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SOME PECULIAR PROPERTIES OF THE RELATIVISTIC OSCILLATOR IN THE POSTGALILEAN APPROXIMATION*

by

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Abstract

In the post-Galilean approximation, the Lagrangians are singular on a submanifold of the phase space. A general analysis of these peculiarities, which differ by the ones considered by Dirac, is, up to day, lacking. The analysis of the dynamics of the one-dimensional and two-dimensional relativistic oscillators of the second tensor rank, is reported. A comparison with the cases of scalar and vector relativistic oscillators, described by a regular Lagrangian, is accomplished. Some equilibrium statistical properties of the relativistic oscillators are also analysed.

1. Introduction

The approximation in which the corrections of order $O(c^{-2})$, c denoting the light velocity, to the Galilei Group are taken in account, is called the *post-Galilean approximation.*

The Postgalilean approximation has a *quasi* classical character. It seems to be sufficient for all known laboratory relativistic systems (e.g. high- temperature plasma in the investigations on the nuclear fusion where $\frac{v^2}{c^2} \cong 10^{-1}$) as well in astrophysics.

A post-Galilean Lagrangian for vectorial particles with Coulomb potential, was constructed in 1920 by C.G.Darwin [1] and post-Galilean Lagrangians for gravitational forces were constructed in 1950-1954 by I.G.Fichtengoltz, V.A.Fock and L.Infeld [2]. To the same period belong the Dirac's formulations of the constraints's method for the quantization of the gravitational field and of the problem of constructing a Hamiltonian theory of relativistic N-body interacting systems [3].

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Then years later it was obtained [4] that, choosing as canonical variables the positions of the bodies and their conjugate momenta, only a not interacting Hamiltonian can generate the *Poincare' - invariant world lines.* This constitutes the main content of the so-called *no interaction theorem.* The same theorem was also stated, later, in the Lagrangian formulation [5].

However interacting Lagrangians can be constructed in the post-Galilean approximations [6], while at the order $O(c^{-4})$ the accelerations can no longer be eliminated from the Lagrangian. Thus, the post-Galilean approximation is singled out and has a *quasi* classical character. This differs from the Galilei group, where the Lagrangians have the form $T(\mathbf{v})+V(\mathbf{q})$, $\mathbf{v}$ and $\mathbf{q}$ denoting the velocities and the positions, and where the corresponding Hessians are not vanishing; the post-Galilean Lagrangians of the form $T(\mathbf{v})+W(\mathbf{q},\mathbf{v})$ can be singular with respect to Lebesgue's measure zero [7]. These peculiarities have a different physical and mathematical nature than the ones, considered by Dirac, whose corresponding Hessians vanish on the whole phase space.

A system of relativistic particles interacting via a pair-potential φ_{ij} is described in the post-Galilean approximation by the Lagrangian [6,7]:

$$\begin{aligned} L = &- mc^2 \sum_i \sqrt{1-\frac{(\mathbf{v}_i)^2}{c^2}} + \sum_{i<j} \Bigg\{ -\varphi_{ij} + \\ &+ \frac{1}{2c^2}\Big(\left[(1-\chi)(\mathbf{v}_i-\mathbf{v}_j)^2 + \mathbf{v}_i\cdot\mathbf{v}_j\right]\varphi_{ij} - \\ &- \frac{(\mathbf{r}_{ij}\cdot\mathbf{v}_i)(\mathbf{r}_{ij}\cdot\mathbf{v}_j)}{r_{ij}}\varphi'_{ij}\Big)\Bigg\} \end{aligned} \tag{1}$$

Here the particles are supposed to have the same mass m, $\mathbf{v}_i$ denotes the velocity, $\mathbf{r}_{ij}$ the relative coordinate of the i and j particles and $\varphi_{ij}(r_{ij})$ the interaction between them. The greek letter χ denotes the tensor rank of the interaction.

In ref. [8,9] it has been remarked that for such a Lagrangian the Hessian:

$$J = \det \left\| \frac{\partial^2 L}{\partial \mathbf{v}_i \partial \mathbf{v}_j} \right\|$$

vanishes on a submanifold of the tangent bundle of the configuration manifold, henceforth called *phase space.*

The general form of the Hessian is not appealing . For a system of N identical particles of masses $m=1$ in the two-dimensional case it reads:

$$J = \begin{vmatrix} D_1 & \psi_{12} & \dots & \psi_{1N} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \psi_{1N} & \psi_{2N} & \dots & D_N \end{vmatrix},$$

where $\forall\ \ i,k \in (1,\dots,N)$:

$$D_i = \frac{c}{\sqrt{1-\frac{v_i^2}{c^2}}} \begin{pmatrix} 1+\frac{(v_i^1)^2}{c^2-v_i^2} & \frac{v_i^1 v_i^2}{c^2-v_i^2} \\ \frac{v_i^1 v_i^2}{c^2-v_i^2} & 1+\frac{(v_i^2)^2}{c^2-v_i^2} \end{pmatrix},$$

$$\psi_{ik} = \begin{pmatrix} \psi_{ik}^{11} & \psi_{ik}^{12} \\ \psi_{ik}^{12} & \psi_{ik}^{22} \end{pmatrix}, \qquad \psi_{ik}^{\alpha\nu} = \frac{1}{2c^2}\left(\varphi_{ik} - \frac{r_{ik}^{\alpha} r_{ik}^{\nu}}{r_{ik}}\varphi'_{ik}\right).$$

Here $\mathbf{v}_i = (v_i^1, v_i^2)$ and $\mathbf{r}_{ik} = (r_{ik}^1, r_{ik}^2)$. It is obvious that all $\psi_{ik}^{\alpha\nu}$ are bounded if φ_{ik} and $r_{ik}\varphi'_{ik}$ are bounded.

In the simple case of the Coulombian interaction between two particles ($\varphi = 1/r$, $N = 2$), for the section $v = 0$, it reduces to:

$$J = \begin{pmatrix} \begin{matrix} c & 0 \\ 0 & c \end{matrix} & \psi \\ \psi & \begin{matrix} c & 0 \\ 0 & c \end{matrix} \end{pmatrix}, \qquad \psi = \frac{1}{2rc^2}\begin{pmatrix} 1 + (r^1)^2/r^2 & 1 + r^1 r^2/r^2 \\ 1 + r^1 r^2/r^2 & 1 + (r^2)^2/r^2 \end{pmatrix},$$

and it is not difficult to see that J is degenerated on some line $\varphi(r^1, r^2) = 0$ of this section. This happens for small values of r.

The analysis of the dynamics on the submanifolds of degeneration is a difficult task in the general case.

A first step can be provided by the analysis of the more simple Lagrangian of the post-Galilean oscillator[1]

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} - kr^2\left[1 + \frac{(\chi - 1)}{2c^2}v^2\right] \tag{1a}$$

As it will be shown, for such a Lagrangian the degeneration manifold corresponds to very large values of r and for $\chi \geq 2$, while in the case of the Coulomb potential to small values of r and for $\chi = 1$ (vector field). The Hessians of the *more physical* scalar and vector oscillators are positive everywhere in the phase-space. However, the example of the Coulomb potential suggests that the form of the potential φ is more relevant than the value of the tensor rank of the interaction χ.

We start with a brief analysis of the motion of scalar and vector oscillators.

2. Scalar and vector oscillators

The analysis is here only limited to the one-dimensional oscillator.

Denoting with q the position of the particle, the momentum p and the energy E associated with the Lagrangian (1a) (with $\mathbf{r} \equiv q$), are given by:

$$p = \frac{\partial L}{\partial v} = \frac{mv}{\sqrt{1 - v^2/c^2}} + \frac{(1-\chi)}{c^2}kq^2 v, \tag{2}$$

$$E = vp - L = \frac{mc^2}{\sqrt{1 - v^2/c^2}} + \left(1 + \frac{1-\chi}{2c^2}\right)kq^2. \tag{3}$$

---

[1] It is worthwhile to observe that different definitions of relativistic oscillator can be given. Here, as the no-interaction theorem[5] does not permit to take in account the interaction terms of the orders $O(c^{-2j})$ with $j = 2, 3, \ldots$, we took the one which follows from the general formula (1)

The corresponding Hessian

$$J = \frac{\partial^2 L}{\partial v^2} = \frac{m}{(1 - v^2/c^2)^{3/2}} + \frac{1-\chi}{c^2} k q^2, \tag{4}$$

is positive for $\chi = 0, 1$.

Vector Oscillator.

The more simple case, corresponding to $\chi = 1$, is called the vector oscillator. Its energy E is given by $E = mc^2(1 - v^2/c^2)^{-1/2} + kq^2$ from which

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \pm \frac{\sqrt{(E - kq^2)^2 - m^2c^4}}{E - kq^2}$$

The integration of this equation, with initial condition $q(0) = 0$, gives

$$-\frac{1}{\sqrt{2 + kQ^2/mc^2}}\; F\left(\arcsin\frac{q}{Q}, \frac{1}{\sqrt{1 + 2mc^2/kQ^2}}\right) +$$
$$+\sqrt{2 + kQ^2/mc^2}\; E^*\left(\arcsin\frac{q}{Q}, \frac{1}{\sqrt{1 + 2mc^2/kQ^2}}\right) = \pm\frac{\omega_0 t}{\sqrt{2}},$$

where Q is the maximal elongation ($q = Q$ when $v = 0$), and $\omega_0 = \sqrt{2k/m}$ the classical frequency. F and E^* denote the elliptic integrals of the first kind and second kind, respectively:

$$F(x,k) = \int_0^x \frac{\mathrm{d}x'}{\sqrt{(1 - x'^2)(1 - kx'^2)}}, \qquad k^2 < 1,$$

$$E^*(x,k) = \int_0^x \sqrt{\frac{1 - kx'^2}{1 - x'^2}}\mathrm{d}x'.$$

The motion is periodic with the period:

$$T = \frac{4\sqrt{2}}{\omega_0}\left[\sqrt{2 + \frac{kQ^2}{mc^2}}\; E^*\left(\frac{\pi}{2}, \frac{1}{\sqrt{1 + 2mc^2/kQ^2}}\right) - \right.$$
$$\left. - \frac{1}{\sqrt{2 + kQ^2/mc^2}}\; F\left(\frac{\pi}{2}, \frac{1}{\sqrt{1 + 2mc^2/kQ^2}}\right)\right].$$

The amplitude Q is derived from the equation

$$E = mc^2 + kQ^2,$$

which, for $E = \lambda mc^2$, gives

$$Q(\lambda) = \pm\sqrt{(\lambda - 1)\frac{m}{k}}\ c = \pm Q(2)\sqrt{\lambda - 1} \tag{5}$$

with $Q(2) = c\sqrt{m/k}$. The condition $kQ^2 \ll mc^2$ gives the classical formulas.

Scalar Oscillator

In next case, corresponding to $\chi = 0$ and called the scalar oscillator, the following law of the motion

$$F\left(\frac{\pi}{2}, \sqrt{\frac{E - mc^2}{2E}}\right) - F\left(\arccos\sqrt{\frac{kq^2}{E - mc^2}}, \sqrt{\frac{E - mc^2}{2E}}\right) = \pm\sqrt{\frac{2k}{E}}\ c\ t$$

can be easily derived.

This also is a periodic motion with period T given by:

$$T = 4\sqrt{\frac{E}{2k}}\ \frac{1}{c}\left[F\left(\frac{\pi}{2}, \sqrt{\frac{E - mc^2}{2}}\right) - F\left(0, \sqrt{\frac{E - mc^2}{2}}\right)\right]$$

The amplitude Q is still given by formula (5).

The law of the motion, for $kQ^2 \ll mc^2$, reduces to the classical one: $q(t) = Q\sin\omega_0 t$.

3. One-dimensional oscillator of tensor rank $\chi = 2$

The Lagrangian, the energy and the Hessian read:

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} - kq^2\left(1 + \frac{v^2}{2c^2}\right),$$

$$E = f(q, v) := \frac{mc^2}{\sqrt{1 - v^2/c^2}} + \left(1 - \frac{v^2}{2c^2}\right)kq^2, \tag{3a}$$

$$J = \frac{m}{(1 - v^2/c^2)^{3/2}} - \frac{kq^2}{c^2}, \tag{4a}$$

So we get the *curve* of degeneration:

$$S(q, v) :=\ kq^2 - \frac{mc^2}{(1 - v^2/c^2)^{3/2}} = 0. \tag{6}$$

Henceforth Q, which loses sometimes, as it will be shown, the meaning of amplitude, will be called a *stop point.*

From ($3a$) and (6) it follows that the stop point Q belongs to S iff $\lambda = 2$.

It is worth to remark that where the Hessian vanish, the Euler-Lagrange equations cannot be put in *normal form*. In other words the acceleration $\dot{v}$ does not exist when $J = 0$. As a matter of fact, the Euler-Lagrange equation:

$$J\dot{v} = -(\partial^2_{qv} L)v + \partial_q L$$

for the given Lagrangian takes the form

$$J\dot{v} = -2kq(1 - v^2/2c^2). \tag{7}$$

The existence of $\dot{v}$ and $J = 0$ would imply $q = 0$. This, obviously, contrasts with (4a) which, for $q = 0$, imply $J = m(1 - v^2/c^2)^{-3/2} > 0$.

In the analysis of the motion, only the points of $S(q, v)$ which also belongs to the curve of constant energy $E = \lambda mc^2$,

$$\begin{cases} \lambda mc^2 - f(q, v) = 0 \\ S(q, v) = 0 \end{cases}.$$

are relevant[2].

The positions, q_d, of degeneration are found from the equation $\lambda mc^2 - f(q, v(q)) = 0$, and the correspondent velocities v_d from the equation $\lambda mc^2 - f(q(v), v) = 0$.

With $\gamma := \frac{1}{\sqrt{(1-v^2/c^2)}}$ we have

$$\gamma^3 + 3\gamma - 2\lambda = 0 \qquad (\lambda \geq 2), \tag{8}$$

.

Then $q_d = \gamma^{3/2} c\sqrt{m/k} = \gamma^{3/2} Q(2)$ and the corresponding value of the velocity can be evaluated from:

$$\lambda mc^2 = E = \frac{2mc^2}{(1 - v^2/c^2)^{3/2}} \left(1 - \frac{3}{4}\frac{v^2}{c^2}\right) \tag{8$'$}$$

The condition $\lambda \geq 2$ in (8) obviously follows observing that the function F in the r.h.s. of equation (8$'$) , defining the velocity at intersection with the surface of fixed energy,

$$F(x) := \frac{2(1 - 3x/4)}{(1 - x)^{3/2}}, \qquad x := \frac{v^2}{c^2},\ x \in [0, 1)$$

has the properties:

$$F(0) = 2,\ \lim_{x \to 1} F(x) = +\infty,\ \ F'(x) = \frac{3}{4}\frac{1 - x/2}{(1 - x)^{5/2}} > 0 \qquad \text{for} \ \ x \in [0, 1)$$

[2] It is worthwhile to observe that, on S, the definition of the energy $E(q, v)$ becomes *formal* as its constancy $\dot{E} = 0 \quad (\dot{E} = [J\dot{v} + 2kq(1 - v^2/2c^2)]v)$ cannot be verified without the use of $\dot{v}$ which does not exist on S.

It is easy to show that for each λ the equation (8) has one real and two complex solutions. Further details on the motion can be obtained considering the acceleration of the particle.

The acceleration

The simmetry of the motion allows to consider the semiaxe $q \geq 0$ only. From the Euler-Lagrange equation (7), then, it follows the acceleration $\dot{v}$ is negative when $J > 0$, does not exist when $J = 0$ and positive when $J < 0$.

Replacing (3a) in (4a), with the energy, expressed, as before, in mc^2 units: $E = \lambda mc^2$, we obtain

$$J = m\frac{F(x) - \lambda}{1 - x/2}, \qquad x \in [0, 1).$$

As $F(x)$ is strongly monotone and greater than 2, we have:
a)for $\lambda < 2$, the peculiar points do not exist and the acceleration is negative,
b)for $\lambda = 2$, $\dot{v} < 0$ everywhere, except the stop-point $Q(2) = c\sqrt{m/k}$, (at $q = Q(2)$, $\dot{v}$ does not exist and $v = 0$).
c) for $\lambda > 2$, the equation (8′) can be written as $\lambda = F(x)$. As F is monotone it has only one solution $x_d = v_d^2/c^2$ for the given λ. The acceleration is positive for $x < x_d$ and negative for $x > x_d$, $\dot{v}$ does not exist at $x = x_d$ ($J < 0$ if $x < x_d$, $J = 0$ if $x = x_d$, $J > 0$ if $x > x_d$).

So (q_d, x_d) is a point of the intersection of the curve S of degeneration with the curve of constant energy in the space of points $(q, v^2/c^2)$.

To better understand the situation let us find the mutual dispositions of the points q_d and the stop-points Q, corresponding to the same energy λ.

As it was shown:
a) For $\lambda < 2$, the point q_d does not exist.
b) For $\lambda = 2$, $q_d = Q$.
c) For any $\lambda > 2$, $|q_d| > |Q|$.

In last case, infact, from (8) and (5) we get

$$Q = \pm\sqrt{\frac{(\lambda - 1)m}{k}}\, c = \sqrt{\frac{\gamma^3 + 3\gamma}{2} - 1}\sqrt{\frac{m}{k}}\, c,$$

and then:

$$q_d = \pm\gamma^{3/2}\sqrt{\frac{m}{k}}\, c.$$

So it is sufficient to show that the function

$$\varphi(\gamma) := \gamma^3 - \left(\frac{\gamma^3 + 3\gamma}{2} - 1\right) = \frac{\gamma^3}{2} - \frac{3}{2}\gamma + 1$$

is, for[3] $\gamma > 1$, positive .

[3] From (8), for $\lambda > 2$ we have $\gamma > 1$

This easily follows observing that $\varphi(1) = 0$ and $\varphi'(\gamma) = \frac{3}{2}(\gamma^2 - 1)$.

We can give to our system the energy $E = \lambda mc^2$ placing the particle at any point (q, v) on the curve $(3a)$. On the non negative semiaxe $q \geq 0$:

For $\lambda < 2$, we have everywhere a periodic motion.

For $\lambda > 2$ and the starting point $(Q, 0)$, the acceleration, as $Q < q_d$, turns out to be negative and the particle begin to move towards the origin $q_0 = 0$.

For $\lambda > 2$ and a starting point (q, v) of the line $(3a)$, with $q > Q$, $v \neq 0$, the particle goes away from the origin, with positive starting acceleration which becomes negative when the velocity overcomes v_d.

The distance $|q|$ as a function of v^2

Further informations, about the points of degeneration of the motion, can be obtained analyzing the function $q^2(v^2)$. It is tedious, but easy, to chek that it has a local maximum at the peculiar points v_d^2.

We have,from $(3a)$ with $E = \lambda mc^2$, $kq^2 = mc^2 g(x)$, $x = v^2/c^2$, with[4]

$$g(x) := \frac{\lambda - 1/\sqrt{1-x}}{1 - x/2}.$$

Then

$$\frac{\mathrm{d}q^2}{\mathrm{d}v^2} = \frac{mc^2}{k} g'(x), \qquad \frac{\mathrm{d}^2 q^2}{\mathrm{d}(v^2)^2} = \frac{m}{kc^2} g''(x)$$

with

$$g'(x) = \frac{-2 + 3x/2 + \lambda(1-x)^{3/2}}{2(1-x/2)^2(1-x)^{3/2}}. \tag{9}$$

and

$$g''(x) = \frac{3(1 - \lambda/\sqrt{1-x})}{4(1-x/2)^2(1-x)^{3/2}} < 0$$

By using $(4a)$, once λ has been expressed, by the equation (8), in terms of x, we get[5]:

$$g'(x_d) = 0, \qquad g''(x_d) < 0$$

with $x_d := v_d^2/c^2 \in [0, 1)$.

For $\lambda > 2$ the particle changes by jump its velocity from v_d to $-v_d$, as, after the minimum, the distance $|q|$ between its position and the origin q_0 must decrease.

For $\lambda = 2$, the particle can either go towards the origin or preserve the velocity $v = 0$. Then either the acceleration is negative, or does not exist. So for $\lambda = 2$, the point $(c\sqrt{m/k}, 0)$ is an attractor.

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[4] It is easy to see that $|q| < \sqrt{2m\lambda/k}\; c = \sqrt{2\lambda}\; Q(2)$

[5] It is possible to prove that $|q| \leq q_d$. So the local maximum is a global one also.

4. Two-dimensional oscillator of tensor rank $\chi = 2$

The Lagrangian $(1a)$, with $r^2 = q_1^2 + q_2^2$ and $v^2 = v_1^2 + v_2^2$, gives the momenta p_i and the energy E:

$$p_1 = \frac{\partial L}{\partial v_1} = \left(\frac{m}{\sqrt{1 - v^2/c^2}} - k\frac{r^2}{c^2} \right) v_1,$$

$$p_2 = \frac{\partial L}{\partial v_2} = \left(\frac{m}{\sqrt{1 - v^2/c^2}} - k\frac{r^2}{c^2} \right) v_2$$

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} + kr^2 \left(1 - \frac{v^2}{2c^2} \right). \qquad (3b)$$

As for the Hessian J we have

$$J = \begin{vmatrix} \partial p_1/\partial v_1 & \partial p_1/\partial v_2 \\ \partial p_2/\partial v_1 & \partial p_2/\partial v_2 \end{vmatrix} =$$

$$= \begin{vmatrix} \frac{m(c^2 - v_2^2)}{(c^2 - v^2)\sqrt{1 - v^2/c^2}} - kr^2/c^2 & \frac{mv_1 v_2}{(c^2 - v^2)\sqrt{1 - v^2/c^2}} \\ \frac{mv_1 v_2}{(c^2 - v^2)\sqrt{1 - v^2/c^2}} & \frac{m(c^2 - v_2^2)}{(c^2 - v^2)\sqrt{1 - v^2/c^2}} - kr^2/c^2 \end{vmatrix} =$$

$$= (kr^2 - \frac{mc^2}{(\sqrt{1 - v^2/c^2})^3})(kr^2 - \frac{mc^2}{\sqrt{1 - v^2/c^2}})$$

Thus we get two submanifolds of degeneration:

$$S_1 := kr^2 - \frac{mc^2}{(\sqrt{1 - v^2/c^2})^3} = 0$$

$$S_2 := kr^2 - \frac{mc^2}{(\sqrt{1 - v^2/c^2})} = 0$$

with the common radius $r_d = r'_d = c\sqrt{m/k}$ which correspond to the energy $E = 2mc^2$ ($\lambda = 2$, $v = 0$). Here S_1 formally[6] coincide with S of the section 3. The expression of the energy also is formally similar to the one of the 1-dimensional case.

Then the equation (8) can be used to find the radiuses of the intersections:

$$\lambda mc^2 - f(r, v) = 0, \qquad S_1(r, v) = 0$$

i.e. $r_d = q_d$. They will be called the *peculiarities of the first type.*

[6] In this case however $q^2 = r^2 = q_1^2 + q_2^2$ and $v^2 = v_1^2 + v_2^2$

The intersections given by:

$$\lambda mc^2 - f(r,v) = 0, \qquad S_2(r,v) = 0$$

will be called the *peculiarities of the second type.*

The intersections of the surface of constant energy with the surface S_2 are found as in the previous section

$E = mc^2 F_1(\frac{v^2}{c^2})$ with $F_1(\frac{v^2}{c^2}) = \frac{1}{\sqrt{1-v^2/c^2}}\left(2 - \frac{v^2}{2c^2}\right)$.

As, $E = mc^2 F_1(x)$, with $F_1(0) = 2$, $\lim_{x\to 1} F_1(x) = +\infty$, and

$$F_1'(x) = x\frac{1 + x^2/2}{(1-x)^{3/2}} > 0, \quad x \in [0,1),$$

the energy, corresponding to the intersection, is greater than $2mc^2$ (or $\lambda \geq 2$).

Owing the limitation $\lambda > 2$, the only peculiar radius of the intersection between S_2 and the submanifold of constant energy $E = \lambda mc^2$, is:

$$r_d' = c\sqrt{\frac{m}{3k}}\sqrt{\lambda + \sqrt{\lambda^2 - 3}} = Q(2)\sqrt{\frac{\lambda + \sqrt{\lambda^2-3}}{3}}, \quad \lambda \geq 2 \tag{10}$$

according with our geometric insight.

The peculiarities of the first type

It was already shown that for the family (10) $r_d' = |Q|$ if $\lambda = 2$. Further for $\lambda > 2$ it turns out that $r_d' < |Q|$.

As a matter of fact, the inequality $r_d' < |Q|$ can be written as (see formula (5))

$$\lambda + \sqrt{\lambda^2 - 3} < 3(\lambda - 1)$$

or

$$(\lambda - 2)^2 > 0.$$

So for a given $\lambda > 2$

$$Q(2) < r_d' < Q < r_d \tag{11}$$

and $r_d' = r_d = Q$ if $\lambda = 2$.

In other words, the function $r^2(v^2)$ has the local maximum at the peculiar points of the first type corresponding to the surface S_1.

We can only conclude that, in the two-dimensional case, S_1 reflects the moving particle in such a way that the velocity $v(t)$ is not smooth at the instant of the reflection.

The peculiarities of the second type

The peculiarities (r_d', x_d') have a different character.

Owing the validity of formula (9) in two-dimensional case too, we that $\partial r^2/\partial v^2 = 0$ when

$$-2 + 3x/2 + \lambda(1-x)^{3/2} = 0, \quad x = v^2/c^2.$$

Replacing here $x(r)$ from the equation $kr^2 = mc^2(1-x)^{-1/2}$, which represents S_2, we get the condition

$$-\frac{1}{2} + \frac{m^2c^4}{k^2r^2}\left(\lambda\frac{mc^2}{kr^2} - \frac{3}{2}\right) = 0$$

For $r = r'_d$ given by formula (10) it takes, with $y = \frac{3}{\lambda+\sqrt{\lambda^2-3}}$, the form

$$2\lambda y^3 - 3y^2 - 1 = 0.$$

We can use the standard formula to express the solution $\tilde{y}(\lambda)$ of the previous equation. It is easy to show that[7] $\tilde{y}(\lambda) \neq \frac{3}{\lambda+\sqrt{\lambda^2-3}}$ if $\lambda > 2$. It turns out that:

$$\forall\lambda > 2, \qquad \partial r^2/\partial v^2 \neq 0 \quad at v = v'_d$$

At this point a special feature of the peculiarities of the second type must be described.

Outside of the peculiar radiuses, the acceleration can be calculated with the help of the Euler-Lagrange equations by:

$$\dot{v}_1 = \Delta_1/J, \; \dot{v}_2 = \Delta_2/J. \tag{12}$$

However, as follows from equation (11), J changes the sign at $r = r'_d < Q$, where Q is the point corresponding to the velocity $v = 0$

In other words, in order to study the behaviour the of the particle in a neighboroud of Q we cannot use the *normal form* (12) of the Euler-Lagrange equations. They must be used in the form:

$$\begin{aligned} \frac{\partial p_1}{\partial v_1}\dot{v}_1 + \frac{\partial p_1}{\partial v_2}\dot{v}_2 &= 2k\left[q_2\frac{v_1v_2}{c^2} - q_1\left(1 + \frac{v_2^2 - v_1^2}{2c^2}\right)\right], \\ \frac{\partial p_2}{\partial v_2}\dot{v}_1 + \frac{\partial p_2}{\partial v_2}\dot{v}_2 &= 2k\left[q_1\frac{v_1v_2}{c^2} - q_2\left(1 + \frac{v_1^2 - v_2^2}{2c^2}\right)\right], \end{aligned} \tag{13}$$

where $\partial p_2/\partial v_j$ are explicitlely written in the expression of the Hessian in section 4.

The functional form of Δ_1 and Δ_2 expressed by the square brackets in (13), is not so simple as in the one-dimensional case and do not permit to easily understand where they are positive and where they are negative.

It is, however, sufficient to evaluate the signes of Δ_1 and Δ_2 only in the small neighbourhood of $v = 0$.

[7] At $\lambda = 2$ we have $\tilde{y}(2) = 1 = \tilde{y}(2) = \frac{3}{\lambda+\sqrt{\lambda^2-3}}\Big|_{\lambda=2}$.

Developing Δ_1 and Δ_2 in series with respect to v_1^2/c^2, v_2^2/c^2 and v^2/c^2, we get[8].

$$\begin{aligned}\Delta_1 &= \begin{vmatrix} -2kq_1 & 0 \\ -2kq_2 & 1-kr^2/mc^2 \end{vmatrix} + \mathrm{O}(v^2/c^2) = -2kq_1[1-r^2/Q^2(2)] + \mathrm{O}(v^2/c^2), \\ \Delta_2 &= -2kq_2[1-r^2/Q^2(2)] + \mathrm{O}(v^2/c^2)\end{aligned} \tag{14}$$

Formulas (14), together with (11), show that $\Delta_1 > 0$ and $\Delta_2 > 0$ in the small neighbourhood of Q defined by $r > r'_d > Q(2)$.

So, as at peculiar radius r'_d the Hessian J changes in sign, the acceleration *necessarily*, becomes negative when the particle approaches to the stop-point Q.

The circle of the radius $r_d = r'_d = Q(2) = c\sqrt{m/k}$ being on $S_1 \cap S_2$ might be invariant, so as the point $q = Q(2)$ in the one-dimentional case.

At the points of this circle the velocity is zero and the acceleration is indetermined. Nevertheless it is easy to show that, for $\lambda > 2$, the peculiar circles of the second type on S_2 cannot be invariant. On these circles, from

$$k{r'_d}^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} \quad \text{or} \quad \frac{\lambda+\sqrt{\lambda^2-3}}{3} = \frac{1}{\sqrt{1-v^2/c^2}},$$

it follows that $|v| = const \neq 0$

Thus the particle, in the case of invariance of this circle, has only one possibility:*to revolve with the constant frequency* ω:

$$q_1 = r'_d \cos\omega t, \;\; q_2 = r'_d \sin\omega t. \tag{15}$$

The accelerations on the manifold either take an infinity of values or are indetermined. In both cases we get a contradiction.[9]

5. One-dimensional equilibrium distributions

For scalar, vector and tensor rank $\chi = 2$ particles, in the one-dimensional case, we have respectively the following *dynamically invariant* Liouville-volums [8, 9]:

$$\mathrm{d}\mu_0 = \left(1-\frac{v^2}{c^2}\right)\mathrm{d}\mu, \;\; \mathrm{d}\mu_1 = \mathrm{d}\mu, \;\; \mathrm{d}\mu_2 = \frac{\mathrm{d}\mu}{1-v^2/c^2},$$

where

$$\mathrm{d}\mu = \frac{\mathrm{d}q\mathrm{d}v}{(1-v^2/c^2)^{3/2}}$$

[8] It is worthwhile to stress that in evaluating Δ_1 and Δ_2 at very small velocities, the term kr^2/mc^2 cannot be negliged as it is greater than 1 when $v \approx 0$. Thus if the energy is large and $E \neq T(\mathbf{v}) + V(\mathbf{q})$, the *relativistic corrections* can be relevant also when $v^2/c^2 \ll 1$.

[9] Formally, if S_2 is invariant and $\dot{v}_1$ exists (any from the infinitely many possible accelerations) we can write: $\dot{p}_1 = \dot{S}_2 v_1 + S_2 \dot{v}_1 = 0$ (see formulas at the beginning of this section). On the other hand replacing (15) in the r.h.s.of the first Euler-Lagrange equations (13) we obtain $\dot{p}_1 = -2kr'_d(1 + r'_d\omega^2/2c^2)\cos\omega t \neq 0$.

is the Poincaré-invariant measure. The *formal* equilibrium distribution functions (the densities of probability), for the above calculated oscillator-energies, takes the forms:

$$\text{for } \chi = 0, \quad \sum_v^0 \sqrt{\frac{\beta k}{\pi}} \left(1 - \frac{v^2}{c^2}\right) \exp\left\{-\beta\left[E_0 + \left(1 + \frac{v^2}{2c^2}\right) kq^2\right]\right\} \mathrm{d}\mu,$$

$$\text{for } \chi = 1, \quad \sum_v^1 \sqrt{\frac{\beta k}{\pi}} \exp\left[-\beta\left(E_0 + kq^2\right)\right] \mathrm{d}\mu,$$

$$\text{for } \chi = 2, \quad \sum_v^2 \sqrt{\frac{\beta k}{\pi}} \left(1 - \frac{v^2}{c^2}\right)^{-1} \exp\left\{-\beta\left[E_0 + \left(1 - \frac{v^2}{2c^2}\right) kq^2\right]\right\} \mathrm{d}\mu,$$

where β denotes the *inverse temperature*, $E_0 = mc^2(1 - v^2/c^2)^{-1/2}$ and

$$\left(\sum_v^0\right)^{-1} = \int_{-c}^{c} \left(1 - \frac{v^2}{c^2}\right)\left(1 + \frac{v^2}{2c^2}\right)^{-1/2} \mathrm{e}^{-\beta E_0} \frac{\mathrm{d}\mu}{\mathrm{d}q},$$

$$\left(\sum_v^1\right)^{-1} = \int_{-c}^{c} \mathrm{e}^{-\beta E_0} \frac{\mathrm{d}\mu}{\mathrm{d}q},$$

$$\left(\sum_v^2\right)^{-1} = \int_{-c}^{c} \left(1 - \frac{v^2}{c^2}\right)^{-1}\left(1 - \frac{v^2}{2c^2}\right)^{-1/2} \mathrm{e}^{-\beta E_0} \frac{\mathrm{d}\mu}{\mathrm{d}q}.$$

The integration over q from $-\infty$ to $+\infty$ gives the equilibrium velocity distributions for $\chi = 0, \quad \chi = 1, \quad \chi = 2,$:

$$\mathrm{d}F_0 = \sum_v^0 \sqrt{\frac{\beta k}{\pi}} \sqrt{\frac{1 - v^2/c^2}{1 + v^2/2c^2}} \mathrm{e}^{-\beta E_0} \mathrm{d}\mu_v = \sum_v^0 \sqrt{\frac{\beta k}{\pi}} F_0(v) \mathrm{d}v,$$

$$\mathrm{d}F_1 = \sum_v^1 \sqrt{\frac{\beta k}{\pi}} \frac{\mathrm{e}^{-\beta E_0}}{\sqrt{1 - v^2/c^2}} \mathrm{d}\mu_v = \sum_v^1 \sqrt{\frac{\beta k}{\pi}} F_1(v) \mathrm{d}v,$$

$$\mathrm{d}F_2 = \sum_v^2 \sqrt{\frac{\beta k}{\pi}} \left(1 - \frac{v^2}{c^2}\right)^{-3/2}\left(1 - \frac{v^2}{2c^2}\right)^{-1/2} \mathrm{e}^{-\beta E_0} \mathrm{d}\mu_v = \sum_v^2 \sqrt{\frac{\beta k}{\pi}} F_2(v) \mathrm{d}v,$$

with

$$\mathrm{d}\mu_v = \frac{\mathrm{d}v}{1 - v^2/c^2}.$$

The general behaviour can be better understood from the derivatives of the functions

F_0, F_1 and F_2 here reported:

$$F_0' = \frac{v}{c^2}\frac{e^{-\beta E_0}}{(1-v^2/c^2)^{3/2}\sqrt{1+v^2/2c^2}}\left[\frac{1+2v^2/c^2}{2+v^2/c^2}-\beta E_0\right],$$
$$F_1' = \frac{v}{c^2}\frac{e^{-\beta E_0}}{(1-v^2/c^2)^{5/2}}\left[3-\beta E_0\right],$$
$$F_2' = \frac{v}{c^2}\frac{e^{-\beta E_0}}{(1-v^2/c^2)^{7/2}\sqrt{1-v^2/2c^2}}\left[\frac{11-6v^2/c^2}{2-v^2/c^2}-\beta E_0\right].$$

Their signs change according to the expressions in the square brackets.

It is enough, owing the similarity of the three cases, to illustrate only the case F_1 which corresponds to the vector interaction ($\chi = 1$).

As $v > 0$ and $\beta mc^2 = 3\sqrt{1-v^2/c^2}$ imply $F_1' < 0$, for $\beta mc^2 \geq 3$, we have in general (for $\beta mc^2 \geq 3$), apart an irrelevant modification, the traditional type of distribution with one maximum at $v = 0$. But for $\beta mc^2 < 3$ (very high temperature) F_1 has the local minimum at $v = 0$ and two symmetric local maximums.

From the dynamical behavior the question arises whether or not it is necessary to correct both the equilibrium and the non-equilibrium statistical distributions.

First of all, in the configuration manifold, we must use the limitations $-Q \leq q \leq Q$ in the case $\lambda < 2$, and the limitations $-q_d \leq q \leq q_d$ in the case $\lambda \geq 2$.

Further we need to take into account the possibility of existence of the invariant points $q = c\sqrt{m/k}$ or of the invariant radius $r = c\sqrt{m/k}$ for $\lambda = 2$, towards to which the particles of the collection of oscillators can accumulate.

In the construction of the Liouville equation, from which the evolution law of the non-equilibrium distributions can be obtained, some difficulties derive from the non existence of the phase-velocity at peculiar points. May be this problem is similar to the one for the billiard's balls.

We were able to analyse the simple case of one-particle conservative system. The case of N-particle systems with exchange of the energy between them, can be still more complicated. We have to take into account the approximative character of the theory which, as in the case of the classical theory, must be stated without inner contradictions.

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