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$*$-STRUCTURES ON BRAIDED SPACES

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Abstract $*$-structures on quantum and braided spaces of the type defined via an R-matrix are studied. These include q-Minkowski and q-Euclidean spaces as additive braided groups. The duality between the $*$-braided groups of vectors and covectors is proved and some first applications to braided geometry are made.

1 Introduction

The programme of q-deforming physics is an important one with possible applications to q-regularisation as well as to models of quantum corrections to geometry. The first stage of this programme, to build algebras suitable for co-ordinates of spacetime and other linear spaces (i.e. deformations of $\mathbb{R}^n$), is fairly complete at an algebraic level. The systematic treatment here is the approach coming out of braided geometry[1][2]. This starts with an addition law (expressed algebraically as a braided coaddition $\underline{\Delta}$) and proceeds with a systematic theory of quantum metric, Poincaré group[3], differentiation[4], integration[5], epsilon tensor and differential forms[6]. There are also natural choices in the braided approach for q-deformed Euclidean[7] and Minkowski[8][9][10] spaces, related in by a quantum Wick rotation. Moreover, the braided approach is compatible with earlier q-deformations of these particular spaces from the point of view of $SO_q(n)$-covariance[11] and spinor decomposition[12][13] respectively. The relation in the latter case is in [14].

Although this theory is fairly complete, it is algebraic. For the next stage, as well as for applications in physics, one needs to understand an abstract approach to the $*$-structure on such spaces too. This is needed in order to be able to properly formulate reality properties,

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Hilbert space representations and other key ingredients for a realisation of such non-commutative algebras in a quantum-mechanical set-up. A general theory of such $*$-structures has been missing until now.

In this note we provide such a theory within the braided groups programme mentioned above, i.e. for deformations of $\mathbb{R}^n$ described by an R-matrix. As usual, our starting point is compatibility with the braided coaddition law. We give appropriate axioms for a $*$-braided group following [8] and then examine its compatibility with the various layers of braided geometry mentioned above.

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2 $*$-Structure on Braided Vectors and Covectors

The most abstract definition of a $*$-braided group is not known, but one which has already been useful in some examples is [8] that the braided group B should be a $*$-algebra in the usual way with an antilinear antihomomorphism $*: B \to B$ and in addition

$$(* \otimes *) \circ \underline{\Delta} = \tau \circ \underline{\Delta}, \quad \overline{\underline{\epsilon}(\ )} = \underline{\epsilon} \circ *, \quad * \circ \underline{S} = \underline{S} \circ * \tag{1}$$

where $\overline{(\ )}$ denotes complex conjugation. These are far from the usual axioms of a Hopf $*$-algebra and so they should be since a braided group is not a Hopf algebra in the usual sense. Recall that for a braided group the coproduct $\underline{\Delta}$ is a braided-homomorphism, i.e. an algebra homomorphism $B \to B \underline{\otimes} B$ where the latter is the braided tensor product algebra

$$(a \otimes c)(b \otimes d) = a\Psi(c \otimes b)d$$

where $\Psi : B \otimes B \to B \otimes B$ is a braiding[15]. We showed in [8] that the usual axioms of a Hopf $*$-algebra become the axioms above under a process of transmutation that converts quantum groups into braided groups. So there are some examples of (1) known.

On the other hand, we want to apply these axioms to fresh examples not obtained (as far as I know) by transmutation. These are the braided vector and covector spaces $V(R',R)=\{v^i\}$ and $V\check{}(R',R)=\{x_i\}$ introduced in [3]. They are defined relative to two R-matrices, with R obeying the QYBE and R' some mixed relations relative to R. The formulae are

$$\mathbf{x}_1\mathbf{x}_2=\mathbf{x}_2\mathbf{x}_1R',\quad \Psi(\mathbf{x}_1\otimes\mathbf{x}_2)=\mathbf{x}_2\otimes\mathbf{x}_1R,\quad \text{i.e.}\quad \mathbf{x}'_1\mathbf{x}_2=\mathbf{x}_2\mathbf{x}'_1R \tag{2}$$

$$\mathbf{v}_1\mathbf{v}_2=R'\mathbf{v}_2\mathbf{v}_1,\quad \Psi(\mathbf{v}_1\otimes\mathbf{v}_2)=R'\mathbf{v}_2\otimes\mathbf{v}_1,\quad \text{i.e.}\quad \mathbf{v}'_1\mathbf{v}_2=R\mathbf{v}_2\mathbf{v}'_1 \tag{3}$$

where the second form gives the relations of the braided tensor product or *braid statistics* directly in the notation $\mathbf{x}\equiv\mathbf{x}\otimes 1,\mathbf{x}'=1\otimes\mathbf{x}$ etc. With these relations, $\mathbf{x}+\mathbf{x}'$ and $\mathbf{v}+\mathbf{v}'$ are again braided covectors and vectors respectively. This corresponds to a linear form of $\underline{\Delta}$ and $\underline{\epsilon}=0$ on the generators.

We will obtain various types of $*$-structure depending on the reality properties of the matrices R,R'. We consider of interest

$$\overline{R^i{}_j{}^k{}_l}=R^l{}_k{}^j{}_i,\quad \text{real type I} \tag{4}$$

$$\overline{R^i{}_j{}^k{}_l}=R^{-1}{}^j{}_i{}^l{}_k,\quad \text{antireal type I} \tag{5}$$

$$\overline{R^i{}_j{}^k{}_l}=R^{\bar k}{}_{\bar l}{}^{\bar i}{}_{\bar j},\quad \text{real type II} \tag{6}$$

$$\overline{R^i{}_j{}^k{}_l}=R^{-1}{}^{\bar i}{}_{\bar j}{}^{\bar k}{}_{\bar l},\quad \text{antireal type II} \tag{7}$$

where the second group assumes that our indexing set is divided into two halves related bijectively by an involution $(\bar{\ })$ on the indices. We use this classification for the reality property of the matrix R' as well.

Lemma 2.1 *If R' is type II we have $*$-algebras*

$$*:V\check{}(R',R)\to V\check{}(R',R),\quad x_i\mapsto x_{\bar i},\quad R'\ \text{type II}$$
$$*:V(R',R)\to V(R',R),\quad v^i\mapsto v^{\bar i},\quad R'\ \text{type II}$$

If R' is type I we have antilinear anti-algebra isomorphisms

$$*:V(R',R)\to V\check{}(R',R),\quad v^i\mapsto x_i,\quad R'\ \text{type I}$$
$$*:V\check{}(R',R)\to V(R',R),\quad x_i\mapsto v^i,\quad R'\ \text{type I}$$

Proof We use only the reality properties of R' as we do not yet consider the braiding. That $*$ as stated is an antilinear anti-algebra homomorphism follows at once. For example the relations $(v^iv^j)^*=(v^bv^a)^*\overline{R'^i{}_a{}^j{}_b}$ requires in the real type II case $v^{\bar j}v^{\bar i}=v^{\bar a}v^{\bar b}R'^{\bar j}{}_{\bar b}{}^{\bar i}{}_{\bar a}$ which are the relations of $V(R',R)$ again. In the real type I case they require $x_jx_i=x_ax_bR'^b{}_j{}^a{}_i$, which are the relations of $V\check{}(R',R)$. The computation for $*$ on the relations of $\mathbf{x}$ is similar. In the antireal cases we have R'^{-1}_{21} in place of R' after complex conjugation, but these define the same algebra. □

Next we suppose that we have a metric η_{ij} with transposed-inverse η^{ij}. It is required to obey

$$\begin{gathered} t^i{}_at^j{}_b\eta^{ba}=\eta^{ji}\\ \eta^{ia}\eta^{jb}R'^k{}_a{}^l{}_b=R'^i{}_a{}^j{}_b\eta^{ak}\eta^{bl},\quad R'^i{}_a{}^j{}_b\eta^{ba}=\eta^{ij}. \end{gathered}\tag{8}$$

The second relation here means precisely that $V(R',R)$ and $V\check{}(R',R)$ are isomorphic by $v^i=x_a\eta^{ai}$. This is clear and was used for example in [9].

Lemma 2.2 *If η obeys*

$$\overline{\eta^{ij}}=\eta_{ji}\tag{9}$$

then we have $*$-algebras

$$\begin{gathered} *:V\check{}(R',R)\to V\check{}(R',R),\quad x_i\mapsto x_a\eta^{ai},\quad R'\ \text{type I}\\ *:V(R',R)\to V(R',R),\quad v^i\mapsto \eta_{ia}v^a,\quad R'\ \text{type I} \end{gathered}$$

Proof It is immediate from the last part of Lemma 2.1 that a real η turns the antilinear antialgebra homomorphism there into the first map $*$ as stated here. We need the additional condition on η to have $*^2=\mathrm{id}$. This is compatible with the R' in (8) for either real type I (as here) or antireal type I. It is also compatible with equations such as (10) needed below which come from covariance when R is type I. Hence we have a $*$-algebra structure on the x_i. Likewise for the $*$-structure on braided covectors. In fact, the latter is such that the isomorphism between covectors and vectors provided by the metric becomes an isomorphism of $*$-algebras. Thus $v^{i*}=(x_a\eta^{ai})^*=\eta_{ia}x_b\eta^{ba}=\eta_{ia}v^a$. Hence there is really only one $*$-braided group here. Later on we will give a second $*$-braided group structure on braided vectors and covectors in the real type I case. □

These elementary lemmas tell us when when our algebras are $*$-algebras in the usual sense of having an antilinear involution $*$. Note that if $(\bar{\ })^2$ or $\eta\bar\eta$ are not the identity, we can still proceed but just lose the condition $*^2=\mathrm{id}$.

We now consider when our $*$-algebras are $*$-braided groups. This depends on the braiding Ψ and is independent of the real or antireal type of $*$-algebra.

Proposition 2.3 *If we have a type II $*$-structure as in Lemma 2.1 and if R is real type II then $V(R',R)$ and $V\check{}(R',R)$ are $*$-braided groups.*

Proof We have to check that the coproduct property in (1) holds. On quadratic elements this is

$$\begin{aligned}
\underline{\Delta}((x_ix_j)^*) &= \underline{\Delta}(x_j^*x_i^*) = (x_{\bar{j}}\otimes 1+1\otimes x_{\bar{j}})(x_{\bar{i}}\otimes 1+1\otimes x_{\bar{i}})\\
&= x_{\bar{j}}x_{\bar{i}}\otimes 1+1\otimes x_{\bar{j}}x_{\bar{i}}+x_{\bar{j}}\otimes x_{\bar{i}}+x_d\otimes x_c R^{\bar{c}}{}_{\bar{j}}{}^{\bar{d}}{}_{\bar{i}}\\
(*\otimes *)\underline{\Delta}(x_ix_j) &= (*\otimes *)(x_ix_j\otimes 1+1\otimes x_ix_j+x_i\otimes x_j+x_b\otimes x_a R^a{}_i{}^b{}_j)\\
&= x_{\bar{j}}x_{\bar{i}}\otimes 1+1\otimes x_{\bar{j}}x_{\bar{i}}+x_{\bar{i}}\otimes x_{\bar{j}}+x_{\bar{b}}\otimes x_{\bar{a}} R^{\bar{b}}{}_{\bar{j}}{}^{\bar{a}}{}_{\bar{i}}
\end{aligned}$$

by our real type II assumption. We see that we have (1) on quadratic elements. We can extend this to higher products by induction. A more formal way to see this is that $\underline{\Delta}$ and $*$ are both extended to products multiplicatively, where $V\check{}\underline{\otimes}V\check{}$ has the braided tensor product algebra and the $*$-structure

$$(x_i\otimes 1)^*=1\otimes x_i^*,\quad (1\otimes x_i)^*=x_i^*\otimes 1.$$

The content of the proof above is to check that this is a well-defined $*$-structure. Since (1) holds on the generators, it then holds on all products too. The proof for braided vectors is similar.

We also have to verify the other half of (1). We have

$$\begin{aligned}
\underline{S}((x_ix_j)^*) &= \underline{S}(x_{\bar{j}}x_{\bar{i}}) = \cdot\Psi(\underline{S}x_{\bar{j}}\otimes \underline{S}x_{\bar{i}}) == x_{\bar{b}}x_{\bar{a}}R^{\bar{a}}{}_{\bar{j}}{}^{\bar{b}}{}_{\bar{i}}\\
\underline{S}(x_ix_j) &= *\circ\cdot\Psi(\underline{S}x_i\otimes \underline{S}x_j) = (x_ax_bR^b{}_i{}^a{}_j)^* = x_{\bar{b}}x_{\bar{a}}\overline{R^b{}_i{}^a{}_j}
\end{aligned}$$

since $\underline{S}(\mathbf{x})=-\mathbf{x}$ extends to products as a braided-antialgebra homomorphism[15]. The two expressions coincide by the real type II assumption on R. The extension to higher powers is by induction using similar techniques. The proof for the braided vectors is analogous. □

Proposition 2.4 *If we have a type I $*$-structure as in Lemma 2.2 and if R is real type I then $V(R',R)$ and $V\check{}(R',R)$ are $*$-braided groups.*

Proof In the type I case we assume the real metric so that we have $*$-algebras from Lemma 2.2. On quadratic elements we have

$$\begin{aligned}\underline{\Delta}((x_ix_j)^*) &= \underline{\Delta}(x_j^*x_i^*) = (x_a\eta^{aj}\otimes 1+1\otimes x_a\eta^{aj})(x_b\eta^{bi}\otimes 1+1\otimes x_b\eta^{bi})\\ &= x_a\eta^{aj}x_b\eta^{bi}\otimes 1+1\otimes x_a\eta^{aj}x_b\eta^{bi}+x_a\eta^{aj}\otimes x_b\eta^{bi}+\eta^{aj}\eta^{bi}x_d\otimes x_cR^c{}_a{}^d{}_b\\ (*\otimes *)\underline{\Delta}(x_ix_j) &= (*\otimes *)(x_ix_j\otimes 1+1\otimes x_ix_j+x_i\otimes x_j+x_b\otimes x_aR^a{}_i{}^b{}_j)\\ &= x_a\eta^{aj}x_b\eta^{bi}\otimes 1+1\otimes x_a\eta^{aj}x_b\eta^{bi}+x_b\eta^{bi}\otimes x_a\eta^{aj}+x_c\eta^{cb}\otimes x_d\eta^{da}R^j{}_b{}^i{}_a\end{aligned}$$

Next we use the identity

$$\eta^{ia}\eta^{jb}R^k{}_a{}^l{}_b = R^i{}_a{}^j{}_b\eta^{ak}\eta^{bl} \tag{10}$$

which follows at once from the covariance condition in the first of (8) by applying the fundamental R-matrix representation to both sides. Using this and comparing our final expressions we see exactly (1) for the coproduct. Once again we use the braided tensor product algebra structure as above for a more formal proof. The computation for braided vectors is analogous.

For the other half of (1) we have

$$\underline{S}((x_ix_j)^*) = \underline{S}(x_b\eta^{bj}x_a\eta^{ai}) = x_cx_d\eta^{bj}\eta^{ai}R^d{}_b{}^c{}_a = x_cx_d\eta^{da}\eta^{cb}R^j{}_a{}^i{}_b = (x_ax_bR^b{}_i{}^a{}_j)^* = *\underline{S}(x_ix_j)$$

using (10) in the middle. Likewise for the braided vectors. □

A general class of examples of type II is provided by the braided matrix $B(R)$ construction in [15]. These are braided versions of $\mathbb{R}^{n^2}$ with generators $\{u^i{}_j\}$ and relations $R_{21}\mathbf{u}_1R\mathbf{u}_2 = \mathbf{u}_2R_{21}\mathbf{u}_1R$. Such relations are among the relations of quantum enveloping algebras in [16], but also arose in [15] as an abstract quadratic algebra for braided matrices. They have a braided matrix coproduct $\underline{\Delta}\mathbf{u} = \mathbf{u}\otimes\mathbf{u}$ and a $*$-structure[8] making $B(R)$ into a $*$-bialgebra in the sense of the first half of (8).

Also introduced in [15] was a multi-index notation $u^{i_0}{}_{i_1}\equiv u_I$ whereby the braided-matrices appeared as an n^2-dimensional quantum plane relative to a bigger matrix $\mathbf{R'}^I{}_J{}^K{}_L$. It was shown in [9] that when R obeys a Hecke condition then there is also a braiding matrix $\mathbf{R}^I{}_J{}^K{}_L$ making $B(R)$ into braided covector space with an additive $\underline{\Delta}$ as above. Explicitly,

$$\mathbf{R'}^I{}_J{}^K{}_L = R^{-1}{}^d{}_{k_0}{}^{j_0}{}_aR^{k_1}{}_b{}^a{}_{i_0}R^{i_1}{}_c{}^b{}_{l_1}\widetilde{R}^c{}_{j_1}{}^{l_0}{}_d,\quad \mathbf{R}^I{}_J{}^K{}_L = R^{j_0}{}_a{}^d{}_{k_0}R^{k_1}{}_b{}^a{}_{i_0}R^{i_1}{}_c{}^b{}_{l_1}\widetilde{R}^c{}_{j_1}{}^{l_0}{}_d. \tag{11}$$

Meyer's additive braid statistics here can be written as $R^{-1}\mathbf{u}'_1 R\mathbf{u}_2 = \mathbf{u}_2 R_{21}\mathbf{u}'_1 R$ if one wants a matrix notation.

Example 2.5 *If R is real type I then $\mathbf{R}'$ in (11) is antireal type II and $\mathbf{R}$ is real type II with $\overline{(i_0, i_1)} = (i_1, i_0)$. Hence in the Hecke case $B(R)$ is a $*$-braided group of the type II in Proposition 2.3. The $*$ structure is $(u^{i_0}{}_{i_1})^* = u^{i_1}{}_{i_0}$.*

Proof The reality properties can be deduced from the form of $\mathbf{R}', \mathbf{R}$ shown and the reality property of R. We then use Proposition 2.3. □

The most familiar case is when R is the $SU_q(2)$ R-matrix with real q. Then $B(R)$ is q-Minkowski space and the $*$-braided group structure is the one announced in [9][14]. The $*$-structure is the Hermitian one, which is also the $*$-structure for the multiplicative braided coproduct introduced in [8]. So $B(R)$ is a $*$-braided group for both coaddition and comultiplication simultaneously.

Closely related to $B(R)$ (by a cocycle twisting) is the algebra $\bar{A}(R)$ in [7]. This also has a matrix $x^{i_0}{}_{i_1} = x_I$ of generators and in the Hecke case also forms a braided covector space with[7]

$$\mathbf{R}'^I{}_J{}^K{}_L = R^{-1}{}^{l_0}{}_{k_0}{}^{j_0}{}_{i_0} R^{i_1}{}_{j_1}{}^{k_1}{}_{l_1}, \quad \mathbf{R}^I{}_J{}^K{}_L = R^{j_0}{}_{i_0}{}^{l_0}{}_{k_0} R^{i_1}{}_{j_1}{}^{k_1}{}_{l_1} \tag{12}$$

The corresponding relations and additive braid statistics can be written as $R_{21}\mathbf{x}_1\mathbf{x}_2 = \mathbf{x}_2\mathbf{x}_1 R$ and $\mathbf{x}'_1\mathbf{x}_2 = R\mathbf{x}_2\mathbf{x}'_1 R$ if one wants a matrix notation.

Example 2.6 *If R is real type I then $\mathbf{R}'$ and $\mathbf{R}$ in (12) are real type I. Hence if R is Hecke and if there is an invariant metric η^{IJ} obeying (9) then $\bar{A}(R)$ is a $*$-braided group of the type I in Proposition 2.4. The $*$-structure is $(x^{j_0}{}_{j_1})^* = x^{i_0}{}_{i_1}\eta^{IJ}$.*

Proof The reality properties are immediate from the form of $\mathbf{R}', \mathbf{R}$ shown. We then use Proposition 2.4. □

The most familiar case is again when R is the $SU_q(2)$ R-matrix with real q. Then $\bar{A}(R)$ is q-Euclidean space and the $*$-braided group structure is

$$\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} d & -q^{-1}c \\ -qb & a \end{pmatrix}$$

as found another way in [7]. We use the construction in Example 2.6 with

$$\eta^{IJ} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -q & 0 \\ 0 & -q^{-1} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is evident that $\eta^2 = \mathrm{id}$ as required for (9) when η is real.

For the specific $SU_q(2)$ R-matrix the above $*$-structures on q-Minkowski and q-Euclidean space agree with those used in other more specific approaches [13] and [11] respectively. On the other hand, the $*$ properties of the braided coaddition have not been considered in any detail before. Moreover, our analysis is quite general and applies just as well to other real type R-matrices, including non-standard and multiparameter ones.

Our analysis also gives something more unexpected in the type I case. The equations (8)–(10) which we used are invariant under η^{ij} replaced by η^{ji}. Since these were all that we really needed to check the $*$-braided group structure, we have:

Corollary 2.7 *When R' is type I and R real typ I with a metric obeying (9), we have second $*$-structures*

$$\star : V\check{}(R', R) \to V\check{}(R', R), \quad x_i \mapsto \eta^{ia} x_a$$

$$\star : V(R', R) \to V(R', R), \quad v^i \mapsto v^a \eta_{ai}$$

also making the braided vectors and covectors into $*$-braided groups.

For example, q-Euclidean space has a second structure

$$\begin{pmatrix} a^\star & b^\star \\ c^\star & d^\star \end{pmatrix} = \begin{pmatrix} d & -qc \\ -q^{-1}b & a \end{pmatrix}.$$

One can check that the first $*$-structure from Lemma 2.2 is such that the usual right coaction of the background quantum group $\mathbf{t}$ is a $*$-algebra homomorphism when we take the usual $*$-structure ${t^i}_j{}^* = S{t^j}_i$ appropriate for R of real type I. On the other hand, this second $\star$ operation does not have this property.

3 Duality Pairing under $*$

We define the duality pairing of vectors and covectors by the braided binomials

$$\langle v^{j_m} \cdots v^{j_2} v^{j_1}, x_{i_1} x_{i_2} \cdots x_{i_r} \rangle = \delta_{m,r} ([m; R]!)^{j_1 j_2 \cdots j_m}_{i_1 i_2 \cdots i_m} \tag{13}$$

where

$$[m; R]! = [2; R]_{m-1\, m} [3; R]_{m-2\, m} \cdots [m; R]_{1 \cdots m}$$

$$[m; R] = 1 + (PR)_{12} + (PR)_{12}(PR)_{23} + \cdots (PR)_{12} \cdots (PR)_{m-1\, m}$$

are the braided factorial and braided integer matrices introduced in [4]. The numerical suffices refer as usual to the copies of a tensor product of matrices. This pairing is an immediate application of the braided-differentiation in the $\mathbf{x}$ co-ordinates in [4]. See also [5]. We will need to consider also

$$[m;R]^{\rm op}! \equiv [m;R]^{\rm op}_{1\cdots m}\cdots[3;R]^{\rm op}_{m-2\,m}[2;R]^{\rm op}_{m-1\,m}$$
$$[m;R]^{\rm op} = 1+(PR)_{12}+(PR)_{23}(PR)_{12}+\cdots+(PR)_{m-1\,m}\cdots(PR)_{12}$$

which is how the pairing comes out if we differentiate in the $\mathbf{v}$ co-ordinates instead.

Lemma 3.1 *For every matrix R obeying the QYBE we have the identities*

$$[m;R]^{\rm op}! = [m;R]! = ([m;R_{21}]!)_{m\cdots 21}.$$

Proof We use the QYBE or braid relations $(PR)_{12}(PR)_{23}(PR)_{12} = (PR)_{23}(PR)_{12}(PR)_{23}$ repeatedly. At order three we have

$$\begin{aligned}
[3;R]^{\rm op}! &= (1+(PR)_{12}+(PR)_{23}(PR)_{12})(1+(PR)_{23})\\
&= 1+(PR)_{12}+(PR)_{23}(PR)_{12}+(PR)_{23}+(PR)_{12}(PR)_{23}+(PR)_{23}(PR)_{12}(PR)_{23}\\
[3;R]! &= (1+(PR)_{23})(1+(PR)_{12}+(PR)_{12}(PR)_{23})\\
&= 1+(PR)_{12}+(PR)_{12}(PR)_{23}+(PR)_{23}+(PR)_{23}(PR)_{12}+(PR)_{23}(PR)_{12}(PR)_{23}
\end{aligned}$$

which holds automatically, while higher m require the braid relations. Likewise

$$\begin{aligned}
[3;R_{21}]!_{321} &= ((1+(RP)_{23})(1+(RP)_{12}+(RP)_{12}(RP)_{23}))_{321}\\
&= (1+(RP)_{21})(1+(RP)_{32}+(RP)_{32}(RP)_{21})\\
&= (1+(PR)_{12})(1+(PR)_{23}+(PR)_{23}(PR)_{12})\\
&= 1+(PR)_{23}+(PR)_{23}(PR)_{12}+(PR)_{12}+(PR)_{12}(PR)_{23}+(PR)_{12}(PR)_{23}(PR)_{12}
\end{aligned}$$

which coincides with $[3;R]!$ above by the braid relations. Similarly for higher m. The formal proof is best done diagramatically. On the other hand, we will give more abstract reasons from the point of braided groups why these non-trivial identities hold. □

Next, we say that two $*$-braided groups are in duality if there is a pairing

$$\overline{\langle b,c\rangle} = \langle b^*,c^*\rangle \tag{14}$$

These axioms appear quite different from the usual ones for Hopf $*$-algebras, in particular not involving the antipode. But they are the appropriate ones if we want consistency with our definition (8) and the definition of duality for the braided group structure. As explained in [2] it is convenient (but not essential) to define this as

$$\langle ab, c\rangle = \langle a, \langle b, c_{\underline{(1)}}\rangle c_{\underline{(2)}}\rangle, \quad \langle a, cd\rangle = \langle a_{\underline{(1)}}, \langle a_{\underline{(2)}}, c\rangle d\rangle, \quad \langle \underline{S}a, c\rangle = \langle a, \underline{S}c\rangle \tag{15}$$

where $\underline{\Delta} c = c_{\underline{(1)}} \otimes c_{\underline{(2)}}$ etc. This is not the usual pairing because we do not move b past $c_{\underline{(1)}}$ to evaluate on $c_{\underline{(2)}}$ etc, as one would usually do. It is possible to define such a more usual pairing by using the braiding Ψ to make the transposition but the result would be equivalent to (15) via the braided antipode, so we avoid such an unnecessary complication. For braided groups paired in this way, (14) is the correct extension to the $*$ case, as one may easily check.

Proposition 3.2 *The $*$-braided groups $V(R', R)$ and $V\check{}(R', R)$ of type II from Proposition 2.3 are in duality.*

Proof We consider the setting with R real type II. Then

$$\begin{aligned}\overline{\langle v^{j_m}\cdots v^{j_1}, x_{i_1}\cdots x_{i_m}\rangle} &= \overline{([m;R]!)^{j_1\cdots j_m}_{i_1\cdots i_m}} = ([m;R_{21}]!)^{\bar{j}_1\cdots\bar{j}_m}_{\bar{i}_1\cdots\bar{i}_m} = ([m;R]!)^{\bar{j}_m\cdots\bar{j}_1}_{\bar{i}_m\cdots\bar{i}_1}\\ &= \langle v^{\bar{j}_1}\cdots v^{\bar{j}_m}, x_{\bar{i}_m}\cdots x_{\bar{i}_1}\rangle = \langle (v^{j_m}\cdots v^{j_1})^*, (x_{i_1}\cdots x_{i_m})^*\rangle\end{aligned}$$

as required. We used the first part of Lemma 3.1 and $*$ from the first part of Lemma 2.1. $\square$

Proposition 3.3 *The $*$-braided group $V(R', R)$ of type I from Corollary 2.7 and $V\check{}(R', R)$ of type I from Proposition 2.4 are in duality.*

Proof We need here the braided vectors with the second $\star$-structure mentioned in Section 2. The same result applies if we take braided covectors with their second $\star$ operation and braided vectors with their original one from Proposition 2.4.

We note first that R real type I means that $\overline{(PR)^i{}_j{}^k{}_l} = (PR)^j{}_i{}^l{}_k$ or $\overline{PR} = (PR)^{t\otimes t}$ where t denotes transposition in the matrix indices. Since transposition also reverses the order of matrix multiplication we see that

$$\overline{[m;R]} = ([m;R]^{\rm op})^{t\otimes\cdots\otimes t}, \quad \overline{[m;R]!} = ([m;R]^{\rm op}!)^{t\otimes\cdots\otimes t}.$$

We also suppose a suitable metric η as above and deduce from repeated applications of (10) that

$$[m;R]^{a_1\cdots a_m}_{b_1\cdots b_m}\eta^{b_1 i_1}\cdots\eta^{b_m i_m} = \eta^{a_1 b_1}\cdots\eta^{a_m b_m}([m;R_{21}]^{\rm op})^{i_1\cdots i_m}_{b_1\cdots b_m} \tag{16}$$

From this we deduce further that

$$([m;R]!)^{a_1\cdots a_m}_{b_1\cdots b_m}\eta^{b_1 i_1}\cdots\eta^{b_m i_m}=\eta^{a_1b_1}\cdots\eta^{a_mb_m}([m;R_{21}]^{\rm op}!)^{i_1\cdots i_m}_{b_1\cdots b_m}=\eta^{a_1b_1}\cdots\eta^{a_mb_m}([m;R]!)^{i_m\cdots i_1}_{b_m\cdots b_1}$$

using (16) repeatedly for the first equality and Lemma 3.1 for the second.

We can now compute

$$\begin{aligned}\overline{\langle v^{j_m}\cdots v^{j_1},x_{i_1}\cdots x_{i_m}\rangle}&=\overline{([m;R]!)^{j_1\cdots j_m}_{i_1\cdots i_m}}\\ &=([m;R]^{\rm op}!)^{i_1\cdots i_m}_{j_1\cdots j_m}=([m;R]!)^{i_1\cdots i_m}_{j_1\cdots j_m}=([m;R]!)^{a_m\cdots a_1}_{b_m\cdots b_1}\eta_{a_1j_1}\cdots\eta_{a_mj_m}\eta^{b_mi_m}\cdots\eta^{b_1i_1}\\ &=\langle v^{a_1}\cdots v^{a_m},x_{b_m}\cdots x_{b_1}\rangle\eta_{a_1j_1}\cdots\eta_{a_mj_m}\eta^{b_mi_m}\cdots\eta^{b_1i_1}=\langle (v^{j_m}\cdots v^{j_1})^\star,(x_{i_1}\cdots x_{i_m})^*\rangle\end{aligned}$$

using the above observations and the definition of $*$ from Lemma 2.2 and $\star$ from Corollary 2.7.

□

This is the abstract reason that the identities in Lemma 3.1 must hold, i.e. one can essentially push these arguments backwards using (15) and (14) to define the dual $*$-braided group to the braided covectors, and computing it on the generators as the braided vectors. Such an argument makes sense in nice cases where the pairing is non-degenerate, which we have not assumed in our direct treatment above.

Finally, we mention what can be done in the case when R is of type I but there is no invariant metric η as we needed for a $*$-braided group structure in Proposition 2.4. In this case we use the maps $*$ in the second part of Lemma 2.1 not as a $*$-structure on one algebra but as a map between the pair consisting of the vectors and covectors. This leads to the notion of what could be called a *braided holomorphic structure*. By definition we consider this to consist of $(B,C,\langle\ ,\ \rangle,*)$ where B,C are two braided groups, $\langle\ ,\ \rangle:B\otimes C\to\mathbb{C}$ is a duality pairing between them and $*$ denotes a mutually inverse pair of antilinear antialgebra homomorphisms $*:B\to C$ and $*:C\to B$ such that (1) holds between the coproducts etc. of B,C and (14) holds in the form

$$\overline{\langle b,c\rangle}=\langle c^*,b^*\rangle. \tag{17}$$

We see by the same techniques as above that if R' is type I and R is real type I then $V(R',R),V\check{}(R',R)$ have such a holomorphic structure. Indeed, we have

$$\overline{\langle v^{j_m}\cdots v^{j_1},x_{i_1}\cdots x_{i_m}\rangle}=\overline{([m;R]!)^{j_1\cdots j_m}_{i_1\cdots i_m}}=([m;R]^{\rm op}!)^{i_1\cdots i_m}_{j_1\cdots j_m}=([m;R]!)^{i_1\cdots i_m}_{j_1\cdots j_m}$$

$$= \langle v^{i_m}\cdots v^{i_1}, x_{j_1}\cdots x_{j_m}\rangle = \langle (x_{i_1}\cdots x_{i_m})^*, (v^{j_m}\cdots v^{j_1})^*\rangle$$

without requiring a metric. This covers the important case of the quantum planes of $GL_q(n)$ type (such as the usual 2-dimensional quantum plane $yx = qxy$). They are known to be braided groups[3].

4 $*$-Properties of Differentials and Related Structures

As a first application of the above duality theory of $*$-braided groups, we consider braided differentiation on braided covectors x_i. This was introduced in [4] as an infinitesimal coproduct,

$$\partial^i f(\mathbf{x}) = \text{coeff } a_i \text{ in} f(\mathbf{a}+\mathbf{x})$$

which, using the braided-binomial theorem[4] gives

$$\partial^i x_{i_1}\cdots x_{i_m} = x_{j_2}\cdots x_{j_m}[m;R]^{ij_2\cdots j_m}_{i_1\cdots i_m}.$$

These operators obey the relations of the vector algebra $V(R',R)$ and a braided-Leibniz rule which can written in the quantum-mechanical form

$$\partial^i x_j - x_a R^a{}_j{}^i{}_b \partial^b = \delta^i{}_j \tag{18}$$

as studied by several authors. Here x_i are multiplication operators from the left. We will also need the right-handed derivatives

$$f(\mathbf{x})\overleftarrow{\partial^i} = f(\mathbf{x}+\mathbf{a})|_{\text{coeff } a_i}, \qquad x_{i_m}\cdots x_{i_1}\overleftarrow{\partial^i} = x_{j_m}\cdots x_{j_2}[m;R_{21}]^{ij_2\cdots j_m}_{i_1\cdots i_m}$$

obtained from the binomial coefficient matrix $\left[{m \atop m-1};R\right] = [m;R_{21}]_{m\cdots 21}$ also given in [4]. The $\overleftarrow{\partial^i}$ obey the relations of $V(R',R)$ when considered acting this time on $f(\mathbf{x})$ from the right. They obey a right-handed braided-Leibniz rule. In terms of operators x_i of multiplication from the right, the latter is

$$x_j\overleftarrow{\partial^i} - \overleftarrow{\partial^b} R^i{}_b{}^a{}_j x_a = \delta^i{}_j. \tag{19}$$

These differentiations can be defined in terms of the pairing of vectors and covectors. Thus

$$\partial^i f = \langle v^i, f_{\underline{(1)}}\rangle f_{\underline{(2)}}, \quad f\overleftarrow{\partial^i} = f_{\underline{(1)}}\langle v^i, f_{\underline{(2)}}\rangle$$

from which their abstract properties (such as the assertions that they are left and right actions of the vector algebra) follow easily. On the other hand, since we know from Section 3 how the pairing behaves under $*$, we deduce at once from Proposition 3.2 or 3.3 that

$$(\partial^i f)^* = \overline{\langle v^i, f_{\underline{(1)}}\rangle}(f_{\underline{(2)}})^* = \langle v^{i*}, f^*{}_{\underline{(2)}}\rangle f^*{}_{\underline{(1)}} = f^* \overleftarrow{\partial^{i*}} = \begin{cases} f^* \overleftarrow{\partial^a} \eta_{ai} & \text{type I} \\ f^* \overleftarrow{\partial^{\bar{i}}} & \text{type II.} \end{cases} \tag{20}$$

We just used (14) and (8). One can verify this directly also using the reality property of R in the type I,II cases and (16) in the first case. Likewise,

$$(f \overleftarrow{\partial^i})^* = (f_{\underline{(1)}})^* \overline{\langle v^i, f_{\underline{(2)}}\rangle} = f^*{}_{\underline{(2)}}\langle v^{i*}, f^*{}_{\underline{(1)}}\rangle = \partial^{i*} f^* = \begin{cases} \eta_{ai}\partial^a f^* & \text{type I} \\ \partial^{\bar{i}} f^* & \text{type II.} \end{cases} \tag{21}$$

If we are in the holomorphic setting where R is type I but without using a metric, we need a different concept, namely that of differentiation (from the right or left) in the v^i co-ordinates. From the right this is

$$f(\mathbf{v})\overleftarrow{\partial_i} = f(\mathbf{v}+\mathbf{w})|_{\mathrm{coeff}\ w_i}, \qquad v^{i_m}\cdots v^{i_1}\overleftarrow{\partial_i} = v^{j_m}\cdots v^{j_2}([m;R]^{\mathrm{op}})^{i_1\cdots i_m}_{i j_2\cdots j_m}.$$

These $\overleftarrow{\partial_i}$ obey the relations of the braided covector algebra when acting from the right, and

$$(\partial^i f(\mathbf{x}))^* = \overline{\langle v^i, f_{\underline{(1)}}\rangle}(f_{\underline{(2)}})^* = f^*{}_{\underline{(1)}}\langle f^*{}_{\underline{(2)}}, x_i\rangle = f(\mathbf{x})^*\overleftarrow{\partial_i} \tag{22}$$

in the holomorphic case. The $GL_q(n)$ R-matrix provides an example and is compatible with the holomorphic calculus in [17] for this example.

Next, the braided exponential $\exp(\mathbf{x}|\mathbf{v})$ can be defined abstractly as coevaluation[5], cf.[4]. By this we mean the canonical element in $V\check{}(R',R)\otimes V(R',R)$ albeit as a formal powerseries since these algebras are not finite-dimensional. This makes sense generically, when the pairing between braided vectors and covectors is non-degenerate so that we can in principle come up with a basis and dual basis. The exact form of the braided exponential depends on the algebras. Nevertheless, without knowing its exact form we deduce from Proposition 3.2 or 3.3 that

$$(*\otimes *)\exp(\mathbf{x}|\mathbf{v}) = \exp(\mathbf{x}|\mathbf{v}) \tag{23}$$

whenever exp can be determined as the canonical element (or coevaluation) for the pairing. In the holomorphic case we have $\tau(\exp(\mathbf{x}|\mathbf{v}))$ instead on the right hand side.

In differential terms we can also characterise exp as the solution of

$$\partial^i \exp(\mathbf{x}|\mathbf{v}) = \exp(\mathbf{x}|\mathbf{v}) v^i, \quad \exp(\mathbf{x}|\mathbf{v}) \overleftarrow{\partial_i} = x_i \exp(\mathbf{x}|\mathbf{v})$$

with the usual conditions at 0 expressed via $\underline{\epsilon}$. These equations are equally well solved by $(*\otimes*)\exp(\mathbf{x}|\mathbf{v})$ or its transpose in the holomorphic case.

For example, if $R' = P$ we can develop exp as a formal powerseries with terms of the form $\mathbf{x}_1 \cdots \mathbf{x}_m([m;R]!)^{-1}\mathbf{v}_m \cdots \mathbf{v}_1$ in a compact notation. In this case we see (23) etc. immediately from Lemma 3.1. On the other hand, our analysis is not tied to this special case.

We have also studied braided Gaussians in [5] as the solution of the equation

$$\partial^i g = -x_a \eta^{ai} g$$

where a metric is supposed. Applying $*$ to this gives the equation

$$g^* \overleftarrow{\partial^i} = -g^* \eta^{ia} x_a \tag{24}$$

provided

$$\overline{\eta^{ij}} = \begin{cases} \eta_{ji} & \text{type I} \\ \eta^{\bar{j}\bar{i}} & \text{type II.} \end{cases} \tag{25}$$

This is a natural condition in the real type II case since the various R-matrix identities obeyed by η are then mapped consistently by complex conjugation. In the real type I case we already demanded the above constraint and for the same reasons. These conditions on the metric are also appropriate when R', R are antireal except that R should be antireal when computed in the quantum group normalisation.

If R, η obey some additional conditions as explained in [5] then one has an explicit form for g as a q-exponential of $\mathbf{x}\cdot\mathbf{x} = x_a x_b \eta^{ba}$. This obeys $(\mathbf{x}\cdot\mathbf{x})^* = \mathbf{x}\cdot\mathbf{x}$ in either type I, type II case when we have the conditions (25) on η. In these cases, which include q-Minkowski and q-Euclidean spaces we have $g^* = g$ for real q. The Euclidean space example agrees with more specific computations in [11].

Next, translation-invariant integration $\int$ was studied in [5] in terms of the Gaussian weighted average $\mathcal{Z}[f(\mathbf{x})] = (\int f(\mathbf{x})g)(\int g)^{-1}$, which we gave using (18) as

$$\mathcal{Z}[1] = 1, \quad \mathcal{Z}[x_i] = 0, \quad \mathcal{Z}[x_i x_j] = \lambda^2 \eta_{ba} R^a{}_{ibj}$$

$$\mathcal{Z}[x_{i_1}\cdots x_{i_m}] = \sum_{r=0}^{m-2} \mathcal{Z}[x_{i_1}\cdots x_{i_r} x_{a_{r+3}}\cdots x_{a_m}]\mathcal{Z}[x_{i_{r+1}} x_{a_{r+2}}][r+2, m; R]^{a_{r+2}\cdots a_m}_{i_{r+2}\cdots i_m} \lambda^{2(m-2-r)}$$

where λ is the quantum group normalisation constant and $[1,m;R]=(PR)_{12}\cdots(PR)_{m-1\,m}$ in the notation of [4]. On the other hand, there are in principle two translation-invariant integrals according to whether we use the coaddition $\underline{\Delta}$ from the left or the right. In differential terms it is natural to characterise them relative to g and g^* (say) as

$$\int \partial^i(f(\mathbf{x})g)=0,\quad \int_R (g^* f(\mathbf{x}))\overleftarrow{\partial^i}=0$$

for polynomials f. An analogous calculation to that in [5] but now using (24) and (19) gives the ratio for $\int_R g^* f(\mathbf{x})$ against $\int_R g^*$ as

$$\mathcal{Z}_R[x_{i_1}\cdots x_{i_m}]=\sum_{r=0}^{m-2}\mathcal{Z}_R[x_{a_1}\cdots x_{a_r}x_{i_{r+3}}\cdots x_{i_m}]\mathcal{Z}[x_{a_{r+1}}x_{i_{r+2}}]([1,r+1;R]^{\rm op})^{a_1\cdots a_{r+1}}_{i_1\cdots i_{r+1}}\lambda^{2r}$$

where $[1,m;R]^{\rm op}=(PR)_{m-1\,m}\cdots(PR)_{12}$. Using these explicit formulae one can see by the same techniques as in the proofs of Proposition 3.2 and 3.3 above that

$$\mathcal{Z}[(x_{i_1}\cdots x_{i_m})^*]=\overline{\mathcal{Z}_R[x_{i_1}\cdots x_{i_m}]}\tag{26}$$

for the real type I or II cases when λ is real. To see this note that this is true for $\mathcal{Z}[x_ix_j]=\mathcal{Z}_R[x_ix_j]$ by the assumptions (25). We then proceed by induction, using the identities

$$\overline{[1,m;R]^{\rm op}}=[1,m;R_{21}]^{\rm op}=[1,m;R]_{m\cdots 21}$$

in the real type II case (with barred indices on the right). In the real type I case we use

$$[1,m;R]^{a_1\cdots a_m}_{b_1\cdots b_m}\eta^{b_1i_1}\cdots\eta^{b_mi_m}=\eta^{a_1b_1}\cdots\eta^{a_mb_m}([1,m;R_{21}]^{\rm op})^{i_1\cdots i_m}_{b_1\cdots b_m},\quad \overline{[1,m;R]}=([1,m;R]^{\rm op})^{t\otimes\cdots\otimes t}.$$

Note that if $g^*=g$ is central and $\int=\int_R$ (the unimodular case) then we can conclude from (26) that the latter is complex-linear as we might expect. On the other hand, let us stress that the ratios $\mathcal{Z},\mathcal{Z}_R$ can be used even when the integrals and Gaussians themselves have no meaning.

Next, we have explained in [6] the braided approach to the calculus of differential forms and ϵ tensors on a braided space. Supposing also that R' obeys the QYBE and other identities, we formally reverse the roles $R\leftrightarrow -R'$. The braided covector space $V\check{}(-R,-R')$ is the algebra of forms with generators $\theta_i=\mathrm{d}x_i$. Hence if R is real type I (with a metric) or real type II, we have a $*$-algebra of forms by Lemma 2.2 or 2.1 respectively. We conclude that the action on forms

θ_i is the same as on x_i, i.e. $(\mathrm{d}x_i)^* = \mathrm{d}(x_i)^*$. The main difference however, between the braided approach and the usual approach to forms based on the axioms of Woronowicz[18][19] is that in the braided approach the operator d is constructed on the algebra generated by θ_i, x_i rather than being posited axiomatically. We have

$$(\theta_{i_1}\cdots\theta_{i_m} f(\mathbf{x}))\overleftarrow{\mathrm{d}} = \theta_{i_1}\cdots\theta_{i_m}\theta_i\partial^i f(\mathbf{x}), \quad \mathbf{x}_1\theta_2 = \theta_2\mathbf{x}_1 R.$$

The exterior algebra here is the braided tensor product Ω_R of forms and co-ordinates. On the other hand, we can equally well define Ω_L as the braided tensor product of co-ordinates and forms instead, and

$$\mathrm{d}\left(f(\mathbf{x})\theta_{i_1}\cdots\theta_{i_m}\right) = f(\mathbf{x})\overleftarrow{\partial^i}\theta_i\theta_{i_1}\cdots\theta_{i_m}, \quad \theta_1\mathbf{x}_2 = \mathbf{x}_2\theta_1 R$$

instead. The operator $\overleftarrow{d}$ is the right-handed exterior derivative in [6] and is a right-handed super-derivation on Ω_R. By reflecting the diagram-proofs there and reversing braid-crossings, or by explicit calculation from (19) one sees that d here is a more usual left-derivation on Ω_L.

On the other hand, neither of our left or right exterior algebras are naturally $*$-algebras in general. Instead, as we noted in the proof of Proposition 2.3 and [8], the natural operation takes us from one braided tensor product algebra to the reversed one:

$$* : \Omega_R \to \Omega_L, \quad (\theta_i \otimes 1)^* = 1 \otimes \theta_i^*, \quad (1 \otimes x_i)^* = x_i^* \otimes 1$$

and similarly for $*$ in the other direction. Then we have

$$(\omega\overleftarrow{\mathrm{d}})^* = \mathrm{d}(\omega^*) \tag{27}$$

for any $\omega \in \Omega_R$. Note that $x_i\overleftarrow{\mathrm{d}} = \mathrm{d}x_i = \theta_i$.

Finally, we defined upper and lower ϵ tensors in [6] by differentiation of the top form in form-space, leading to general formulae for them in terms of braided-factorials $[n; -R']!$. From Lemma 3.1 we conclude at once that if R' is of real type I then

$$\overline{\epsilon_{i_1 i_2 \cdots i_n}} = \epsilon^{i_n \cdots i_2 i_1} \tag{28}$$

in the construction of [6].

5 Concluding Remarks

There are several open problems in q-deformed physics which deserve serious attention and where an abstract understanding of the $*$-structure along the lines of the above should be useful. Probably the most important is to find an appropriate $*$-structure on the Weyl (or quantum mechanics) algebra (18). Likewise to find an appropriate $*$-structure on the Poincaré Hopf algebra generated by ∂^i and q-Lorentz transformations.

The Weyl algebra here was studied for the $GL_q(n)$ R-matrices in [20][17] and elsewhere, while the general R-matrix or braided point of view was given in [4]. In the latter we showed that this algebra is abstractly a braided semidirect product of the braided group of vectors acting on the braided covectors. Since we have obtained natural $*$-structures on these objects above, one can expect that there is a suitable theory of semidirect products by $*$-braided groups to apply here. It is hoped to present such a theory elsewhere.

The Poincaré algebra was studied in the Minkowski case by hand in [13], with a general braided groups construction giving the Minkowski, Euclidean and other Poincaré group quantum function algebras, due to the author in [3]. The general construction is again by means of an abstract semidirect product procedure (bosonisation) applied to the braided group of covectors. The above $*$-braided group structure on the braided covectors, and a standard Hopf $*$-algebra structure on the rotations or Lorentz sector imply together a $*$-structure for the resulting Poincaré quantum group. But because it is a hybrid object built partly from a braided group and partly from a quantum one (for the rotations etc.) one need not expect it to be a usual Hopf $*$-algebra. Instead it obeys some hybrid axioms to be elaborated elsewhere.

Once the appropriate $*$-structure and its properties are known in these two cases, one can proceed to the consideration of Hilbert space representations of these structures. These are some directions for further work.

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