# A Note on Improving the Rate of Convergence of 'High Order Finite Elements' on Polygons

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# A NOTE ON IMPROVING THE RATE OF CONVERGENCE OF 'HIGH ORDER FINITE ELEMENTS' ON POLYGONS

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ABSTRACT. Let u and  $u_{V_n}$  be the solution and, respectively, the finite element solution of the Poisson's equation  $\Delta u = f$  with zero boundary conditions. We construct for any  $m \in \mathbb{N}$  and any polygon  $\mathbb{P}$  a sequence of finite dimensional subspaces  $V_n$  such that  $||u - u_{V_n}||_{H^1} \leq C \dim(V_n)^{-m/2} ||f||_{H^{m-1}}$ , where  $f \in H^{m-1}(\mathbb{P})$  is arbitrary and C is a constant that depends only on  $\mathbb{P}$ (we do not assume  $u \in H^{m+1}(\mathbb{P})$ ). Although the final result is in terms of the "usual" Sobolev spaces, the proof relies on estimates for the Poisson problem in Sobolev spaces with weights. Other " $h^m$ "-type approximation results are also obtained. This is an announcement, but some sketches of the proofs of the main results are provided. Full details of the proofs and complete references will be provided in a different paper.

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# INTRODUCTION

Let  $\mathbb{P}$  be a polygonal domain in the plane and  $m \in \mathbb{N} = \{1, 2, ...\}$ . In this paper we will construct a class of triangulations (or adaptive meshes) of  $\mathbb{P}$  that provide an asymptotic order of convergence = m.

Denote by  $H^m(\mathbb{P})$  the *m*th order Sobolev space on  $\mathbb{P}$ , with norm  $||u||_{H^m}$ . We shall consider the *Poisson problem* 

$$\begin{cases} \Delta u = g\\ u|_{\partial \mathbb{P}} = 0 \end{cases}$$

Let  $H_0^1(\mathbb{P})$  be the subspace of distributions in  $H^1(\mathbb{P})$  with vanishing trace on  $\mathbb{P}$ . The inner product of  $f_1, f_2 \in H^0(\mathbb{P}) = L^2(\mathbb{P})$  will be denoted by  $\langle f_1, f_2 \rangle$ . It will be

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convenient to write Poisson's problem in the form

(1) 
$$a(u, v) := \langle \nabla u, \nabla v \rangle = \langle f, v \rangle, \quad \forall v \in H_0^1(\mathbb{P}).$$

Let  $V \subset H^1_0(\mathbb{P})$  be a subspace, we shall denote by  $u_V$  the solution to the problem

(2) 
$$a(u_V, v) = \langle f, v \rangle, \quad \forall v \in V$$

It is a basic problem to construct finite dimensional subspaces  $V \subset H_0^1(\mathbb{P})$  such that the error  $||u - u_V||_{H^1}$  is small. We also want to achieve this in an "economic" way, that is, with dim(V) not too large.

To better explain our results, let us recall the following basic result in approximation theory [9, 10]. Let  $\mathcal{T} = (T_j)$  be a triangulation of  $\mathbb{P}$  with triangles. Let  $V = V(\mathcal{T}, m+1)$  be the finite element space of piecewise polynomials of order  $\leq m$ on  $\mathcal{T}$ . For any function  $u \in C(\mathbb{P})$ , we shall denote by  $u_I \in V = V(\mathcal{T}, m+1)$  the *interpolating* function associated to u. Then, if we consider on  $H^1$  the norm  $|\cdot|_1$ defined by the form  $a(\cdot, \cdot)$ , we have the standard inequality

(3) 
$$|u - u_V|_1 \le |u - u_I|_1$$

Using the equivalence of the  $\|\cdot\|_{H^1}$  and  $|\cdot|_1$ , we obtain

**Theorem 0.1.** Let  $V = V(\mathcal{T}, m + 1)$ . Assume that all triangles  $T_j$  of the triangulation  $\mathcal{T} = (T_j)$  of  $\mathbb{P}$  have angles  $\geq \alpha$  and edges of length  $\leq h$  and  $\geq ah$ . Then there exists an absolute constant  $C_1 = C_1(\alpha, m)$  such that

(4) 
$$c^{-1} \|u - u_V\|_{H^1} \le \|u - u_I\|_{H^1} \le C_1 h^m \|u\|_{H^{m+1}}$$

for any  $u \in H^{m+1}(\mathbb{P})$ . Similarly, there exists an absolute constant  $C_2 = C_2(\alpha, a, m)$  such that

(5) 
$$c^{-1} \|u - u_V\|_{H^1} \le \|u - u_I\|_{H^1} \le C_2 \left(area(\mathbb{P})/\dim(V)\right)^{m/2} \|u\|_{H^{m+1}}$$

The constants  $C_1$  and  $C_2$  above are "absolute" in the sense that they do not depend on the domain  $\mathbb{P}$ , its triangulation  $\mathcal{T}$ , or the function u. The constant c however depends on the polygon  $\mathbb{P}$ .

One can argue, based on Weyl's theorem on the asymptotic of eigenvalues of the Laplace operator with Dirichlet boundary conditions [15], that the estimates obtained by combining Theorem 0.1 and Equation (3) are optimal as far as the asymptotic order of convergence m, if  $u \in H^{m+1}(\mathbb{P})$ . However, it is *not* true, in general, that  $u \in H^{m+1}(\mathbb{P})$ , even if  $f = \Delta u \in \mathcal{C}^{\infty}(\mathbb{P})$ , because the boundary of  $\mathbb{P}$ is not smooth [11, 13]. On the other hand, Weyl's theorem mentioned above does not prevent similar asymptotic rates of convergence for polygons. However, it is known [3] that the only hope to achieve similar rates of convergence is to choose carefully the triangulation  $\mathcal{T}$ . It is the purpose of this paper to provide conditions on the triangulation  $\mathcal{T}$  under which such higher asymptotic rates of convergence are obtained.

More precisely, we shall construct for any  $\mathbb{P}$  a class  $\mathcal{C}(m, h, \epsilon, \kappa, \alpha, b)$  of partitions  $\mathcal{T}$  of  $\mathbb{P}$ , depending on  $m \in \mathbb{N}$  and some parameters  $h, \kappa, \alpha, \epsilon, b$ , such that the following theorem holds.

**Theorem 0.2.** There exists a constant  $B = B(\kappa, \alpha, l, \epsilon, b)$  such that for any polygon  $\mathbb{P}$  with any two vertices at distance  $\geq l$  and any partition  $\mathcal{T}$  in  $\mathcal{C}(m, h, \epsilon, \kappa, \alpha, b)$ we have

$$||u - u_V||_{H^1} \le B \dim(V)^{-m/2} ||f||_{H^{m-1}}, \quad \forall f \in H^{m-1}(\mathbb{P}).$$

Here  $V = V(\mathcal{T}, m+1)$  and u and  $u_V$  are the solutions to Equations (1) and (2).

The precise meaning of the constants  $h, \kappa, \alpha, \epsilon$ , and b is explained in Section 4. Suffices to say now that m is the asymptotic rate of convergence, h is the largest admissible length of the sides of the triangles in the partition,  $\kappa$  controls the decay of the triangles as they approach a vertex,  $\alpha$  is the minimum admissible angle of a triangle in the partition,  $0 < \epsilon < \pi/\alpha_1$ , where  $\alpha_1$  is the largest angle of the polygon, and b controls the ratio of the size of close triangles. The constants  $h, \kappa, \alpha, \epsilon$ , and bmust satisfy certain conditions for the class  $\mathcal{C}(m, h, \kappa, \alpha, \epsilon, b)$  to be non-empty. The following result is therefore relevant.

**Theorem 0.3.** For any polygon  $\mathbb{P}$  there exist  $0 < \epsilon \leq 1$ ,  $\alpha > 0$ , 1 > b > 0 and a sequence  $h_n \to 0$  such that, if  $\kappa = 2^{-m/\epsilon}$ , the class  $\mathcal{C}(m, h_n, \kappa, \alpha, \epsilon, b)$  is not empty.

The constants  $\alpha$  and b can be written explicitly in terms of  $\epsilon$  and the geometry of  $\mathbb{P}$ .

We should point out at this time that our results are not yet in the most suitable form for applications. This is because it is important to chose  $\alpha$  and  $\kappa$  large and b < 1 close to 1 in order to decrease the constant  $B = B(\kappa, \alpha, l, \epsilon, b)$ . Too small or too large values for  $\epsilon$  will increase the error. Also, it is not clear when the class  $C(m, \kappa, \alpha, \epsilon, b)$  is not empty. We hope to deal with this issues in later papers.

The proofs use some estimates on the Dirichlet problem in Sobolev spaces with weights [8, 13, 14]. These estimates follow from the results in [14], see Section 2.

We suggest that the reader consults also the very nice paper of Babuška, Kellogg, and Pitkäranta [3] for some related results. In fact, our results when m = 1 can be recovered from the results in that paper. Our method is, however, different because we do not use "singular functions."

Throughout this paper, "x := y" will mean that "x" is defined to be equal to "y," as customary.

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## 1. Sobolev spaces with weights

We now recall the definition of Sobolev spaces with weights [8, 12, 13] and establish some properties of these spaces needed for our results.

1.1. Notation. We shall use the standard notation and denote by  $\partial^{\alpha} := \partial_{1}^{\alpha_{1}} \partial_{2}^{\alpha_{2}}$ , a constant coefficient differential monomial on  $\mathbb{R}^{2}$ , for any multi-index  $\alpha = (\alpha_{1}, \alpha_{2}) \in \mathbb{Z}_{+}^{2}$ . Also,  $|\alpha| := \alpha_{1} + \alpha_{2}$ . By  $L^{2}(\Omega)$ , we shall denote the space of square integrable functions on an a subset set  $\Omega \subset \mathbb{R}^{2}$  with respect to the usual Lebesgue measure, with norm  $||u||_{L^{2}}^{2} := \int_{\Omega} |u(x)|^{2} dx$ . Also, by  $L^{2}_{loc}(\Omega)$  we shall denote the space of functions on  $\Omega$  whose restriction to any compact subset K of  $\Omega$  is in  $L^{2}(K)$ .

First, let us recall that the *m*th Sobolev space  $H^m(\Omega)$ ,  $\Omega \subset \mathbb{R}^2$  open, is defined by

$$H^{m}(\Omega) := \{ u \in L^{2}(\Omega), \ \partial^{\alpha} u \in L^{2}(\Omega), \ \forall |\alpha| \leq m \},\$$

and is endowed with the norm

$$||u||_{H^m}^2 := \sum_{|\alpha| \le m} ||\partial^{\alpha} u||_{L^2}^2.$$

(Compare with the definition of the norm on Sobolev spaces with weights below.) We agree that  $||u||_{H^m} = \infty$  if  $u \notin H^m(\Omega)$ . Also, by  $H^m_{loc}(\Omega)$  we shall denote the

space of functions on  $\Omega$  whose restriction to any relatively compact, open subset V of  $\Omega$  is in  $H^m(V)$ .

1.2. Weighted Sobolev spaces. A crucial role in our proof is played by a certain modification of the usual definition above of Sobolev spaces. For the simplicity of the presentation, we shall assume from now on that  $\mathbb{P}$  is a triangle all of whose angles are acute. The general case is similar, but the notation is more complicated. To make a choice, we shall decide that  $\mathbb{P}$  is an open set.

We now introduce some notation that will remain fixed throughout the paper. Let l be the length of the shortest edge of  $\mathbb{P}$ . We denote by  $Q_1, Q_2$ , and  $Q_3$  the vertices of  $\mathbb{P}$ . Let  $\mathbb{P}_{\delta}$  be the union of the three isosceles triangles that have an angle in common with  $\mathbb{P}$  and equal sides of length  $= \delta$ . The complement  $\mathbb{P} \setminus \mathbb{P}_{\delta}$  is a hexagon precisely when  $\delta < l/2$ . (In fact, we shall only need this construction when  $\delta \leq l/4$ .)

Fix in what follows a smooth function  $\rho : \overline{\mathbb{P}} \to [0, \infty)$  such that  $\rho(x) =$  the distance from x to the closest vertex of  $\mathbb{P}$ , for  $x \in \mathbb{P}_{l/4}$ , and  $l/4 \leq \rho(x) \leq l$ , for  $x \in \mathbb{P} \setminus \mathbb{P}_{l/4}$ . We are ready now to recall the following definition [8, 13].

**Definition 1.1.** Let  $m \in \mathbb{Z}_+$  and  $a \in \mathbb{R}$ . The *m*th Sobolev space with weight  $\rho^a$  on  $\Omega \subset \mathbb{P}$ ,  $\Omega$  open, is the space  $\mathcal{K}_a^m(\Omega)$  defined by

$$\mathcal{K}^m_a(\Omega) := \{ u \in L^2_{loc}(\Omega), \ \rho^{|\alpha| - a - 1} \partial^{\alpha} u \in L^2(\Omega), \ \forall |\alpha| \le m \}, \quad \alpha \in \mathbb{Z}^2_+.$$

The norm on  $\mathcal{K}_a^m$  is

$$||u||_{\mathcal{K}^m_a}^2 := \sum_{|\alpha| \le m} ||\rho^{|\alpha| - a - 1} \partial^{\alpha} u||_{L^2}^2.$$

Standard arguments show that  $\mathcal{K}_a^m(\mathbb{P})$  is complete (and hence a Hilbert space).

Our notation is slightly different from the one in above mentioned papers, in that the value of the weight parameter a is shifted:  $\mathcal{K}_a^m = V_{2,m-a-1}^m$ . This simplifies certain formulas.

1.3. Some lemmas. We now record some properties of the spaces  $\mathcal{K}_a^m(\mathbb{P})$  that will be needed in what follows. All the following properties follow from straightforward calculations and are, for the most part, well known.

Since most of the functions spaces used below are defined on  $\mathbb{P}$ , we shall often omit  $\mathbb{P}$  from the notation. We shall thus write  $\mathcal{K}_a^m := \mathcal{K}_a^m(\mathbb{P}), L^2 = L^2(\mathbb{P})$ , and so on. We shall denote  $\rho^a W = \{\rho^a f, f \in W\}$ , for any space of functions W. Below, an isomorphism of Banach spaces is a continuous bijection.

**Lemma 1.2.** The function  $\rho^{|\beta|-a}\partial^{\beta}\rho^{a}$  is bounded on  $\mathbb{P}$ .

This gives:

**Lemma 1.3.** We have  $\mathcal{K}_{-1}^0 = L^2$  and  $\rho^a \mathcal{K}_b^m = \mathcal{K}_{a+b}^m$ . Moreover, multiplication by  $\rho^a$  gives rise to an isomorphism  $\mathcal{K}_b^m \to \mathcal{K}_{a+b}^m$ .

We have the following inclusions:

**Lemma 1.4.** Let  $m \ge m'$  and  $a \ge a'$ . We have: (a)  $\|u\|_{\mathcal{K}^{m'}_{a'}} \le l^{a-a'} \|u\|_{\mathcal{K}^{m'}_{a'}}$ . (b)  $\|u\|_{\mathcal{K}^{m'}_{a'}} \le \delta^{a-a'} \|u\|_{\mathcal{K}^{m'}_{a'}}$ , if  $u \in \mathcal{K}^{m}_{a}(\mathbb{P}_{\delta})$ ,  $0 < \delta \le l/4$ . (c)  $\mathcal{K}^{m}_{a} \subset \mathcal{K}^{m'}_{a'}$ . **Lemma 1.5.** We have  $||u||_{H^m} \leq M ||u||_{\mathcal{K}_{m-1}^m}$  and  $||u||_{\mathcal{K}_{-1}^m} \leq M ||u||_{H^m}$ , where  $M = \max\{1, l^m\}$ .

Corollary 1.6.

$$\mathcal{K}_{m+a-1}^m \subset \rho^a H^m \subset \mathcal{K}_{a-1}^m.$$

The following lemma asserts that the  $H^m$  and  $\mathcal{K}^m_a$ -norms are equivalent on  $H^m(\Omega)$ , for any region  $\Omega$  on which  $\rho$  is bounded from below. More precisely, we have.

**Lemma 1.7.** Let  $0 < \delta < l/4$  and let  $\Omega \subset \mathbb{P}$  be an open subset such that  $\rho \geq \delta$  on  $\Omega$ . Then  $\|u\|_{H^m} \leq M_1 \|u\|_{\mathcal{K}^m_a}$  and  $\|u\|_{\mathcal{K}^m_a} \leq M_2 \|u\|_{H^m}$ , for any  $u \in H^m(\Omega)$ , where  $M_1 := \max\{l^{a+1}, l^{-m+a+1}, \delta^{a+1}, \delta^{-m+a+1}\}$  and, similarly,  $M_2 := \max\{l^{-a-1}, l^{-m+a+1}, \delta^{-a-1}, \delta^{m-a-1}\}$ .

The following lemma compares the weighted Sobolev spaces to the usual Sobolev spaces close to the vertices.

**Lemma 1.8.** Let  $0 < \delta < \min\{l/4, 1\}$  and  $\Omega \subset \mathbb{P}_{\delta}$  be an open subset. Then  $\|u\|_{H^m} \leq \delta^{a-m+1} \|u\|_{\mathcal{K}^m_a}$ , if  $a \geq m-1$ , and  $\|u\|_{\mathcal{K}^m_a} \leq \delta^{-a-1} \|u\|_{H^m}$ , if  $a \leq -1$ . Also,  $\|u\|_{\mathcal{K}^m_t} \leq \delta^{a-a'} \|u\|_{\mathcal{K}^m_a}$ , for any  $a \geq a'$ .

One of the main reasons for using the weighted Sobolev spaces is the homogeneity of their norms. We first need to introduce dilations for certain functions defined on  $\mathbb{P}_{\delta}$ . Assume for a moment that  $Q_1 = O := (0,0)$ , the origin of the coordinate system. Let  $\lambda > 0$  and let  $\Omega \subset \mathbb{P}_{l/4} \cap \mathbb{P}_{\lambda l/4}$  be completely contained in the triangle closest to  $Q_1$ . Also, let v be a function defined on  $\Omega$ . Then we define  $v_{\lambda}(x) := v(\lambda x)$ for any  $x \in \lambda^{-1}\Omega$ . (The conditions on  $\Omega$  are formulated so that this definition makes sense.) In general, if  $\Omega \subset \mathbb{P}_{l/4} \cap \mathbb{P}_{\lambda l/4}$ , but is not necessarily contained in a single connected component of  $\mathbb{P}_{l/4} \cap \mathbb{P}_{\lambda l/4}$ , we define  $v_{\lambda}(x)$  by translating each  $Q_j$ , j = 1, 2, 3 to the origin first.

**Lemma 1.9.** Let  $\lambda > 0$  and let  $\Omega \subset \mathbb{P}_{l/4} \cap \mathbb{P}_{\lambda l/4}$  be an open subset. Then  $\|u_{\lambda}\|_{\mathcal{K}^m_a} = \lambda^a \|u\|_{\mathcal{K}^m_a}$  for any  $u \in \mathcal{K}^m_a(\Omega)$ .

The above lemma also explains why we are choosing a different normalization for the weight factor  $\rho^a$ . (See the comment after our definition of weighted Sobolev spaces.)

We shall need a well known alternative definition of the Sobolev spaces with weights. Assume again, for a moment, that  $Q_1 = O := (0, 0)$ , the origin of the coordinate system. Let  $\Omega \subset \mathbb{P}_{l/4}$  be completely contained in the triangle closest to  $Q_1$ . Then the vector fields  $\partial_r$  and  $\partial_{\theta}$  are defined using polar coordinates on  $\Omega$ . For general  $\Omega \subset \mathbb{P}_{l/4}$ , we define these vector fields by translations (or, which is the same thing, by considering polar coordinates centered at either of the vertices  $Q_j$ ). Then  $\rho = r$ ,  $\partial_r \rho = 1$  and  $\partial_{\theta} \rho = 0$  on  $\mathbb{P}_{l/4}$ , by definition.

Lemma 1.10. We have

$$\mathcal{K}^m_a(\mathbb{P}) = \{ u \in H^m_{loc}(\mathbb{P}), \ r^{-a-1}(r\partial_r)^i \partial^j_\theta u \in L^2(\mathbb{P}_{l/4}), \ \forall i+j \le m \}.$$

We conclude our list of lemmas on the weighted Sobolev spaces with the following result, which is also well known.

**Lemma 1.11.** Let P be a constant coefficient differential operator of order k on  $\mathbb{R}^d$ . Then P defines a continuous map  $P : \mathcal{K}^m_a(\mathbb{P}) \to \mathcal{K}^{m-k}_{a-k}(\mathbb{P}), m \geq k$ .

### 2. Estimates for Poisson's equation

We shall need the following estimates on the solutions of Poisson's equation. First, let us notice that  $\mathcal{K}_a^m(\mathbb{P}) \subset H^m_{loc}(\mathbb{P})$ . Thus, if  $m \geq 1$ , the trace

(6) 
$$u|_{\partial \mathbb{P}} \in H^{m-1/2}_{loc}(\partial \mathbb{P}), \quad u \in \mathcal{K}^m_a$$

is defined. (One can give a more precise description of the range of this trace, or restriction, map, [2, 14], but this will not be needed in what follows. See also [1].) The following theorem is a slight variation on a result in [16].

**Theorem 2.1.** Let  $\mathbb{P}$  be a polygon in the place. Then the map

(7) 
$$\Delta: \mathcal{K}_0^{m+2}(\mathbb{P}) \cap \{ u \in H^1_{loc}(\mathbb{P}), \ u|_{\partial \mathbb{P}} = 0 \} \to \mathcal{K}_{-2}^m(\mathbb{P}), \quad m \ge 0,$$

is an isomorphism.

We shall write  $\mathcal{K}_0^{m+2}(\mathbb{P}) \cap \{u|_{\partial \mathbb{P}} = 0\} := \mathcal{K}_0^{m+2}(\mathbb{P}) \cap \{u \in H^1_{loc}(\mathbb{P}), u|_{\partial \mathbb{P}} = 0\}$  in what follows, for simplicity.

*Proof.* Let  $\Delta_C$  be the Laplace operator associated to the metric  $g_C := \rho^{-2}g_E$ , where  $g_E$  is the Euclidean metric on  $\mathbb{R}^2$ . Then the main result of [14] asserts that  $\Delta_C$  defines an isomorphism

$$\Delta_C: \mathcal{K}_0^{m+2}(\mathbb{P}) \to \mathcal{K}_{-2}^m(\mathbb{P})$$

The result then follows from  $\Delta = \rho^{-2} \Delta_C$ .

The above theorem can be found in a slightly different form in [16].

**Corollary 2.2.** For any polygon  $\mathbb{P}$ , there exists a constant  $\eta > 0$ , depending on  $\mathbb{P}$ , such that

(8) 
$$\Delta: \mathcal{K}^{m+2}_{\epsilon}(\mathbb{P}) \cap \{u|_{\partial \mathbb{P}} = 0\} \to \mathcal{K}^{m}_{\epsilon-2}, \quad m \ge 0$$

is an isomorphism for any  $|\epsilon| < \eta$ .

It is possible to show that  $\eta = \pi/\alpha_1$ , where  $\alpha_1$  is the largest angle of  $\mathbb{P}[16]$ .

# 3. A modified Bramble-Hilbert Lemma

As we have explained in the introduction, we are looking for extensions of the well known Theorem 0.1.

Let  $M(l, \delta, \alpha) := C(\alpha)M_1M_2$ , where  $C(\alpha)$  is as in Theorem 0.1 and  $M_1$  and  $M_2$  are as in Lemma 1.7.

**Theorem 3.1.** Fix  $\alpha > 0$  and  $0 < \delta < l/4$ . Let  $\mathbb{P}$  be a triangle with the shortest edge = l and  $\Omega \subset \mathbb{P}$  be a polygonal domain such that  $\rho \ge \delta$  on  $\Omega$ . Let  $\mathcal{T} = (T_j)$  be a triangulation as of  $\Omega$  with triangles with angles  $\ge \alpha$  and sides  $\le h$ . Then

(9) 
$$\|u - u_I\|_{\mathcal{K}^1_{\alpha}} \le M(l,\delta,\alpha)h^m \|u\|_{\mathcal{K}^{m+1}_{\alpha}}$$

for any  $u \in \mathcal{K}^{m+1}_{\epsilon}(\Omega)$ .

*Proof.* This follows from Theorem 0.1 and the equivalence of the  $H^m$  and  $\mathcal{K}_a^{m-1}$  norms on  $\Omega$  (Lemma 1.7).

We now extend Theorem 3.1 to trapezoids of the form  $\mathbb{P}_{\delta} \setminus \mathbb{P}_{\kappa\delta}$ . Let  $C_1(\kappa) = M(l, \kappa l/8, \alpha) (l/4)^m$ .

**Theorem 3.2.** Let  $\kappa, \alpha > 0$ , and  $0 < \delta < l/4$ . Let  $\mathcal{T} = (T_i)$  be a triangulation of  $\Omega := \mathbb{P}_{\delta} \setminus \mathbb{P}_{\kappa\delta}$  with triangles with angles  $\geq \alpha$  and edges  $\leq h$ . Then

(10) 
$$\|u - u_I\|_{\mathcal{K}^1_0} \le C_1(\kappa)\delta^{\epsilon}(h/\delta)^m \|u\|_{\mathcal{K}^{m+1}_{\epsilon}}$$

for any  $u \in \mathcal{K}^{m+1}_{\epsilon}(\Omega)$ .

*Proof.* We use Lemma 1.9 with  $\lambda = 4\delta/l$  to conclude that  $||u - u_I||_{\mathcal{K}_{0}^{m}} = ||u_{\lambda} - u_I||_{\mathcal{K}_{0}^{m}}$  $u_{I\lambda}||_{\mathcal{K}^m}$ . Then we notice that  $u_{I\lambda} = u_{\lambda I}$  (that is, dilation commutes with interpolation. Therefore, we can apply Theorem 3.1 to the region  $\lambda^{-1}\Omega = \mathbb{P}_{l/4} \setminus \mathbb{P}_{\kappa l/4}$ , the triangulation  $\lambda^{-1}\mathcal{T}$ , and the function  $u_{\lambda}$  to obtain that

$$\|u_{\lambda} - u_{\lambda I}\|_{\mathcal{K}_{0}^{1}} \leq M(l, \kappa l/8, \alpha) \left(\frac{hl}{4\delta}\right)^{m} \|u_{\lambda}\|_{\mathcal{K}_{0}^{m+1}}$$

This then gives

$$\begin{aligned} \|u - u_I\|_{\mathcal{K}_0^1} &= \|u_\lambda - u_{\lambda I}\|_{\mathcal{K}_0^1} \le C_1(\kappa)(h/\delta)^m \|u_\lambda\|_{\mathcal{K}_0^{m+1}} \\ &\le C_1(\kappa)\delta^\epsilon(h/\delta)^m \|u_\lambda\|_{\mathcal{K}_\epsilon^{m+1}}, \end{aligned}$$
here the last inequality is provided by Lemma 1.4b.

where the last inequality is provided by Lemma 1.4b.

#### 4. The main results

We now introduce the class of triangulations for which we will prove our main results, namely the class  $\mathcal{C}(m, h, \epsilon, \kappa, \alpha, b)$ . Then we prove our main approximations results, including the Theorems 0.2 and 0.3 announced in the introduction.

4.1. The class  $\mathcal{C}(m,h,\epsilon,\kappa,\alpha,b)$ . The following definition is formulated for the case  $\mathbb{P}$  an acute angle triangle, for simplicity. However, the case of a polygon is completely similar and will be discussed in the paper that will include the details of the proofs.

We continue to denote by l the shortest edge of the triangle  $\mathbb{P}$ . Also, recall the constant  $M(l, \delta, \alpha)$  introduced in Theorem 3.1.

**Definition 4.1.** Fix  $m \in \mathbb{N} = \{1, 2...\}$  and let  $\epsilon \in (0, 1]$ , h > 0,  $\kappa, b \in (0, 1)$ , and  $\alpha \in (0, \pi/2)$  be parameters. We define  $\mathcal{C}(m, h, \epsilon, \kappa, \alpha, b)$  to be the set of triangulations  $\mathcal{T}$  defined as follows. Choose *n* such that

 $\kappa^{n\epsilon} < M(l, \kappa l/8, \alpha) h^m.$ 

We decompose  $\mathbb{P}$  as the union of  $\Omega_0 := \mathbb{P} \smallsetminus \mathbb{P}_{l/4}, \, \Omega_1 := \mathbb{P}_{l/4} \smallsetminus \mathbb{P}_{\kappa l/4}, \, \dots \, , \, \Omega_n :=$  $\mathbb{P}_{\kappa^{n-1}l/4} \setminus \mathbb{P}_{\kappa^{n}l/4}$ , and  $\widetilde{\Omega}_{n+1} := \mathbb{P}_{\kappa^{n}l/4}$ . For each  $j = 0, \ldots, n$ , we triangulate  $\Omega_j$ with triangles with all angles  $\geq \alpha$ , and edges of length at most

(11) 
$$h_{n,j} = h \kappa^{(1-\epsilon/m)j}$$

and at least  $bh_{n,j}$ . Then  $\mathcal{T}$  is the union of the triangles appearing in the triangulations of  $\Omega_i$ ,  $j \leq n$ , and of the three triangles forming  $\Omega_{n+1}$ .

We begin with the following " $h^m$ "-approximation result for the triangulations in  $\mathcal{C}(m, h, \epsilon, \kappa, \alpha, b)$ .

**Theorem 4.2.** There exists a constant  $B_0 = B_0(l, \kappa, \alpha)$  such that

 $||u - u_I||_{\mathcal{K}^1_0} \le B_0 (2h)^m ||u||_{\mathcal{K}^{m+1}_1},$ (12)

for any triangulation  $\mathcal{T} \in \mathcal{C}(m, h, \epsilon, \kappa, \alpha, b)$  and any  $u \in \mathcal{K}^{m+1}_{\epsilon} \cap \{u|_{\mathbb{P}} = 0\}$ .

*Proof.* It is enough to establish the corresponding estimate for  $||(u - u_I)|_{\Omega_j}||_{\mathcal{K}^1_0}$ , for  $j = 0, 1, \ldots, n+1$  and  $B_0 = \max\{1, l\}M(l, \kappa l/8, \alpha)$ .

For j = 0, we use Theorem 3.1 for  $\Omega = \Omega_0$ . For j = 1, 2, ..., n, we use Theorem 3.2 for  $\Omega = \Omega_j$ . Then we notice that  $u_I = 0$  on  $\widetilde{\Omega}_{n+1}$  and hence

(13) 
$$\|(u-u_I)|_{\tilde{\Omega}_{n+1}}\|_{\mathcal{K}^1_0} = \|u|_{\tilde{\Omega}_{n+1}}\|_{\mathcal{K}^1_0} \le \delta^{\epsilon} \|u|_{\tilde{\Omega}_{n+1}}\|_{\mathcal{K}^{m+1}_{\epsilon}}$$

by Lemma 1.8, with  $\delta = \kappa^n l/4$ . The result then follows by adding the squares of all these estimates, using also Equation (11).

From this we obtain the following estimate on the error  $||u - u_V||_{H^1}$  of the Finite Element solution. From now on we shall assume that  $0 < \epsilon < 1$  is chosen such that

$$\Delta: \mathcal{K}_{\epsilon}^{m+1} \cap \{u|_{\mathbb{P}} = 0\} \to \mathcal{K}_{\epsilon}^{m-1}$$

be an isomorphism. This is possible due to Corollary 2.2.

**Theorem 4.3.** There exists a constant  $B'_0 = B'_0(l, \kappa, \alpha)$  such that

(14) 
$$\|u - u_V\|_{H^1} \le B_0'(2h)^m \|f\|_{H^{m-1}}$$

for any  $\mathcal{T} \in \mathcal{C}(m, h, \epsilon, \kappa, \alpha, b)$  and any  $f \in \mathcal{K}_{\epsilon}^{m-1}$ , where  $u \in \mathcal{K}_{\epsilon}^{m+1} \cap \{u|_{\mathbb{P}} = 0\}$  is the unique solution of  $\Delta u = f$ .

*Proof.* Let  $\nu_{\epsilon}$  be the norm of  $\Delta^{-1} : \mathcal{K}_{\epsilon}^{m-1} \to \mathcal{K}_{\epsilon}^{m+1} \cap \{u|_{\mathbb{P}} = 0\}$ . We have

$$\begin{aligned} \|u - u_V\|_{H^1} &\leq c \|u - u_I\|_{H^1} \\ &\leq M c \|u - u_I\|_{\mathcal{K}_0^1} \\ &\leq M c B_0 (2h)^m \|u\|_{\mathcal{K}_{\epsilon}^{m+1}} \\ &\leq \nu_{\epsilon} M c B_0 (2h)^m \|f\|_{\mathcal{K}_{\epsilon-1}^{m-1}} \\ &\leq \nu_{\epsilon} M^2 c B_0 (2h)^m \|f\|_{H^{m-1}} \end{aligned}$$

For the first inequality we first replace the norm  $||u||_{H^1}$  with the equivalent norm  $|u|_{H^1} = ||\nabla u||_{L^2}$  (use Poincaré's inequality) and use that  $u_V$  is the projection of u onto V in the inner product defined by  $|\cdot|_{H^1}$ . The second and fifth inequalities are obtained using Lemma 1.5 (use also  $\epsilon \leq 1$ ). The third inequality is obtained from Theorem 4.2. The fourth inequality is obtained from the invertibility of  $\Delta$  on the corresponding spaces.

The above theorems are satisfactory, except for one feature, namely that they do not give a bound for the dimension of the interpolating spaces  $V := V(\mathcal{T}, m+1)$ . This is remedied by the following result.

**Theorem 4.4.** There exists a constant  $B_1 = B_1(l, \epsilon, \kappa, \alpha, b)$  such that

(15) 
$$\|u - u_I\|_{\mathcal{K}^1_0} \le B_1 \dim(V)^{m/2} \|u\|_{\mathcal{K}^{m+1}_{\epsilon}}.$$

for any partition  $\mathcal{T} \in \mathcal{C}(m, h, \epsilon, \kappa, \alpha, b)$  and any  $u \in \mathcal{K}_{\epsilon}^{m+1}$ .

*Proof.* The area of each triangle in the triangulation of  $\Omega_j$  is bounded from below because it has all sides of length  $\geq bh_{n,j}$  and all angles  $\geq \alpha$ . The dimension of V is then bounded from above by estimating the minimum area of the triangles in the partition, which must be  $\leq area(\mathbb{P})$ .

4.2. The two main theorems. We now prove the two main theorems stated in the introduction. We begin with Theorem 0.2.

*Proof.* The proof of Theorem 0.2 is similar to that of Theorem 4.3, but using Theorem 4.4 instead of Theorem 4.2.  $\Box$ 

It remains to prove Theorem 0.3. This is achieved through the following example.

*Proof.* Fix  $\mathbb{P}$  and  $m \in \mathbb{N} = \{1, 2, ...\}$  arbitrary. Also, choose  $0 < \epsilon \leq 1$  such that  $\Delta : \mathcal{K}_{\epsilon}^{m+1} \cap \{u|_{\mathbb{P}} = 0\} \to \mathcal{K}_{\epsilon-2}^{m-1}$ 

be an isomorphism. Then choose  $\kappa = 2^{-m/\epsilon}$ .

For any triangulation  $\mathcal{T} = (T_j)$  of a polygonal domain  $\Omega$ , we shall denote by  $2^{-k}\mathcal{T}$  the triangulation of  $\Omega$  obtained by dividing each triangle  $T_j$  of  $\mathcal{T} = (T_j)$  into  $2^{2k}$  equal triangles, all similar to  $T_j$ . Also, if  $\mathcal{T} = (T_j)$  is a triangulation of  $\mathbb{P}_{\delta} \setminus \mathbb{P}_{\kappa\delta}$ , then we shall denote by  $(\mathcal{T})_{\lambda}$  the triangulation of  $\mathbb{P}_{\lambda\delta} \setminus \mathbb{P}_{\kappa\lambda\delta}$  obtained by applying a suitable similarity of ratio  $\lambda$  to each of the triangles of  $\mathcal{T}$  (the center of each similarity is the closest vertex to the triangle that is transformed).

We shall use the notation of Definition 4.1. Let  $\mathcal{T}_0$  be the triangulation of  $\Omega_0$ obtained by joining the baricenter of  $\Omega_0$  to each of the six vertices of  $\Omega_0$ . Write  $\Omega_1 = U_1 \cup U_2 \cup U_3$ , the union of its connected components. We triangulate each  $U_j$  into 5 triangles by joining its baricenter to the middle of the long basis and to each of its four vertices. We shall denote by  $\widetilde{\Omega}_{n+1}$  the triangulation of  $\widetilde{\Omega}_{n+1}$  into its three connected components.

Next, we define  $\alpha$  to be the least of the angles appearing in  $\mathcal{T}_0 \cup \mathcal{T}_1 \cup \Omega_{n+1}$ . Let h' be the shortest edge of  $\mathcal{T}_0 \cup \mathcal{T}_1$  and  $h = \kappa^{\epsilon/m-1}h'$ . Let  $b_0 < 1$  be the ratio of the shortest and the longest edges in  $\mathcal{T}_0$ . Define  $b_1$  similarly, as the ratio of the shortest and the longest edges in  $\mathcal{T}_0$ . We complete our set of choices by taking  $b = \min\{b_0, b_1\}\kappa^{\epsilon/m-1}$ .

We define

$$\mathcal{T} = 2^{-n} \mathcal{T}_0 \cup 2^{-n+1} \mathcal{T}_1 \cup_{j=2}^n 2^{-n+j} (\mathcal{T}_1)_{\kappa^{j-1}} \cup \widetilde{\Omega}_{n+1}.$$

Then  $\mathcal{T} \in \mathcal{C} := \mathcal{C}(m, 2^{-n}h, \epsilon, 2^{-n/\epsilon}, \alpha, b)$ . Hence  $\mathcal{C}$  is not empty. The statement of the theorem is obtained by taking  $h_n = 2^{-n}h$ .

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