ESI **The Erwin Schrödinger International Institute for Mathematical Physics**

Pasteurgasse 4/7
A-1090 Wien, Austria

C^*–dynamical systems for which the tensor product formula for entropy fails

Heide Narnhofer
Erling Størmer
Walter Thirring

Vienna, Preprint ESI 129 (1994)

September 8, 1994

Supported by Federal Ministry of Science and Research, Austria
Available via WWW.ESI.AC.AT

C*-dynamical systems for which the tensor product formula for entropy fails

by

Heide Narnhofer(*)
Erling Størmer(**)(+)
Walter Thirring(*)

(*) Institut für Theoretische Physik, Universität Wien, A-1090 Wien, Austria, and International Erwin Schrödinger Institute for Mathematical Physics, A-1090 Wien, Pasteurgasse 6, Austria.

(**) Department of Mathematics, University of Oslo, P.b. 1053, Blindern, N-0316 Oslo, Norway.

(+) Partially supported by The Norwegian Science Foundation.

C*-dynamical systems for which the tensor product formula for entropy fails

by

H. Narnhofer, E. Størmer and W. Thirring

1. Introduction.

In the present paper we study C*-dynamical systems which are highly non-asymptotically abelian. More specifically we consider a unital C*-algebra A with an automorphism α such that there is a self-adjoint subset S of A which together with the identity spans a dense subset of A, and with the property that the anticommutators $[\alpha^n(w), w^*]_+$ converge to 0 for some properly chosen sequence of n's depending on w for $w \in S$. It turns out that then there exists a unique α-invariant state ϕ, ϕ restricted to S is zero, and the entropy of α with respect to ϕ in the sense of [ST] in zero. Hence if A is nuclear the entropy in the sense of [CNT] vanishes, thus we have another example of a highly nonabelian C*-dynamical system with vanishing entropy.

Examples of systems as above can be found among the C*-algebras introduced by Powers [P], see also [Pr], in the study of binary shifts of the hyperfinite II_1-factor. The set S will consist of finite products of self-adjoint unitary operators $\{s_i\}_{i\in\mathbf{Z}}$ with the property that $s_i s_j = \pm s_j s_i$, and α is the shift $\alpha(s_i) = s_{i+1}$. If $w_1, w_2 \in S$ then $w_1 w_2 = \pm w_2 w_1$, hence the C*-subalgebra C of $A \otimes A$ generated by $w \otimes w, w \in S$, is abelian. As pointed out in [AN] $\alpha \otimes \alpha$ restricted to C is the baker's transform so has entropy $\log 2$. It follows that $h_{\phi\otimes\phi}(\alpha \otimes \alpha) \geq h_{\phi\otimes\phi|C}(\alpha \otimes \alpha|c) = \log 2 > 0 = h_\phi(\alpha) + h_\phi(\alpha)$, hence the tensor product formula "$h_{\phi\otimes\psi}(\alpha \otimes \beta) = h_\phi(\alpha) + h_\phi(\beta)$", see [SV], is false in general.

2. General results.

Let A be a unital C*-algebra, $\alpha \in \text{Aut } A$, and ϕ an α-invariant state. We shall show that if the C*-dynamical system (A, α) is highly nonasymtotically abelian then the entropy $H_\phi(\alpha)$ in the sense of [ST] in zero. For our purposes it is unnecessary to repeat the definition, only the following. Let B be an abelian C*-algebra with an automorphism β, μ a β-invariant state on B, and λ an $\alpha\otimes\beta$-invariant state on $A\otimes B$ such that $\lambda(a\otimes 1) = \phi(a), \lambda(1\otimes b) = \mu(b)$ for $a \in A, b \in B$. If $\lambda = \phi \otimes \mu$ then by [ST, 2.1] the "mutual information" $\varepsilon_\lambda(A, B) = 0$, hence the "conditional entropy" $H_\lambda(B|A) = H_\mu(B)$, so by [ST, 3.1] the entropy $h'(B, \lambda) = 0$. If $\lambda = \phi \otimes \mu$ is the only $\alpha \otimes \beta$-invariant state as above then by [ST, Lemma 3.2] the entropy $H_\phi(\alpha) = 0$.

It was also shown in [ST, Prop. 4.1] that if A is nuclear then $H_\phi(\alpha) = h_\phi(\alpha)$, where $h_\phi(\alpha)$ is the entropy of α with respect to ϕ in the sense of [CNT].

Recall that a subset $S \subset A$ is said to be *self-adjoint* if $a \in S$ implies $a^* \in S$, and *total* if its linear span is norm dense in A. We shall use the notation $[a,b]_+$ for the anticommutator

$$[a,b]_+ = ab + ba, \; a, b \in A.$$

Theorem 2.1. Let A be a unital C*-algebra and $\alpha \in \mathrm{Aut}A$. Suppose S is a self-adjoint subset of A such that $S \cup \{1\}$ is total in A and for which the following condition holds:

$$(*) \qquad \forall w \in S, \; \forall \varepsilon > 0, \; \forall N \in \mathbf{N} \text{ there exist } n_1, \cdots, n_N \in \mathbf{N}$$

such that if $i \neq j$ then

$$\|[\alpha^{n_i}(w^*), \alpha^{n_j}(w)]_+\| < \varepsilon, \quad i,j = 1, \cdots, N.$$

Then there exists a unique α-invariant state ϕ. ϕ satisfies $\phi|S = 0$, and the entropy $H_\phi(\alpha) = 0$.

The theorem generalizes [NT] and is an easy consequence of the following lemma, where we use the notation $\|x\|_{2,\phi} = \phi(x^*x)^{\frac{1}{2}}$ if $x \in A$ and ϕ is a state.

Lemma 2.2. Let A be a unital C*-algebra, $\alpha \in \mathrm{Aut}A$, and ϕ an α-invariant state. Suppose S is a self-adjoint subset of A for which $S \cup \{1\}$ is total in A and such that the following condition holds:

$$(**) \qquad \forall w \in S, \; \forall \varepsilon > 0, \; \forall N \in \mathbf{N} \text{ there exist } n_1, \cdots, n_N \in \mathbf{N}$$

such that if $i \neq j$ then

$$\|[\alpha^{n_i}(w^*), \alpha^{n_j}(w)]_+\|_{2,\phi} < \varepsilon, \quad i,j = 1, \cdots, N.$$

Suppose B is an abelian C*-algebra, $\beta \in \mathrm{Aut}B$, and μ a β-invariant state on B. Let λ be an $\alpha \otimes \beta$ invariant state on $A \otimes B$ such that $\lambda(a \otimes 1) = \phi(a), \lambda(1 \otimes b) = \mu(b), a \in A, b \in B$. Then $\lambda|S \otimes B = 0$ and $\lambda = \phi \otimes \mu$.

Proof. Note that if $a \in A, b \in B$ then by the Cauchy-Schwarz inequality

$$\begin{aligned} |\lambda(a \otimes b)| &= |\lambda((1 \otimes b)(a \otimes 1))| \leq \lambda((1 \otimes b)(1 \otimes b)^*)^{\frac{1}{2}} \lambda((a \otimes 1)^*(a \otimes 1))^{\frac{1}{2}} \\ &= 20 = \mu(bb^*)^{\frac{1}{2}} \phi(a^*a)^{\frac{1}{2}} = \|a\|_{2,\phi} \|b\|_{2,\mu}, \end{aligned}$$

=46urthermore, again since B is abelian

$$[a^* \otimes b^*, a \otimes b]_+ = [a^*, a]_+ \otimes b^*b.$$

Let now $w \in S, b \in B$, where we for simplicity assume $\|b\| \leq 1$. Let $\varepsilon > 0$, and choose $N \in \mathbf{N}$ so large that

$$\frac{1}{N} \lambda([w^*, w]_+ \otimes b^*b) < \varepsilon.$$

Using the inequality $|\rho(x)|^2 \leq \frac{1}{2}\rho([x^*,x]_+)$ for a state ρ, we have for $n_1,\cdots,n_N$ as given by $(**)$

$$\begin{aligned}
|\lambda(w\otimes b)|^2 &= \Big|\frac{1}{N}\lambda(\sum_{i=1}^{N}\alpha^{n_i}(w)\otimes\beta^{n_i}(b))\Big|^2 \\
&\leq \frac{1}{2N^2}\lambda([\sum_i \alpha^{n_i}(w^*)\otimes\beta^{n_i}(b^*), \sum_j \alpha^{n_j}(w)\otimes\beta^{n_j}(b)]_+) \\
&= \frac{1}{2N^2}\sum_{i,j}\lambda([\alpha^{n_i}(w^*),\alpha^{n_j}(w)]_+\otimes\beta^{n_i}(b^*)\beta^{n_j}(b)) \\
&= \frac{1}{2N^2}\sum_{i}\lambda([\alpha^{n_i}(w^*),\alpha^{n_i}(w)]_+\otimes\beta^{n_i}(b^*b)) \\
&+ \frac{1}{2N^2}\sum_{i\neq j}\lambda([\alpha^{n_i}(w^*),\alpha^{n_j}(w)]_+\otimes\beta^{n_i}(b^*)\beta^{n_j}(b)) \\
&\leq \frac{1}{2N}\lambda([w^*,w]_+\otimes b^*b) + \frac{1}{2N^2}N(N-1)\|[\alpha^{n_i}(w^*),\alpha^{n_j}(w)]_+\|_{2,\phi} \\
&\times \|\beta^{n_i}(b^*)\beta^{n_j}(b)\|_{2,\mu} \\
&< \varepsilon/2 + \frac{1}{2}\varepsilon\|b\|\mu(b^*b)^{\frac{1}{2}} \\
&\leq \varepsilon.
\end{aligned}$$

Since ε is arbitrary, $\lambda(w\otimes b)=0$, hence $\lambda|S\otimes B=0$. In particular $\phi(w)=\lambda(w\otimes 1)=0$. Thus if $c\in\mathbf{C}$ then

$$\begin{aligned}
\lambda((c1+w)\otimes b) &= c\lambda(1\otimes b) = c\mu(b) = \phi(c1+w)\mu(b) \\
&= \phi\otimes\mu((c1+w)\otimes b).
\end{aligned}$$

Since $(S\cup\{1\})\otimes B$ is total in $A\otimes B$, $\lambda=\phi\otimes\mu$. QED.

Proof of Theorem 2.1. Since the group $\{\alpha^n : n\in\mathbf{Z}\}$ is amenable there exists an α-invariant state ϕ on A. For all $x\in A$ $\|x\|_{2,\phi}\leq\|x\|$, so that condition $(**)$ of Lemma 2.2 follows from $(*)$. If we apply Lemma 2.2 to the case $B=\mathbf{C}$ we conclude that $\phi|S=0$, so ϕ is unique by the assumption that $S\cup\{1\}$ is total in A. The conclusion of Lemma 2.2 holds for all triples (B,β,μ) and all λ, hence from the discussion preceding the statement of the theorem, $H_\phi(\alpha)=0$. QED.

Corollary 2.3. If in Theorem 2.1 A is nuclear then $h_\phi(\alpha)=0$.

Proof. As remarked before $h_\phi(\alpha)=H_\phi(\alpha)$ if A is nuclear.

Remark 2.4. If we as in Lemma 2.2 assume the existence of the invariant state satisfying $(**)$ then as in the proof of Theorem 2.1 we obtain $H_\phi(\alpha)=0$.

Remark 2.5. If we in Theorem 2.1 assume S has the property that $z, w \in S$ implies $zw \in S \cup \mathbf{C}1$, and $zw \in \mathbf{C}1$ implies $zw = wz$, then ϕ is a trace. Indeed, if $a, b \in \mathbf{C}$ then

$$\phi((a1 + w)(b1 + z)) = ab + \phi(wz) = \begin{cases} ab & \text{if } wz \in S \\ ab + wz = ba + zw & \text{if } wz \in \mathbf{C}1 \end{cases}$$

hence the assertion follows from the totality of $S \cup \{1\}$.

Remark 2.6. If there exists a tracial state τ on A (e.g. if A is a unital AF-algebra) then the unique invariant state is tracial. Indeed, we can take an invariant mean over $\tau \circ \alpha^n$ to obtain the invariant state.

3. A number theoretic lemma.

In order to find sequences (n_i) as in Theorem 2.1 we need a result on the existence of certain sequences in $\mathbf{N}$. We use the notation

$$[k, m] = \{k, k+1, \cdots, m\} \text{ when } k \leq m \text{ in } \mathbf{Z}$$

Lemma 3.1. For each $j \in \mathbf{N}$ there exists a sequence $(x_s^j)_{s \in \mathbf{N}}$ in $\mathbf{N}$ such that $x_1^j = j, j + (j+2)^s < x_s^j - x_{s-1}^j$, and if

$$W_j = \{x_s^j - x_t^j + i : \ i \in [-j, j], \ 1 \leq t < s, \ s, t \in \mathbf{N}\}$$

then the sets W_j are pairwise disjoint.

We shall need an estimate for the growth of the cardinality of unions of sets of the form $W_j \cap [1, m]$ as above.

Lemma 3.2. If $W_1, \cdots, W_k$ are constructed as in Lemma 3.1 then

$$\text{card}(\bigcup_{j=1}^{k} W_j \cap [1, m]) \leq ((k+1)\log m)^2, \quad m \in \mathbf{N}.$$

Proof. Fix m and let $1 \leq j \leq k$. Let s be chosen as the minimal natural number such that $x_{s+1}^j - x_r^j - j > m$ for all $1 \leq r \leq s$. Then

$$\begin{aligned} \text{card}(W_j \cap [1, m]) &\leq \sum_{r=2}^{s} \text{card}\{x_r^j - x_t^j + i : \ i \in [-j, j], \ 1 \leq t < r\} \\ &= (2j+1)\sum_{r=2}^{s}(r-1) < (j+1)s^2. \end{aligned}$$

By choice of s

$$x_s^j - x_{s-1}^j - j \leq m$$

by minimality of this element among the numbers $x_s^i - x_t^j + i, \ i \in [-j, j]$. By assumption then,

$$(j+2)^s \leq x_s^j - x_{s-1}^j - j \leq m,$$

hence

$$s \leq \frac{\log m}{\log(j+2)} < \log m,$$

using that $\log(j+2) \geq \log 3 > 1$. Thus

$$\operatorname{card}(W_j \cap [1, m]) < (j+1)(\log m)^2.$$

It follows that

$$\begin{aligned} \operatorname{card}(\bigcup_{j=1}^{k} W_j \cap [1, m]) &\leq \sum_{j=1}^{k} \operatorname{card}(W_j \cap [1, m]) \\ &< (\log m)^2 \sum_{j=1}^{k} (j+1) \\ &< ((k+1) \log m)^2. \end{aligned}$$

QED.

Proof of Lemma 3.1. We shall construct the sets W_j by induction on j. If $j = 1$ put $x_1^1 = 1$ and choose $x_s^1 \in \mathbf{N}$ such that $3^s + 1 < x_s^1 - x_{s-1}^1$, and put

$$W_1 = \{x_s^1 - x_t^1 + i : \ i \in [-1, 1], \ 1 \leq t < s, s, t \in \mathbf{N}\}$$

Suppose the sequences $(x_s^j)_{s \in \mathbf{N}}, \ j = 1, \cdots, p-1$, are chosen such that $x_s^j - x_{s-1}^j > (j+2)^s + j$ and such that the sets $W_1, \cdots, W_{p-1}$ are pairwise disjoint.

Put $x_1^p = p$. We first seek x_2^p such that

(i) $[x_2^p - 2p, x_2^p] \cap \bigcup_{j=1}^{p-1} W_j = \emptyset$

(ii) $x_2^p > x_1^p + (p+2)^2 + p = p^2 + 6p + 4$.

By Lemma 3.2 card $(\bigcup_{j=1}^{p-1} W_j \cap [1, m]) \leq (p \log m)^2$ for all m, hence for m sufficiently large

$$(2p+1)\operatorname{card}(\bigcup_{j=1}^{p-1} W_j \cap [1, m]) \leq m - (p^2 + 6p + 4)$$

Thus there exists $x_2^p \in \mathbf{N}$, $x_2^p \leq m$ satisfying (ii) such that

$$[x_2^p - 2p,\ x_2^p] \cap \bigcup_{j=1}^{p-1} W_j = \emptyset,$$

from which (i) follows.

Let $r \geq 2$ and suppose $x_1^p, \cdots, x_r^p$ are constructed such that $x_s^p - x_{s-1}^p > (p+2)^s + p$ and such that the sets

$$\{x_s^p - x_t^p + i :\ i \in [-p,p],\ 1 \leq t < s\}, \quad 2 \leq s \leq r$$

are disjoint from $\bigcup_{j=1}^{p-1} W_j$.

Let $m_0 = x_r^p + (p+2)^{r+1} + p + 1$. By Lemma 3.2

$$\text{(1)} \qquad \operatorname{card}([m_0, m] \setminus \bigcup_{j=1}^{p-1} W_j) \geq m - m_0 - (p \log m)^2.$$

Choose by Lemma 3.2 m so large that

$$(x_r^p + p + 1)\operatorname{card}(\bigcup_{j=1}^{p-1} W_j \cap [1,m]) \leq m - m_0 - (p \log m)^2.$$

Then by (1) we can find $x_{r+1}^p \in [m_0, m]$ such that

$$\text{(2)} \qquad \{x_{r+1}^p - n :\ n \in [0, x_r^p + p]\} \cap \bigcup_{j=1}^{p-1} W_j = \emptyset\,.$$

If $1 \leq t \leq r$ we find for $i \in [-p,p]$

$$x_{r+1}^p \geq x_{r+1}^p - x_t^p + i \geq x_{r+1}^p - x_r^p - p = x_{r+1}^p - (x_r^p + p).$$

Thus by (2)

$$\{x_{r+1}^p - x_t^p + i :\ i \in [-p,p], 1 \leq t \leq r\} \cap \bigcup_{j=1}^{p-1} W_j = \emptyset\,.$$

This completes the induction, since by choice of m_0, $x_{r+1}^p - x_r^p > (p+2)^s + p$.=20 QED.

4. The C*-algebras of Powers.

In [P] Powers introduced a class of C*-algebras obtained from binary shifts of the hyperfinite II_1-factor, see also [Pr]. The definition is as follows. Let $X \subset \mathbf{N}$ be a subset considered as a subset of $\mathbf{Z}$, and let g be its characteristic function. Put

$$\sigma(n) = (-1)^{g(n)}.$$

Changing Powers' definition slightly we let $(s_i)_{i\in\mathbf{Z}}$ be a sequence of self-adjoint unitary operators satisfying the commutation relations

$$s_i s_j = \sigma(|i-j|) s_j s_i.$$

We denote by

$$I = \{i_1 < \cdots < i_r\}$$

the ordered subset $\{i_k : k \in [1,r],\ i_1 < i_2 < \cdots < i_r\}$, and we denote by

$$w_I = s_{i_1} s_{i_2} \cdots s_{i_r},\ w_\emptyset = 1.$$

If $J = \{j_1 < \cdots < j_s\}$ an easy calculation yields

$$w_I w_J = \prod_{k,l} \sigma(|i_k - j_l|) w_\tau w_I$$

Put

$$S = \{w_I : I = \{i_1 < \cdots < i_r\},\ I \neq \emptyset\}.$$

Let $A(X)$ denote the C*-algebra generated by the set of $s_i,\ i \in \mathbf{Z}$. Then $S \cup \{1\}$ is total in $A(X)$. We define $\alpha \in \text{Aut}\ A(X)$ to be the shift $\alpha(s_i) = s_{i+1}$.

Theorem 4.1. With the above notation there exists $X \subset \mathbf{N}$ such that the C*-dynamical system $(A(X), \alpha)$ satisfies the assumptions of Theorem 2.1.

Proof. For each $j \in \mathbf{N}$ let $(x_s^j)_{s\in\mathbf{N}}$ be the sequence found in Lemma 3.1. Put

$$U_j = \{x_s^j - x_t^j + j\ :\ 1 \leq t < s\,,\ s,t \in \mathbf{N}\},$$

and put

$$X = \bigcup_{j=1}^{\infty} U_j.$$

Let $I = \{i_1 < \cdots < i_r\}$ and $N \in \mathbf{N}$. We shall find $n_1 < n_2 < \cdots < n_N \in \mathbf{N}$ such that

$$[\alpha^{n_s}(w_I),\ \alpha^{n_t}(w_I)]_+ = 0 \text{ if } s \neq t. \tag{3}$$

Note that this is sufficient since $w_I^* = \pm w_I$. Since (3) holds if we replace n_s and n_t by $n_s + n$ and $n_t + n$ for any $n \in \mathbf{Z}$, we may assume $1 \leq i_1 < \cdots < i_r$. Since also $[a,b]_+ = [b,a]_+$ for all a, b, it suffices to show (3) for $n_s > n_t$. Put $j = i_r - i_1$, $n_s = x_s^j$, $n_t = x_t^j$. Then with W_j as in Lemma 3.1 we have

$$\begin{aligned} n_s - n_t + i_\ell - i_m &\in W_j \setminus U_j \text{ if } i_\ell - i_m < j, \\ = 20 n_s - n_t + i_r - i_1 &\in U_j. \end{aligned}$$

By Lemma 3.1 the sets W_k are pairwise disjoint, so the only contribution to σ applied to the numbers $n_s - n_t + i_l - i_m$ comes from $U_j \subset W_j$. Thus we have

$$\begin{aligned} \prod_{l,m} \sigma(|n_s - n_t + i_\ell - i_m|) &= \prod_{l,m} \sigma(n_s - n_t + i_\ell - i_m) \\ &= \sigma(n_s - n_t + i_r - i_1) = -1. \end{aligned}$$

Thus (3) holds whenever $n_s = x_s^j$, $n_t = x_t^j$. This completes the proof. QED.

By Remark 2.5 or by [P] the unique invariant state ϕ found in Theorem 2.1 is a trace. Also by [P] $A(X)$ is an AF-algebra, hence is nuclear, so the entropies $H_\phi(\alpha)$ and $h_\phi(\alpha)$ coincide. We shall now prove that with X and α as in Theorem 4.1 the tensor product formula for entropy fails for $\alpha \otimes \alpha$ and $\phi \otimes \phi$.

Theorem 4.2. Let $A(X)$ and α be as in Theorem 4.1, and let ϕ be the unique α-invariant trace. Then the tensor product formula fails for $\alpha \otimes \alpha$ and $\phi \otimes \phi$. More specifically we have

$$h_{\phi\otimes\phi}(\alpha \otimes \alpha) \geq \log 2, \; h_\phi(\alpha) = 0.$$

Proof. Let A_0 denote the C*-subalgebra of $A(X) \otimes A(X)$ generated by operators of the form $w_I \otimes w_I$. Since $w_I w_J = \pm w_J w_I$, A_0 is abelian. Since A_0 is generated by the self-adjoint unitaries $s_i \otimes s_i$, and $\alpha \otimes \alpha$ is the shift, and the invariant state $\phi \otimes \phi$ vanishes on each $s_i \otimes s_i$, the dynamical system $(A_0, \; \alpha \otimes \alpha, \; \phi \otimes \phi)$ is isomorphic to the two shift, or equivalently to the baker's transform, see [AN], and has entropy $\log 2$. Thus

$$h_{\phi\otimes\phi}(\alpha \otimes \alpha) \geq h_{\phi\otimes\phi}(\alpha \otimes \alpha | A_0) = \log 2.$$

By Theorem 2.1 $h_\phi(\alpha) = 0$, so that

$$h_{\phi\otimes\phi}(\alpha \otimes \alpha) > h_\phi(\alpha) + h_\phi(\alpha),$$

proving the theorem. QED.

By [P, Theorem 3.9] the trace ϕ in the above theorem is a factor state. Since in the GNS-representation of ϕ the entropy of [CNT] equals that of [CS] we have

Corollary 4.3. There exists an automorphism α of the hyperfinite II_1-factor such that the tensor product formula fails for $\alpha \otimes \alpha$.

Remark 4.4. In a recent paper [V] Voiculescu has introduced some alternative definitions of entropy in AF and hyperfinite von Neumann algebras based on approximation of given operators by operators in finite dimensional C*-subalgebras. For all these entropies Voiculescu showes the inequality

$$\text{“}h'_{\tau\otimes\sigma}(\alpha\otimes\beta) \leq h'_\tau(\alpha) + h'_\sigma(\beta)\text{”},$$

hence by Theorem 4.2 and Corollary 4.3 his entropies are in general different from the entropies of [CS] and [CNT] considered in the present paper.

Acknowledgement: The authors want to thank the Erwin Schrödinger Institute in Vienna, which made this collaboration possible.

References

[AN] *R. Alicki, H. Narnhofer*, Comparison of dynamical entropies for the non-commutative shifts, Letters Math. Phys. (1994), to appear.

[CNT] *A. Connes, H. Narnhofer, W. Thirring*, Dynamical entropy of C*-algebras and von Neumann algebras, Commun. Math. Phys. 112 (1987), 691-719.

[CS] *A. Connes, E. Størmer*, Entropy of automorphisms of II_1 von Neumann algebras, Acta Math. 134 (1975), 289-306.

[NT] *H. Narnhofer, W. Thirring*, Clustering for algebraic K-systems, Letters Math. Phys. 30 (1994), 307-316.

[P] *R.T. Powers*, An index theory for semigroups of $*$-endomorphisms of $B(H)$ and type II_1 factors, Canad. J. Math. 40 (1988), 86-114.

[Pr] *G.L. Price*, Shifts on II_1 factors, Canad. J. Math. 39 (1987), 492-511.

[ST] *J-L. Sauvageot, J-P. Thouvenot*, Une nouvelle definition de l'entropic dynamique des systems non commutatifs, Commun. Math. Phys. 145 (1992), 411-423.

[SV] *E. Størmer, D. Voiculescu*, Entropy of Bogoliubov automorphisms of the canonical anticommutation relations, Commun. Math. Phys. 133 (1990), 521-542.

[V] *D. Voiculescu*, Dynamical approximation entropies and topological entropy in operator algebras, to appear.