

## Duality Principle and Braided Geometry

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# DUALITY PRINCIPLE AND BRAIDED GEOMETRY

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**Abstract** We give an overview of a new kind symmetry in physics which exists between observables and states and which is made possible by the language of Hopf algebras and quantum geometry. It has been proposed by the author as a feature of Planck scale physics. More recent work includes corresponding results at the semi-classical level of Poisson-Lie groups and at the level of braided groups and braided geometry.

Keywords: quantum groups – noncommutative geometry – braided geometry – integrable systems – Poisson structures – Mach principle –  $\kappa$ -deformation –  $q$ -deformation

## 1 Introduction

I did not know Feza Gürsey personally, but I would guess that he understood very well one of the principles close to my own heart: the deep interdependence of theoretical physics and pure mathematics. A fairly conventional view, expressed neatly by Pierre Ramond in his talk, is that we theoretical physicists are probing into higher and higher energy scales building finer and more accurate theories in the process. Some say that we can learn in this task from pure mathematicians to supply us with the right conceptual structures, others say that they can learn from us to find examples. My own view, however, is more radical but I will try nevertheless to outline it here and to reflect it in my talk. The starting point is that after all Nature does not care what mathematics has been discovered until now. So it would be rather naive to think that the maths that we know now is going to be enough to reach higher and higher levels until we reach the Planck scale or beyond to the ultimate theory of physics. Most likely if we did meet someone who knew this ultimate theory of physics and asked them to tell us, we would not understand a word of it because of *completely new* mathematical concepts that are centuries from being invented. Put another way, we physicists should not only try to apply known or fashionable mathematics, we should be ready and willing to invent whatever we think Nature uses as we go

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along in our search: we should be equally good *pure* mathematicians (in the creative sense, not necessarily in terms of rigour) as we are calculators and applied ones. Sometimes one can forget that mathematics is about creating natural ideas and concepts, and not about theorems and lemmas *per se*, though these are how a mathematician knows that the concepts are worthwhile. Physicists have perhaps other intuitive ways of knowing what is worthwhile.

Put another way, the naive view is that Nature exists and is out there, and maths exists and is out there, and we just have to apply the latter to describe the former. By contrast, I would like to argue that the word ‘exists’ is questionable in both cases, and that in fact each creates and justifies the other as we climb to higher and higher energy scales in our understanding. I mean that Nature and Language (which for me means mathematics) define each other and are at the very least interdependent.

I would like to describe here how this philosophy works in my own approach to unifying physics, which includes, as for many physicists, the goal of unifying quantum mechanics and gravity. It concerns a new kind of geometry more general than Riemannian and powerful enough to not break down in the quantum domain. Planck scale physics is the reason that I first became interested in Hopf algebras (quantum groups) and quantum or non-commutative geometry. It has nothing to do with the reason that quantum groups subsequently became very fashionable and important, which has more to do with work on inverse scattering and applications to knot and three-manifold invariants. It is gratifying that the same kind of new mathematics is emerging from all these different directions. The goal in this lecture is to convince you, a general physics audience, of two things:

a) There already *is* today a fairly complete theory of quantum geometry, at least as a new mathematical language. There are several approaches and the one I describe here and which I consider fairly complete is my own one which I called the *braided approach* (or *braided geometry*[1][2]). There are braided lines[3], planes[4], matrices[5][6], differentials and plane waves (exponentials)[7], Gaussians and integration[8], forms and epsilon tensors[9] and more known in this approach in association with a general  $R$ -matrix, i.e. a general solution of the celebrated quantum Yang-Baxter equations (QYBE). These equations are usually associated with the quantum groups of Faddeev, Jimbo and Drinfeld[10][11][12] and these do indeed play a role in the braided approach as a background covariance of our constructions, but not as the geometry itself. I will begin with the more conventional (but not so complete) approach in which quantum groups are geometrical objects too, and then move on to the systematic braided theory. This approach is quite different also from the approach to non-commutative geometry coming

out of the work of A. Connes and others.

b) This new language of quantum or braided geometry does indeed make it possible to formulate new and previously unsayable concepts in theoretical physics. The duality principle which is the title of my lecture will be such a new concept. This in turn allows the possibility to realise this new concept experimentally and thereby to discover new and previously inconceivable physical phenomena. This is what I mean by the deep interdependence of theoretical physics and pure mathematics. One can develop this point of view into a Kantian or Hegelian perspective on the concept of reality in physics[13].

The new and previously unsayable physical phenomenon which I want to concentrate on here is a generalisation of wave-particle duality or position-momentum symmetry. I would like to formulate this mathematically as a consequence of something a bit deeper, which I call a *quantum-geometry transformation*. This is the assertion that the symmetry algebra (or generalised momentum) of a system could also be isomorphic to the coordinate algebra of some geometry. That geometry would be momentum space as a geometrical object. This is certainly possible in flat space where

$$U(\mathbb{R}^n) = \mathbb{C}[\mathbf{p}] = \mathbb{C}(\mathbb{R}^n). \quad (1)$$

Here  $U(\mathbb{R}^n)$  is the enveloping algebra of the momentum generators  $\mathbf{p} = \{p^i\}$  which is the symmetry point of view, while  $\mathbb{C}(\mathbb{R}^n)$  is the algebra of coordinate functions  $\{p^i\}$  on momentum space as a geometrical object. They are obviously isomorphic since both are polynomials in  $n$ -variables. In fact, this isomorphism is as *Hopf algebras*. The axioms of a Hopf algebra will be given next in Section 2 but they have a coproduct map  $\Delta$  which on the left is trivial (on the left the Lie algebra structure of  $\mathbb{R}^n$  is reflected in the product of the enveloping algebra). On the right the same generators are viewed as coordinates and have a coproduct which corresponds to addition in momentum space, while the algebra is the trivial one. So the isomorphism is perhaps remarkable. The right hand side is the way that we will always deal with geometry in this lecture, in terms of the algebra of coordinates. It is a geometrical picture of momentum against a picture as differential operators  $\frac{\partial}{\partial x_i}$ .

The algebra  $\mathbb{C}(\mathbb{R}^n) = \mathbb{C}[\mathbf{x}]$  as generated by position coordinates  $\{x_i\}$  is how we think of position space. So (1) allows us to think of the momentum in the same language as we think of position, but with  $\mathbf{p}$  in the role of  $\mathbf{x}$ . We can also say it in the reverse way,

$$\mathbb{C}(\mathbb{R}^n) = \mathbb{C}[\mathbf{x}] = U(\mathbb{R}^n) \quad (2)$$

whereby we think of  $x_i = \frac{\partial}{\partial p^i}$  as differential operators on momentum space. This is wave-particle

duality.

As well as being isomorphic, the position and momentum are connected in a symmetrical way by this differentiation. This can be formulated as the assertion that the two Hopf algebras  $\mathbb{C}[\mathbf{x}]$ ,  $\mathbb{C}[\mathbf{p}]$  are dually paired by

$$\langle \phi(\mathbf{p}), \psi(\mathbf{x}) \rangle = (\phi(\partial)\psi)(0) \quad (3)$$

making them dual as Hopf algebras. An element of one defines a linear functional on the other. We will see the precise definition in Section 2 but it should be clear that the dual Hopf algebra to  $\mathbb{R}^n$  is  $\hat{\mathbb{R}}^n \cong \mathbb{R}^n$  again because  $\mathbb{R}^n$  is a self-dual Abelian group. A basis and dual basis are

$$\phi^{m_1 \dots m_n} = (p^1)^{m_1} \dots (p^n)^{m_n}, \quad \psi_{m_1 \dots m_n} = \frac{x_1^{m_1} \dots x_n^{m_n}}{m_1! m_2! \dots m_n!}.$$

The map (3) is the evaluation  $\mathbb{C}[\mathbf{p}] \otimes \mathbb{C}[\mathbf{x}] \rightarrow \mathbb{C}$  as dual linear spaces. But there is also a canonical element called the *coevaluation* and defined simply as the element in  $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{p}]$  corresponding to the identity map  $\mathbb{C}[\mathbf{x}] \rightarrow \mathbb{C}[\mathbf{x}]$  in the usual way since  $\mathbb{C}[\mathbf{p}]$  is dual to  $\mathbb{C}[\mathbf{x}]$ . In our case it is a formal powerseries (rather than living in the algebraic tensor product) and comes out as

$$\text{coev} = \sum_{m_i} \psi_{m_1 \dots m_n} \otimes \phi^{m_1 \dots m_n} = e^{\mathbf{p} \cdot \mathbf{x}}. \quad (4)$$

This is a rather modern way of thinking about exponentials and Fourier theory, but one that generalises well to quantum geometry[8].

So conceptually,  $\mathbb{C}[\mathbf{p}]$  and  $\mathbb{C}[\mathbf{x}]$  are dual to each other, but they are also isomorphic (i.e. self-dual). This is our formulation of wave-particle duality. It works just as well for any self-dual Abelian group (such as  $\mathbb{Z}_n$ ). It also works when the group is not self-dual but still Abelian. Then the Fourier conjugate space is the dual group, e.g.  $\hat{\mathbb{Z}} = S^1$ . The dual point of view with position and momentum interchanged is no longer isomorphic but both are still possible and in a dual relationship.

On the other hand, this duality phenomenon completely fails to get off the ground when the symmetry or generalised momentum group is non-Abelian. If  $\mathfrak{g}$  is a non-Abelian Lie algebra and we ask for a geometrical picture

$$U(\mathfrak{g}) = \mathbb{C}(?) \quad (5)$$

as functions on some ordinary manifold, we can never succeed simply because the left hand side is non-commutative, while the pointwise multiplication of functions on a manifold on the right hand side is always commutative. So for a geometrical picture we need some more general

non-commutative or quantum geometry. To give an elementary example, consider the sphere  $S^2$  with a standard Poisson bracket. Its Kirillov-Kostant quantisation is

$$[x_i, x_j] = \lambda \epsilon_{ij}^k x_k, \quad \sum_i x_i^2 = 1.$$

This is just  $U(su_2)$  modulo a ‘constant distance’ relation. So at least in this case we can think geometrically of  $U(su_2)$  as a quantum space even though the  $x_i$  are non-commuting coordinates. Moreover, with the greater generality of non-commutative or quantum geometry we can continue to pursue our position-momentum symmetry or duality phenomenon.

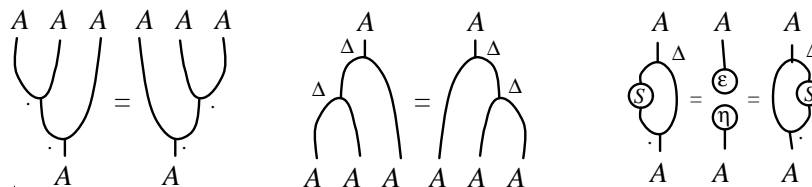
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## 2 What IS a Hopf algebra or quantum group?

Probably the reader has seen the definition of a Hopf algebra or quantum group elsewhere. There are several points of view which, remarkably, all lead to the same set of axioms. Here I want to give the axioms from the duality point of view, which is the one that interests us here.

In this case a Hopf algebra  $A$  is just an algebra equipped with further structure such that the dual linear space  $A^*$  of functionals on  $A$  is also an algebra in a compatible way. In terms of  $A$  this additional structure is a *coproduct*  $\Delta : A \rightarrow A \otimes A$ . It looks like a product but goes in the opposite direction. While an algebra is of course associative, the coproduct is *coassociative*. These conditions are



where we also show axioms for an *antipode* or ‘linearised inverse’ map  $S : A \rightarrow A$ . In addition, we require that  $\Delta$  is an algebra homomorphism. We also require a *counit* map  $\epsilon : A \rightarrow \mathbb{C}$  obeying  $(\epsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \epsilon) \circ \Delta$ .

The nice thing about these axioms is their input-output symmetry: Turn them up-side-down and you get the same axioms with the roles of  $\cdot$  and  $\Delta$  interchanged! The counit axioms become

the ones for the inclusion of the identity element  $\mathbb{C} \rightarrow A$  which we usually take for granted. This input-output symmetry is a deep feature of quantum geometry in the form that comes out of Hopf algebras. For our present purposes we realise it as follows: *If  $A$  is a Hopf algebra (say finite-dimensional) then so is  $A^*$ .* The structures correspond according to

$$\langle \phi\psi, a \rangle = \langle \phi \otimes \psi, \Delta a \rangle, \quad \langle \phi, ab \rangle = \langle \Delta\phi, a \otimes b \rangle, \quad \langle \phi, Sa \rangle = \langle S\phi, a \rangle$$

for all  $\phi, \psi \in A^*$  and  $a, b \in A$ . In the non-finite dimensional case we can still look for such a *duality pairing* between two Hopf algebras.

Thus  $\mathbb{C}[\mathbf{x}]$  and  $\mathbb{C}[\mathbf{p}]$  are Hopf algebras which are dually paired in this way. The Hopf algebra structure is

$$\Delta x_i = x_i \otimes 1 + 1 \otimes x_i, \quad \epsilon x_i = 0, \quad Sx_i = -x_i$$

and similarly for  $\mathbf{p}$ . For a general function  $a(\mathbf{x})$  the coproduct  $\Delta a(\mathbf{x})$  is a function of two variables with value  $(\Delta a)(x, y) = a(x + y)$ . So the coproduct expresses the addition law on  $\mathbb{R}^n$  in terms of the co-ordinate functions. The addition law is of course the basis for the geometry of  $\mathbb{R}^n$ .

More generally if  $G$  is a suitably nice group there will be a Hopf algebra  $\mathbb{C}(G)$  generated by coordinates on  $G$  and again with a coproduct corresponding to the group law. If the group is non-Abelian the geometry here typically has curvature. It corresponds to the output of  $\Delta$  not being symmetric or cocommutative. We also have another Hopf algebra  $U(\mathfrak{g})$  generated by the Lie algebra of  $G$ . This is dual

$$U(\mathfrak{g}) = \mathbb{C}(G)^*$$

in the loose sense that we have a duality pairing between these Hopf algebras. Since enveloping algebras are typically part of the algebra of quantum observables of a quantum system, we see that Hopf algebras suggest a general principle that *quantum theory and geometry are in a dual relationship*, implemented in the simplest case by Hopf algebra duality.

When we look away from commutative examples such as  $\mathbb{C}(G)$  we can still think of a Hopf algebra as a generalisation of the functions on a group manifold, even when there is no manifold in the usual sense. Likewise, when we look away from cocommutative examples such  $U(\mathfrak{g})$  we can still think of a Hopf algebra as like an enveloping algebra or a Lie algebra. Because the axioms of a Hopf algebra are self dual, one algebraic concept contains both points of view. This means that the category of Hopf algebras unifies two concepts in physics: one linked to quantum mechanics and the other to geometry. This is why the language of Hopf algebras opens up the

possibility to solve (5) by generalising both sides slightly and why we call it a *quantum-geometry transformation*.

This also suggests that Hopf algebras are a natural category in which to develop some simple models in which quantum theory and gravity are unified. This is the approach to Planck scale physics introduced in the author's thesis[14][15][16][13] on the basis of the above duality considerations. Obviously a natural source of non-commutative algebras  $A$  is as quantum algebras of observables, which are non-commutative versions of the classical algebra of functions on phase space. So we ask:

- When is a quantum algebra of observables a Hopf algebra?
- If so, what does it mean physically?

The answer to the first question is that it happens quite often; demanding it is a non-trivial but interesting constraint to put on a physical system. I will describe a large class of such physical systems associated to certain homogeneous spaces[14][16]. The answer to the second question is two fold:

(a) We now have geometry of phase space (in the crude form of a group law) even in the quantum setting, i.e. a unification of quantum theory and geometry. A general quantum system would include in its algebra of observables the enveloping algebra  $U(\mathfrak{g})$  for any generalised momentum generators; so to render such an algebra as like functions on a manifold involves for the momentum part exactly the quantum-geometry transformation (5). We now have non-commutativity coming from the momentum-position cross relations which we also have to deal with quantum geometry.

(b) We have the possibility of a new kind of symmetry in physics in which we reinterpret  $A^*$  as the algebra of observables of a dual system, and  $A \subset A^{**}$  as containing the states of the dual system. Thus we think of

$$\langle \phi, a \rangle = \phi(a) = \sum_i \rho_i \langle \phi_i | a | \phi_i \rangle$$

as the expectation value of observable  $a$  in mixed state  $\phi = \sum \rho_i \langle \phi_i | | \phi_i \rangle$  while the dual system considers the same number as  $\langle \phi, a \rangle = a(\phi)$ , the expectation value of the dual-observable  $\phi$  in the dual-state  $a$ . Note that we work with states in the usual way in mathematics as (say positive) linear functionals on the algebra of observables. These are typically convex linear combinations of expectations against Hilbert-space states, as indicated here. Thus we have the possibility of

a second physical system with observables  $A^*$  dual to our original one. This is a rather radical phenomenon in the quantum world which we propose to call *observable-state* or *micro-macro* duality[16][15]. It is also connected as we have seen with input-output symmetry, a point which is developed further in [17].

The simplest example in 1 dimension is the Hopf algebra  $\mathbb{C}[x] \blacktriangleleft \mathbb{C}[p]$  generated by  $x, p$  with the relations and Hopf structure[15][18]

$$[p, x] = -\frac{A}{B}(1 - e^{-Bx}), \quad \Delta x = x \otimes 1 + 1 \otimes x, \quad \Delta p = p \otimes e^{-Bx} + 1 \otimes p$$

$$\epsilon x = \epsilon p = 0, \quad Sx = -x, \quad Sp = -pe^{Bx}$$

depending on two real parameters  $A, B$ . It is the most general solution of the problem in 1-dimensions within the bicrossproduct construction in the next section. If we identify  $\hbar = -\frac{A}{B}$  we have a deformation of usual quantum mechanics. If we keep  $p$  as a conserved momentum and keep Hamiltonian  $\frac{p^2}{2m}$  so that the particle is in free-fall then the modified commutation relation results in modified dynamics. We use conventions in which  $p$  is antihermitian, i.e.  $\frac{-ip}{m}$  is the velocity. Then

$$\frac{dx}{dt} = \frac{i}{\hbar}[-\frac{p^2}{2m}, x] = (\frac{-ip}{m})(1 - e^{-Bx}) + O(\hbar), \quad \frac{dp}{dt} = 0.$$

Hence if we consider a particle falling in from  $\infty$  we see that as the particle approaches the origin  $x = 0$  it goes more and more slowly. In fact, it takes an infinite amount of time to reach the origin, which therefore behaves in some ways like a black-hole event horizon. This analogy should not be pushed too far since our present treatment is in nonrelativistic quantum mechanics, but it gives us an estimate  $B = \frac{c^2}{MG}$  as introducing the distortion in the geometry comparable to a gravitational mass  $M$ . Here  $G$  is Newton's constant.

On the other hand we get a commutative algebra if we take the limit  $\hbar \rightarrow 0$ . In this case  $\mathbb{C}[x] \blacktriangleleft \mathbb{C}[p] \cong \mathbb{C}(\mathbb{R} \ltimes \mathbb{R})$  where  $\mathbb{R} \ltimes \mathbb{R}$  has a non-Abelian group law corresponding to the coproduct above. It has curvature related to  $B$ . Combining with the above we see that our Hopf algebra has two limits[15][18]:

$$\begin{array}{l} mM \ll m_P^2 \rightarrow \mathbb{C}[x] \bowtie \mathbb{C}[p] \text{ usual quantum mechanics} \\ \mathbb{C}[x] \blacktriangleleft \mathbb{C}[p] \begin{array}{l} \nearrow \\ \searrow \end{array} \\ mM \gg m_P^2 \rightarrow \mathbb{C}(X) \text{ usual curved geometry} \end{array}$$

where  $m_P$  is the Planck mass. In the first limit the particle motion is not detectably different from usual quantum mechanics outside the Compton wavelength from the origin. Of course, there is still a singularity in our model right at the origin. In the second limit the non-commutativity would not show up for length scales larger than the background gravitational scale set by  $M$ .

This demonstrates our goal of unifying quantum mechanical and gravitational effects within a single model.

The dual Hopf algebra is generated by linear functionals  $\phi, \psi$  defined by

$$\langle \phi, : a(x, p) : \rangle = \left( \frac{\partial a}{\partial x} \right) (0, 0), \quad \langle \psi, : a(x, p) : \rangle = \left( \frac{\partial a}{\partial p} \right) (0, 0)$$

where  $: a(x, p) := \sum a_{n,m} x^n p^m$  is the normal-ordered form of a function in the two variables  $x, p$ . Then we have

$$\begin{aligned} [\psi, \phi] &= \hbar^{-1} (1 - e^{-A\psi}), \quad \Delta\phi = \phi \otimes 1 + e^{-A\psi} \otimes \phi, \quad \Delta\psi = \psi \otimes 1 + 1 \otimes \psi \\ \epsilon\phi &= \epsilon\psi = 0, \quad S\phi = -e^{A\psi}\phi, \quad S\psi = -\psi \end{aligned}$$

which is of just the same type as the above:

$$(\mathbb{C}[x] \blacktriangleleft \mathbb{C}[p])^* = \mathbb{C}[\phi] \blacktriangleleft \mathbb{C}[\psi] \cong \mathbb{C}[p] \blacktriangleleft \mathbb{C}[x].$$

So this particular Hopf algebra is self-dual. It means that the observable-state duality is indeed realised in this model, the dual quantum system being of a similar form. This is a new kind of symmetry principle which one can propose as a speculative idea for the structure of Planck scale physics[16].

### 3 Bicrossproduct Hopf algebras

The example at the end of the last section is one of a large class of bicrossproduct models[16] which we describe now. We consider first the finite group case and then move on to the Lie version.

The data for the bicrossproduct construction is a *group factorisation*. This means a group  $X$  with subgroups  $G, M \subset X$  such that  $G \times M \rightarrow X$  given by multiplying within  $X$  is an isomorphism. In this case multiplying in the group and projecting down to  $M$  and  $G$  gives mutual actions

$$\triangleleft : M \times G \rightarrow M, \quad \triangleright : M \times G \rightarrow G$$

from the right and left respectively, obeying

$$\begin{aligned} s\triangleleft e &= s, \quad (s\triangleleft u)\triangleleft v = s\triangleleft(uv); \quad e\triangleleft u = e, \quad (st)\triangleleft u = (s\triangleleft(t\triangleright u)) (t\triangleleft u) \\ e\triangleright u &= u, \quad s\triangleright(t\triangleright u) = (st)\triangleright u; \quad s\triangleright e = e, \quad s\triangleright(uv) = (s\triangleright u) ((s\triangleleft u)\triangleright v) \end{aligned} \tag{6}$$

for all  $u, v \in G$  and  $s, t \in M$ . The first two in each line say that we have an action, and the second say that the action is almost by automorphisms, but twisted by the other action.

This data is just what it takes to obtain a Hopf algebra  $\mathbb{C}(M) \blacktriangleright \blacktriangleleft \mathbb{C}G$  built on the vector space with basis  $G \times M$ . We write the basis elements as labelled squares  $s \square_u = s \square_u^{\triangleright u}$  where the convention is that if we label the left and lower edges then the other two are labelled by the values transformed by the actions  $\triangleright, \triangleleft$ . In this notation, the Hopf algebra structure is [18]

$$\begin{aligned} \Delta(s \square_u) &= \sum_{ab=s} \begin{array}{c} a \square \\ \otimes \\ b \square \\ u \end{array} & s \square_u \cdot \square_v &= s \square_{uv} \\ \varepsilon(s \square_u) &= \delta_{s,e} & I &= \sum_s s \square_e & S(s \square_u) &= \square_{u^{-1}} s^{-1} \end{aligned}$$

The product consists of gluing the squares horizontally whenever the edges are suitably matched. The coproduct by contrast consist of *ungluing* vertically, i.e. it is the sum of all pairs of squares which when glued vertically would give the square we began with. The dual Hopf algebra is  $(\mathbb{C}(M) \blacktriangleright \blacktriangleleft \mathbb{C}G)^* = \mathbb{C}M \blacktriangleright \blacktriangleleft \mathbb{C}(G)$  and has just the same form with vertical gluing and horizontal ungluing. The roles of  $\triangleright, \triangleleft$  and  $G, M$  are interchanged.

In mathematical terms the algebra here is a *cross product algebra*. This is a more or less standard way to quantise particles moving under the action of a group [19][20]. In nice cases the action  $\triangleleft$  of  $G$  on  $M$  induces a metric with the particle then moving on geodesics. The Lie algebra  $\mathfrak{g}$  of  $G$  plays the role of momentum since its elements generate the geodesics or 1-parameter flows. This works for any action  $\triangleleft$ . The main result of [14] is that if in this setting of particles on homogeneous spaces there is a *backreaction*  $\triangleright$  obeying (6) then the resulting quantum algebra of observables is a Hopf algebra. Not every action  $\triangleleft$ , i.e. not every homogeneous space, admits such a backreaction, i.e. it is a strong constraint on the system. We have seen the flavour of the constraint in the example  $\mathbb{C}[x] \blacktriangleright \blacktriangleleft \mathbb{C}[p]$ : it forces non-linear dynamics. One can think of it within this class of models as some kind of ‘Einstein’s equation’ [16] since it is an (integrated) second order constraint on  $\triangleleft$ . We see then that the duality principle which leads us to look for a Hopf algebra structure of this type forces something a bit like Einstein’s equation.

Note that we gave the Hopf algebras above for finite groups, while for physics we need to work with Lie groups and topological spaces. This can be done with Hopf-von Neumann algebras [21] as well as algebraically in the form  $\mathbb{C}(M) \blacktriangleright \blacktriangleleft U(\mathfrak{g})$  with dual  $U(\mathfrak{m}) \blacktriangleright \blacktriangleleft \mathbb{C}(G)$  where  $\mathfrak{g}, \mathfrak{m}$  are the respective Lie algebras. There are examples for all  $\mathfrak{g}$  compact semisimple, with  $\mathfrak{m} = \mathfrak{g}^{*\text{op}}$  a suitable solvable Lie algebra coming from the Iwasawa decomposition [22].

For the simplest 3-dimensional example we take  $\mathfrak{g} = \mathfrak{su}_2$  and  $M = SU(2)^{*\text{op}}$ . Then  $\mathbb{C}(SU(2)^{*\text{op}}) \blacktriangleright \blacktriangleleft U(\mathfrak{su}_2)$  is generated by position co-ordinates  $\{x_1, x_2, x_3, (x_3 + 1)^{-1}\}$  and  $\mathfrak{su}_2$  gen-

erators  $\{e_1, e_2, e_3\}$  with

$$\begin{aligned} [x_i, x_j] &= 0, \quad \Delta x_i = x_i \otimes 1 + (x_3 + 1) \otimes x_i, \quad \epsilon x_i = 0, \quad S x_i = -\frac{x_i}{x_3+1} \\ [e_i, e_j] &= \epsilon_{ij}^k e_k, \quad [e_i, x_j] = \epsilon_{ij}^k x_k - \frac{\epsilon_{ij3}}{2} \frac{x^2}{x_3+1}, \quad \epsilon e_i = 0 \\ \Delta e_i &= e_i \otimes \frac{1}{x_3+1} + e_3 \otimes \frac{x_i}{x_3+1} + 1 \otimes e_i, \quad S e_i = e_3 x_i - e_i (x_3 + 1). \end{aligned}$$

That this is a Hopf algebra is rather hard to check directly, and is best done via the theory above. We start with the action

$$e_i \triangleright x_j = \epsilon_{ij}^k \left( x_k - \frac{\delta_{k3}}{2} \frac{x^2}{x_3 + 1} \right)$$

of  $su_2$  on  $\mathbb{C}(SU(2)^{\text{op}})$  induced by  $\triangleleft$  above, and then make a cross product  $\mathbb{C}(SU(2)^{\text{op}}) \rtimes U(su_2)$  to obtain the algebra. For the coalgebra we follow the dual construction: we construct from  $\triangleright$  in (6) a *coaction*

$$\beta(e_i) = e_i \otimes \frac{1}{x_3 + 1} + e_3 \otimes \frac{x_i}{x_3 + 1}$$

and make a *cross coproduct* construction  $\mathbb{C}(SU(2)^{\text{op}}) \blacktriangleleft U(su_2)$  to obtain  $\Delta$ . The axioms for a coaction  $\beta$  here are like the axioms for an action but with the arrows reversed. For experts, there is also a  $*$ -structure  $e_i^* = -e_i$  and  $x_i^* = x_i$  making the above into a Hopf  $*$ -algebra. We gave the Hopf-von Neumann algebra version in [21].

The manifold  $SU(2)^{\text{op}}$  is the region  $\{\mathbf{s} \in \mathbb{R}^3, \quad s_3 > -1\}$  equipped with a deformed (non-Abelian) group law[16][22]. The action  $\triangleleft$  is a deformation of the usual action of  $su_2$  by rotations. Its orbits are non-concentrically nested spheres in this region, accumulating as  $s_3 = -1$ . Particles quantised on these orbits are described by the above Hopf algebra. It is a deformed quantum top. The dual system consists of the Lie algebra  $su_2^{\text{op}}$  acting on the group manifold  $SU(2)$  with orbits which are in fact the symplectic leaves in  $SU(2)$  for the Sklyanin bracket, i.e. an equally interesting system.

Recently, a four dimensional example similar to this one has been found in [23] with  $M$  a non-Abelian version of  $\mathbb{R}^{1,3}$  and  $g = so(1, 3)$ . Writing  $\{x_\mu\}$  for the position-coordinates and  $\{N_i, M_i\}$  for the Lorentz generators we have the Hopf algebra  $U(so(1, 3)) \blacktriangleleft \mathbb{C}(M)$  as [23]

$$\begin{aligned} [x_\mu, x_\nu] &= 0, \quad [M_i, M_j] = \epsilon_{ij}^k M_k, \quad [N_i, N_j] = \epsilon_{ij}^k N_k, \quad [M_i, N_j] = 0 \\ [x_0, M_i] &= 0, \quad [x_i, M_j] = \epsilon_{ij}^k x_k \\ [x_0, N_i] &= -x_i, \quad [x_i, N_j] = -\delta_{ij} \left( \frac{\kappa}{2} (1 - e^{-\frac{2x_0}{\kappa}}) + \frac{x^2}{2\kappa} \right) + \frac{x_i x_j}{\kappa} \\ \Delta x_0 &= x_0 \otimes 1 + 1 \otimes x_0, \quad \Delta x_i = x_i \otimes 1 + e^{-\frac{x_0}{\kappa}} \otimes x_i \\ \Delta M_i &= M_i \otimes 1 + 1 \otimes M_i, \quad \Delta N_i = N_i \otimes 1 + e^{-\frac{x_0}{\kappa}} \otimes N_i + \frac{\epsilon_{ij}^k}{\kappa} x_j \otimes M_k \end{aligned}$$

along with a suitable counit, antipode and  $*$ -structure. According to the above, we consider this the quantum algebra of observables of a system consisting of particles moving in orbits in our curved  $\mathbb{R}^{1,3}$  under a deformed Lorentz transformation.

On the other hand, it is obvious that both this example and the preceding  $\mathbb{R}^3$  example from [16][22][21] could be considered instead as deformed enveloping algebras of 3-dimensional or 4-dimensional Poincaré Lie algebras

$$\mathbb{C}(SU(2)^{\text{op}}) \bowtie U(su_2) \equiv U_\kappa(\mathfrak{p}_3), \quad U(\mathfrak{so}(1,3)) \bowtie \mathbb{C}(M) \cong U_\kappa(\mathfrak{p}(1,3)).$$

In the first case we introduce a parameter  $\kappa$  in the formulae above. In the second case we recover a Hopf algebra isomorphic [23] to the  $\kappa$ -deformed Poincaré Hopf algebra introduced by other means in [24]. We regard the  $x_i$  or  $x_\mu$  generators now as momentum generators. This demonstrates once again the quantum geometry transformation (5), this time dualising the interpretation of the position co-ordinates to view them as momentum.

## 4 Bicrossproduct Poisson structures and Lie bialgebras

This section announces new material in which we look at the above ideas at the semiclassical level. The semiclassical notion of a Hopf algebra is a *Poisson-Lie group*. At the Lie algebra level it is a *Lie bialgebra* [25]. These notions are useful in the theory of classical inverse scattering where Lie algebra splittings lead to solutions of the Classical Yang-Baxter equations. The examples we give in this section are however, quite different from this standard theory: they are by contrast the actual Poisson brackets whose quantisation is the bicrossproduct quantum algebras of observables described in the last section. The latter are likewise far removed from the usual quantum groups  $U_q(\mathfrak{g})$  of Drinfeld and Jimbo, being a different origin of quantum groups and Hopf algebras as coming out of ideas for Planck scale physics in [16]. Further details of the results in this section will be given in [26, Chapter 8].

Briefly, a Lie bialgebra is a Lie algebra  $\mathfrak{g}$  such that  $\mathfrak{g}^*$  is also a Lie algebra in a compatible way. In terms of  $\mathfrak{g}$  it means that there is an additional structure  $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}$  called the *Lie cobracket* and required to obey

$$\delta = -\tau \circ \delta, \quad (\text{id} \otimes \delta) \circ \delta\xi + \text{cyclic} = 0, \quad \delta([\xi, \eta]) = \text{ad}_\xi(\eta) - \text{ad}_\eta(\xi) \quad (7)$$

for all  $\xi, \eta \in \mathfrak{g}$ . Here  $\tau$  is the usual transposition and  $\text{ad}$  is the adjoint action extended to  $\mathfrak{g} \otimes \mathfrak{g}$  as a derivation. The dual of a finite-dimensional Lie bialgebra is also a Lie bialgebra with the role of  $[\ , \ ]$  and  $\delta$  interchanged.

The last condition in (7) can be understood as Lie algebra cocycle. Hence if  $G$  is a connected simply connected Lie group with Lie algebra  $\mathfrak{g}$  then  $\delta$  exponentiates to a group cocycle  $D : G \rightarrow \mathfrak{g} \otimes \mathfrak{g}$ . This in turn defines a Poisson bracket on  $G$  by

$$\{f, g\}(u) = (R_*D)(f \otimes g)(u), \quad \forall f, g \in C^\infty(G)$$

where  $R_{*u}$  is the right translation  $g = T_e G \rightarrow T_u G$  applied to the tensor square  $\mathfrak{g} \otimes \mathfrak{g}$  to give the tensor-field  $R_*D$ . We evaluate this 2-tensor field as a differential operator on the functions  $f$  and  $g$ . This is the geometrical meaning of a Lie bialgebra[25]. It corresponds to a class of Poisson brackets on the group manifold. The Poisson brackets in this class behave well with respect to the group product.

We find such a structure now in the semiclassical part of our bicrossproduct Hopf algebras. These are quantisations of  $C^\infty(X)$  where  $X$  is the classical phase space and in our case is a Lie group corresponding to the coproduct structure. The commutator in our quantum algebra of observables to lowest order in  $\hbar$  gives us the Poisson bracket. To describe the result let us start with the bicrossproduct data (6) in Lie algebra form. This data is a pair  $\mathfrak{g}, \mathfrak{m}$  of Lie algebras acting on each other by

$$\triangleleft : \mathfrak{m} \otimes \mathfrak{g} \rightarrow \mathfrak{m}, \quad \triangleright : \mathfrak{m} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$$

obeying the matching conditions[15]

$$\begin{aligned} [\phi, \psi] \triangleright \xi &= \phi \triangleright (\psi \triangleright \xi) - \psi \triangleright (\phi \triangleright \xi), & \phi \triangleleft [\xi, \eta] &= (\phi \triangleleft \xi) \triangleleft \eta - (\phi \triangleleft \eta) \triangleleft \xi \\ \phi \triangleright [\xi, \eta] &= [\phi \triangleright \xi, \eta] + [\xi, \phi \triangleright \eta] + (\phi \triangleleft \xi) \triangleright \eta - (\phi \triangleleft \eta) \triangleright \xi & (8) \\ [\phi, \psi] \triangleleft \xi &= [\phi \triangleleft \xi, \psi] + [\phi, \psi \triangleleft \xi] + \phi \triangleleft (\psi \triangleright \xi) - \psi \triangleleft (\phi \triangleright \xi) \end{aligned}$$

for all  $\xi, \eta \in \mathfrak{g}$  and  $\phi, \psi \in \mathfrak{m}$ .

It was shown in [22] that subject to a completeness condition for some resulting vector fields, such Lie algebras acting on each other exponentiate to Lie groups as in (6). Moreover, we have such a pair whenever a Lie algebra splits as a vector space into sub-Lie algebras  $\mathfrak{g}, \mathfrak{m}$ . This bigger Lie algebra is reconstructed as a *double cross sum*  $\mathfrak{g} \bowtie \mathfrak{m}$  it is built on  $\mathfrak{g} \oplus \mathfrak{m}$  with the Lie bracket in [15]. It is generated by  $\mathfrak{g}, \mathfrak{m}$  as sub-Lie algebras and the cross-bracket

$$[\phi, \xi] = \phi \triangleright \xi + \phi \triangleleft \xi.$$

The construction can be useful for generating more and more solutions of (8) in view of the proposition:

**Proposition 4.1** *Let  $(g, m, \triangleleft, \triangleright)$  be a Lie algebra matched pair as in (8). Then  $(g^{\text{op}} \oplus m, g \bowtie m)$  is also a Lie algebra matched pair, where  $g \bowtie m$  acts by the left adjoint action and  $g^{\text{op}} \oplus m$  acts by  $-\triangleleft$  for the action of  $g^{\text{op}}$  and  $-\triangleright$  for the action of  $m$ .*

Explicitly,  $g^{\text{op}}$  is  $g$  with the negated Lie bracket. The action of  $g \bowtie m$  and matching ‘backreaction’ of  $g^{\text{op}} \oplus m$  are

$$\begin{aligned}(\xi \oplus \phi) \triangleright (\eta \oplus \psi) &= ([\xi, \eta] + \phi \triangleright \eta - \psi \triangleright \xi) \oplus ([\phi, \psi] + \phi \triangleleft \eta - \psi \triangleleft \xi) \\(\xi \oplus \phi) \triangleleft (\eta \oplus \psi) &= (-\psi \triangleright \xi) \oplus (-\phi \triangleleft \eta)\end{aligned}$$

when both are built on the vector space  $g \oplus m$ , using the Lie brackets of  $g, m$  and the original actions. These new actions obey (8) for our bigger system. We then obtain a new Lie algebra  $(g^{\text{op}} \oplus m) \bowtie (g \bowtie m)$ , a construction which can clearly be iterated.

By contrast to these double cross sum Lie algebras, we now obtain from the same data (8) a Lie bialgebra.

**Proposition 4.2** *Let  $(g, m, \triangleleft, \triangleright)$  be a Lie algebras matched pair as in (8). There is a bicross-sum Lie bialgebra  $m^* \blacktriangleright g$  generated by  $m^*$  with zero Lie bracket,  $g$  and the cross relations and Lie coalgebra*

$$[\xi, f] = \xi \triangleright f, \quad \delta f = \langle f, [e_a, e_b] \rangle e^a \otimes e^b, \quad \delta \xi = e_a \triangleright \xi \otimes e^a - e^a \otimes e_a \triangleright \xi$$

for all  $f \in m^*$  and  $\xi \in g$ . Here  $\{e_a\}$  is a basis of  $m$  and  $\{e^a\}$  a dual basis.

This is built as a vector space on  $m^* \oplus g$  with the Lie bracket etc., defined by the above on the two components. The proof is a matter of some detailed calculations to check the axioms (7) for a Lie bialgebra. To explain this construction in detail one has to introduce the notion of Lie bialgebra coactions  $\beta$  dual to the notion of a Lie algebra action. Then the Lie bracket is the usual semidirect sum Lie bracket for the action of  $g$  on  $m^*$  induced by  $\triangleleft$ , while the cobracket  $\delta$  is a co-semidirect sum by a Lie coaction  $\beta$  induced by  $\triangleright$ . The dual Lie bialgebra is

$$(m^* \blacktriangleright g)^* = m \blacktriangleleft g^*$$

built in an analogous way as another bicross-sum with the roles of  $m, g$  interchanged.

These constructions, like  $g \bowtie m$ , all have generalisations to the case when  $g, m$  are Lie bialgebras to begin with. This more general theory is the semiclassical part of the double cross product and bicrossproduct of general Hopf algebras in [15, Sec. 3]. We do not try to cover it here, but see [26]. Instead, we give some examples of Proposition 4.2 and its dual.

For the first bicrossproduct example in Section 3, the Lie algebra data is as follows. We have  $g = su_2 = \{e_i\}$  and  $m = su_2^{*op} = \{x^i\}$  say, with

$$[x^1, x^2] = 0, \quad [x^3, x^1] = x^1, \quad [x^3, x^2] = x^2 \quad (9)$$

$$x^i \triangleleft e_j = \epsilon^i_{jk} x^k, \quad x^i \triangleright e_j = e_3 \delta^i_j - \delta^i_3 e_j \quad (10)$$

which solve (8). Both Lie algebras here are 3-dimensional and  $m \bowtie g^*$  is six-dimensional. It has the structure (9) and

$$\begin{aligned} [e^i, e^j] &= 0, & [x^i, e^j] &= \delta^i_3 e^j - e^i \delta^j_3 \\ \delta e^i &= \epsilon^i_{jk} e^j \otimes e^k, & \delta x^i &= \epsilon^i_{jk} (e^j \otimes x^k - x^k \otimes e^j) \end{aligned}$$

where  $\{e^i\}$  is a dual basis to the  $su_2$  basis. Its associated Lie group is a six-dimensional manifold  $X$  on which the cobracket  $\delta$  defines a natural Poisson structure as explained above. Its quantisation is the bicrossproduct Hopf algebra in Section 3. From another point of view, the same Hopf algebra is a deformation of the enveloping algebra of a 3-dimensional Poincaré Lie bialgebra  $m^* \bowtie g$  with structure from Proposition 4.2 given by

$$\begin{aligned} [x_i, x_j] &= 0, & [e_i, e_j] &= \epsilon_{ij}^k e_k, & [e_i, x_j] &= \epsilon_{ij}^k x_k \\ \delta x_i &= x_3 \otimes x_i - x_i \otimes x_3, & \delta e_i &= e_3 \otimes x_i - x_i \otimes e_3 + x_3 \otimes e_i - e_i \otimes x_3 \end{aligned}$$

where  $\{x_i\}$  is a dual basis to the  $\{x^i\}$ .

The same theory applies to the example at the end of Section 3. We have  $g = so(1, 3) = \{M_i, N_j\}$  and  $m = \{x^\mu\}$  say, with [23]

$$[x^i, x^j] = 0, \quad [x^i, x^0] = x^i \quad (11)$$

$$\begin{aligned} M_i \triangleleft x^0 &= M_i \triangleleft x^j = 0, & N_i \triangleleft x^0 &= -N_i, & N_i \triangleleft x^j &= \epsilon^{ijk} M_k \\ M_i \triangleright x^0 &= 0, & M_i \triangleright x^j &= \epsilon^{ijk} x^k, & N_i \triangleright x^0 &= -\delta_{ij} x^j, & N_i \triangleright x^j &= -\delta_{ij} x^0. \end{aligned} \quad (12)$$

One can check that  $(m, g, \triangleleft, \triangleright)$  is again a Lie algebra matched pair. The Lie bialgebra  $g^* \bowtie m$  is ten-dimensional and has the structure from Proposition 4.2 given by (11) and

$$\begin{aligned} [M^i, M^j] &= [N^i, N^j] = [M^i, N^j] = [x^i, N^j] = [x^0, M^i] = 0 \\ [x^0, N^i] &= -N^i, & [x^i, M^j] &= \epsilon^{ijk} N^k, & \delta N^i &= \epsilon^i_{jk} N^j \otimes N^k \\ \delta M^i &= \epsilon^i_{jk} M^j \otimes M^k, & \delta x^0 &= (N^i \otimes x^j - x^j \otimes N^i) \delta_{ij} \\ \delta x^i &= N^i \otimes x^0 - x^0 \otimes N^i + \epsilon^i_{jk} (M^j \otimes x^k - x^k \otimes M^j) \end{aligned}$$

where  $\{N^i, M^i\}$  are a dual basis to the Lorentz basis. Its associated Lie group is a ten-dimensional manifold  $X$  on which the cobracket  $\delta$  defines a natural Poisson structure as explained. Its quantisation is the bicrossproduct Hopf algebra at the end of Section 3. From another point of view, the same Hopf algebra is a deformation of the enveloping algebra of a 4-dimensional Poincaré Lie bialgebra  $g \bowtie m^*$  with structure

$$\begin{aligned} [x_\mu, x_\nu] &= [x_0, M_i] = [M_i, N_j] = 0, & [M_i, M_j] &= \epsilon_{ij}{}^k M_k, & [N_i, N_j] &= \epsilon_{ij}{}^k N_k \\ [x_i, M_j] &= \epsilon_{ij}{}^k x_k, & [x_i, N_j] &= -\delta_{ij} x_0, & [x_0, N_i] &= -x_i \\ \delta x_0 &= 0, & \delta x_i &= x_i \otimes x_0 - x_0 \otimes x_i \\ \delta M_i &= 0, & \delta N_i &= N_i \otimes x_0 - x_0 \otimes N_i + \epsilon_i{}^{jk} (x_j \otimes M_k - M_k \otimes x_j) \end{aligned}$$

where  $\{x_\mu\}$  is a dual basis to the  $\{x^\mu\}$ .

## 5 Braided Geometry and $q$ -Minkowski space

Now we outline another and more complete approach to quantum geometry which has emerged recently in [1][5] and subsequent works by the author and collaborators. In this approach the deformation of usual geometry is not connected conceptually with quantisation, but with *braided statistics*. We want to explore our Hopf algebra duality ideas in this context as well.

The idea of this *braided geometry* is to start with super-geometry and replace the its usual Bose-Fermi statistics  $\pm 1$  by something more general. This can be an arbitrary factor  $q$  [27] or more generally it can be a matrix solution  $R$  of the quantum Yang-Baxter equation (QYBE). This equation is connected historically with quantum groups[10] and quantum inverse scattering, but we will not use it in this historical way. I.e., this is a new approach to  $q$ -deformation. There are already two big reviews by the author[2][28] so we shall be brief.

The starting point for braided geometry is the concept of a *braided group*[1]. In mathematical terms a braided group  $B$  is defined like a Hopf algebra or quantum group as in Section 2, but the coproduct  $\underline{\Delta} : B \rightarrow B \otimes B$  is no longer multiplicative as it was there. Instead, it is *braided-multiplicative* in the sense

$$\underline{\Delta}(ab) = (\underline{\Delta}a)(\underline{\Delta}b), \quad (a \otimes c)(b \otimes d) = a\Psi(c \otimes b)d \tag{13}$$

for all  $a, b, c, d \in B$ . In diagrammatic form we use an operator  $\Psi = \bowtie : B \otimes B \rightarrow B \otimes B$  called the braiding in those places where diagrammatically we would need to write a braid crossing, i.e. in those places where there is a transposition in our constructions. We are working in fact

in a kind of braided mathematics. The familiar case on which we model everything is the super case where

$$\Psi(c \otimes b) = b \otimes c(-1)^{|c||b|}$$

on homogeneous elements of degree  $| \cdot |$ .

The simplest new example is the braided line  $B = \mathbb{C}[x]$  with structure

$$\underline{\Delta}x = x \otimes 1 + 1 \otimes x, \quad \underline{\epsilon}x = 0, \quad \underline{S}x = -x, \quad \Psi(x^m \otimes x^n) = q^{mn} x^n \otimes x^m$$

The new part is  $\Psi$  and means for example that

$$\underline{\Delta}x^m = \sum_{r=0}^m [m; q]_r x^r \otimes x^{m-r}, \quad [m; q]_r = \frac{[m; q]!}{[r; q]![m-r; q]!}, \quad [m; q] = \frac{1 - q^m}{1 - q}$$

which is the origin in braided geometry of the  $q$ -integers  $[m, q]$  familiar in working with  $q$ -deformations.

The next simplest example is the braided plane  $B$  generated by  $x, y$  with

$$\begin{aligned} yx &= qxy, & \underline{\Delta}x &= x \otimes 1 + 1 \otimes x, & \underline{\Delta}y &= y \otimes 1 + 1 \otimes y \\ \underline{\epsilon}x &= \underline{\epsilon}y = 0, & \underline{S}x &= -x, & \underline{S}y &= -y \\ \Psi(x \otimes x) &= q^2 x \otimes x, & \Psi(x \otimes y) &= qy \otimes x, & \Psi(y \otimes y) &= q^2 y \otimes y \\ \Psi(y \otimes x) &= qx \otimes y + (q^2 - 1)y \otimes x \end{aligned}$$

The algebra here is sometimes called the ‘quantum plane’; the new part is the coproduct  $\underline{\Delta}$  and the braiding  $\Psi$ . The latter is the same one that leads to the Jones knot polynomial in a more standard context.

The general construction of which these are examples is associated to a pair of invertible matrices  $R, R' \in M_n \otimes M_n$  obeying

$$\begin{aligned} R_{12}R_{13}R_{23} &= R_{23}R_{13}R_{12} \\ R_{12}R_{13}R'_{23} &= R'_{23}R_{13}R_{12}, & R'_{12}R_{13}R_{23} &= R_{23}R_{13}R'_{12} \\ (PR + 1)(PR' - 1) &= 0, & R_{21}R' &= R'_{21}R \end{aligned}$$

where  $P$  is the permutation matrix and  $R_{12} = R \otimes \text{id}$  in  $M_n^{\otimes 3}$ , etc. Given such data we define the associated braided covector algebra  $B$  with generators  $\{x_i\}$  as[4]

$$\begin{aligned} x_i x_j &= x_b x_a R'^a{}_i{}^b{}_j, \quad \text{i.e.,} \quad \mathbf{x}_1 \mathbf{x}_2 = \mathbf{x}_2 \mathbf{x}_1 R' \\ \underline{\Delta}x_i &= x_i \otimes 1 + 1 \otimes x_i, \quad \underline{\epsilon}x_i = 0 \quad \underline{S}x_i = -x_i \\ \Psi(x_i \otimes x_j) &= x_b \otimes x_a R^a{}_i{}^b{}_j, \quad \text{i.e.,} \quad \Psi(\mathbf{x}_1 \otimes \mathbf{x}_2) = \mathbf{x}_2 \otimes \mathbf{x}_1 R. \end{aligned}$$

We use here and below a shorthand notation where the suffices on bold-face vectors etc., refer to the position of the indices in a tensor product of matrices.

We can proceed to develop geometry in this general setting, starting with this  $\underline{\Delta}$ , which we call ‘coaddition’ since it has the additive form. We can bring this out by using a notation  $\mathbf{x} \equiv \mathbf{x} \otimes 1$ ,  $\mathbf{x}' \equiv 1 \otimes \mathbf{x}$ . In our braided tensor product in (13) these two do not commute but rather obey the braid-statistics

$$\mathbf{x}'_1 \mathbf{x}_2 = \mathbf{x}_2 \mathbf{x}'_1 R$$

corresponding to  $\Psi$ . The braided-homomorphism property of  $\underline{\Delta}$  is then just the statement that  $\mathbf{x}'' = \mathbf{x} + \mathbf{x}'$  obey the same relations of  $B$  provided we use these braid statistics between our two independent copies.

In this notation, we define braided-differentiation as[7]

$$\partial^i f(\mathbf{x}) = \left( a_i^{-1} (f(\mathbf{a} + \mathbf{x}) - f(\mathbf{x})) \right)_{\mathbf{a}=0} \equiv \text{coeff of } a_i \text{ in } f(\mathbf{a} + \mathbf{x})$$

and find that these operators obey the relations  $\partial_1 \partial_2 = R' \partial_2 \partial_1$  of a *braided vector algebra*. This is defined like our covectors  $B$  but with upper indices. We have a braided-Leibniz rule

$$\partial^i (bc) = (\partial^i b)c + \cdot \Psi^{-1}(\partial^i \otimes b)c, \quad \forall b, c \in B$$

where there is a natural braiding between vectors and covectors defined also by  $R$ . In braided geometry all independent objects enjoy braid statistics with respect to each other.

On monomials the braided differentiation comes out as

$$\partial^i (\mathbf{x}_1 \cdots \mathbf{x}_m) = \mathbf{e}^i_1 \mathbf{x}_2 \cdots \mathbf{x}_m [m; R]_{1 \dots m}$$

where  $\mathbf{e}^i$  is a basis covector  $(\mathbf{e}^i)_j = \delta^i_j$  and

$$[m; R] = 1 + (PR)_{12} + (PR)_{12}(PR)_{23} + \cdots + (PR)_{12} \cdots (PR)_{m-1,m}$$

is a certain *braided integer matrix*[7]. In the 1-dimensional case the latter become the usual  $[m, q]$  and  $\partial^i$  becomes the celebrated Jackson  $q$ -derivative. Using the braided integers we can define braided-binomial coefficient matrices[7] from which the coproduct of  $x_{i_1} \cdots x_{i_m}$  can be recovered much as in the 1-dimensional case above.

It is also possible to define a braided-exponential or ‘plane wave’ following the same lines as in (4): we define a pairing between the braided covectors and vectors of the form (3) with  $\partial$  the braided one, and define  $\exp$  to be the corresponding coevaluation as a formal powerseries. It is an eigenfunction of the  $\partial^i$  as well as an eigenfunction with respect to braided differentiation  $\frac{\partial}{\partial p^i}$ . We

have a braided wave-particle duality. Note that if the braided integer matrices are all invertible then  $\exp$  takes a simple form using  $([m; R]!)^{-1}$  in the powerseries. In the one-dimensional case we recover the celebrated  $q$ -exponential  $\sum_{m=0}^{\infty} \frac{x^m p^m}{[m, q]!}$ .

We can also define a braided Gaussian[8] as the solution  $g_\eta$  of the equation

$$\partial^i g_\eta = -x_a \eta^{a i} g_\eta, \quad \underline{\epsilon}(g_\eta) = 1$$

again as a formal powerseries. Here  $\eta^{ij}$  is a *braided metric* defined in such a way that  $x_a \eta^{a i}$  behaves like a braided vector. It obeys a number of identities with  $R, R'$ . If  $\eta$  obeys some further identities, the Gaussian takes a nice form involving the 1-dimensional  $q$ -exponential of  $x_j x_i \eta^{ij}$ .

One can also define translation-invariant integration  $\int$ . More precisely, it turns out to be more natural to define a linear functional  $\mathcal{Z} : B \rightarrow \mathbb{C}$  where[8]

$$\mathcal{Z}[f(\mathbf{x})] = \left( \int f(\mathbf{x}) g_\eta \right) \left( \int g_\eta \right)^{-1}; \quad \mathcal{Z}[x_i] = 0, \quad \mathcal{Z}[x_i x_j] = \eta_{ba} R^a{}_i{}^b{}_j \lambda^2$$

etc. Here  $\lambda$  is a certain constant depending on  $R$ , the quantum group normalisation constant[29].

Finally, we add to our assumptions on  $R$  the additional equations

$$\begin{aligned} R_{12} R'_{13} R'_{23} &= R'_{23} R'_{13} R_{12}, & R'_{12} R'_{13} R_{23} &= R_{23} R'_{13} R'_{12} \\ R'_{12} R'_{13} R'_{23} &= R'_{23} R'_{13} R'_{12} \end{aligned}$$

which gives a kind of symmetry  $R \leftrightarrow -R'$ . Using it, we can define forms  $\theta_i = dx_i$  as like the above braided covectors but with  $-R$  in place of  $R'$ . This leads to the relations

$$\begin{aligned} d\mathbf{x}_1 d\mathbf{x}_2 &= -d\mathbf{x}_2 d\mathbf{x}_1 R, & \mathbf{x}_1 d\mathbf{x}_2 &= d\mathbf{x}_2 \mathbf{x}_1 R \\ d(dx_{i_1} \cdots dx_{i_p} f(\mathbf{x})) &= dx_{i_1} \cdots dx_{i_p} dx_a \frac{\partial}{\partial x_a} f(\mathbf{x}) \end{aligned}$$

for exterior differentials in the braided approach. As this point we make contact with similar formulae for  $dx_i$  imposed in other approaches[30] by consistency arguments. We also define the epsilon tensor as[9]

$$\epsilon^{i_1 i_2 \cdots i_n} = \frac{\partial}{\partial \theta_{i_1}} \cdots \frac{\partial}{\partial \theta_{i_n}} \theta_1 \cdots \theta_n = ([n; -R]!)_{12 \cdots n}^{i_n \cdots i_1}$$

where we braided-differentiate in form-space.

This is our survey of braided geometry. We also have a natural candidate in this approach for braided-Minkowski space. We just use the braided matrices  $B(R)$  introduced in [5] with generators  $\{u^i{}_j\}$  and relations  $R_{21} \mathbf{u}_1 R \mathbf{u}_2 = \mathbf{u}_2 R_{21} \mathbf{u}_1 R$ . These relations can be put in the form of a braided covector space with  $u_I = u^i{}_{i_1}$  in a multi-index notation and  $\mathbf{R}'$  a suitable matrix

built from  $R$ . We did this in [5] while the correct  $\mathbf{R}$  for the additive braiding  $\Psi$  was found by U. Meyer in [31] and corresponds to the braid statistics  $R^{-1}\mathbf{u}'_1R\mathbf{u}_2 = \mathbf{u}_2R_{21}\mathbf{u}'_1R$ . It assumes that  $R$  obeys a certain Hecke condition. There is also a third matrix  $\mathbf{R}$ . corresponding to a different braiding  $\Psi$  needed for a braided group with multiplicative  $\underline{\Delta}\mathbf{u} = \mathbf{u} \otimes \mathbf{u}$  as introduced in [5]. In short, we have natural matrix-like braided groups with both coaddition and comultiplication.

Moreover, for  $R$  obeying a certain reality condition, there is a natural  $*$ -structure defined by  $u^{i,*} = u^j_i$ , i.e. our braided matrices are hermitian[32]. Then it is obvious that  $2 \times 2$  braided hermitian matrices using, for example, the standard Jones polynomial R-matrix are a good definition of  $q$ -Minkowski space. It has four generators  $\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and relations

$$\begin{aligned} ba &= q^2ab, & ca &= q^{-2}ac, & da &= ad, & bc &= cb + (1 - q^{-2})a(d - a) \\ db &= bd + (1 - q^{-2})ab, & cd &= dc + (1 - q^{-2})ca. \end{aligned}$$

Previously [33][34] proposed a similar algebra as  $q$ -Minkowski space on the basis of tensoring two copies of the quantum plane as spinors. The braided approach on the other hand means that we get all the structure above: braid statistics, coaddition, differentiation, exponentiation, Gaussians, integration, forms and the epsilon tensor. We refer to the literature and the review [28] for details. See also[35][36].

Finally, we return to the braided theory and ask about braided-Lie algebras and braided-enveloping algebras. There is such a theory introduced by the author and based on the axioms[37]

where our braided Lie-algebra  $\mathcal{L}$  has a bracket  $[ , ]$  and an additional ‘sharing out’ map  $\Delta$  which we usually take for granted when working with ordinary Lie algebras. There is also a map  $\epsilon$ . We refer to [37] for details and for the theorem that every braided-Lie algebra has a universal enveloping braided group  $U(\mathcal{L})$ . Also in [37] is a general class of examples of the form

$$\begin{aligned} \mathcal{L} &= \text{span}\{u^i_j\}, & \Delta\mathbf{u} &= \mathbf{u} \otimes \mathbf{u}, & [\mathbf{u}_1, R\mathbf{u}_2] &= R_{21}^{-1}\mathbf{u}_2R_{21}R \\ & & \Psi(R^{-1}\mathbf{u}_1 \otimes R\mathbf{u}_2) &= \mathbf{u}_2R^{-1} \otimes \mathbf{u}_1R. \end{aligned}$$

One has  $U(\mathcal{L}) = B(R)$  for this braided-Lie algebra. We can also generate it by the generators

$$\chi^i_j = u^i_j - \delta^i_j, \quad R_{21}\chi_1R\chi_2 - \chi_2R_{21}\chi_1R = R_{21}R\chi_2 - \chi_2R_{21}R.$$

Again the Jones polynomial R-matrix can be fed into this construction and gives us the braided-Lie algebra  $gl_{2,q}$ . It has basis  $h, x_+, x_-, \gamma$  with braided-Lie bracket

$$\begin{aligned} [h, x_+] &= (q^{-2} + 1)q^{-2}x_+ = -q^{-2}[x_+, h] \\ [h, x_-] &= -(q^{-2} + 1)x_- = -q^2[x_-, h], \quad [x_+, x_-] = q^{-2}h = -[x_-, x_+] \\ [h, h] &= (q^{-4} - 1)h, \quad [\gamma, \begin{cases} h \\ x_+ \\ x_- \end{cases}] = (1 - q^{-4}) \begin{cases} h \\ x_+ \\ x_- \end{cases} \end{aligned}$$

and zero for the others. We see that as  $q \rightarrow 1$  the  $\gamma$  mode decouples and we have the Lie algebra  $su_2 \oplus u(1)$ , but for  $q \neq 1$  these are unified. On the other hand, the enveloping bialgebra  $U(\mathcal{L})$  comes out as the isomorphism[37]

$$U(gl_{2,q}) \cong \mathbb{R}_q^{1,3}, \quad \begin{pmatrix} h \\ x_+ \\ x_- \\ \gamma \end{pmatrix} = (q^2 - 1)^{-1} \begin{pmatrix} a - d \\ c \\ b \\ q^{-2}a + d - (q^{-2} + 1) \end{pmatrix}$$

which is another example of a quantum-geometry transformation, now in our braided setting! What is remarkable about this is that when  $q \rightarrow 1$  the left hand side is the enveloping algebra of  $su(2) \oplus u(1)$  and is non-commutative, but the right hand side is the coordinate algebra of Minkowski space and becomes commutative, i.e. the isomorphism is only possible in the braided  $q \neq 1$  world! Such an isomorphism between two of the key ingredients of the standard model in particle physics suggests a deep application of q-deformation in this context.

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