda_i^2 = 1$. This shows that $\tau\theta(\tau)^{-1}$ belongs to the center of J_δ. Let $S_\pm \subset \{1, \cdots, m\}$ be the subset of indices such that $\lambda_i = \pm 1$ and let us denote by $m_\pm$ its cardinal. Observe that m_- is always even. One has

$$J_\delta \simeq GL(m_+) \times GL(m_-)$$

unless m is odd and $m_- = 2$ where

$$J_\delta \simeq GL(m_+) \times GL(1) \times GL(1) \ .$$

But since we assume $J_\delta = \tilde{G}$ we must have only one factor; we have three cases:

(a) $m = m_+ = 2n+1$, $\delta = 1 \rtimes \theta$, then $I_\delta = SO(2n+1)$

(b) $m = m_+ = 2n$, $\delta = 1 \rtimes \theta$, then $I_\delta = Sp(2n)$

(c) $m = m_- = 2n$, $\delta = t \rtimes \theta^*$ with

$$t = \begin{pmatrix} -\mathbf{1}_n & 0 \\ 0 & \mathbf{1}_n \end{pmatrix}$$

then $I_\delta = SO(2n)$ while the full centralizer is $O(2n)$.

3 – The root system is of type D_n, and $\ell = 2$. We take $\tilde{G} = SO(2n)$ and let $\tilde{T}$ be a maximal torus. The roots can be written $\pm e_i \pm e_j$ with $1 \leq i < j \leq n$. The automorphism θ is induced by $e_n \mapsto -e_n$. For $t \in \tilde{T}$ let $\lambda_i = t^{e_i}$ for $i \leq n-1$ and $\lambda_n = 1$. Since δ^2 is central we have $\lambda_i^2 = 1$ for all i. We take $\tau^{e_i} = \lambda_i$ for all i. Clearly $\tau\theta(\tau)^{-1} = 1$. There is $a \in \tilde{T}$ such that

$$(a\,t\,\theta(a)^{-1})^{e_n} = 1 \ .$$

Then for such a choice we have $a\,t\,\theta(a)^{-1} = \tau$. The roots in P_δ are of the form $\alpha = \pm e_i \pm e_j$ with $i < j \leq n-1$ and $\lambda_i = \lambda_j$ or $\alpha = \pm e_i \pm e_n$ with $i < n$. The stable centralizer of $\delta' = t' \rtimes \theta$ is the product

$$SO(2n_+ + 1) \times SO(2n_- + 1) \subset SO(2n)$$

where $n_\pm$ is the cardinal of the $\lambda_i = \pm 1$ and $i < n$. In particular $n_+ + n_- = n - 1$. We have to study the particular case $n_+ = n_-$ which implies $n = 2m+1$. We may view $SO(2n) \rtimes < 1, \theta >$ as $O(2n)$. Since n is odd, we may take

$$\delta = \begin{pmatrix} -\mathbf{1}_n & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{1}_n \end{pmatrix} \in O(2n)$$

and its stable centralizer is $SO(2n) \cap (O(n) \times O(n))$ embedded diagonally while the element

$$w = \begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix} \in SO(2n)$$

flips the two copies of $O(n)$ and centralize δ up to the center: $w\delta w^{-1} = -\delta$. In the projective group the centralizer of the image of δ is generated by the images of I_δ and w.

4 – The root system is of type D_4 and $\ell = 3$. We shall not use the standard Bourbaki description of the roots; we shall use instead a basis better adapted to the study of the automorphism of order 3. We consider a real euclidean vector space $V = \tilde{V}$ with orthonormal basis $\{e_0, e_1, e_2, e_3\}$; the roots are of the following form:

$$\pm e_i \qquad \text{and} \qquad \frac{1}{2}[\pm e_0 \pm e_1 \pm e_2 \pm e_3] \ .$$

A set of simple roots is given by

$$\alpha_0 = \frac{1}{2}[e_0 - e_1 - e_2 - e_3], \qquad \alpha_1 = e_1, \qquad \alpha_2 = e_2, \qquad \alpha_3 = e_3 \ .$$

The automorphism θ is induced by the cyclic permutation of $\{e_1, e_2, e_3\}$. The fixed positive roots are

$$\alpha_0, \qquad \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = \frac{1}{2}[e_0 + e_1 + e_2 + e_3], \quad \text{and} \quad 2\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = e_0$$

and the norm $N\alpha$ for non trivial positive orbits are

$$N\alpha_1 = \alpha_1 + \alpha_2 + \alpha_3, \qquad N(\alpha_0 + \alpha_1) = 3\alpha_0 + N\alpha_1 \quad \text{and} \quad N(\alpha_0 + \alpha_1 + \alpha_2) = 3\alpha_0 + 2N\alpha_1 \ .$$

Let $\lambda_0 = t^{\alpha_0}$ and $\lambda_1 = t^{N\alpha_1}$. Observe that

$$(a\, t\, \theta(a)^{-1})^{e_i} = t^{e_i} a^{e_i - e_{\phi(i)}}$$

for $i \in \{1, 2, 3\}$; this shows that given μ_i for $i \in \{1, 2, 3\}$ with $\mu_1 \mu_2 \mu_3 = \lambda_1$ we may find a such that

$$(a\, t\, \theta(a)^{-1})^{e_i} = \mu_i \ .$$

We shall always choose a such that $\mu_1 = \mu_2 = \mu_3$, and hence $\mu_i^3 = \lambda_1$. We now discuss subcases.

(i) The set P_δ contains exactly one positive root which is θ-fixed then one and only one of these equality holds:

$$\lambda_0 = 1\,, \qquad \lambda_0 \lambda_1 = 1 \quad \text{or} \quad \lambda_0^2 \lambda_1 = 1\ .$$

Then t, as well as $a\, t\, \theta(a)^{-1}$ for any a, are in the center of J_δ but $J_\delta \neq G$.

(ii) The set P_δ contains two positive fixed roots; we necessarily have $\lambda_0 = \lambda_1 = 1$ this shows and P_δ is the set of all roots and hence $J_\delta = G$. We choose a such that $\mu_i = 1$ for $i \in \{1, 2, 3\}$ and for such a choice $\tau = a\, t\, \theta(a)^{-1} = 1$.

(iii) The set P_δ contains only one non trivial positive orbit; then one and only one of these equality holds:

$$\lambda_1 = 1\,, \qquad \lambda_0^3 \lambda_1 = 1 \quad \text{or} \quad \lambda_0^3 \lambda_1^2 = 1\ .$$

We choose the μ_i as follows: $\mu_i = 1$ if $\lambda_1 = 1$, $\lambda_0 \mu_i = 1$ if $\lambda_0^3 \lambda_1 = 1$ and $\mu_i = \lambda_0 \lambda_1$ if $\lambda_0^3 \lambda_1^2 = 1$. Again with such a choice $\tau = a\, t\, \theta(a)^{-1}$ is in the center of J_δ but $J_\delta \neq G$.

(iv) The set P_δ contains one fixed root β_0 and a non trivial orbit of some root β_1. If it contains no other positive root then β_0 and β_1 are orthogonal; we may proceed as in case (iii) above. Otherwise, $\lambda_0 = \lambda_1 = 1$ and we may proceed as in case (ii).

(v) The set P_δ contains two non trivial positive orbits but no fixed root. This implies $\lambda_0^3 = \lambda_1 = 1$ with $\lambda_0 = j \neq 1$ is a primitive 3rd-root of unity. In such a case $J_\delta = G$. By choosing a such that the $\mu_i = 1$ we have $a\, t\, \theta(a)^{-1} = \tau$ with $\tau^3 = 1$ and $\tau = \theta(\tau)$.

Then $J_\delta = G$ occurs in case (ii) with $t_1 = 1$ then the stable centralizer I_δ is a group of type G_2 or in case (v) where the group I_δ is of type A_2: its root system is generated by the short roots in the G_2 root system.

5 – The root system is of type E_6 and $\ell = 2$. We consider a real euclidean vector space $\tilde{V}$ of dimension 8, with orthonormal basis $\{e_0, e_1, \cdots, e_6, e_7\}$; the roots are in the subspace V defined by the equations $x_0 + x_7 = 0$ and $x_1 + \cdots + x_6 = 0$. They are of the following form:

$\pm(e_i - e_j)$ for $1 \leq i < j \leq 6$ or $i = 0$ and $j = 7$.

$\pm\frac{1}{2}[e_0 - e_7 \pm e_1 \pm e_2 \cdots \pm e_6]$ with an equal number of $+$ and $-$ signs inside the bracket.

A set of simple roots is given by

$$\alpha_1 = \frac{1}{2}[e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7] \qquad \text{and} \qquad \alpha_{i+1} = [e_{i+1} - e_i] \qquad \text{for } 1 \le i \le 5 \ .$$

The automorphism is induced by $\theta(e_i) = -e_{(7-i)}$; it sends α_{i+1} to α_{7-i} for $1 \le i \le 5$ and fixes α_1 and α_4. The reader should be warned that this is not the standard Bourbaki notation: we have exchanged the numbering of the simple roots α_1 and α_2 and used a different orthonormal basis. Here is the correspondence:

Our notation		Bourbaki's notation
$\alpha_1 = \frac{1}{2}[e_0 + e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7]$	$\longleftrightarrow$	$\alpha_2 = \epsilon_1 + \epsilon_2$
$\alpha_2 = e_2 - e_1$	$\longleftrightarrow$	$\alpha_1 = \frac{1}{2}[\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8]$
$\alpha_{i+1} = e_{i+1} - e_i$	$\longleftrightarrow$	$\alpha_{i+1} = \epsilon_i - \epsilon_{i-1}$ for $2 \le i \le 5$

The correspondence between the orthonormal basis is given by

$$e_1 = \eta - \epsilon_8 \qquad \text{and} \qquad e_{i+1} = \epsilon_i - \eta \qquad \text{for} \qquad 1 \le i \le 5$$

where

$$\eta = \frac{1}{4}[\epsilon_1 + \epsilon_2 + \cdots + \epsilon_8] \qquad \text{and} \qquad e_0 - e_7 = \frac{1}{2}[\epsilon_1 + \epsilon_2 + \cdots + \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8] \ .$$

The θ-fixed roots are

$$\beta_i = (e_i - e_{7-i}) \quad \text{and} \quad \gamma_A = \alpha_1 - \sum_{i \in A} \beta_i = \frac{1}{2}[\pm(e_0 - e_7) \pm (e_1 - e_6) \pm (e_2 - e_5) \pm (e_3 - e_4)]$$

where A is any subset of $\{0, 1, 2, 3\}$. They can be rewritten

$$\pm\beta_i \quad \text{for } 0 \le i \le 3 \qquad \text{and} \qquad \frac{1}{2}[\pm\beta_0 \pm \beta_1 \pm \beta_2 \pm \beta_3]$$

The roots with non trivial θ-orbits have norms

$$N\alpha = [\pm(e_i - e_{7-i}) \pm (e_j - e_{7-j})] = [\pm\beta_i \pm \beta_j]$$

with $0 \le i < j \le 3$. We observe that $NE_6 = F_4$. We consider the torus $\tilde{T}$ whose group of characters is the lattice generated by the e_i and α_1 in $\tilde{V}$ and we denote by T the quotient torus whose group of caracters is the root lattice in V. Let

$$\Lambda_A = t^{\gamma_A} \quad \text{and} \quad \lambda_i = t^{\beta_i} \quad \text{for} \quad 0 \le i \le 7 \ .$$

Observe that $\delta^2 = 1$ implies $\lambda_i^2 = 1$ and $\Lambda_A^2 = 1$ for all i and all A; in particular

$$\Lambda_\emptyset^2 = \lambda_0 \lambda_1 \lambda_2 \lambda_3 = 1 \ .$$

Hence, the set

$$B = \{i \,|\, 0 \le i \le 3 \,,\, \lambda_i = -1\}$$

has an even cardinal. Now define $\tau \in \tilde{T}$ by $\tau^{e_i} = \lambda_i$ for $0 \le i \le 3$. Let $\tau^{e_i} = 1$ otherwise. Moreover choose τ^{γ_A} in order that

$$t^{\gamma_A} = \Lambda_A = \tau^{\gamma_A}$$

for some A. Then this will hold true for all A. We have $\tau^2 = 1$. Then $\tau = \varepsilon\theta(\tau)$ where ε is in the kernel of the map $\tilde{T} \to T$. Observe that for $a \in T$

$$(a\,t\,\theta(a)^{-1})^{e_i} = t^{e_i} a^{e_i + e_{7-i}} \ .$$

Hence one can find $a \in t$ such that

$$a\,t\,\theta(a)^{-1} = \tau \ .$$

If all $\lambda_i = 1$ and $\Lambda_\emptyset = 1$ then $\delta = 1 \rtimes \theta$ and I_δ is of type F_4. The case where all $\lambda_i = 1$ and $\Lambda_\emptyset = -1$ need not be considered since then $J \neq G$. If all $\lambda_i = -1$ then I_δ is of type C_4. If half of the $\lambda_i = -1$ then I_δ is of type $B_3 \cup A_1$ (in this case J_δ may not be equal to G).

□

II.3.3. Corollary. – *Assume that J_δ is simple. The group of outer automorphism of I_δ induced by elements of J_δ is of order 2 in cases (2c), (3) with $n_+ = n_-$ and (4b). It is trivial otherwise.*

Proof: In fact, in case (2c), we have $J_\delta = PGL(2m)$; the centralizer is the projective image of $O(2m)$ while the stable centralizer is the image of $SO(2m)$. In case (3) with $m = n_+ = n_-$ the group is the projective image of $SO(4m+2)$ while the stable centralizer is the image of $SO(2m+1)\times SO(2m+1)$; but the centralizer also contains an element whose adjoint action flips the two $SO(2m+1)$ subgroups. In case (4b), $G^* = J$ of type D_4, $\ell = 3$ and $\delta^* \neq 1\rtimes\theta^*$ the stable centralizer is of type A_2. The non trivial outer automorphism is induced by the symmetry with respect to the θ-stable simple root α_0. In all other cases the classification II.3.2 shows that I_δ has no non trivial outer automorphism.

□

II.4 – Absolute endoscopic groups

Consider a connected reductive group G and a Borel pair (B,T). A coroot $\check\alpha$ for T and G defines a cocharacter of T that factors through T_{sc}:

$$\check\alpha : \mathbb{G}_m \to T_{sc} \to T$$

and the set of simple coroots form a basis of the $\mathbb{Z}$-module $\check X(T_{sc})$ of cocharacters of T_{sc}. Conversely, if we are given a torus T, a based root system R, viewed as a subset of the character group $X(T)$, and the based coroot system $\check R$, viewed as a subset of the cocharacter group $\check X(T)$, we recover the group G and the Borel pair (B,T) up to an inner isomorphism. The role of roots and coroots is exchanged by considering the dual torus $\check T$ in the dual group $\check G$. The order on the roots defined by B yields an order on the coroots and hence defines a Borel subgroup $\check B$ in the dual group. Choose a splitting in G and in $\check G$ containing the Borel pairs (B,T) and $(\check B,\check T)$. Consider an automorphism θ preserving the splitting in G. Let $\check\theta$ be the automorphism preserving the splitting in $\check G$ and inducing the same permutation of roots systems.

By composition with the conorm map $T \to T_\theta$, a coroot defines a cocharacter

$$\check\alpha_\theta : \mathbb{G}_m \to T_\theta$$

which depends only on the θ-orbit of $\check\alpha$. The set of all such cocharacters form a non necessarily reduced root system in the real vector space $\check X(T_{sc})_\theta \otimes \mathbb{R}$. In fact they can be seen as the system of restricted roots for the adjoint dual group $(\check G)_{ad}$ equipped with the dual automorphism $\check\theta$. Given a coroot $\check\alpha$ one defines $N\check\alpha$ as the sum over the θ-orbit. This is a cocharacter for T_{sc} that factors through T_{sc}^θ since $N\check\alpha$ belongs to the free $\mathbb{Z}$-module $\check X(T_{sc})^\theta$ of θ-invariant cocharacters of T_{sc}. A root α and its associated coroot $\check\alpha$ have the same type, in particular $\epsilon_\alpha = \epsilon_{\check\alpha}$.

II.4.1. Lemma. – *The coroot of α_{res} is $N\check\alpha$ if α is of type 1 or 3 and $2N\check\alpha$ if α is of type 2.*

Proof: Let $\beta = \alpha_{res}$, its coroot is of the form $\check\beta = c_\alpha N\check\alpha$. We must have $<\beta|\check\beta>= 2$ and hence

$$<\beta|\check\beta>=<\alpha|c_\alpha N\check\alpha>= c_\alpha \sum_{i=0}^{\ell_\alpha-1} <\alpha|\theta^i(\check\alpha)>= 2\ .$$

A case by case checking shows that the coroots $\check\alpha$ and $\theta^i(\check\alpha)$ are orthogonal for all $i \not\equiv 0$ modulo ℓ_α except when α is of type 2 and $i = \ell_\alpha/2$ in which case $<\alpha|\theta^i(\check\alpha)>= -1$. Hence

$$\sum_{i=0}^{\ell_\alpha-1} <\alpha|\theta^i(\check\alpha)>=<\alpha|\check\alpha>= 2$$

when α is of type 1 or 3 and then $c_\alpha = 1$ while

$$\sum_{i=0}^{\ell_\alpha-1} <\alpha|\theta^i(\check\alpha)>=<\alpha|\check\alpha>+<\alpha|\theta^{\ell_\alpha/2}\check\alpha>= 2-1 = 1$$

when α is of type 2 and then $c_\alpha = 2$.

□

Let $\overline{\kappa}$ be a character of $\check{X}(T_{sc})^{\theta}$. We shall define four sets of characters and cocharacters. Let $R_{\overline{\kappa}}$ be the set of characters of T_θ defined by $\beta = N\alpha$ if α is of type 1 or 3 or $\beta = 2N\alpha$ if α is of type 2, and $\overline{\kappa}(N\check{\alpha}) = \epsilon_\alpha$ in all cases. Let $\check{R}_{\overline{\kappa}}$ be the set of cocharacters $\check{\alpha}_\theta$ of T_θ defined by coroots $\check{\alpha}$ such that $\overline{\kappa}(N\check{\alpha}) = \epsilon_\alpha$.

Let $R'_{\overline{\kappa}}$ be the set of restricted roots α_{res} where either α is of type 1 or 3 and $\overline{\kappa}(N\check{\alpha}) = 1$ or α is of type 2 and $\overline{\kappa}(N\check{\alpha}) = -1$. Let $\check{R}'_{\overline{\kappa}}$ be the set of cocharacters of T^θ of the form $\check{\beta} = N\check{\alpha}$ if $\overline{\kappa}(N\check{\alpha}) = 1$ and α of any type, or $\check{\beta} = 2N\check{\alpha}$ if $\overline{\kappa}(N\check{\alpha}) = -1$ and α of type 2 or 3.

II.4.2. Lemma. – *The set $R_{\overline{\kappa}}$ is a reduced root system dual to $\check{R}_{\overline{\kappa}}$. It is called the endoscopic root system. The set $R'_{\overline{\kappa}}$ is also a reduced root system dual to $\check{R}'_{\overline{\kappa}}$. The four Weyl groups are isomorphic to a group $\Omega_{\overline{\kappa}}$ called the $\overline{\kappa}$-endoscopic Weyl group.*

Proof: Consider the dual picture: the torus $(T_{sc})\check{}$ is a maximal torus in the adjoint quotient of $\check{G}$. The character $\overline{\kappa}$ defines an element in the group of complex points of $\check{\theta}$-coinvariants of $(T_{sc})\check{}$; it is the image of some $s \in \check{T}(\mathbb{C}) \subset \check{G}(\mathbb{C})$. As observed in II.2.2 the set $\check{R}_{\overline{\kappa}}$ is isomorphic to the set of roots of the connected centralizer $\check{H}$ of $s \rtimes \check{\theta}$ in $\check{G}$. The root system dual to the coroot system $\check{R}_{\overline{\kappa}}$ is the set of $\beta = cN\alpha$ with $< \check{\beta}|\beta >= 2$ where $\check{\beta} = \check{\alpha}_\theta$. Now II.4.1 shows that $R_{\overline{\kappa}}$ is the root system dual to $\check{R}_{\overline{\kappa}}$. There is a canonical bijection between $R_{\overline{\kappa}}$ and $R'_{\overline{\kappa}}$ and each root in $R_{\overline{\kappa}}$ is proportional to an element in $R'_{\overline{\kappa}}$ since

$$N\alpha = \ell_\alpha \ \alpha_{res} \ ,$$

but the two sets may not be homothetic. To prove that $R'_{\overline{\kappa}}$ is a root system we assume G simple and we use the classification; the general case follows easily. If θ is trivial then $R_{res} = NR$ and all roots are of type 1; hence $R'_{\overline{\kappa}} = R_{\overline{\kappa}}$. Now assume $\theta \neq 1$ and that all roots are of type 1; since all roots in R have the same length $R \simeq \check{R}$ and it follows that $R_{\overline{\kappa}}$ and $\check{R}'_{\overline{\kappa}}$ are isomorphic sub-root systems of $NR \simeq N\check{R}$. It remains to consider the case where G is of type A_{2n} and θ is of order 2; in such a case the bijection $\beta \mapsto \beta'$ between $R_{\overline{\kappa}}$ and $R'_{\overline{\kappa}}$ is such that $\beta = 2\beta'$ and hence $R'_{\overline{\kappa}}$ is an isomorphic root system. We have checked that $R'_{\overline{\kappa}}$ is a root system; now, that $\check{R}'_{\overline{\kappa}}$ is its dual coroot system follows from II.4.1. The roots in $R'_{\overline{\kappa}}$ and $R_{\overline{\kappa}}$ being proportional their Weyl groups are isomorphic.

□

II.4.3. Definition. – *The endoscopic group H associated to $(T, \overline{\kappa})$ is the connected reductive group (defined over the algebraic closure) with maximal torus, root and coroot system $(T_H, R_H, \check{R}_H)$ isomorphic to $(T_\theta, R_{\overline{\kappa}}, \check{R}_{\overline{\kappa}})$. This group is well defined up to inner automorphism induced by elements of T_θ. The quasi-connected reductive group K defined by $(T^\theta, R'_{\overline{\kappa}}, \check{R}'_{\overline{\kappa}})$ will be called a pre-endoscopic group.*

Galois structures will be discussed later on. When $\theta = 1$ or more generally when $L = G \rtimes \theta$ is induced from $G_0 \rtimes 1$ the two groups K and H are isomorphic.

Let $\check{X}^{\overline{\kappa}}_{sc}$ be the sublattice in $\check{X}^{\theta}_{sc}$, generated by the $\check{\beta} \in \check{R}'_{\overline{\kappa}}$. Denote by $T_{\overline{\kappa}}$ the associated torus. The image of the small complex $[T_{\overline{\kappa}} \to T^{\theta^*}]$ in $\mathcal{DT}$ is K_{ab}. We observe that in general there is no homomorphism of K into G extending the inclusion $T^\theta \to T$. Nevertheless there is a canonical morphism in $\mathcal{DT}$ from K_{ab} to G_{ab} and the object

$$[K_{ab} \to G_{ab}]$$

can be represented by the bicomplex of diagonalizable groups

$$\begin{bmatrix} T_{\overline{\kappa}} & \to & T_{sc} \\ \downarrow & & \downarrow \\ T^\theta & \to & T \end{bmatrix} .$$

II.4.4. Lemma. – *Let $\delta \in T \rtimes \theta$ and let $I = I_\delta$ be its stable centralizer. We denote by I_{SC} the simply connected cover of the derived group of I. Let S_{sc} be the maximal torus in I_{SC} that projects into $S = T^\theta \subset I$.*

Assume that the image of $\check{X}(S_{sc})$ *in* $\check{X}(T_{sc})^\theta$ *is in the kernel of* $\overline{\kappa}$*. The map* $S_{sc} \to T_{sc}^\theta$ *factors through* $T_{\overline{\kappa}}$ *and the map* $I_{ab} \to G_{ab}$ *factors in* $\mathcal{DT}$ *through* K_{ab}*:*

$$I_{ab} \to K_{ab} \to G_{ab} \ .$$

Let γ *be the image of* δ *in* T_θ*. There is a canonical bijection between the set of roots of* T_θ *in the connected centralizer* I_γ *of* γ *in* H *and the set of roots of* T^θ *in* I_δ*. This bijection preserves angles between roots and induces an isomorphism of Weyl groups:* $\Omega(I_\delta, T^\theta) \simeq \Omega(I_\gamma, T_\theta)$*.*

Proof: Let $\delta = t \rtimes \theta$. The roots of T^θ in I_δ are the $\beta = \alpha_{res}$ for roots α such that $t^{N\alpha} = \epsilon_\alpha$. According to II.4.1 the coroots $\check{\beta}$ are the $N\check{\alpha}$ if α is of type 1 or 3 or $2N\check{\alpha}$ if α is of type 2. They form a basis of $\check{X}(S_{sc})$. Since the image of $\check{X}(S_{sc})$ is in the kernel of $\overline{\kappa}$, then $\overline{\kappa}(N\check{\alpha}) = 1$ if $\check{\alpha}$ is of type 1 or 3 and $\overline{\kappa}(2N\check{\alpha}) = 1$ if $\check{\alpha}$ is of type 2. Hence

$$\check{X}(S_{sc}) \subset \check{X}_{sc}^{\overline{\kappa}} \subset \check{X}_{sc}^{\theta^*} \ .$$

Recall that I_{ab} can be represented by the small complex $[S_{sc} \to S]$. This proves the first assertion. Consider the following correspondence between roots:

(i) For a root α of type 1 or 2 such that $t^{N\alpha} = 1$ and $\overline{\kappa}(N\check{\alpha}) = 1$, let $\alpha_1 = \alpha$.

(ii) For a root α of type 2 such that $t^{N\alpha} = 1$ and $\overline{\kappa}(N\check{\alpha}) = -1$, take for α_1 a root of type 3 such that $N\check{\alpha}_1 = N\check{\alpha}$.

(iii) For a root α of type 3 such that $t^{N\alpha} = -1$ and $\overline{\kappa}(N\check{\alpha}) = 1$, take for α_1 a root of type 2 such that $N\alpha_1 = N\alpha$.

Observe that the θ-orbit of α_1 depends only on the θ-orbit of α, and conversely. For $\beta = \alpha_{res}$ a root of I consider a root α_1 that corresponds to α as above and let $\beta_1 = N\alpha_1$ if α_1 is of type 1 or 3 and $\beta_1 = 2N\alpha_1$ if α_1 is of type 2. We have $\beta_1 \in R_{\overline{\kappa}}$ and $t^{\beta_1} = 1$. Hence, β_1 is a roots of T_θ in I_γ. The correspondence $\beta \mapsto \beta_1$ is the expected bijection between root for I_δ and root for I_γ. Now it remains to observe that β and β_1 are proportional to $N\alpha = N\alpha_1$ and hence the Weyl groups defined by those two root systems are isomorphic. □

II.4.5. Corollary. – *There is a natural injective map*

$$H^\gamma / I_\gamma \to G^\delta / I_\delta \ .$$

This is a bijection when $\overline{\kappa} = 1$*.*

Proof: In II.1.4 we have seen that

$$G^\delta / I_\delta \simeq \Omega(G^\delta, T^\delta) / \Omega(I_\delta, T^\delta)$$

and

$$H^\gamma / I_\gamma \simeq \Omega(H^\gamma, T_H) / \Omega(I_\gamma, T_H) \ .$$

Moreover, II.4.4 shows that the Weyl groups for I_δ and H_γ are in natural bijection:

$$\Omega(I_\delta, T^\delta) \simeq \Omega(I_\gamma, T_H) \ .$$

Since $\delta = t \rtimes \theta$ preserves the Borel pair (B, T) we know that $T^\delta = T^\theta$ and that $\Omega(G^\delta, T^\delta)$ injects into $\Omega(G, T)^\theta$ which in turn is isomorphic to $\Omega(G^\theta, T^\theta)$ since θ preserves a splitting containing T. Now consider $w \in \Omega(G^\delta, T^\delta)$; it is represented by some $n \in G^\delta$ which normalizes T, then $n \delta n^{-1} = \delta$ is equivalent to $n\, t\, \theta(n)^{-1} = t$. Using the injection

$$\Omega(G^\delta, T^\delta) \to \Omega(G^\theta, T^\theta)$$

we see that $n = un_1$ with $u \in T$ and $\theta(n_1) = n_1$ and hence, $n_1\, t\, u^{1-\theta}\, n_1^{-1} = t$. If we denote by γ the class of t in T_θ this is equivalent to $w(\gamma) = \gamma$. Let H_1 be the endoscopic group attached to the trivial endoscopic character. Using the isomorphism

$$\Omega(H_1, T_\theta) \to \Omega(G^\theta, T^\theta)$$

we get further isomorphisms

$$\Omega(H_1^\gamma, T_\theta) \to \Omega(G^\delta, T^\delta) \qquad \text{and} \qquad H_1^\gamma / I_\gamma \to G^\delta / I_\delta \ .$$

More generally one can view H^γ / I_γ as a subgroup of H_1^γ / I_γ and hence of G^δ / I_δ.

□

III – Stable conjugacy, inner forms and prenorms

III.1 – Stable conjugacy

Let L be a twisted G-space defined over some field F. Consider two elements δ and η in L that are conjugate (over $\overline{F}$):

$$x^{-1} \delta x = \eta$$

for some $x \in G(\overline{F})$. If $\delta \in L(F)$ then η is also rational if and only if the Galois cocycle $\sigma \mapsto x\sigma(x)^{-1}$ takes its values in the centralizer G^δ. Stable conjugacy is a slight variant of this using the stable centralizer.

III.1.1. Definition. – *Two elements δ and η in $L(F)$ are said to be stably conjugate if there exist $x \in G(\overline{F})$ such that*

$$x^{-1} \delta x = \eta \qquad \textit{and} \qquad x\sigma(x)^{-1} \in I_\delta \qquad \textit{for all } \sigma \in \mathrm{Gal}(\overline{F}/F) \ .$$

III.1.2. Lemma. – *Let $\mathcal{S}(\delta)$ be the stable conjugacy class of δ. The map $x \mapsto x^{-1}\delta x$ induces a surjection from $\mathbf{H}^0(F, I_\delta \backslash G)$ onto $\mathcal{S}(\delta)$. The fiber of this map above δ is $\mathbf{H}^0(F, I_\delta \backslash G^\delta)$. Consider $\eta \in \mathcal{S}(\delta)$ then I_η is an inner form of I_δ.*

Proof: The first two assertions are clear. Consider the last one. By assumption there is $x \in G$ such that $x^{-1}\delta x = \eta$ with $a_\sigma = x\sigma(x)^{-1} \in I_\delta$. Denote by $\psi : I_\eta \to I_\delta$ the isomorphism over the algebraic closure defined by $t \mapsto xtx^{-1}$. It suffices to observe that $\psi\sigma(\psi)^{-1}$ is induced by the restriction of $\mathrm{Ad}_G(a_\sigma)$ to I_δ with $a_\sigma \in I_\delta$.

□

III.1.3. Corollary. – *The number $\tilde{\imath}_G(\delta)$ of elements in $\mathbf{H}^0(F, I_\delta \backslash G^\delta)$ is constant when δ vary in a stable conjugacy class.*

Proof: If η is stably conjugate to δ, III.1.2 shows that the G^η and I_η are inner form of G^δ and I_δ respectively and that the twisting cocycle takes its values in I_δ. Hence there is an F-isomorphism

$$I_\eta \backslash G^\eta \to I_\delta \backslash G^\delta \ .$$

□

III.2 – Inner forms of twisted spaces

Let G be an algebraic group and let L be a twisted G-space over F. We have seen in I.3 that the group $\mathrm{Int}\,(L)$ of inner automorphism of L sits in an exact sequence

$$1 \to Z_G \xrightarrow{v} G \times Z_G \to \mathrm{Int}\,(L) \to 1$$

with $v : z \mapsto (z \times z^{1-\theta})$. This shows that the map of complexes

$$[Z_G \xrightarrow{v} G \times Z_G] \to [1 \to \mathrm{Int}\,(L)]$$

is a quasi-isomorphism. A class

$$\alpha \in \mathbf{H}^1(F, Z_G \xrightarrow{v} G \times Z_G) \simeq \mathbf{H}^1(F, \mathrm{Int}\,(L))$$

can be represented by a hyper-cochain $(u \times z, \xi)$ where

$$\xi : (\sigma, \tau) \mapsto \xi_{\sigma,\tau}$$

is a closed 2-cochain on $\Gamma = \mathrm{Gal}(\overline{F}/F)$ with values in Z_G, $u : \sigma \mapsto u_\sigma$ is a 1-cochain on Γ with values in G whose coboundary,

$$\partial u_{\sigma,\tau} := u_\sigma \sigma(u_\tau) u_{\sigma\tau}^{-1} \ ,$$

verifies $\partial u = \xi$, while z is a 1-cochain with values in Z_G such that $\partial z = \xi^{1-\theta}$. When G is reductive, it follows from I.3.1 that

$$\mathbf{H}^1(F, Z_{sc} \xrightarrow{v} G_{SC} \times Z_G) \to \mathbf{H}^1(F, \mathrm{Int}\,(L))$$

is also an isomorphism. This shows that, in this case, a class $\alpha \in \mathbf{H}^1(F, \mathrm{Int}\,(L))$ can be represented by a hyper-cochain $(u \times z, \xi)$ where u and ξ take their value in G_{SC} and Z_{sc} respectively.

A class $\alpha \in \mathbf{H}^1(F, \mathrm{Int}\,(L)))$ defines an inner form of L, as follows. Observe first that the natural map

$$\mathbf{H}^1(F, \mathrm{Int}\,(L)) \to \mathbf{H}^1(F, \mathrm{Int}\,(G))$$

sends α to a class defining an inner form G_α of G. We consider an isomorphism of twisted spaces $\varphi : L_\alpha \to L$ over $\overline{F}$ with a companion isomorphism $\psi : G_\alpha \to G$. The Galois action on L_α is defined so that the cocycle $\sigma \mapsto \varphi\sigma(\varphi)^{-1}$ is in the class α : if α is represented by a hyper-cocycle $(u \times z, \xi)$, for $\sigma \in \Gamma$, $x \in G_\alpha$ and $\delta \in L_\alpha$ we put

$$\psi(\sigma(x)) = u_\sigma\, \sigma(\psi(x))\, u_\sigma^{-1} \qquad \text{and} \qquad \varphi(\sigma(\delta)) = z_\sigma^{-1} u_\sigma\, \sigma(\varphi(\delta))\, u_\sigma^{-1} \ .$$

We shall say that L_α is a central form of L if α is the class defined by a 1-cocycle $\sigma \mapsto z_\sigma$ with values in the center of G acting by translations:

$$\varphi(\sigma(\delta)) = z_\sigma^{-1}\, \sigma(\varphi(\delta)) \ .$$

Let G be a connected reductive group over F. We shall say that L is quasi-split if L is isomorphic over F to $G \rtimes \theta$ where G is quasi-split and θ is an automorphism fixing an F-splitting of G. The following lemma is due to Kottwitz and Shelstad.

III.2.1. Lemma. – *Let G be a connected reductive group over F. Any twisted G-space L is an inner form of a quasi-split twisted space $L^* = G^* \rtimes \theta^*$.*

Proof: Let G^* be the quasi-split inner form of the group G. Choose a 1-cochain u on Γ with values in G^* whose coboundary $\partial u = \xi$ takes its values in the center of G^*, defining G as an inner twist of G^*. Recall that this means there is an isomorphism $\psi : G \to G^*$ over the algebraic closure and the Galois actions on G and G^* are connected by the formula

$$\psi(\sigma(g)) = u_\sigma\, \sigma(\psi(g))\, u_\sigma^{-1}\ .$$

Let $\widetilde{\psi} : \mathrm{Aut}\,(G) \to \mathrm{Aut}\,(G^*)$ be the isomorphism between groups of automorphisms defined by

$$\widetilde{\psi}(\theta) = \psi \circ \theta \circ \psi^{-1}\ .$$

Choose an F-splitting $(B, T, \{X_\alpha\})$ of G^*, then, there is a unique automorphism θ^* in the image of $\widetilde{\psi} \circ \mathrm{Ad}_L$, that preserves the splitting. In particular θ^* is rational and almost semisimple. Choose $\delta_0 \in L$ such that

$$\theta^* = \widetilde{\psi}(\mathrm{Ad}_L(\delta_0))\ .$$

The element δ_0 is only defined up to translation by the center. Consider the quasi-split twisted space $L^* = G^* \rtimes \theta^*$. Consider the isomorphism of twisted spaces over the algebraic closure $\varphi : L \to L^*$ defined by

$$\varphi(x\delta_0) = \psi(x) \rtimes \theta^*\ .$$

For any $\sigma \in \Gamma$ we have $\sigma(\theta^*) = \theta^*$ and hence

$$\mathrm{Ad}_{G^*}(u_\sigma)^{-1} \circ (\mathrm{Ad}_{L^*}(\varphi(\sigma(\delta_0)) \circ \mathrm{Ad}_{G^*}(u_\sigma) = \mathrm{Ad}_{L^*}(\varphi(\delta_0))$$

and this is equivalent to

$$u_\sigma^{-1}\varphi(\sigma(\delta_0))u_\sigma = z_\sigma^{-1}\varphi(\delta_0) = z_\sigma^{-1} \rtimes \theta^*$$

where z is a 1-cochain with values in the center. This implies that

$$\partial z = \xi^{1-\theta^*} \qquad \text{with} \qquad \xi = \partial u\ .$$

A change in the choice of the splitting and of δ_0 modifies the isomorphism φ but the class of $\sigma \mapsto \varphi\sigma(\varphi)^{-1}$ is independent of these choices. The twisted space L is the inner form of L^* defined by this class.

□

III.2.2. Lemma. – *Consider $\delta \in L$, $x \in G^*$ and $\delta^* = x\varphi(\delta)x^{-1} \in L^*$. Then we have $\delta \in L(F)$ if and only if*

$$\sigma(\delta^*) = z_\sigma a_\sigma^{-1} \delta^*\, a_\sigma \qquad \textit{with} \qquad a_\sigma = x\, u_\sigma \sigma(x)^{-1}\ .$$

Proof: If $\sigma(\delta) = \delta$ we have $\varphi(\delta) = z_\sigma^{-1} u_\sigma\, \sigma(\varphi(\delta))\, u_\sigma^{-1}$ and conversely.

□

III.3 – Admissible tori and prenorms

Let G^* be a quasi-split reductive group and θ^* an automorphism that fixes an F-splitting $(B^*, T^*, \{X_\alpha\})$. Let G^1 be the connected component of $(G^*)^{\theta^*}$. The simply connected cover of the derived group of G^1 is denoted G^1_{SC}.

A maximal torus T in G^* will be said to be admissible if T is defined over F and belongs to a θ^*-stable Borel pair (B, T). Let T be a F-torus in G^* and let S be an F-torus with an isomorphism $\psi : S \to T$ over $\overline{F}$. Then for σ in the Galois group $w_\sigma = \psi\sigma(\psi)^{-1}$ is an automorphism of T.

III.3.1. Lemma. – *Let T be an admissible torus. Assume that $w_\sigma = \psi\sigma(\psi)^{-1}$ belongs to the Weyl group $\Omega(G^*, T)$ for all σ in the Galois group and preserves T^{θ^*}. Then, there is an $x \in G^1$ such that $U = xTx^{-1}$ is an admissible torus in G^* such that*

$$\mathrm{Ad}(x) \circ \psi : S \to U$$

is an F-isomorphism.

Proof: This is essentially Corollary 2.2 of [Ko1]. Since w_σ preserves T^{θ^*} lemma II.2.1 shows that w_σ belongs to $\Omega(G^1, T^1)$. Consider T^1_{sc} the preimage of $T^1 = T \cap G^1$ in G^1_{SC}. Using that

$$\Omega(G^1, T^1) \simeq \Omega(G^1_{SC}, T^1_{sc})$$

we get a cocycle with values in $\Omega(G^1_{SC}, T^1_{sc})$ again denoted w_σ. Let S^1_{sc} be the torus defined as the fibered product of S and G^1_{SC} above G^*:

$$S^1_{sc} = S \underset{G^*}{\times} G^1_{SC}$$

Then, since w_σ belongs to $\Omega(G^1_{SC}, T^1_{sc})$ the torus S^1_{sc} is defined over F. The map ψ lifts to a map again denoted ψ between S^1_{sc} and T^1_{sc}. Consider $s \in S^1_{sc}(F)$ and let $t = \psi(s)$; since $w_\sigma = \psi\sigma(\psi)^{-1}$ we have

$$\sigma(t) = w_\sigma^{-1}(t) = n_\sigma^{-1}\, t\, n_\sigma$$

for some $n_\sigma \in G^1_{SC}$. Hence, the conjugacy class of t is rational in G^1_{SC}. Steinberg's theorem on rational points in semisimple conjugacy classes for semisimple simply connected quasi-split groups [St1], shows that there exists $x \in G^1_{SC}$ such that $s_1 = xtx^{-1}$ is rational. Then $x\, n_\sigma\, \sigma(x)^{-1}$ centralize s_1. By abuse of notation we shall use the same letter to denote the image of x in G^1. Choose s so that t is strongly regular, then the centralizer of s_1 in G^* is a torus U defined over F since s_1 is rational. Hence we have $x\, n_\sigma\, \sigma(x)^{-1} \in U$. This implies that $\mu = \mathrm{Ad}(x) \circ \psi$ verifies $\sigma(\mu) = \mu$ which means that U is F-isomorphic to S. Since $U = xTx^{-1}$ with $x \in G^1$ it is admissible.

□

Let L be an inner form of the quasi-split space $L^* = G^* \rtimes \theta^*$. We consider as above an isomorphism $\varphi : L \to L^*$ over the algebraic closure, the twisting of the Galois action being given by a hyper-cochain $(u \times z, \xi)$.

III.3.2. Definition. – *We say that a pair (δ^*, T) is L-admissible if $\delta^* = t \rtimes \theta^* \in T \rtimes \theta^*$ where T is an admissible torus in G^* and if there is a 1-cochain $\sigma \mapsto b_\sigma$ with values in T such that*

$$\sigma(\delta^*) = z_\sigma\, b_\sigma^{-1}\, \delta^*\, b_\sigma\ .$$

In other words, we have

$$\delta^* \in T \rtimes \theta^* \subset B \rtimes \theta^* \subset L^*$$

and the T-conjugacy class of δ^* is rational for the natural F-structure on $T \rtimes \theta^*$ twisted by z. We observe that L is only involved through the cochain z, and that the cochain b is uniquely defined by δ^* and z, up to multiplication by a 1-cochain with values in T^{θ^*}. Observe also that $T^{\theta^*} = T^{\delta^*}$.

Since the cochain b takes its values in T then, although δ^* may not be rational, the group J_{δ^*} is defined over F. This can be seen also as follows: we have $\delta^* = t \rtimes \theta^*$ and

$$t\sigma(t)^{-1} = b_\sigma^{1-\theta^*} z_\sigma^{-1}$$

and hence

$$(t\sigma(t)^{-1})^{N\alpha} = 1 \qquad \text{or, equivalently} \qquad t^{N\alpha} = \sigma(t)^{N\alpha}$$

for all $\alpha \in R(G^*,T)$. By definition $\alpha \in P_{\delta^*}$ means that $t^{N\alpha} = \epsilon_\alpha$ and hence we also have $\sigma(t)^{N\alpha} = \epsilon_\alpha$ for any element σ in the Galois group Γ. Recall that $\epsilon_\alpha = 1$ unless α is of type 3 where $\epsilon_\alpha = -1$. The Galois action preserves the type of roots. This shows that P_{δ^*} is stable under the Galois action and again we see that J_{δ^*} is defined over F.

Consider $\delta \in L(F)$ such that there is $x \in G^*$ with $x\varphi(\delta)x^{-1} = \delta^*$ where (δ^*,T) is L-admissible. By definition of admissibility there is a 1-cochain b such that

$$\sigma(\delta^*) = z_\sigma\, b_\sigma^{-1}\, \delta^*\, b_\sigma\ .$$

Now since δ is rational $a_\sigma = xu_\sigma\sigma(x)^{-1}$ is such that

$$\sigma(\delta^*) = z_\sigma\, a_\sigma^{-1}\, \delta^*\, a_\sigma$$

(cf. III.2.2) and hence $a_\sigma = c_\sigma b_\sigma$ where c_σ centralizes δ^*. We shall define prenorms using a slightly stricter condition.

III.3.3. Definition. – *We shall say that an L-admissible pair (δ^*,T) attached to a 1-cochain b is a prenorm for $\delta \in L(F)$ if there exists $x \in G^*$ and a torus $T(\delta)$ in G defined over F and that belongs to a δ-stable Borel pair, such that*

$$x\varphi(\delta)x^{-1} = \delta^* \qquad x\psi(T(\delta))x^{-1} = T \qquad \text{and} \qquad a_\sigma = xu_\sigma\sigma(x)^{-1} = c_\sigma b_\sigma$$

with $c_\sigma \in I_{\delta^*}$. In particular the cochain a takes its values in J_{δ^*}. We shall say that it is a strict prenorm if a_σ takes its values in T.

III.3.4. Proposition. – *Any almost semisimple element $\delta \in L(F)$ has a strict prenorm.*

Proof: The connected centralizer G_δ of δ is a reductive group defined over F. Choose a Borel pair (B_δ,T_δ) in G_δ with T_δ defined over F. The centralizer $T(\delta)$ of T_δ in G is a maximal torus defined over F that belongs to a δ-stable Borel pair $(B(\delta),T(\delta))$ in G such that $B_\delta = G_\delta \cap B(\delta)$. This is due to Steinberg [St2] (see also [KS, Thm 1.1.A]). The image via ψ of the Borel pair $(B(\delta),T(\delta))$ is conjugate to the Borel pair (B^*,T^*) in G^*:

$$x_1\,\psi((B(\delta),T(\delta)))\,x_1^{-1} = (B^*,T^*)\ .$$

Let G^1 be the connected component of $(G^*)^{\theta^*}$ and $T^1 = T^* \cap G^1$. Then $\mathrm{Ad}(x_1)\circ\psi$ induces an isomorphism

$$\psi_1 : T_\delta \to T^1 \qquad \text{with} \quad \psi_1\sigma(\psi_1)^{-1} = w_\sigma \in \Omega(G^*,T^*)^{\theta^*} \simeq \Omega(G^1,T^1)$$

since θ^* preserves a splitting. Now III.3.1 shows there exists $y \in G^1$ such that $y\,T^1\,y^{-1}$ is a torus isomorphic to T_δ over F. Let $x = yx_1$, then $\mathrm{Ad}(x)\circ\psi$ is an F-isomorphism between $T(\delta)$ and $T = yT^*y^{-1}$. This implies that the cochain $\sigma \mapsto a_\sigma := xu_\sigma\sigma(x)^{-1}$ takes its values in T. Moreover, since y belongs to G^1

$$(B,T) := y\,(B^*,T^*)\,y^{-1}$$

is a θ^*-stable Borel pair. Let $\delta^* = x\varphi(\delta)x^{-1}$ then

$$\sigma(\delta^*) = z_\sigma\, a_\sigma^{-1}\, \delta^*\, a_\sigma\ .$$

This shows that (δ^*,T) is a strict prenorm.

□

Two L-admissible pairs (δ^*, T) and (η^*, S) associated to cochains b and d will be said to be equivalent if there is $y \in G^*$ such that

$$\eta^* = y^{-1}\,\delta^*\,y \qquad \text{and} \qquad y\;d_\sigma\sigma(y)^{-1} = e_\sigma b_\sigma$$

with $e_\sigma \in I_{\delta^*}$. Observe that $c \in I_{\delta^*}$ implies $b_\sigma\sigma(c)b_\sigma^{-1} \in I_{\delta^*}$. Thus b defines a F-structure on I_{δ^*}. Two conjugate L-admissible pairs (δ^*, T) and (η^*, S) are equivalent, if and only if the stable centralizers I_{δ^*} and I_{η^*} with the F-structures defined by b and d are inner forms. We say they are strongly equivalent if moreover one can choose y so that $S = y^{-1}\,T\,y$.

III.3.5. Lemma. – *Equivalent pairs are strongly equivalent*

Proof: Let (δ^*, T) and (η^*, S) be L-admissible equivalent pairs. Consider $y \in G^*$ such that

$$\eta^* = y^{-1}\,\delta^*\,y \qquad \text{and} \qquad y\;d_\sigma\sigma(y)^{-1} = e_\sigma b_\sigma$$

with $e_\sigma \in I_{\delta^*}$. Since $T^{\theta^*} = T^{\delta^*}$ and $S^{\theta^*} = S^{\eta^*}$, the tori $y\,S\,y^{-1} \cap I^0_{\delta^*}$ and $T \cap I^0_{\delta^*}$ are maximal tori in $I^0_{\delta^*}$ and hence are conjugate by $v \in I^0_{\delta^*} \subset I_{\delta^*}$:

$$v(y\,S\,y^{-1} \cap I^0_{\delta^*})v^{-1} = T \cap I^0_{\delta^*}\ .$$

Now let $y_1 = vy$. Since T is the centralizer of $T \cap I^0_{\delta^*}$ we have $y_1 S y_1^{-1} = T$ and $y_1\eta^* y_1^{-1} = \delta^*$ moreover

$$y_1 d_\sigma\sigma(y_1)^{-1} = ve_\sigma b_\sigma\sigma(v)^{-1} = ve_\sigma(b_\sigma\sigma(v)^{-1}b_\sigma^{-1})b_\sigma\ .$$

To conclude we recall that v and e_σ belong to I_{δ^*} and we observe that since $b_\sigma\sigma(\delta^*)b_\sigma^{-1} = z_\sigma\delta^*$ we have $b_\sigma\sigma(I_{\delta^*})b_\sigma^{-1} = I_{\delta^*}$ and in particular $b_\sigma\sigma(v)b_\sigma^{-1}$ belongs to I_{δ^*}.

$\square$

III.3.6. Lemma. – *Two elements δ and η in $L(F)$ are stably conjugate if and only if they have equivalent prenorms.*

Proof: Assume that δ and η in $L(F)$ have equivalent prenorm (δ^*, T) and (η^*, S). Let $\delta^* = t\rtimes\theta^*$ and $\eta^* = s\rtimes\theta^*$. Then there is x , x' and y in G^* with

$$x\,\varphi(\delta)\,x^{-1} = \delta^* \quad , \qquad y^{-1}\,\delta^*\,y = \eta^* \qquad \text{and} \qquad \eta^* = x'\,\varphi(\eta)\,(x')^{-1}$$

the cochains $a_\sigma = x\,u_\sigma\sigma(x\,)^{-1}$ and $a'_\sigma = x'u_\sigma\sigma(x')^{-1}$ being such that

$$a_\sigma = c_\sigma b_\sigma \qquad \text{and} \qquad a'_\sigma = c'_\sigma b'_\sigma$$

with $b_\sigma \in T$, $b'_\sigma \in S$ and

$$y\;b'_\sigma\sigma(y)^{-1} = e_\sigma b_\sigma$$

where c_σ, $y\,c'_\sigma y^{-1}$ and e_σ belongs to I_{δ^*}. Let $p = x^{\,-1}y\,x'$, then

$$p\,u_\sigma\sigma(p)^{-1}u_\sigma^{-1} = x^{\,-1}(y\,a'_\sigma\,\sigma(y)^{-1})\,a_\sigma^{-1}x\, = x^{\,-1}[(y\,c'_\sigma\,y^{-1})e_\sigma^{-1}c_\sigma^{-1}]x$$

belongs to

$$x^{\,-1}(I_{\delta^*})x\, = \psi(I_\delta)\ .$$

Now if $p = \psi(r)$ we have $\eta = r^{-1}\delta r$ and

$$\psi(r\sigma(r)^{-1}) = p\,u_\sigma\sigma(p)^{-1}u_\sigma^{-1} \in \psi(I_\delta)$$

which shows that δ and η are stably conjugate. Conversely, consider two stably conjugate elements δ and η in $L(F)$: there is $r \in G$ such that $\eta = r^{-1}\delta r$ with $r\sigma(r)^{-1} \in I_\delta$. Now III.3.4 shows that prenorms (δ^*, T) and (η^*, S) exist. Reversing the above computation we see that they are equivalent, the equivalence being induced by

$$y = x\,\psi(r)\,(x')^{-1}\ .$$

$\square$

III.4 – Special pairs

Consider a twisted space $L = G \rtimes \theta$ and an almost semisimple point $\delta \in L$. Not to overload notation let $J := J_\delta$ and denote by Z_J its center.

III.4.1. Definition. – *We shall say that $\delta \in L$ is special in L, if J is defined over F and if there is a 1-cochain $\sigma \mapsto \zeta_\sigma$ with values in Z_J, such that*

$$\sigma(\delta) = \zeta_\sigma \, \delta \ .$$

III.4.2. Lemma. – *If $\delta \in L$ is special, the stable centralizer I_δ is defined over F. If moreover there is a cochain a with values in J and a cochain z with values in Z_J such that*

$$\sigma(\delta) = z_\sigma \, a_\sigma^{-1} \delta a_\sigma$$

then a_σ normalizes I_δ.

Proof: If $c \in J$ centralizes δ then $\sigma(c)$ centralizes $\sigma(\delta)$ and since δ is special

$$\sigma(c) \, \zeta_\sigma \, \delta \sigma(c^{-1}) = \zeta_\sigma \, \delta \ .$$

But since ζ is in the center of J which is defined over F we see that $\sigma(c)$ centralizes δ. This shows that the centralizer of δ in J_δ is defined over F. Since the connected centralizer G_δ is a subgroup of J_δ it is defined over F. Now, since Z_L is defined over F so is I_δ. Now if

$$\sigma(\delta) = z_\sigma \, a_\sigma^{-1} \delta a_\sigma$$

and if c centralizes δ, we have

$$z_\sigma \, a_\sigma^{-1} \delta a_\sigma = \sigma(\delta) = c \sigma(\delta) c^{-1} = z_\sigma \, c \, a_\sigma^{-1} \delta a_\sigma c^{-1}$$

and hence a_σ normalizes the centralizer and hence also the stable centralizer.

□

Consider an inner form L of L^*. We say that an L-admissible pair (δ^*, T) is special if δ^* is special in L^*.

III.4.3. Lemma. – *Let (δ^*, T) be a L-admisible pair. Consider the properties:*

(i) The pair is special.

(ii) The stable centralizer I_{δ^*} is defined over F.

(iii) There is a cochain β_σ with values in Z_J such that $\sigma(\delta^*) = z_\sigma \, \beta_\sigma^{-1} \, \delta^* \, \beta_\sigma$.

The three properties are equivalent.

Proof: Lemma III.4.2 shows that (i) implies (ii). Since (δ^*, T) is L-admisible one has

$$\sigma(\delta^*) = z_\sigma \, b_\sigma^{-1} \, \delta^* \, b_\sigma \ .$$

where b_σ takes its values in T. Let $c \in I_{\delta^*}$ then $b_\sigma \sigma(c) b_\sigma^{-1}$ centralize δ^* and hence it belongs to I_{δ^*}. If I_{δ^*} is defined over F this shows that b_σ normalize I_{δ^*} and since b_σ takes its values in T it induces an inner automorphism of I_{δ^*}. But then II.2.4 shows that one may write $b = c\beta$ where β takes its values in Z_J and c takes its values in I_{δ^*}. This shows that (ii) implies (iii). Clearly (iii) implies (i).

□

III.4.4. Lemma. – *Let G^* be an absolutely simple quasi-split group over F, and θ^* an automorphism that preserves a splitting $(B, T, ...)$ over F. Let $\delta^* = t \rtimes \theta^* \in T \rtimes \theta^*$ such that $G^* = J_{\delta^*}$. Then up to T-conjugacy we may choose δ^* such that*

$$\sigma(\delta^*) = \delta^* \qquad \textit{or} \qquad \sigma(\delta^*) = \overline{\delta^*}$$

where $\overline{\delta^*} := t^{-1} \rtimes \theta^*$, for any $\sigma \in \mathrm{Gal}(\overline{F}/F)$.

Proof: We have proved in lemma II.3.2 that, up to T-conjugacy, we may assume $(\delta^*)^\ell = 1$ and $\delta^* = \theta^*(\delta^*)$. But then $\delta^* = t \rtimes \theta^*$ is such that $t^\ell = 1$ and the t^α are ℓ-th roots of unity for all α. If $\delta^* = 1 \rtimes \theta^*$, in particular if $\ell = 1$, the assertion is trivial. Assume now $\delta^* \neq 1 \rtimes \theta^*$. Assume first that G^* is split. Clearly, $\delta^* = t \rtimes \theta^*$ is rational if $\ell = 2$. If $\ell = 3$ and $t \neq 1$ the θ^*-fixed simple root is such that $j = t^\alpha$ is a primitive 3rd-root of unity, while we may assume that $t^\beta = 1$ if β is a non fixed simple root. The unique θ^*-fixed simple root must be Galois invariant hence

$$\sigma(t)^\alpha = \sigma(t^\alpha) = \sigma(j)$$

for all σ. This shows that $\sigma(\delta^*) = \delta^*$ or $\overline{\delta^*} = t^{-1} \rtimes \theta^*$ since $\sigma(j) = j^{\pm 1}$. The same holds if G^* is quasi-split, but non split, since the twist is by powers of θ^* and they fixe δ^*.

□

III.4.5. Proposition. – *Assume that G^* is absolutely simple quasi-split. Any L-admissible pair is equivalent to a special pair.*

Proof: Let (δ_1, T_1) be an L-admissible pair and put $J_1 = J_{\delta_1}$. Let $J^* \rtimes \theta_J^*$ be the quasi-split inner form of $J_1 \rtimes \theta^*$. Let S be a torus in a θ_J^*-stable F-splitting for J^*. The isomorphism over the algebraic closure

$$\psi_J : J^* \to J_1$$

can be chosen so that $T_1^{\theta^*} = \psi_J(S^{\theta_J^*})$. Hence $\psi_J \sigma(\psi_J)^{-1} = \mathrm{Ad}_{J_1}(n_\sigma)$ with n_σ in the normalizer of $T_1^{\theta^*}$ in $J_1 \subset G^*$. Using III.3.1 we get $y_1 \in G^1$ such that $T = y_1\, T_1\, y_1^{-1}$ is F-isomorphic to S. The group $J := y_1 J_1 y_1^{-1}$ is also defined over F and is quasi-split since $T \subset J$ is F-isomorphic to S. Now let $\delta^* = y_1 \delta_1\, y_1^{-1}$ we have $\delta^* = t \rtimes \theta^* \in T \rtimes \theta^*$. Let

$$a_\sigma := y_1\, b_\sigma \sigma(y_1)^{-1}$$

then

$$\sigma(\delta^*) = z_\sigma\, a_\sigma^{-1}\, \delta^*\, a_\sigma\ .$$

Since

$$\mathrm{Ad}_{G^*}(y_1 \sigma(y_1)^{-1}) = \mathrm{Ad}_{G^*}(y_1 n_\sigma^{-1} y_1^{-1})$$

we see that $y_1 \sigma(y_1)^{-1}$ belongs to the normalizer of T in J. Now, since

$$a_\sigma = (y_1\, b_\sigma\, y_1^{-1})\, y_1 \sigma(y_1)^{-1}$$

we see that a_σ also belongs to the normalizer of T in J. The automorphism θ^* stabilizes J and preserves a Borel pair (B_J, T) in J. Denote by θ_J an automorphism of J that differs from θ^* restricted to J by an inner automorphism and fixes a splitting in J containing (B_J, T):

$$\theta^* = \mathrm{Ad}_J(t_J) \circ \theta_J \qquad \text{for some } t_J \in T.$$

It allows to consider δ^* as an element in the quasi-split twisted space $J \rtimes \theta_J$. Since G^* is assumed to be simple the order of θ^* and hence of θ_J is prime. Consider the quotient of $J \rtimes \theta_J$ by the center Z_J of J and decompose this quotient into irreducible factors. We have classified the irreducible factors in II.3.2. Assume for a while that there is no factors of type (4b); then III.4.4 shows that

$$\sigma(\delta^*) = \zeta_\sigma \delta^*$$

with $\zeta_\sigma \in Z_J$. This means that $\delta^* = t \rtimes \theta^*$ defines a special pair (δ^*, T). This implies that

$$z_\sigma \, a_\sigma^{-1} \, \delta^* \, a_\sigma = \zeta_\sigma \delta^* \ .$$

Hence, III.4.2 shows that the group I_{δ^*} is defined over F and that a_σ belongs to the normalizer of I_{δ^*} in J. The cochain $\sigma \mapsto a_\sigma$ defines an other rational structure on I_{δ^*} say I. Observe also that an element of J that induces a trivial automorphism of I_{δ^*} must belong to the center Z_J of J and that $Z_J \subset T$. Assume first that I_{δ^*} is an inner form of I, then II.2.4 and the preceeding remark show that the pair (δ^*, T) is admissible and equivalent to $({\delta^*}_1, T_1)$; the assertion is proved in this case. Now consider the case where I_{δ^*} is not an inner form of I: the cocycle $\sigma \mapsto a_\sigma$ induces a non trivial outer twist of the Galois action on I_{δ^*} induced by elements in J. According to the classification II.3.3 a non trivial outer automorphism of I_{δ^*} induced by an element of J can occur only for factors of type (2c) or (3) with $n_+ = n_-$ (recall that (4b) is excluded). Since G is simple, a case by case inspection shows that at most one such factor can occur and hence it is defined over F. The twist involves a quadratic extension and an outer automorphism that can be defined by a Weyl group element of order 2, that is induced by an element in the centralizer of the image of δ^* in the adjoint group of J. In fact, in case (2c), for a factor $PGL(2m)$ the centralizer is the image in $PGL(2m)$ of $O(2m)$ while the stable centralizer is the image of $SO(2m)$. In case (3) with $m = n_+ = n_-$ the factor is the projective image of $SO(4m+2)$ while the stable centralizer is the image of $SO(2m+1) \times SO(2m+1)$; but the centralizer also contains an element whose adjoint action flips the two $SO(2m+1)$ subgroups. This shows that in fact $a_\sigma = n'_\sigma \zeta'_\sigma$ where $n'_\sigma \in J$ normalizes the torus T and centralizes δ^* while ζ'_σ belongs to Z_J. Again using III.3.1 we see there is $y \in G^1$ such that $\mathrm{Ad}(y^{-1}\sigma(y)) = \mathrm{Ad}(n'_\sigma)$ and the conjugate pair $y\,(\delta^*, T)\,y^{-1}$ has the expected properties. We are left to study the case of a factor of type (4b): i.e. a factor of type D_4, $\ell = 3$ and $\delta^* \neq 1 \rtimes \theta^*$. There is an outer automorphism of the stable centralizer induced by s_0 the symmetry with respect to the θ-stable simple root α_0; in particular $t^{\alpha_0} = j$ and $s_0(t)^{\alpha_0} = j^{-1}$. At worst δ^* is rational in J_{ad} on a quadratic extension E/F. Let σ be the non trivial element of the Galois group of E/F. There are two possibilities: either $\sigma(j) = j$ or $\sigma(j) = j^{-1}$. In the first case δ^* is rational over F, and hence a takes its values in the centralizer. But the centralizer of δ^* in the adjoint group of type D_4 is connected. Hence the twisting induced by a is inner and II.2.4 shows that the pairs $({\delta^*}_1, T_1)$ and (δ^*, T) are equivalent. In the second case δ^* is rational for the Galois action twisted by s_0. Using III.3.1 we see as above that there is $y \in G^1$ such that the pair $({\delta^*}_1, T_1)$ is equivalent to the pair $y\,(\delta^*, T)\,y^{-1}$.

□

Remark – Our construction yields a special point whose stable centralizer I_{δ^*} is quasi-split.

III.4.6. Theorem. – *Any L-admissible pair (δ^*, T) is equivalent to a special pair.*

Proof: To prove the theorem we only need to study the adjoint actions and hence we may assume that G is adjoint. This allows us to decompose L^* into a product of irreducible spaces. Then the group is a product of simple groups and we are reduced to consider the case where the Dynkin diagram of G^*, over the algebraic closure, is a single orbit of irreducible Dynkin diagrams under the group generated by θ and the Galois group $\Gamma = \mathrm{Gal}(\overline{F}/F)$. Lemma I.3.2 and Shapiro's lemma reduce us to study the "primitive case" where both θ^* and Γ act transitively on the set of irreducible Dynkin diagram. We assume from now on that G^* is of this type with G_0^* absolutely simple quasi-split. According to I.3.3 we may assume L^* induced from $L_0^* = G_0^* \rtimes \theta_0^*$. Denote by ℓ_0 the order of θ_0^* and by m the number of factors; then $\ell = \ell_0 m$ is the order of θ^*. Start with an L-admissible pair $(t \rtimes \theta^*, T)$ for L^*; since T is admissible we have

$$T = T_0 \times \cdots \times T_0 \ .$$

Then I.3.2 shows that up to equivalence we may take $t = (1,\cdots,1,t_0)$ where $(t_0\rtimes\theta_0^*, T_0)$ is an admissible pair in $G_0^*\rtimes\theta_0^*$. Up to a further equivalence, III.4.5 shows that we may assume that this pair is special for $G_0^*\rtimes\theta_0^*$. As usual denote by J the group generated by the stable centralizer and the torus; it is of the form

$$J = J_0\times\cdots\times J_0\ .$$

Denote by $\tilde t$, $\tilde T$, ... the image of t, T, ... in the adjoint group J_{ad} of J. Let θ be the induced automorphism on $\tilde T$, then I.3.3 shows that the Galois action on $\tilde T$ is of the form

$$\sigma : (\tilde t_1,\cdots,\tilde t_m)\mapsto \theta^{n(\sigma)}(\sigma_0(\tilde t_1),\cdots,\sigma_0(\tilde t_m))$$

where $n:\sigma\mapsto n(\sigma)$ is a homomorphism

$$n:\Gamma\to\mathbb{Z}/\ell\mathbb{Z}$$

and $\tilde t_0\mapsto\sigma_0(\tilde t_0)$ is a Galois action on $\tilde T_0$ that commutes with θ_0. Since we have assumed that Γ permutes transitively the irreducible components of the Dynkin diagram of G^* the map $\sigma\mapsto n(\sigma)$ has an image generated by $\ell_2\in\mathbb{Z}$ where ℓ_2 is prime to m. Then $\ell_0=\ell_1\ell_2$, but since $\ell_0 = 1$, 2 or 3, either $\ell_2=\ell_0$ and m is prime to ℓ_0, or $\ell_2 = 1$. We shall denote by E the fixed field under the kernel of $\sigma\mapsto n(\sigma)$; this is a cyclic extension of degree $\ell_1 m$ of F. Since we were considering an L-admissible pair that yields a special pair for $G_0^*\rtimes\theta_0^*$ we may take $\tilde t\in\tilde T\subset J_{ad}$ with:

$$\tilde t = (1,\cdots,1,\tilde t_0)\quad ,\qquad \tilde t_0 = \sigma_0(\tilde t_0)\qquad\text{and}\qquad \theta_0(\tilde t_0) = \tilde t_0\ .$$

We observe that $(1,\cdots,1,\tilde t_0)$ is θ-conjugate in $\tilde T$ to τ with

$$\tau = (\sigma_0^{(m-1)}(\tau_1),\cdots,\sigma_0(\tau_1),\tau_1)$$

provided that $\tau_1\in\tilde T_0(E)$ is such that

$$\tau_1\,\sigma_0(\tau_1)\cdots\sigma_0^{m-1}(\tau_1) = \tilde t_0\qquad\text{and}\qquad \sigma_0^m(\tau_1) = \tau_1\ .$$

If $\ell_2=\ell_0$ then m is prime to ℓ_0 and we may take $\tau_1 = \tilde t_0^v$ with $vm\equiv 1 \bmod \ell_0$. Since $\tilde t_0 = \sigma(\tilde t_0)$ and $\tilde t_0 = \theta_0^*(\tilde t_0)$, the same is true for $\tau_1 = \tilde t_0^v$. This shows that $\tau = \sigma(\tau)$ when $\ell_2 = \ell_0$. Assume now that $\ell_2 = 1$, and choose σ with $n(\sigma) = \ell_2 = 1$. Let E_0 be the subfield of E fixed by σ^m. The Galois group of E/E_0 is cyclic of order $\ell_0 = 1$, 2 or 3. We know that $\tilde t_0^{\ell_0} = 1$. This shows that $\sigma^m(\tilde t_0) = \tilde t_0$ and that $N_{E/E_0}(\tilde t_0) = 1$. Hilbert's theorem 90 tells us that for any $\eta_0\in E$ with $N_{E/E_0}(\eta_0) = 1$ there exists $\eta\in E$ such that

$$\eta\sigma^m(\eta)^{-1} = \eta_0\ .$$

Now let $\eta_1 = \eta\sigma(\eta)^{-1}$ then

$$\eta_1\sigma(\eta_1)\cdots\sigma^{m-1}(\eta_1) = \eta_0$$

and

$$\sigma^m(\eta_1) = \sigma^m(\eta)\sigma^{m+1}(\eta)^{-1} = \eta_0^{-1}\eta\sigma(\eta_0^{-1}\eta)^{-1} = \eta_1\ .$$

This shows that one can find $\xi\in\tilde T_0(E)$ such that $\xi = \theta(\xi)$ and $\xi/\sigma_0^m(\xi) = \tilde t_0$. Now we get a solution by taking

$$\tau_1 = \xi\,\sigma_0(\xi)^{-1}\ .$$

It is enough to prove that $\tau = \sigma(\tau)$ when $n(\sigma) = \ell_2 = 1$. But, in this case

$$\tau\sigma(\tau)^{-1} = (1,\cdots,1,\tau_1\theta_0(\tau_1)^{-1})$$

and we are left to observe that by construction $\tau_1 = \theta_0(\tau_1)$. Then $(\tau\rtimes\theta,\tilde T)$ is the image in J_{ad} of a special L-admissible pair equivalent to the given one.

□

III.4.7. Corollary. – *Any $\delta \in L(F)$ has a special prenorm (δ^*, T). The stable centralizer I^* of δ^* is an inner form of I_δ.*

Proof: That any $\delta \in L(F)$ has a special prenorm (δ^*, T) is an immediate consequence of III.3.4, III.3.5 and III.4.6. The groups I_δ and I^* are isomorphic over the algebraic closure

$$x\,\psi(I_\delta)\,x^{-1} = I^*$$

and the Galois actions differ by a twist given by the cochain $a_\sigma = x\,u_\sigma\,\sigma(x)^{-1}$. But III.4.3 shows that $a_\sigma = c_\sigma \beta_\sigma$ with $c_\sigma \in I^*$ and β_σ belongs to Z_J and hence the twist is inner.

□

IV – Endoscopy

IV.1 – Some cohomological results

Let F be a field and L be a twisted space. Let δ be an almost semisimple element in $L(F)$ and let I be the stable centralizer of δ. Let S_{sc} be a maximal torus in I_{SC} the simply connected cover of the derived group of I. Let S be the centralizer of the image of S_{sc} in I and let T be the centralizer of S in G. The torus T is a δ-stable maximal torus. The crossed set associated to the commutative diagram

$$\left[\begin{tikzcd} I_{SC} \arrow[r] \arrow[d] & G_{SC} \arrow[d] \\ I \arrow[r] & G \end{tikzcd}\right]$$

is quasi-isomorphic to the sub-bicomplex of diagonalizable groups

$$\left[\begin{tikzcd} S_{sc} \arrow[r] \arrow[d] & T_{sc} \arrow[d] \\ S \arrow[r] & T \end{tikzcd}\right]$$

and hence it defines an element in $\mathcal{DT}$ denoted $[I_{ab} \to G_{ab}]$. The abelianized cohomology group for $I\backslash G$ defined by

$$\mathbf{H}^*_{ab}(F, I\backslash G) := \mathbf{H}^*(F, I_{ab} \to G_{ab})$$

can be computed using the above bicomplex of abelian groups or better using a quasi-isomorphic complex of tori.

Let F be a global field. For the definition and the properties of adelic cohomology the reader is refered to [LBC]. Let us consider

$$\mathfrak{D}(I, G; \mathbb{A}_F) = \text{ coker } [\mathbf{H}^0(\mathbb{A}_F, G) \to \mathbf{H}^0(\mathbb{A}_F, I\backslash G)]\ .$$

In this non abelian setting the cokernel is defined to be the quotient of the pointed set $\mathbf{H}^0(\mathbb{A}_F, I\backslash G)$ by the group $G(\mathbb{A}_F) = \mathbf{H}^0(\mathbb{A}_F, G)$ acting by right translations. Observe that one has also

$$\mathfrak{D}(I, G; \mathbb{A}_F) = \ker[\mathbf{H}^1(\mathbb{A}_F, I) \to \mathbf{H}^1(\mathbb{A}_F, G)]\ .$$

We also need the variants where the ring of adeles $\mathbb{A}_F$ is replaced by a global field F or a finite product of local fields, and their abelianized avatars

$$\mathfrak{E}(I, G; \mathbb{A}_F) = \text{ coker } [\mathbf{H}^0_{ab}(\mathbb{A}_F, G) \to \mathbf{H}^0_{ab}(\mathbb{A}_F, I\backslash G)]$$

etc... But, in the case of adele classes we put

$$\mathfrak{E}(I,G;\mathbb{A}_F/F) := \text{ coker } [\mathbf{H}^0_{ab}(\mathbb{A}_F,G) \to \mathbf{H}^0_{ab}(\mathbb{A}_F/F,I\backslash G)] \ .$$

Notice that this is $\mathbf{H}^0_{ab}(\mathbb{A}_F,G)$ and not $\mathbf{H}^0_{ab}(\mathbb{A}_F/F,G)$ we use here.

IV.1.1. Lemma. – *Let F be a global field. The localization map*

$$\mathbf{H}^0_{ab}(F,I\backslash G) \to \mathbf{H}^0_{ab}(F_\infty,I\backslash G)$$

has a dense image.

Proof: Choose S and T as above so that S_{sc} is a fundamental torus in I_{SC}. Recall that S is the centralizer of S_{sc} in I and $S = T^{\theta^*}$ where T is the centralizer of S in G. We have an exact sequence

$$\mathbf{H}^0_{ab}(F_\infty,S\backslash G) \to \mathbf{H}^0_{ab}(F_\infty,I\backslash G) \to \mathbf{H}^2(F_\infty,S_{sc}) \ .$$

But $\mathbf{H}^2(F_\infty,S_{sc})$ is trivial for a fundamental torus S_{sc} (see the appendix of [Ko3]) and hence the first map is surjective. Let $V = T/S$; up to a shift due to different conventions for degrees in hyper-cohomology, Lemma C.5.A of [KS] shows that the map

$$\mathbf{H}^0(F,T_{sc}\to V) \to \mathbf{H}^0(F_\infty,T_{sc}\to V)$$

has dense image. In our notation this says that the map

$$\mathbf{H}^0_{ab}(F,S\backslash G) \to \mathbf{H}^0_{ab}(F_\infty,S\backslash G)$$

has dense image. To conclude it suffices to consider the commutative diagram

$$\begin{tikzcd}
\mathbf{H}^0_{ab}(F,S\backslash G) \arrow[r] \arrow[d] & \mathbf{H}^0_{ab}(F,I\backslash G) \arrow[d] & \\
\mathbf{H}^0_{ab}(F_\infty,S\backslash G) \arrow[r] & \mathbf{H}^0_{ab}(F_\infty,I\backslash G) \arrow[r] & \mathbf{H}^2(F_\infty,S_{sc})
\end{tikzcd}$$

□

IV.1.2. Lemma. – *Let $\mathbb{A}_F^\infty$ be the ring of finite adèles of F. Then we have an exact sequence.*

$$\mathbf{H}^0(\mathbb{A}_F^\infty,I\backslash G) \to \mathfrak{E}(I,G;\mathbb{A}_F/F) \to \ker^1_{ab}(F,I\backslash G) \to 1 \ .$$

Proof: According to [LBC] section 1.8, we have an exact sequence

$$\mathfrak{E}(I,G;F) \to \mathfrak{E}(I,G;\mathbb{A}_F) \to \mathfrak{E}(I,G;\mathbb{A}_F/F) \to \ker^1_{ab}(F,I\backslash G) \to 1$$

and a surjective morphism

$$\mathbf{H}^0(\mathbb{A}_F,I\backslash G) \to \mathfrak{E}(I,G;\mathbb{A}_F)$$

It suffices to invoque IV.1.1 and the finiteness of the local groups $\mathfrak{E}(I,G;F_v)$

□

We say that I is L-elliptic if the quotient I/Z_L contains an F-anisotropic maximal torus. We say that $\delta \in L(F)$ is elliptic if its stable centralizer I is elliptic. Consider the group

$$D(I,G) = \text{ coker } [\mathbf{H}^1_{ab}(\mathbb{A}_F/F, I) \to \mathbf{H}^1_{ab}(\mathbb{A}_F/F, G)] \ .$$

This is a finite group; we denote by $d(I,G)$ its order.

IV.1.3. Lemma. – *Let I_δ be the stable centralizer of an elliptic element $\delta \in L(F)$. The number $d(I_\delta, G)$ is independent of δ. It will be denoted $d(L)$.*

Proof: Let δ and η be such that $I_\delta \subset I_\eta$. Let S be the centralizer in I_δ of a maximal torus of its neutral component I^0_δ; let R be its image in I_η. Let S_{sc} and R_{sc} be the preimage of S and R in the simply connected cover of the derived groups of the stable centralizers. The map $S \to R$ is an isomorphism and hence there is an isomorphism

$$\mathbf{H}^1_{ab}(\mathbb{A}_F/F, I_\delta \to I_\eta) \xrightarrow{\sim} \mathbf{H}^2(\mathbb{A}_F/F, S_{sc} \to R_{sc}) \ .$$

Since δ is elliptic we may choose S so that S_{sc} and R_{sc} are anisotropic; then

$$\mathbf{H}^2(\mathbb{A}_F/F, S_{sc} \to R_{sc})$$

is trivial. This remark and the exactness of the sequence

$$\mathbf{H}^1_{ab}(\mathbb{A}_F/F, I_\delta) \to \mathbf{H}^1_{ab}(\mathbb{A}_F/F, I_\eta) \to \mathbf{H}^1_{ab}(\mathbb{A}_F/F, I_\delta \to I_\eta)$$

shows that the map

$$\mathbf{H}^1_{ab}(\mathbb{A}_F/F, I_\delta) \to \mathbf{H}^1_{ab}(\mathbb{A}_F/F, I_\eta)$$

is surjective. As a consequence we get that

$$d(I_\delta, G) = d(I_\eta, G) \ .$$

To finish the proof we shall use the existence of special prenorms (III.4.7). The abelianized cohomology is insensitive to inner twistings and hence we may replace $\delta \in L$ by a special prenorm $\delta^* \in L^*$ without changing this number: in fact, although δ^* may not be rational, its centralizer is defined over F and is an inner form of I_δ and hence

$$d(I_\delta, G) = d(I_{\delta^*}, G^*) \ .$$

It remains to observe that, at least when δ^* is regular, i.e. when there is an admissible torus T such that $I_{\delta^*} = T^{\theta^*}$, we have

$$I_{\delta^*} = T^{\theta^*} \subset (G^*)^{\theta^*} = I_{1 \rtimes \theta^*}$$

and hence

$$d(I_{\delta^*}, G^*) = d(I_{1 \rtimes \theta^*}, G^*) \ .$$

□

Consider a connected reductive group G^* defined over F and I^* the stable centralizer of a special point δ^*. Recall that I^* is defined over F.

IV.1.4. Lemma. – *Let v be place of F and assume that G_v and I_v are inner forms of G_v^* and I_v^*, the twisting being defined by a cochain with values in I_v^*. There is a natural map*

$$\mathbf{H}^0(F_v, I_v\backslash G_v) \to \mathbf{H}^0_{ab}(F_v, I^*\backslash G^*)\ .$$

If v is a finite place, this is an isomorphism.

Proof: The map is obtained by composing two maps: the abelianization map

$$\mathbf{H}^0(F_v, I_v\backslash G_v) \to \mathbf{H}^0_{ab}(F_v, I_v\backslash G_v)$$

and the map defined by the inner twisting

$$\mathbf{H}^0_{ab}(F_v, I_v\backslash G_v) \to \mathbf{H}^0_{ab}(F_v, I^*\backslash G^*)\ .$$

Since inner twistings act trivially on the abelianized cohomology this second map is always an isomorphism while Kneser's theorem shows that, at finite places, the abelianization map is an isomorphism (cf. [LBC, Proposition 1.6.7]).

□

IV.2 – The Langlands obstruction

Let F be a global field and let L be an inner form of a quasi-split twisted space $L^* = G^* \rtimes \theta^*$ over F associated to a class in

$$\mathbf{H}^1(F, Z_{sc} \to G^*_{SC} \times Z_G)$$

defined by a cochain $(u \times z, \xi)$ with $\partial u = \xi$ and $\partial z = \xi^{1-\theta^*}$. Consider an L-admissible special pair (δ^*, T). Lemma III.4.3 shows that there is a cochain b_σ with values in $Z_J(\overline{F})$ such that

$$\sigma(\delta^*) = z_\sigma\, b_\sigma^{-1}\, \delta^*\, b_\sigma\ .$$

Not to overload notation we shall denote by I^* instead of I_{δ^*} the stable centralizer of δ^*, and by I^*_{SC} the simply connected cover of its derived group. We write J for J_{δ^*} and Z_J is its center. These groups are defined over F.

IV.2.1. Lemma. – *If (δ^*, T) is special prenorm pair for $\delta \in L(F)$ there is a canonical isomorphism $\mathfrak{C}(I_\delta, G; \mathbb{A}_F/F) \to \mathfrak{C}(I^*, G^*; \mathbb{A}_F/F)$ of compact abelian groups.*

Proof: Since abelianized cohomology is insensitive to inner twistings, this is an immediate consequence of III.4.7.

□

Recall that to compute adelic cohomology using Galois cohomology or even to get quasi-isomorphisms for complexes of $\overline{\mathbb{A}}_F$-points from quasi-isomorphic complexes of algebraic groups over F, it is necessary to use complexes of connected groups. If Z_L is not connected the groups I^* may not be connected. In such a case we shall embed the complex $I^* \to G^*$ as a quasi-isomorphic subcomplex of $I_1^* \to G_1^*$ a complex of connected groups and the twisted space L in a twisted space L_1, constructed as follows: let Z_0 be a torus containing a diagonalizable subgroup isomorphic to Z_L and let L_1 be the quotient of $L \times Z_0$ by Z_L acting diagonally, similarly let G_1^* and I_1^* be the quotient of $G^* \times Z_0$ and $I^* \times Z_0$ by Z_L. The embedding

$$[I^* \to G^*] \to [I_1^* \to G_1^*]$$

is a quasi-isomorphism and it induces an isomorphism

$$\mathbf{H}^i_{ab}(\mathbb{A}_F/F, I^*\backslash G^*) \to \mathbf{H}^i_{ab}(\mathbb{A}_F/F, I_1^*\backslash G_1^*) \ .$$

Observe that I_1^* is the image of $G^*_{\delta^*} \times Z_0$ and hence is connected. The complex $[I_1^* \to G_1^*]$ has a better behaviour when dealing with adelic cohomology.

Assume for a while that Z_L is connected. Consider a special L-admissible pair (δ^*, T) which is a prenorm of $\delta \in L(\mathbb{A}_F)$ locally everywhere. More precisely assume there is $x \in G(\overline{\mathbb{A}}_F)$ such that

$$x\,\varphi(\delta)\,x^{-1} = \delta^* \qquad \text{and} \qquad a_\sigma := x\,u_\sigma \sigma(x)^{-1} = c_\sigma b_\sigma$$

where c_σ takes its values in $I^*(\overline{\mathbb{A}}_F)$ and b takes its values in $Z_J(\overline{F})$. Hence a takes its values in the group $I^*(\overline{\mathbb{A}}_F)Z_J(\overline{F})$ and

$$\partial a = \partial c\,\partial b = \partial u \ .$$

The relations

$$\sigma(x) = a_\sigma^{-1} x\,u_\sigma \quad , \quad \partial a = \xi \quad \text{and} \quad \partial u = \xi$$

mean that the cochain $e = (x\,, u, a, \xi)$ defines a 0-cocycle for the crossed set defined by the commutative diagram

$$\begin{tikzcd}
Z_{sc}(\overline{F}) \arrow[r] \arrow[d] & G^*_{SC}(\overline{F}) \arrow[d] \\
I^*(\overline{\mathbb{A}}_F)Z_J(\overline{F}) \arrow[r] & G^*(\overline{\mathbb{A}}_F)
\end{tikzcd}$$

We have a morphism from the previous crossed set to the crossed set defined by

$$\begin{tikzcd}
I^*_{SC}(\overline{\mathbb{A}}_F) \times Z_{sc}(\overline{F}) \arrow[r] \arrow[d] & G^*_{SC}(\overline{\mathbb{A}}_F) \arrow[d] \\
I^*(\overline{\mathbb{A}}_F)Z_J(\overline{F}) \arrow[r] & G^*(\overline{\mathbb{A}}_F)
\end{tikzcd}$$

which is its abelianized companion. This crossed set is quasi-isomorphic to the sub-bi-complex of abelian groups

$$\begin{tikzcd}
S_{sc}(\overline{\mathbb{A}}_F) \times Z_{sc}(\overline{F}) \arrow[r] \arrow[d] & T_{sc}(\overline{\mathbb{A}}_F) \arrow[d] \\
S(\overline{\mathbb{A}}_F)Z_J(\overline{F}) \arrow[r] & T(\overline{\mathbb{A}}_F)
\end{tikzcd}$$

where $S = T^{\theta^*}$ and S_{sc} is the preimage of S in I^*_{SC}. Since we assume Z_L connected, I^* is connected and S is a torus; the cohomology groups $\mathbf{H}^0_{ab}(\mathbb{A}_F/F, I^*\backslash G^*)$ can be computed using the Galois cohomology of the bicomplex

$$\begin{tikzcd}
S_{sc}(\overline{\mathbb{A}}_F/\overline{F}) \arrow[r] \arrow[d] & T_{sc}(\overline{\mathbb{A}}_F/\overline{F}) \arrow[d] \\
S(\overline{\mathbb{A}}_F/\overline{F}) \arrow[r] & T(\overline{\mathbb{A}}_F/\overline{F})
\end{tikzcd}$$

This shows that the cocycle e defines a class $[e]$ in $\mathbf{H}^0_{ab}(\mathbb{A}_F/F, I^*\backslash G^*)$. The set of all cocycles $e = (x\,, u, a, \xi)$ attached to the pair (δ, δ^*) defines a set of cohomology classes in $\mathbf{H}^0_{ab}(\mathbb{A}_F/F, I^*\backslash G^*)$ we denote Obs_δ; its image in

$$\mathfrak{E}(I^*, G^*; \mathbb{A}_F/F) := \text{ coker } [\mathbf{H}^0_{ab}(\mathbb{A}_F, G^*) \to \mathbf{H}^0_{ab}(\mathbb{A}_F/F, I^*\backslash G^*)]$$

is denoted obs_δ. (These sets may contain more than one element when the stable centralizer differs from the full centralizer). Return now to the general case where Z_L may be disconnected. Using the embedding

$$[I^* \to G^*] \to [I_1^* \to G_1^*]$$

and the isomorphisms induced in cohomology the above construction is easily extended to the general case.

IV.2.2. Theorem. – *The set* obs_δ *contains the trivial element if and only if* δ *is* $G(\mathbb{A}_F)$*-conjugate to a rational element* $\delta' \in L(F)$ *with prenorm* δ^*.

Proof: This is essentially [LBC, Théorème 2.6.3]. For the convenience of the reader we recall the main steps in the proof. Assume for a while that I^* is connected. First, it is clear that, if δ^* is the prenorm of a rational element, all cochains (x, u, a, ξ) can be taken with values in groups over $\overline{F}$ and $[e]$ is trivial. Conversely, to say that the set obs_δ contains the trivial element means that one can choose x such that $[e]$ has a trivial image in $\mathfrak{E}(I^*, G^*; \mathbb{A}_F/F)$. Then, following [LBC, 1.6.12] the sequence of pointed sets

$$1 \to \ker^1(F, I^*) \to \mathbf{H}^1(F, I^*) \to \mathbf{H}^1(\mathbb{A}_F, I^*) \to \mathbf{H}^1_{ab}(\mathbb{A}_F/F, I^*)$$

is exact. This implies that the next sequence of pointed sets is also exact

$$\ker^1(F, I^*) \to \mathbf{H}^1(\Gamma, Z_{sc}(\overline{F}) \to I^*(\overline{F})Z_J(\overline{F})) \to$$
$$\mathbf{H}^1(\Gamma, Z_{sc}(\overline{F}) \to I^*(\overline{\mathbb{A}}_F)Z_J(\overline{F})) \to \mathbf{H}^1_{ab}(\mathbb{A}_F/F, I^*)\ .$$

The natural map

$$\mathfrak{E}(I^*, G^*; \mathbb{A}_F/F) \to \mathbf{H}^1_{ab}(\mathbb{A}_F/F, I^*)$$

sends the image of $[e]$ to the trivial element in $\mathbf{H}^1_{ab}(\mathbb{A}_F/F, I^*)$. Hence the image of a in this group is also trivial. This shows that the class of a in

$$\mathbf{H}^1(\Gamma, [Z_{sc}(\overline{F}) \to I^*(\overline{\mathbb{A}}_F)Z_J(\overline{F})]$$

is in the image of

$$\mathbf{H}^1(\Gamma, [Z_{sc}(\overline{F}) \to I^*(\overline{F})Z_J(\overline{F})]$$

and hence we may choose x such that a takes its values in $I^*(\overline{F}) \subset G^*(\overline{F})$. Then,

$$a_\sigma u_\sigma^{-1} = x\ u_\sigma \sigma(x^{-1}) u_\sigma^{-1}$$

is the image via ψ of a cochain in $\ker^1(F, G)$. Recall that, since the group G is connected, the maps

$$\ker^1(F, G) \to \ker^1_{ab}(F, G) \to \ker^1_{ab}(F, G^*)$$

are bijective and that

$$\ker^1(F, I^*) \to \ker^1_{ab}(F, I^*)$$

is surjective. According to [LBC1.8.4] we have an exact sequence

$$\ker^1_{ab}(F, I^*) \to \ker^1_{ab}(F, G^*) \to \mathfrak{E}(I^*, G^*; \mathbb{A}_F/F) \to \mathbf{H}^1_{ab}(\mathbb{A}_F/F, I^*)\ .$$

This shows that one can again modify the choices so that the image of $a_\sigma u_\sigma^{-1}$ in $\ker^1(F,G)$ is trivial, which means that there exists $x_1 \in G(\overline{F})$ such that

$$a_\sigma u_\sigma^{-1} = \psi(x_1)\ u_\sigma \sigma(\psi(x_1)^{-1}) u_\sigma^{-1} = \psi(x_1 \sigma(x_1)^{-1}) \ .$$

But this implies that $x = \psi(x_1 x_2)$ with $x_2 \in G(\mathbb{A}_F)$. Recall that we have started with $\delta \in L(\mathbb{A}_F)$, $\delta^* \in L^*(\overline{F})$ and $x \in G^*(\overline{\mathbb{A}}_F)$ such that $x\,\varphi(\delta)\,x^{-1} = \delta^*$. Then

$$\delta' = x_2 \delta x_2^{-1} \in L(\mathbb{A}_F)$$

is such that $\varphi(x_1\,\delta'\,x_1^{-1}) = \delta^* \in L^*(\overline{F})$ with $x_1 \in G(\overline{F})$. This shows that δ' belongs to the intersection

$$L(\mathbb{A}_F) \cap L(\overline{F}) = L(F)$$

and has δ^* as prenorm. This establishes the theorem when I^* is connected. Return now to the general case. We have to replace L by L_1 and $[I^* \to G^*]$ by $[I_1^* \to G_1^*]$ as already explained. Now, observe that L injects in L_1 and is invariant under G_1-conjugacy; the conclusion follows since $L(F) = L(\mathbb{A}_F) \cap L_1(\overline{F})$.

□

IV.3 – Endoscopic spaces

If T is an admissible torus in G^* we denote by T_{sc} its preimage in G^*_{SC}. We denote by $\mathfrak{K}(T,\theta^*;F)$ the Pontyagin dual of the compact abelian group

$$\mathbf{H}^0(\mathbb{A}_F/F, T^{\theta^*} \times T_{sc} \to T) = \mathbf{H}^0(\mathbb{A}_F/F, T^{\theta^*} \to G^*_{ab}) \ .$$

Recall that in our conventions T and G^*_{ab} are in degree 0. The notation $\mathfrak{K}(T,\theta^*;F)$ is borrowed from [KS, section 6.4]. In fact this is how they denote the Pontyagin dual of

$$\mathbf{H}^1(\mathbb{A}_F/F, T_{sc} \to V)$$

where $V = T^{1-\theta^*} \simeq T/T^{\theta^*}$ is put in degree -1; but the map of complexes, without a shift of degree,

$$[T^{\theta^*} \times T_{sc} \to T] \to [T_{sc} \to V]$$

is a quasi-isomorphism.

Geometric endoscopic pairs are pairs (T,κ) where T is an admissible torus in G^* and $\kappa \in \mathfrak{K}(T,\theta^*;F)$. We say that an endoscopic pair is elliptic if T is L^*-elliptic which means that $T_{sc}^{\theta^*}$ is F-anisotopic. There is a natural injective map

$$\mathbf{H}^1(\mathbb{A}_F/F, T_{sc}^{\theta^*}) \to \mathbf{H}^0(\mathbb{A}_F/F, T^{\theta^*} \to G^*_{ab}) \ .$$

Observe that $T_{sc}^{\theta^*}$ is a torus; we denote by $\check{X}_{sc}^{\theta^*}$ the free $\mathbb{Z}$-module of its cocharacters. Let E be a finite Galois extension of F that splits T_{sc}. We have the Tate-Nakayama isomorphism

$$\widehat{\mathbf{H}}^{-1}(\Gamma_{E/F}, \check{X}_{sc}^{\theta^*}) \to \mathbf{H}^1(\mathbb{A}_F/F, T_{sc}^{\theta^*}) \ .$$

There is an exact sequence

$$0 \to \widehat{\mathbf{H}}^{-1}(\Gamma_{E/F}, \check{X}_{sc}^{\theta^*}) \to \mathbf{H}_0(\Gamma_{E/F}, \check{X}_{sc}^{\theta^*}) \to \mathbf{H}^0(\Gamma_{E/F}, \check{X}_{sc}^{\theta^*})$$

Assume now that the endoscopic pair is elliptic. This is equivalent to the vanishing of $\mathbf{H}^0(\Gamma_{E/F}, \check{X}_{sc}^{\theta^*})$. In such a case we have a surjective map

$$\check{X}_{sc}^{\theta^*} \to \widehat{\mathbf{H}}^{-1}(\Gamma_{E/F}, \check{X}_{sc}^{\theta^*}) \ .$$

By composition of the maps

$$\check{X}_{sc}^{\theta^*} \to \widehat{\mathbf{H}}^{-1}(\Gamma_{E/F}, \check{X}_{sc}^{\theta^*}) \to \mathbf{H}^1(\mathbb{A}_F/F, T_{sc}^{\theta^*}) \to \mathbf{H}^0(\mathbb{A}_F/F, T^{\theta^*} \to G_{ab}^*)$$

and via Pontryagin duality, we get a map

$$\mathfrak{K}(T, \theta^*; F) \to \mathrm{Hom}(\check{X}_{sc}^{\theta^*}, \mathbb{C}^\times) \ .$$

We shall denote by $\overline{\kappa}$ the image of κ by this map. Following the prescriptions of II.4 we get a triple $(T_\theta, R_{\overline{\kappa}}, \check{R}_{\overline{\kappa}})$.

The endocopic group attached to $\mathcal{E} = (T, \kappa)$ is the F-quasi-split group denoted H^* (or ${H^*}_{\mathcal{E}}$ if some confusion may arise) with a maximal torus, a root and a coroot system $(T_{H^*}, R_{H^*} \check{R}_{H^*})$ that are F-isomorphic to $(T_{\theta^*}, R_{\overline{\kappa}}, \check{R}_{\overline{\kappa}})$. Similarly we get a quasi-split pre-endoscopic group K^* (or $K_{\mathcal{E}}^*$) using $(T^{\theta^*}, R'_{\overline{\kappa}}, \check{R}'_{\overline{\kappa}})$. Endoscopic spaces are attached to $\mathcal{E}$ and L. The endoscopic space H is a central form of $H^* \rtimes 1$ the twist for the Galois action being defined as follows. Recall that L is an inner form of $L^* = G^* \rtimes \theta^*$ defined by a hyper-cochain $(u \times z, \xi)$. Then H is the inner form of $H^* \rtimes 1$ defined by the hyper-cochain $(1 \times [z], 1)$ where $[z]$ is the 1-cocycle image in $Z_{\theta^*} \subset Z_{H^*}$ of the 1-cochain z. If we denote by φ_H the isomorphism over the algebraic closure $\varphi_H : H \to H^* \rtimes 1$ the Galois structures are related by the equation

$$\varphi_H(\sigma(h)) = [z_\sigma]^{-1} \sigma(\varphi_H(h)) \qquad \text{for } h \in H \text{ and } \sigma \in \Gamma \ .$$

We have attached to $\overline{\kappa}$ a torus $T_{\overline{\kappa}}$ (see II.4). It is defined over F and is anisotropic since T^{θ^*} is L-elliptic. We have an exact sequence

$$\mathbf{H}^1(\mathbb{A}_F/F, T_{\overline{\kappa}}) \to \mathbf{H}^0(\mathbb{A}_F/F, T^{\theta^*} \to G_{ab}^*) \to \mathbf{H}^0(\mathbb{A}_F/F, K_{ab}^* \to G_{ab}^*) \to \mathbf{H}^2(\mathbb{A}_F/F, T_{\overline{\kappa}}) \ .$$

Since $T_{\overline{\kappa}}$ is anisotropic $\mathbf{H}^2(\mathbb{A}_F/F, T_{\overline{\kappa}})$ is trivial and since κ induces a trivial character on $\mathbf{H}^1(\mathbb{A}_F/F, T_{\overline{\kappa}})$ we see that κ defines a character of the compact group

$$\mathbf{H}^0(\mathbb{A}_F/F, K_{ab}^* \to G_{ab}^*) \ .$$

Consider two endoscopic pairs $\mathcal{E} = (T, \kappa)$ and $\mathcal{F} = (S, \nu)$. Assume there is an $x \in G^*$ such that

$$x^{-1}(T^{\theta^*}, R_{\overline{\kappa}})\, x = (S^{\theta^*}, R_{\overline{\nu}})$$

then the cocycle $\sigma \mapsto x\sigma(x)^{-1}$ takes its values in the normalizer of T^{θ^*}. If defines a cocycle w_σ with values in $\Omega(G^*, T)^{\theta^*}$ (see II.2.1). We identify $\Omega_{\overline{\kappa}}$ the Weyl group of K with a subgroup of $\Omega(G^*, T)^{\theta^*}$. If the cocycle w_σ takes its values in $\Omega_{\overline{\kappa}}$, it acts trivially on $K_{ab} = K_{\mathcal{E},ab}$ and $\mathrm{Ad}(x)$ induces an F-isomorphism $K_{\mathcal{F},ab}^* \to K_{\mathcal{E},ab}^*$.

IV.3.1. Definition. – *We say that the two endoscopic pairs (T, κ) and (S, ν) are equivalent if there is an $x \in G^*$ such that*

$$x^{-1}(T^{\theta^*}, R_{\overline{\kappa}})\ x = (S^{\theta^*}, R_{\overline{\nu}})$$

and such that the image w_σ of $x\sigma(x)^{-1}$ in the Weyl group of G^* belongs to $\Omega_{\overline{\kappa}}$ the Weyl group of $K_{\mathcal{E}}^*$. We ask moreover that the characters κ and ν correspond to each other via the isomorphism

$$\mathbf{H}^0(\mathbb{A}_F/F, K_{\mathcal{E},ab}^* \to G_{ab}^*) \simeq \mathbf{H}^0(\mathbb{A}_F/F, K_{\mathcal{F},ab}^* \to G_{ab}^*) \ .$$

IV.3.2. Lemma. – *Equivalent endoscopic pairs give rise to isomorphic endoscopic groups. All maximal F-tori in an endoscopic group arise from equivalent endoscopic pairs.*

Proof: Consider two equivalent endoscopic pairs $\mathcal{E} = (T, \kappa)$ and $\mathcal{F} = (S, \nu)$. The pair $\mathcal{E} = (T, \kappa)$ defines an endoscopic group H^* together with a maximal torus T_{H^*} and an isomorphism

$$T_{\theta^*} \to T_{H^*} \ .$$

Recall that the Weyl group of H^* can be seen as a subgroup of the Weyl group of G^*. If x induces the equivalence between $\mathcal{E}$ and $\mathcal{F}$, denote by w_σ the cocycle image of $x\sigma(x)^{-1}$ in the Weyl group of H^*. It follows from III.3.1 that one can find $y \in H^*$ such that $U = yT_{H^*}\,y^{-1}$ is a maximal torus in H^*, isomorphic over F to S_{θ^*}; the associated root system is F-isomorphic to the endoscopic root system defined by ν. This proves the first assertion. Conversely an F-torus U in H^* is of the form $yT_{H^*}\,y^{-1}$ for some $y \in H^*$ and $y^{-1}\sigma(y)$ defines an elements in the Weyl groups of H^* which is a subgroup of the Weyl group of G^*. Using again III.3.1 we get an $x \in G^*$ that defines an equivalent endoscopic pair $\mathcal{F} = (S, \nu)$ with U isomorphic over F to S_{θ^*}.

□

Let (δ^*, T) be a special L-admissible pair. Let I^* be the stable centralizer of δ^*. Assume that T^{θ^*} is L-elliptic. There is a surjection

$$\mathbf{H}^0(\mathbb{A}_F/F, T^{\theta^*} \to G^*_{ab}) \to \mathbf{H}^0_{ab}(\mathbb{A}_F/F, I^*\backslash G^*) \ .$$

If we denote by $\mathfrak{K}(I^*, G^*; F)$ the Pontryagin dual of $\mathbf{H}^0_{ab}(\mathbb{A}_F/F, I^*\backslash G^*)$ we get an injection

$$\mathfrak{K}(I^*, G^*; F) \to \mathfrak{K}(T, \theta^*; F) \ .$$

We denote again by κ the image of $\kappa \in \mathfrak{K}(I^*, G^*; F)$ by this injection, and we associate to $\mathcal{E} = (T, \kappa)$ an endoscopic space H. Such a κ is said to be an endoscopic character for δ^*. It follows from II.4.4 that the surjective map

$$\mathbf{H}^0(\mathbb{A}_F/F, T^{\theta^*} \to G^*_{ab}) \to \mathbf{H}^0(\mathbb{A}_F/F, K^*_{ab} \to G^*_{ab})$$

factors through $\mathbf{H}^0_{ab}(\mathbb{A}_F/F, I^*\backslash G^*)$:

$$\mathbf{H}^0(\mathbb{A}_F/F, T^{\theta^*} \to G^*_{ab}) \to \mathbf{H}^0_{ab}(\mathbb{A}_F/F, I^*\backslash G^*) \to \mathbf{H}^0(\mathbb{A}_F/F, K^*_{ab} \to G^*_{ab}) \ .$$

The image $[\delta^*]$ of δ^* in $T_{H^*} \simeq T_{\theta^*}$ defines a semisimple element γ in $H(F)$. We say that γ is the conorm of δ^*. If δ^* is itself the prenorm of $\delta \in L(F)$ we say that γ is a norm for δ.

The group Aut$(\mathcal{E})$ of automorphisms of an endoscopic pair $\mathcal{E} = (T, \kappa)$ is the subgroup of $w \in \Omega(G^*, T)^{\theta^*}$ that stabilizes $(T^{\theta^*}, R_{\overline{\kappa}})$ and verifies

$$w\sigma(w)^{-1} \in \Omega_{\overline{\kappa}} \qquad \text{and} \qquad w(\kappa) = \kappa \ .$$

We denote by $\Lambda(\mathcal{E})$ the group of outer automorphisms of $\mathcal{E}$ which, by definition, is the quotient of Aut$(\mathcal{E})$ by

$$\mathrm{Aut}\,(\mathcal{E}) \cap \Omega_{\overline{\kappa}} \ .$$

Let $\lambda(\mathcal{E})$ be the order of $\Lambda(\mathcal{E})$.

Consider $\gamma \in H(F)$ semisimple and whose centralizer I_γ contains T_{H^*}. We denote by Aut$(\mathcal{E}, \gamma)$ the subgroup of $w \in$ Aut$(\mathcal{E})$ such that

$$w(\gamma) = \gamma \qquad \text{and} \qquad w\sigma(w)^{-1} \in \Omega(I_\gamma, T_{H^*}) \ .$$

The quotient of $\mathrm{Aut}\,(\mathcal{E},\gamma)$ by the subgroup of inner automorphisms is denoted by $\Lambda(\mathcal{E},\gamma)$. We denote by $\lambda(\mathcal{E},\gamma)$ the order of $\Lambda(\mathcal{E},\gamma)$. We have

$$\lambda(\mathcal{E}) = M(\mathcal{E},\gamma)\lambda(\mathcal{E},\gamma)$$

where $M(\mathcal{E},\gamma)$ is the number of stable conjugacy classes in $H(F)$ in the orbit of γ under $\mathrm{Aut}\,(\mathcal{E})$.

IV.3.3. Proposition. – *Consider an L-admissible pair (δ^*,T) and let $\gamma \in H(F)$ be the conorm of δ^*. Let κ be an endoscopic character for δ^* and let H the associated endoscopic space. Let $\iota_{G^*}(\delta^*)$ be the order of $\mathbf{H}^0(F,G^{\delta^*}/I_{\delta^*})$ and $\iota_{H^*}(\gamma)$ be the order of $\mathbf{H}^0(F,H^\gamma/I_\gamma)$. Let $N(T,\delta^*,\kappa)$ be the number of endoscopic characters ν for δ^* such that $\mathcal{F}=(T,\nu)$ is equivalent to $\mathcal{E}=(T,\kappa)$ where the equivalence is induced by an x that defines an auto-equivalence of (δ^*,T). Then*

$$\iota_{G^*}(\delta^*) = N(T,\delta^*,\kappa)\lambda(\mathcal{E},\gamma)\iota_{H^*}(\gamma)\ .$$

Proof: Not to overload notation let $G^{\delta^*}=(G^*)^{\delta^*}$ and $G^{\theta^*}=(G^*)^{\theta^*}$. Recall that, according to II.1.4,

$$G^{\delta^*}/I_{\delta^*} \simeq \Omega(G^{\delta^*},T^{\delta^*})/\Omega(I_{\delta^*},T^{\delta^*})$$

and that H^γ/I_γ can be identified with a subgroup of $G^{\delta^*}/I_{\delta^*}$ (cf. II.4.5). Observe that if $x\in G^{\delta^*}$ induces an equivalence between $\mathcal{F}=(T,\nu)$ and $\mathcal{E}=(T,\kappa)$ then x normalize T^{θ^*} and hence defines a $w\in\Omega(G^{\delta^*},T^{\delta^*})$. If moreover x induces an auto-equivalence of (δ^*,T) then we must have $w\sigma(w)^{-1}\in\Omega(I_{\delta^*},T^{\delta^*})$. This is equivalent to say that x projects on a rational point $[w]$ of the group $G^{\delta^*}/I_{\delta^*}$. The subgroup of those $[w]\in\mathbf{H}^0(F,G^{\delta^*}/I_{\delta^*})$ that fix $\mathcal{E}$ is the quotient of $\mathrm{Aut}\,(\mathcal{E},\gamma)$ by

$$\Omega(I_\gamma,T_{H^*}) \simeq \Omega(I_{\delta^*},T^{\delta^*})\ .$$

Now, inner automorphisms in $\mathrm{Aut}\,(\mathcal{E},\gamma)$ are induced by $w\in\Omega(H^\gamma,T_{H^*})$ with $[w]\in\mathbf{H}^0(F,H^\gamma/I_\gamma)$. This shows that

$$\iota_{G^*}(\delta^*) = N(T,\delta^*,\kappa)\lambda(\mathcal{E},\gamma)\iota_{H^*}(\gamma)\ .$$

□

IV.3.4. Corollary. – *Let $\mathcal{E}=(T,\kappa)$ be an endoscopic pair and let*

$$V(\mathcal{E},\gamma) = \sum_{\{\mu\}} \frac{1}{\lambda(\mathcal{E})\iota_{H^*}(\mu)}$$

where the sum is over representatives μ of stable conjugacy classes in $H(F)$ where μ is the conorm of η^* and (T,η^*,κ) is an L-admissible triple equivalent to (T,δ^*,κ). Then

$$V(\mathcal{E},\gamma) = \frac{1}{\lambda(\mathcal{E},\gamma)\iota_{H^*}(\gamma)} = \frac{N(T,\delta^*,\kappa)}{\tilde\iota_{G^*}(\delta^*)}\ .$$

Proof: Observe that $\iota_{H^*}(\mu)=\iota_{H^*}(\gamma)$ if η^* is equivalent to δ^*. Hence

$$V(\mathcal{E},\gamma) = \frac{M(\mathcal{E},\gamma)}{\lambda(\mathcal{E})\iota_{H^*}(\gamma)}\ .$$

Recall that $\lambda(\mathcal{E}) = M(\mathcal{E},\gamma)\lambda(\mathcal{E},\gamma)$. The conclusion now follows from IV.3.3.

□

V – Stabilisation of the elliptic trace

V.1 – Pres-stabilization

Let G be a reductive algebraic group defined over a number field F and let L be a twisted G-space over F. Assume that L has an almost semisimple rational point δ_0. Let

$$\theta_0 = \mathrm{Ad}_L(\delta_0)\,.$$

There is a natural map

$$\pi : L(\mathbb{A}_F) \to G(F)\backslash G(\mathbb{A}_F)$$

defined as follows. Any $\delta \in L(\mathbb{A}_F)$ is of the form $\delta = \delta_0\, y$ with $y \in G(\mathbb{A}_F)$ and we define $\pi(\delta) \in G(F)\backslash G(\mathbb{A}_F)$ by

$$\pi(\delta) = \pi(\delta_0\, y) = \dot{y}$$

where $\dot{y}$ is the image of y in $G(F)\backslash G(\mathbb{A}_F)$. The image is independent of the chosen F-rational point δ_0. The map π defines a natural right action of the set $L(\mathbb{A}_F)$ on the quotient $G(F)\backslash G(\mathbb{A}_F)$

$$G(F)\backslash G(\mathbb{A}_F) \times L(\mathbb{A}_F) \to G(F)\backslash G(\mathbb{A}_F)$$

by $(x,\delta) \mapsto \pi(x\delta)$.

Let A_G be the split component of the center in $\mathrm{Res}_{F/\mathbb{Q}}G$. This is the maximal $\mathbb{Q}$-split torus in $\mathrm{Res}_{F/\mathbb{Q}}Z_G$. Let $\mathfrak{A}_G$ be the Lie algebra of $A_G(\mathbb{R})$ identified, via the exponential map, with the connected component of this real Lie group. We denote by $\mathfrak{A}_L$ the group of θ-fixed points in $\mathfrak{A}_G$, this is the Lie algebra of $A_L(\mathbb{R})^0$ where A_L is the maximal $\mathbb{Q}$-split torus in $\mathrm{Res}_{F/\mathbb{Q}}Z_L$.

Let f be a smooth compactly supported function on $L(\mathbb{A}_F)$ we will need its companions

$$f^1(x) = \int_{\mathfrak{A}_G} f(xz)dz \qquad \text{and} \qquad f^0(x) = \int_{\mathfrak{A}_L} f(xz)dz\ .$$

The function f defines an operator $\rho(f)$ on the Hilbert space $L^2(\mathfrak{A}_G G(F)\backslash G(\mathbb{A}_F))$: given ϕ in the Hilbert space we define $\rho(f)\phi$ by

$$(\rho(f)\phi)(x) = \int_{L(\mathbb{A}_F)} \phi(\pi(x\delta))f(\delta)d\delta = \int_{L(\mathbb{A}_F)} f(x^{-1}\delta)\,\phi(\pi(\delta))d\delta = \int_{G(\mathbb{A}_F)} f(x^{-1}\delta_0 y)\,\phi(y)dy$$

which can be written

$$(\rho(f)\phi)(x) = \int_{\mathfrak{A}_G G(F)\backslash G(\mathbb{A}_F)} \sum_{\delta\in L(F)} f^1(x^{-1}\delta y)\,\phi(y)\,dy\,.$$

This shows that the operator $\rho(f)$ has a kernel on $\mathfrak{A}_G G(F)\backslash G(\mathbb{A}_F)$ given by

$$K(x,y) = \sum_{\delta\in L(F)} f^1(x^{-1}\delta y)\ .$$

The general twisted trace formula not only involves working with twisted G-spaces but also with characters of $G(\mathbb{A}_F)$ trivial on $G(F)$. Let ω be a characters of the compact abelian group

$$\mathrm{coker}\ [\mathbf{H}^0_{ab}(F,G) \to \mathbf{H}^0_{ab}(\mathbb{A}_F,G)] = \ker[\mathbf{H}^0_{ab}(\mathbb{A}_F/F,G) \to \mathbf{H}^1_{ab}(F,G)]\ .$$

By composition with the natural homomorphism

$$G(\mathbb{A}_F) \to \mathbf{H}^0_{ab}(\mathbb{A}_F,G)$$

we get a character of $G(\mathbb{A}_F)$ trivial on $G(F)$ again denoted ω. Conversely, any character of $G(\mathbb{A}_F)$ trivial on $G(F)$ can be extended, in a unique way, to a character of $\mathbf{H}^0_{ab}(\mathbb{A}_F, G)$ trivial on the image of $\mathbf{H}^0_{ab}(F, G)$. This is a consequence of the following observation: since G is connected the finite abelian groups

$$M_1 = \text{ coker } [G(F) \to \mathbf{H}^0_{ab}(F, G)] \qquad \text{and} \qquad M_2 = \text{ coker } [G(\mathbb{A}_F) \to \mathbf{H}^0_{ab}(\mathbb{A}_F, G)]$$

are isomorphic (cf [Lab]). The general operator $\rho(f, \omega)$ is the composition of $\rho(f)$ followed by the multiplication by ω:

$$(\rho(f, \omega)\phi)(x) = \omega(x) \int_{L(\mathbb{A}_F)} \phi(x\delta) f(\delta) d\delta$$

and its kernel is then

$$K(x, y) = \sum_{\delta \in L(F)} \omega(x) f^1(x^{-1}\delta y) \ .$$

We denote by L_e the set of elliptic elements in $L(F)$. The elliptic part of the restriction to the diagonal of the kernel

$$K_e(x, x) = \sum_{\delta \in L_e} \omega(x)\, f^1(x^{-1}\delta\, x)$$

once integrated, yields the “elliptic trace” i.e. the contribution of elliptic conjugacy classes to the geometric side of the twisted trace formula:

$$\mathbf{T}_e(f, \omega) = \int_{\mathfrak{A}_G G(F)\backslash G(\mathbb{A}_F)} K_e(x, x)\ dx \ .$$

This integral is absolutely convergent. As usual, one can split this expression into a sum over conjugacy classes.

Remark – In [KS] the notation $\mathbf{T}_e$ is used with a slightly different meaning: there, the summation that defines K_e is restricted to strongly regular elliptic elements, while here we allow all elliptic elements.

Recall that an almost semisimple element $\delta \in L(F)$ is elliptic if its stable centralizer I_δ is L-elliptic, which means that the maximal F-split torus in the center of I_δ equals the maximal F-split torus in Z_L and hence, the map

$$\mathfrak{A}_L \to \mathfrak{A}_{I_\delta}$$

is an isomorphism. For $\delta \in L_e$ let $\iota_G(\delta)$ be the index of $I_\delta(F)$ in $G^\delta(F)$ and

$$a^L(\delta) = \iota_G(\delta)^{-1} \text{ vol } (\mathfrak{A}_L I_\delta(F)\backslash I_\delta(\mathbb{A}_F)) \ .$$

Assume now that all groups are endowed with Tamagawa measures; the definition of Tamagawa measures for quasi-connected groups is given in [Lab]. As usual the Tamagawa number of a quasi-connected group I is denoted $\tau(I)$:

$$\tau(I) = \text{ vol } (\mathfrak{A}_I I(F)\backslash I(\mathbb{A}_F)) \ .$$

Let $c(I_\delta, L)$ be the ratio of Haar measures on the isomorphic real vector spaces $\mathfrak{A}_L$ and $\mathfrak{A}_{I_\delta}$.

V.1.1. Lemma. – *The number $c(I_\delta, L)$ is independent of $\delta \in L_e$.*

Proof: We have to show that the Haar measure on the real vector space $\mathfrak{A}_L$ associated to the Tamagawa measure on I_δ via the isomorphism

$$\mathfrak{A}_L \to \mathfrak{A}_{I_\delta}$$

is independent of δ. Let S be the centralizer in I_δ of a maximal torus S^0 in $G_\delta = I^0_\delta$ and let T be the centralizer of S in G. We know that $S = T^\delta$. Recall that the measure on $\mathfrak{A}_{I_\delta}$ is defined using the $\mathbb{Z}$-module $\mathbf{H}^0(\Gamma, X(I_\delta))$ (see [Lab]). Since δ is elliptic the natural map

$$\mathbf{H}^0(\Gamma, X(G)_\theta) \to \mathbf{H}^0(\Gamma, X(I_\delta))$$

is an isomorphism and hence this $\mathbb{Z}$-module is independent of δ.

□

The measure on $\mathfrak{A}_L$ can be chosen so that $c(I_\delta, L) \equiv 1$ as follows from V.1.1. Then, for such a choice,

$$a^L(\delta) = \iota_G(\delta)^{-1}\tau(I_\delta) \ .$$

Let

$$J(L) = |\det\,(1-\theta\,|\,\mathfrak{A}_G/\mathfrak{A}_L)|$$

and

$$\mathrm{O}_{\delta,\omega}(f) = \int_{I_\delta(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \omega(x) f(x^{-1}\delta x)dx \ .$$

V.1.2. Proposition. – *Let us denote by $\{L_e\}$ a set of representatives for elliptic conjugacy classes in $L(F)$. Then*

$$\mathbf{T}_e(f,\omega) = J(L) \sum_{\delta\in\{L_e\}} \frac{\tau(I_\delta)}{\iota_G(\delta)} \mathrm{O}_{\delta,\omega}(f^0) \ .$$

Proof: An elementary calculation yields

$$\mathbf{T}_e(f,\omega) = J(L) \sum_{\delta\in\{L_e\}} a^L(\delta)\ \mathrm{O}_{\delta,\omega}(f^0) \ .$$

It suffices to observe that thanks to V.1.1 we have $a^L(\delta) = \tau(I_\delta)\,\iota_G(\delta)^{-1}$.

□

Consider an almost semisimple point $\delta \in L(\mathbb{A}_F)$. We denote by I_δ its stable centralizer. This is a group scheme over $\mathbb{A}_F$, but it may not be defined over F. We denote by $\mathfrak{K}(I_\delta, G; \mathbb{A}_F)$ the Pontryagin dual of the locally compact abelian group $\mathbf{H}^0_{ab}(\mathbb{A}_F, I_\delta\backslash G)$. This is a group of endoscopic characters. Given $\kappa \in \mathfrak{K}(I_\delta, G; \mathbb{A}_F)$, we denote by $[\kappa]$ the character of $\mathbf{H}^0_{ab}(\mathbb{A}_F, G)$ obtained by composing κ with the map

$$\mathbf{H}^0_{ab}(\mathbb{A}_F, G) \to \mathbf{H}^0_{ab}(\mathbb{A}_F, I_\delta\backslash G) \ .$$

Let $x \in G(\overline{\mathbb{A}}_F)$ with $x\sigma(x)^{-1} \in I_\delta$. Such an x defines a class in $\mathbf{H}^0(\mathbb{A}_F, I_\delta\backslash G)$. We denote by $x \mapsto \kappa(x)$ the function defined by the composition of the natural map:

$$\mathbf{H}^0(\mathbb{A}_F, I_\delta\backslash G) \to \mathbf{H}^0_{ab}(\mathbb{A}_F/F, I_\delta\backslash G)$$

with the character κ. Let $\delta_x := x^{-1}\delta x$ and $I_x = x^{-1}I_\delta\, x$; let us denote by $e(I_x)$ the product of local Kottwitz signs as defined in [Ko2] (see also [LBC, Définition I.7.1]):

$$e(I_x) = \prod_v e(x_v^{-1} I_\delta\, x_v) \ .$$

Now consider $\delta \in L_e$. Then I_δ is defined over F. Let $\mathfrak{K}(I_\delta, G; F)$ be the Pontryagin dual of the compact abelian group $\mathbf{H}^0_{ab}(\mathbb{A}_F/F, I_\delta\backslash G)$. This is a discrete subgroup of $\mathfrak{K}(I_\delta, G; \mathbb{A}_F)$. A Tamagawa measure on $\mathfrak{D}(I_\delta, G; \mathbb{A}_F)$ and on $\mathbf{H}^0(\mathbb{A}_F, I_\delta\backslash G)$ has been defined in [Lab]. This allows to define κ-orbital integral by

$$\mathrm{O}^\kappa_\delta(f) = \int_{\mathbf{H}^0(\mathbb{A}_F, I_\delta\backslash G)} e(I_x)\kappa(x) f(\delta_x)dx = \int_{\mathfrak{D}(I_\delta, G;\mathbb{A}_F)} e(I_x)\kappa(x)\mathrm{O}_{\delta_x,[\kappa]}(f^0)d\dot{x} \ .$$

Let

$$\tilde{\iota}_G(\delta) = \#\mathbf{H}^0(F, I_\delta\backslash G^\delta) \qquad \text{and} \qquad j(\delta) = \#\ker[\mathbf{H}^1(F, I_\delta) \to \mathbf{H}^1(F, G^\delta)] \ .$$

Observe that $\tilde\iota_G(\delta)=\iota_G(\delta)j(\delta)$. Finally recall that we have introduced in IV.1.3 an integer $d(L)$ which is the common value of the $d(I_\delta,G)$. Let

$$\tau(L)=\frac{J(L)\,\tau(G)}{d(L)}\ .$$

The notation makes sense since $J(G)=d(G)=1$.

V.1.3. Theorem. – *Let f be a smooth compactly supported function on $L(\mathbb{A}_F)$. Let $\{\{L_e\}\}$ be a set of representatives of stable conjugacy classes in L_e. The elliptic trace can be expanded as a sum of κ-orbital integrals:*

$$\mathbf{T}_e(f,\omega)=\tau(L)\sum_{\delta\in\{\{L_e\}\}}\frac{1}{\tilde\iota_G(\delta)}\sum_{\{\kappa\in\mathfrak{K}(I_\delta,G;F)\ \mid\ [\kappa]=\omega\}}\mathrm{O}^\kappa_\delta(f^0)\ .$$

Proof: We know by V.1.2 that

$$\mathbf{T}_e(f,\omega)=J(L)\sum_{\delta\in\{L_e\}}\frac{\tau(I_\delta)}{\iota_G(\delta)}\ \mathrm{O}_{\delta,\omega}(f^0)\ .$$

Grouping together conjugacy classes in the same stable class, this can be rewritten

$$\mathbf{T}_e(f,\omega)=J(L)\sum_{\delta\in\{\{L_e\}\}}\ \sum_{\eta\in\mathfrak{D}(I_\delta,G;F)}\frac{\tau(I_{\delta_\eta})}{j(\delta_\eta)\iota_G(\delta_\eta)}\mathrm{O}_{\delta_\eta,\omega}(f^0)\ .$$

Observe that, according to III.1.3 the number $\tilde\iota_G(\delta)=\iota_G(\delta)j(\delta)$ is constant on stable conjugacy classes. Hence, we have

$$\mathbf{T}_e(f,\omega)=J(L)\sum_{\delta\in\{\{L_e\}\}}\frac{1}{\tilde\iota_G(\delta)}\sum_{\eta\in\mathfrak{D}(I_\delta,G;F)}\tau(I_{\delta_\eta})\mathrm{O}_{\delta_\eta,\omega}(f^0)\ .$$

Let $\mathfrak{K}(I_\delta,G;F)_1$ be the Pontryagin dual of $\mathfrak{C}(I_\delta,G;\mathbb{A}_F/F)$. Now theorem 3.9 in [Lab], which is an elaboration of the Poisson summation formula, shows that

$$\sum_{\eta\in\mathfrak{D}(I_\delta,G;F)}\tau(I_{\delta_\eta})\,\mathrm{O}_{\delta_\eta,\omega}(f^0)=\frac{\tau(G)}{d(I_\delta,G)}\sum_{\kappa_1\in\mathfrak{K}(I_\delta,G;F)_1}\mathrm{O}^{\kappa_1\kappa_2}_\delta(f^0)$$

where κ_2 is any endoscopic character in $\mathfrak{K}(I_\delta,G;F)$ such that $[\kappa_2]=\omega$. Since any character $\kappa\in\mathfrak{K}(I_\delta,G;F)$ with $[\kappa]=\omega$ is of the form $\kappa=\kappa_1\kappa_2$ with $\kappa_1\in\mathfrak{K}(I_\delta,G;F)_1$ we also have

$$\sum_{\eta\in\mathfrak{D}(I_\delta,G;F)}\tau(I_{\delta_\eta})\,\mathrm{O}_{\delta_\eta,\omega}(f^0)=\frac{\tau(G)}{d(I_\delta,G)}\sum_{\{\kappa\in\mathfrak{K}(I_\delta,G;F)\ \mid\ [\kappa]=\omega\}}\mathrm{O}^{\kappa}_\delta(f^0)\ .$$

To conclude we recall that $d(I_\delta,G)=d(L)$ (cf. IV.1.3).

□

V.2 – Stabilization: the second step

Consider an L-admissible special pair (δ^*,T). Recall that this means that T is admissible and that $\delta^*\in T(\overline{F})\rtimes\theta^*$ verifies

$$\sigma(\delta^*)=z_\sigma^{-1}b_\sigma^{1-\theta^*}\delta^*$$

with b_σ in the center of $J_{\delta^*}(\overline{F})$. As usual denote by I^* the stable centralizer of δ^*. Let $\mathfrak{A}(\mathbb{A}_F,\delta^*,L)$ be the set of classes of $x\in G^*(\overline{\mathbb{A}}_F)$ modulo $I^*(\overline{\mathbb{A}}_F)$ such that

$$a_\sigma = x\,u_\sigma\,\sigma(x)^{-1} = c_\sigma b_\sigma$$

with $c_\sigma\in I^*(\overline{\mathbb{A}}_F)$. For such an x it follows from III.2.2 that there is a $\delta_x\in L(\mathbb{A}_F)$ such that $\varphi(\delta_x)=x^{-1}\delta^*x$ and δ^* is locally everywhere a prenorm for δ_x. We denote by $I_x(\mathbb{A}_F)$ the stable centralizer in $G(\mathbb{A}_F)$ of δ_x. Observe that I_x is a group scheme over $\mathbb{A}_F$ but it need not be defined over F.

Assume that $\mathfrak{A}(\mathbb{A}_F,\delta^*,L)$ is non empty. Consider x and x' representing classes in $\mathfrak{A}(\mathbb{A}_F,\delta^*,L)$. Then $x'=x\psi(y)$ with

$$\psi(y\sigma(y)^{-1})\in x^{-1}I^*(\overline{\mathbb{A}}_F)x\ .$$

This shows that the map $y\mapsto x\psi(y)$ induces a bijection

$$\mathbf{H}^0(\mathbb{A}_F,I_x\backslash G)\to\mathfrak{A}(\mathbb{A}_F,\delta^*,L)\ .$$

This can be used to define a locally compact topology on $\mathfrak{A}(\mathbb{A}_F,\delta^*,L)$. We have also defined in [Lab] Tamagawa measures on $\mathfrak{E}(I^*,G^*;\mathbb{A}_F)$. Since $I_x(\mathbb{A}_F)$ is locally everywhere an inner form of $I^*(\mathbb{A}_F)$ there is an isomorphism

$$\mathfrak{E}(I_x,G;\mathbb{A}_F)\to\mathfrak{E}(I^*,G^*;\mathbb{A}_F)\ .$$

For the same reason $I_x(\mathbb{A}_F)$ can be endowed with a canonical measure using the Tamagawa measure on $I^*(\mathbb{A}_F)$. This yields a measure on $\mathfrak{D}(I_x,G;\mathbb{A}_F)$ viewed as a finite covering of an open subset of $\mathfrak{E}(I_x,G;\mathbb{A}_F)$, using that

$$\mathfrak{D}(I_x,G;\mathbb{A}_F)\to\mathfrak{E}(I_x,G;\mathbb{A}_F)$$

is an isomorphism at finite places. Finally recall that there is an exact sequence of pointed sets

$$1\to I_x(\mathbb{A}_F)\backslash G(\mathbb{A}_F)\to\mathbf{H}^0(\mathbb{A}_F,I_x\backslash G)\to\mathfrak{D}(I_x,G;\mathbb{A}_F)\to 1\ .$$

Thus we get a canonical measure on $\mathbf{H}^0(\mathbb{A}_F,I_x\backslash G)$ and on $\mathfrak{A}(\mathbb{A}_F,\delta^*,L)$ we call the Tamagawa measure. As already seen in IV.2 there is a natural map

$$\mathfrak{A}(\mathbb{A}_F,\delta^*,L)\to\mathfrak{E}(I^*,G^*;\mathbb{A}_F/F)\ .$$

Given a character κ of $\mathfrak{E}(I^*,G^*;\mathbb{A}_F/F)$ we define a new kind of adelic κ-orbital integrals:

$$\mathrm{O}^\kappa_{\delta^*}(f)=\int_{\mathfrak{A}(\mathbb{A}_F,\delta^*,L)}\kappa(x)e(I_x)f(\delta_x)\,dx\ .$$

If $\mathfrak{A}(\mathbb{A}_F,\delta^*,L)$ is empty $\mathrm{O}^\kappa_{\delta^*}(f)=0$. If $\mathfrak{A}(\mathbb{A}_F,\delta^*,L)$ is non empty choose an x as above. By abuse of notation, we denote again by κ the function on $\mathbf{H}^0(\mathbb{A}_F,I_x\backslash G)$ obtained via the composed map

$$\mathbf{H}^0(\mathbb{A}_F,I_x\backslash G)\to\mathbf{H}^0_{ab}(\mathbb{A}_F,I_x\backslash G)\to\mathbf{H}^0_{ab}(\mathbb{A}_F,I^*\backslash G^*)\to\mathfrak{E}(I^*,G^*;\mathbb{A}_F/F)$$

(see IV.1.4). Then, for $x\in G^*(\overline{\mathbb{A}}_F)$ defining a point in $\mathfrak{A}(\mathbb{A}_F,\delta^*,L)$ we have

$$\mathrm{O}^\kappa_{\delta^*}(f)=\kappa(x)e(I_x)\mathrm{O}^\kappa_{\delta_x}(f)\ .$$

The next proposition is the generalization to arbitrary elliptic elements of a result of [KS] for strongly regular elements; we borrow their proof. A similar statement is given in [LBC, Proposition 2.7.3] but the proof there is incorrect.

V.2.1. Proposition. – *Consider the sum*

$$S = \sum_{\{\kappa \in \mathfrak{K}(I^*, G^*; F) \ | \ [\kappa]=\omega\}} \mathrm{O}^{\kappa}_{\delta^*}(f) \ .$$

Then $S = 0$ unless δ^* is the prenorm of some $\delta \in L(F)$ in which case

$$S = \sum_{\{\kappa \in \mathfrak{K}(I_\delta, G; F) \ | \ [\kappa]=\omega\}} \mathrm{O}^{\kappa}_{\delta}(f)$$

Proof: Assume the sum S does not vanish, then $\mathfrak{A}(\mathbb{A}_F, \delta^*, L)$ is non empty. Consider $x_0 \in \mathfrak{A}(\mathbb{A}_F, \delta^*, L)$; it defines a $\delta_0 \in L(\mathbb{A}_F)$ with prenorm δ^* locally everywhere. Recall that we have an exact sequence

$$\mathbf{H}^0_{ab}(\mathbb{A}_F, I^* \backslash G^*) \to \mathfrak{E}(I^*, G^*; \mathbb{A}_F/F) \to \ker^1_{ab}(F, I^* \backslash G^*) \to 1 \ .$$

Let us denote by $\mathfrak{K}(I^*, G^*; F)_0$ the subgroup of $\mathfrak{K}(I^*, G^*; F)$ which is the Pontryagin dual of $\ker^1_{ab}(F, I^* \backslash G^*)$. This is a finite group (see [Lab] Lemme 2.5). For each place, the image of the map in IV.1.4 lies in the kernel of $\kappa_0 \in \mathfrak{K}(I^*, G^*; F)_0$. Hence, for $\kappa_0 \in \mathfrak{K}(I^*, G^*; F)_0$ we have

$$\mathrm{O}^{\kappa_0 \kappa}_{\delta^*}(f) = \kappa_0(x_0) \mathrm{O}^{\kappa}_{\delta^*}(f) \ .$$

Since S does not vanish this implies that the finite sum

$$\sum_{\kappa_0 \in \mathfrak{K}(I^*, G^*; F)_0} \kappa_0(x_0)$$

does not vanish and hence the class of x_0 in $\mathfrak{E}(I^*, G^*; \mathbb{A}_F/F)$ lies in the image of $\mathbf{H}^0_{ab}(\mathbb{A}_F, I^* \backslash G^*)$. But by IV.1.2 we know that this image equals the image of the cohomology group over the finite adeles $\mathbf{H}^0_{ab}(\mathbb{A}^{\infty}_F, I^* \backslash G^*)$. Using that at finite places the maps in IV.1.4 are isomorphisms, we see that up to changing x_0 to x_1 and replacing δ_0 by a locally everywhere stably conjugate $\delta_1 \in L(\mathbb{A}_F)$ we may assume that the class of x_1 in $\mathfrak{E}(I^*, G^*; \mathbb{A}_F/F)$ is trivial. It follows from IV.2.2 that this is possible only if δ_1 is $G(\mathbb{A}_F)$-conjugate to $\delta = \delta_x \in L(F)$ with $x \in G^*(\overline{F})$. In such a case, thanks to the reciprocity law for Kottwitz signs and using IV.2.1 to identify $\mathfrak{E}(I_\delta, G; \mathbb{A}_F/F)$ with $\mathfrak{E}(I^*, G^*; \mathbb{A}_F/F)$, we see that :

$$\mathrm{O}^{\kappa}_{\delta^*}(f) = \mathrm{O}^{\kappa}_{\delta}(f) \ .$$

□

We say that a triple (T, δ^*, κ) is L-admissible if (δ^*, T) is an L-admissible special pair and κ is an endoscopic character for δ^* i.e. a character of the compact group $\mathbf{H}^0(\mathbb{A}_F/F, I^* \backslash G^*)$ where I^* is the stable centralizer of δ^*. Two admissible triples (T, δ^*, κ) and (S, η^*, ν) are said to be equivalent if there is an $x \in G^*$ that induces an equivalence between the special L-admissible pairs (δ^*, T) and (η^*, S) and between the endoscopic pairs (T, κ) and (S, ν). Let (T, δ^*, κ) be an L-admissible triple and let $\mathcal{E} = (T, \kappa)$. We introduced in IV.3.3 the number $N(T, \delta^*, \kappa)$. This is the number of L-admissible triples (T, δ^*, ν) equivalent to (T, δ^*, κ).

V.2.2. Theorem. – *The elliptic trace can be expanded as a sum over the equivalence classes of L-admissible triples (T, δ^*, κ), such that $[\kappa] = \omega$:*

$$\mathbf{T}_e(f, \omega) = \tau(L) \sum_{\{(T, \delta^*, \kappa) \ | \ [\kappa]=\omega\}} \frac{N(T, \delta^*, \kappa)}{\tilde{\iota}_{G^*}(\delta^*)} \mathrm{O}^{\kappa}_{\delta^*}(f^0) \ .$$

Proof: This is an immediate consequence of V.1.3 and V.2.1.

□

This generalizes the theorem 6.4.C in [KS] where they consider only strongly regular elements. Observe that for a strongly regular element δ^* one has

$$N(T,\delta^*,\kappa)=\tilde{\imath}_{G^*}(\delta^*)=1\ .$$

V.3 – Endoscopic transfer and end of the stabilization

In [KS] the definition of the transfer involves twisted stable orbital integrals on endoscopic groups (see Lemma 7.3.C). Here we use instead stable orbital integrals on endoscopic spaces; this is equivalent. To state the transfer assumption in full generality it may be necessary to consider central extensions of endoscopic spaces and to use functions on endoscopic spaces that transform according to a character of the abelian group involved in the central extension. For the sake of simplicity we shall ignore this difficulty here and we refer the reader to [KS] for a discussion of this point.

V.3.1. Definition. – *Consider $f\in\mathcal{C}_c^\infty(L(\mathbb{A}_F))$ and an endoscopic pair $\mathcal{E}=(T,\kappa)$. Let H be the endoscopic space defined by $\mathcal{E}$. We say that a function $f\in\mathcal{C}_c^\infty(L(\mathbb{A}_F))$ satisfies the transfer assumption for $\mathcal{E}$ if there exist $f_\mathcal{E}\in\mathcal{C}_c^\infty(H(\mathbb{A}_F))$, such that*

$$\mathrm{O}^1_\gamma(f_\mathcal{E})=\mathrm{O}^\kappa_{\delta^*}(f)$$

whenever $\gamma\in H(F)$ is the conorm of δ^*.

Observe that, by definition of $\mathrm{O}^\kappa_{\delta^*}(f)$, we have $\mathrm{O}^1_\gamma(f_\mathcal{E})=0$ unless γ is locally everywhere the norm of a $\delta\in L(\mathbb{A}_F)$.

Remark – A detailed discussion of the transfer assumption would involve the definition of the local transfer factors and the fundamental lemma. As regards transfer factors we refer the reader to [KS] for their definition in the case of strongly regular elements. As regards the fundamental lemma this is an open problem in most cases.

Let H be a central form of $H^*\rtimes 1$ where H^* is a quasi-split connected reductive group. Given $f\in\mathcal{C}_c^\infty(H(\mathbb{A}_F))$, the elliptic part of the geometric side of the stable trace formula for H is, by definition, the expression

$$\mathbf{ST}^H_e(f)=\tau(H)\sum_{\gamma\in\{\{H_e\}\}}\frac{1}{\tilde{\imath}_{H^*}(\gamma)}\,\mathrm{O}^1_\gamma(f^0)\ .$$

If H is an endoscopic space defined by an endoscopic pair $\mathcal{E}=(T,\kappa)$ for a twisted space L we consider a slight variant of this expression: we introduce

$$\mathbf{ST}^\mathcal{E}_e(f)=\tau(H)\sum_{\gamma\in\{\{H_e\}\}_L}\frac{1}{\tilde{\imath}_{H^*}(\gamma)}\,\mathrm{O}^1_\gamma(f^0)\ .$$

where the sum is over a set $\{\{H_e\}\}_L$ of representatives of stable conjugacy classes of elements γ in H_e such that there exists a special L-admissible pair (δ^*,T) above γ and κ is an endoscopic character for δ^*. Moreover we attach to $\mathcal{E}$ the scalar:

$$a(\mathcal{E})=\frac{\tau(L)}{\lambda(\mathcal{E})\,\tau(H)\,c(L,H)}=\frac{J(L)\,\tau(G)}{\lambda(\mathcal{E})\,\tau(H)\,c(L,H)\,d(L)}$$

where $\lambda(\mathcal{E})$ is the cardinal of the group of outer automorphisms of the endoscopic pair $\Lambda(\mathcal{E})$ and $c(L,H)$ is the ratio of Haar measures on the isomorphic vector spaces $\mathfrak{A}_L$ and $\mathfrak{A}_H$, this last isomorphism being induced by the conorm map n_θ. In particular, given $\phi \in \mathcal{C}_c^\infty(\mathfrak{A}_H)$ we have

$$\int_{\mathfrak{A}_H} \phi(\zeta) d_H\zeta = c(L,H) \int_{\mathfrak{A}_L} \phi(n_\theta(z)) d_L z$$

We may now state and prove the final theorem. It generalizes the formula 7.4.4 in [KS]. Recall that our $\mathbf{T}_e$ contains the contribution of all elliptic elements while in [KS] only strongly regular elements are taken into account.

V.3.2. Theorem. – *Let $\mathcal{E}(\omega)$ be a set of representatives of equivalence classes of endoscopic pairs $\mathcal{E} = (T,\kappa)$ such that $[\kappa] = \omega$. If f satisfies the transfer assumption V.3.1 the elliptic part of the geometric side of the trace formula can be written*

$$\mathbf{T}_e(f,\omega) = \sum_{\mathcal{E}\in\mathcal{E}(\omega)} a(\mathcal{E}) \ \mathbf{ST}_e^{\mathcal{E}}(f_{\mathcal{E}})$$

Proof: By definition

$$\mathbf{ST}_e^{\mathcal{E}}(f) = \tau(H) \sum_{\gamma\in\{\{H_e\}\}_L} \frac{1}{\tilde\iota_{H^*}(\gamma)} \ \mathrm{O}_\gamma^1(f^0) \ .$$

Then

$$\sum_{\mathcal{E}\in\mathcal{E}(\omega)} a(\mathcal{E}) \ \mathbf{ST}_e^{\mathcal{E}}(f_{\mathcal{E}}) = \sum_{\mathcal{E}\in\mathcal{E}(\omega)} \frac{\tau(L)}{\lambda(\mathcal{E}) \ c(L,H)} \sum_{\gamma\in\{\{H_e\}\}_L} \frac{1}{\tilde\iota_{H^*}(\gamma)} \ \mathrm{O}_\gamma^1(f_{\mathcal{E}}^0) \ .$$

The sum on the right hand side is over a set of representatives of equivalence classes of endoscopic pairs (T,κ) such that $[\kappa] = \omega$. Since f satisfies the transfer assumption we have

$$\mathrm{O}_\gamma^1(f_{\mathcal{E}}^0) = c(L,H) \ \mathrm{O}_{\delta^*}^\kappa(f^0)$$

where γ is the conorm of δ^* and hence

$$\sum_{\mathcal{E}\in\mathcal{E}(\omega)} a(\mathcal{E}) \ \mathbf{ST}_e^{\mathcal{E}}(f_{\mathcal{E}}) = \tau(L) \sum_{\mathcal{E}\in\mathcal{E}(\omega)} \ \sum_{\gamma\in\{\{H_e\}\}_L} \frac{1}{\lambda(\mathcal{E}) \, \tilde\iota_{H^*}(\gamma)} \ \mathrm{O}_{\delta^*}^\kappa(f^0) \ .$$

It follows from IV.3.2 that given $\gamma \in H_e$, and up to equivalence on the pair $\mathcal{E} = (T,\kappa)$, we may assume that T_{H^*} contains γ. But then (T,δ^*,κ) is an L-admissible triple. Hence we may rewrite the right hand side of last expression as

$$\tau(L) \sum_{\{(\mathcal{E},\gamma)\}} \frac{1}{\lambda(\mathcal{E}) \, \tilde\iota_{H^*}(\gamma)} \ \mathrm{O}_{\delta^*}^\kappa(f^0)$$

where the sum is over pairs $(\mathcal{E},\gamma)$ that comes from an L-admissible triple (T,δ^*,κ) up to equivalence on endoscopic pairs $\mathcal{E} = (T,\kappa)$ such that $[\kappa] = \omega$ and up to stable conjugacy for γ in H. On the other hand, according to V.2.2 we have

$$\mathbf{T}_e(f,\omega) = \tau(L) \sum_{\{(T,\delta^*,\kappa) \mid [\kappa]=\omega\}} \frac{N(T,\delta^*,\kappa)}{\tilde\iota_{G^*}(\delta^*)} \ \mathrm{O}_{\delta^*}^\kappa(f^0) \ ,$$

where the sum is over equivalence classes of L-admissible triples (T,δ^*,κ) such that $[\kappa] = \omega$. The theorem now follows from IV.3.4.

□

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