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REFLECTION GROUPS ON RIEMANNIAN MANIFOLDS

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Abstract. We investigate discrete groups G of isometries of a complete connected Riemannian manifold M which are generated by reflections, in particular those generated by disecting reflections. We show that these are Coxeter groups, and that the the orbit space M/G is isometric to a Weyl chamber C which is a Riemannian manifold with corners and certain angle conditions along intersections of faces. We can also reconstruct the manifold and its action from the Riemannian chamber and its equipment of istropy group data along the faces. We also discuss these results from the point of view of Riemannian orbifolds.

1. Introduction

The aim of this paper is to study the discrete groups G generated by reflections with respect to hypersurfaces (shortly, reflection groups) on a Riemannian manifold M. If $M = E^n$ is the Euclidean space, then the classification of all reflection groups was given in a fundamental paper by Coxeter [11]. This implies also the classification of reflection groups on the sphere S^n. There are many results about reflection groups in hyperbolic space, see Vinberg [32], [34], [33], and [35], but the complete classification is missing. In all these cases the appropriate fundamental domain C of a reflection group G (called Weyl chamber) is a Coxeter polyhedron, i.e., a convex polyhedron where any two neighbour walls (codimension 1 faces F_i, F_j with codimension 2 intersection) have angle $\pi/n_{i,j}$ for $n_{i,j} \in \mathbb{N}$. We call this the *Coxeter property.* Conversely, any Coxeter polyhedron C in a space of constant curvature $M = S^n, E^n, H^n$ is the fundamental domain of the reflection group G which is generated by the reflections $s_i = s_{F_i}$ with respect to the walls F_i of C. The group G is a Coxeter group, i.e., a group with a set $S = \{s_1, \dots, s_l\}$ of generators, and relations $s_i^2 = 1$, $(s_i s_j)^{n_{i,j}} = 1$ for $n_{i,j} \in \mathbb{N} \cup \{\infty\}$. In our case, $n_{i,j}$ is defined by the angle between the walls F_i and F_j as above.

The manifold M with the action of G can be reconstructed from the Weyl chamber C (which is homeomorphic to the orbit space M/G) by the universal construction of Vinberg [32]: Define the equivalence relation in $G \times C$ by

$$(x, g) \sim (y, h) \iff x = y, \ \ g^{-1}h \in G_x$$

where $G_x = \langle s_{F_i} : x \in F_i \rangle$ is the subgroup generated by all reflections with respect to walls containing x. Then the quotient space

$$\mathcal{U}(G, C) = G \times C/\sim$$

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has the structure of a space of constant curvature such that the natural action of G on $\mathcal{U}(G,C) = M$ is isometric and G is the reflection group of M with fundamental domain C.

More generally, if G is a Coxeter group with a set $S = \{s_1,\dots,s_l\}$ of standard generators, and relations $s_i^2 = 1$, $(s_i s_j)^{n_{i,j}} = 1$, where $i,j = 1,\dots,l$ and $n_{ij} \in \mathbb{N} \cup \{\infty\}$, and if C is a topological space with closed subspaces $P_1,\dots,P_l$ (called panels), then the Vinberg construction with $G_x = \langle s_i : x \in P_i \rangle$ gives a topological space $\mathcal{U}(G,C)$ with a continuous action of the group G and orbit space C. The topological G-space $\mathcal{U}(G,C)$ is called the *universal space* of the Coxeter group G, and it satisfies the following following universal property [32]:

> If G acts in a topological space X and if $\varphi : C \to X$ is a continuous map such that $s_i.\varphi(x) = \varphi(x)$ for $x \in P_i$ then there exists a unique extension of φ to a G-equivariant continuous map $\tilde\varphi : \mathcal{U}(G,C) \to X$ such that $\tilde\varphi[1,x] = \varphi(x)$ for any $x \in C$.

Davis [12] found necessary and sufficient conditions that $\mathcal{U}(G,C)$ is a topological manifold and G is a topological reflection group of $\mathcal{U}(G,C)$, i.e., any generator s_i acts on $\mathcal{U}(G,C)$ as a topological reflection (an involutive transformation whose fixed point set $\mathcal{U}(G,C)^{s_i}$ separates $\mathcal{U}(G,C)$). These conditions are that C is a topological 'nice' manifold with corners and that each panel P_i is a disjoint union of walls such that for any $x \in C$ the subgroup $G_x = \langle s_i : x \in P_i \rangle$ is finite. Conversely, let G be a discrete group of transformations of a topological manifold M generated by topological reflections, and let C be its Weyl chamber (the closure of a connected component of the set $M_{\text{reg}} = \{x \in M : G_x = \{1\}\}$ of regular points). Let $s_1,\dots s_l$ be reflections in G such that $M^{s_i} \cap C$ contains a codimension 1 component. Let P_i be the union of all codimension 1 components $M^{s_i} \cap C$. Then G is a Coxeter group with standard generators $s_1,\dots,s_l$ and the G-manifold M is G-homeomorphic to the universal G-manifold $\mathcal{U}(G,C)$ defined by the panels $P_1,\dots,P_l$.

One of the aims of this paper is to describe the structure of the Weyl chamber $C \cong M/G$ of a Riemannian manifold M with a discrete group G generated by reflections, and to get a similar description of such G-manifolds M in terms of 'abstract Riemannian chambers' C, which are Riemannian manifolds with corners such that any two neighbouring walls F_i, F_j satisfy the Coxeter property, i.e., the corresponding angle has constant value π/n_{ij} along $F_i \cap F_j$.

In section (2) we fix terminology and describe general properties of reflections of a Riemannian manifold M and of a discrete group G generated by reflections. We discuss the relations between a Dirichlet domain D of the group G and its Weyl chamber C which is defined as the closure of a connected component of the set M_{reg} of regular points of G. We give an example when a Weyl chamber is larger than a Dirichlet domain. We prove that for a simply connected manifold M, any reflection s is disecting, i.e. its fixed point set M^s is a connected totally geodesic hypersurface which decomposes M into two parts. We observe that a reflection group G on a Riemannian manifold M can be lifted canonically to a reflection group $\tilde G$, which is an extension of G, on the universal covering $\tilde M$ of M. As an interesting example of Riemannian manifold with a group generated by non disecting reflections, we consider the maximal torus of the group $SU(n)$ for $n > 2$ with the action of the Weyl group.

Starting from section (3), we mostly consider a Riemannian manifold M with a reflection group G generated by *disecting* reflections. Such a G-manifold is called a Coxeter manifold. Following M. Davis [12], we derive from a lemma of Bourbaki [4] that the group G acts simply transitively on the set of Weyl chambers C of a Coxeter G-manifold M. This implies that Weyl chambers coincide with Dirichlet domains of regular points and hence are homeomorphic to the orbit space, and that the reflection group G is a Coxeter group with reflections s_i with respect to walls F_i of C as standard generators. Moreover, the Weyl chamber C has the structure of a Riemannian manifold with corners and any two neighbouring walls F_i, F_j of C satisfy the Coxeter property and yield a Coxeter relation $(s_i s_j)^{n_{ij}} = 1$. We prove that in the simply connected case these relations generate all relations of G. In the general case, we give a geometric description of the fundamental group $\pi_1(M)$.

In section (4) we recall the notion and main the properties of a (smooth) manifold M with corners and we define the concept of a Coxeter equipment of M. This is an order reversing mapping of the poset of faces of M into the poset of Coxeter subgroups of a given Coxeter system (G, S) (where S is the set of standard generators of a Coxeter group G) which satisfies the Vinberg finiteness condition, see [34], [13]. We define a notion of *Riemannian chamber* C as a manifold with corners C equipped with an appropriate Riemannian metric such that walls W_i of C are totally geodesic and neighbouring walls satisfy the Coxeter property. Any Riemannian chamber carries a universal Coxeter equipment.

The Weyl chamber C of a Coxeter G-manifold M has the natural structure of a Riemannian chamber with an admissible (in some rigorous sense) Coxeter equipment. Moreover, this equipment is universal if and only if $\pi_1(M) = \pi_1(C)$. Conversely, if C is a Riemmanian chamber with an admissible Coxeter (G, S)-equipment then the universal space $M = \mathcal{U}(G, C)$ has the structure of a Coxeter G-manifold with Weyl chamber C. We prove also that if C is a manifold with corners and s is a Coxeter equipment of M then there exist a Riemannian metric γ such that (M, γ) is a Riemannian chamber and the equipment s is admissible. Hence any manifold with corners C with a Coxeter equipment determines a Coxeter G-manifold M, where the metric of M depends on the admissible metric on C and any Coxeter manifold can be obtained by this construction.

In section (5) and (6) we discuss another approach for reconstructing the Coxeter manifold from its Weyl chamber C which can be identified with the orbit space M/G based on the Thurston construction [31] of the universal covering orbifold. We recall this construction in section (5) and we derive from the main theorem of [23] that an orbifold structure of a space X can be reconstructed from the sheaf S_X of its smooth functions. In section (6) we define the notion of a Coxeter orbifold as an orbifold whose local groups are finite linear Coxeter groups. An example of Coxeter orbifold is the Weyl chamber C of a Coxeter manifold M. We prove that any Coxeter orbifold is such a Weyl chamber. More precisely, the universal covering $M = \tilde{C}$ of a Coxeter orbifold C admits a structure of (smooth) Coxeter G-manifold such that C is isomorphic to the Weyl chamber of the isometry group G. In particular, this shows that any Coxeter orbifold is *good* in the sense of Thurston.

In the last section we described all Coxeter equipments of an n-simplex Δ_n. This gives a classification of Coxeter orbifold structures on Δ and a classification of Coxeter manifolds with orbit space Δ_n up to a diffeomorphism.

2. Groups of isometries generated by reflections

2.1 Lemma. *Let M be a connected complete Riemannian manifold, and let $G \subseteq \operatorname{Isom}(M)$ be a group of isometries. Then G is a discrete subgroup in the Lie group $\operatorname{Isom}(M)$ if and only if each orbit of G in M is discrete.*

We shall say that G acts discretely on M.

Proof. The pointwise-open topology on the Lie group $\operatorname{Isom}(M)$ of all isometries coincides with the compact open topology.

If G is a discrete subgroup in $\operatorname{Isom}(M)$ then it is closed and acts properly on M so the action admits slices, and the orbit $G.x$ through $x \in M$ is homeomorphic to G/G_x where G_x is the isotropy group of x. Thus each orbit is discrete.

Conversely, suppose that each orbit is discrete. Since G consists of isometries, each discrete orbit is closed. We consider the closure $\bar{G}$ of G in $\operatorname{Isom}(M)$. Since G-orbits are closed, $\bar{G}.x = G.x$ for each $x \in M$. The action of the closed group $\bar{G}$ of isometries is proper, so there exist slices. Let x_0 be a regular point for the $\bar{G}$-action. Since $\bar{G}.x_0$ is discrete, the slice S_{x_0} through x_0 is open in M, and the isotropy group $\bar{G}_{x_0}$ acts trivial on S_{x_0}. Thus $\bar{G}_{x_0}$ acts trivial on M and equals $\{e\}$. Then $G.x_0 = \bar{G}.x_0 \cong \bar{G}$, thus $\bar{G} = G$ and is discrete in $\operatorname{Isom}(M)$. □

2.2. Dirichlet domains and central hypersurfaces. Let $G \subset \operatorname{Isom}(M)$ be a group which acts isometrically and discretely on a connected complete Riemannian manifold. Let x_0 be a regular point. The *closed Dirichlet domain* for this point is the set

$$D(x_0) := \{y \in M : d(y, x_0) \le d(y, g.x_0) \text{ for all } g \in G\},$$

where d is the geodesic distance on M. The open interior $D(x_0)^o$ is called the *open Dirichlet domain* for the regular orbit $G.x_0$, and we can find a *fundamental domain* F for the action of G satisfying $D(x_0)^o \subseteq F \subset D(x_0)$, i.e., a set F which meets each orbit in exactly one point, since

$$M = \bigcup_{g\in G} g.D(x_0).$$

For any two different points $y_0, y_1 \in M$ the *central hypersurface* is given by

$$H_{y_0,y_1} := \{y \in M : d(y, y_0) = d(y, y_1)\}.$$

It disects M in the sense that $M \setminus H_{y_0,y_1}$ is the disjoint union of the two open sets $\{x \in M : d(x, y_0) > d(x, , y_1)\}$ and $\{x \in M : d(x, y_0) < d(x, , y_1)\}$. Note that if M is a simply connected space of constant curvature then H_{y_0,y_1} is a totally geodesic submanifold, since it is the fixed point set of a symmetry, but that in general H_{y_0,y_1} is not a submanifold: On an elongated 2-torus it can be a figure 8.

Lemma. *For $x \in H_{y_0,y_1}$ let c_0 be a minimal geodesic from y_0 to x. Then c_0 meets H_{y_0,y_1} only at x.*

Proof. Let $c_0(t_x) = x$ and suppose for contradiction that $c_0(t) \in H_{y_0,y_1}$ for $t < t_x$. Let c_2 be a minimal geodesic from x to y_1. Then $t_x = d(y_0, c_0(t)) + d(c_0(t), x) = d(y_1, c_0(t)) + d(c_0(t), x) < d(y_1, x) = t_x$ unless c_2 equals the minimal geodesic $s \mapsto c_1(t - s)$ and hence $y_0 = y_1$, both a contradiction. □

2.3. Lemma. *Let $D = D(x_0)$ be the closed Dirichlet domain of a regular point x_0 for a discrete action of a group $G \subset \operatorname{Isom}(M)$. Then we have:*

(1) *If $g.D = D$ then $g = e$ in G.*

(2) *The open Dirichlet domain* $D(x_0)^o$ *is the connected component containing* x_0 *of*
$$M \setminus \bigcup_{e\neq g\in G} H_{x_0,g.x_0} \subset M_{reg}.$$
Here M_{reg} *denotes the set of all regular points, i.e. those points with trivial stabilizers.*

(3) G *acts simply transitively on the set* $\{D(g.x_0) : g \in G\}$ *of all Dirichlet domains.*

Proof. The isotropy group G_{x_0} is trivial: See the proof of (2.1).

(1) If $g.D = D$ then $g.x_0 \in D^o$, thus $d(g.x_0, x_0) \leq d(g.x_0, h.x_0)$ for each $h \neq e$ in G. If $g \neq e$, putting $h = g$, we get $g.x_0 = x_0$, a contradiction.

(2) If $x \notin \bigcup_{e\neq g\in G} H_{x_0,g.x_0}$ then $d(x, x_0) \neq d(x, g.x_0)$ for each $e \neq g \in G$. So if $g.x = x$ for $g \neq e$ then $d(x, x_0) = d(g.x, x_0) = d(x, g^{-1}.x_0) \neq d(x, x_0)$, a contradiction. Thus the isotropy group G_x is trivial and x is regular. The connected component of $M \setminus \bigcup_{e\neq g\in G} H_{x_0,g.x_0}$ containing x_0 is the set of all $x \in M$ with $d(x, x_0) < d(x, g.x_0)$ for all $e \neq g \in G$ which is D^o.

(3) Transitivity was seen in (2.2) and simple transitivity follows from (1). □

2.4. Walls of Dirichlet domains. Let $G \subset \mathrm{Isom}(M)$ be a discrete subgroup. For a regular point $x_0 \in M$ the set $F := H_{x_0,g.x_0} \cap D(x_0)$ is called a *wall* of the closed Dirichlet domain $D(x_0)$ if it contains an open non-empty subset of $H_{x_0,g.x_0}$.

Two closed Dirichlet domains are called *neighbors* if they contain a common wall.

2.5. Lemma of Poincaré. *Let* $D = D(x_0)$ *be a closed Dirichlet domain of a regular point* x_0*, and let* $g_1.D, g_2.D, \dots$ *be all the neighbors of* D*. Then the elements* $g_1, g_2, \dots$ *generate the group* G*.*

See fig. 4 for a Dirichlet domain with countably many walls.

Proof. *Claim.* For each $g \in G$ there exists a sequence $e = h_0, h_1, \dots, h_n = g$ such that $D(h_i.x_0)$ and $D(h_{i+1}.x_0)$ are neighbors for each i. We call this a Dirichlet neighbors chain from x_0 to $g.x_0$.

The claim proves the lemma as follows. Since $D(h_1.x_0)$ is a neighbor of $D = D(x_0)$ we have $D(h_1.x_0) = g_{i_1}.D$ for some i_1. Then $g_{i_1}.g_{i_2}.D$ is the neighbor $D(h_2.x_0)$ of $g_{i_1}.D$. Finally $g_{i_1}\dots g_{i_n}.D$ is the neighbor $D(h_n.x_0) = D(g.x_0)$ of $g_{i_1}\dots g_{i_{n-1}}.D = D(h_{n-1}.x_0)$. By (2.3) we have $g = g_{i_1}\dots g_{i_n}$.

We prove the claim by induction on $\{d_g := d(x_0, g.x_0) : g \in G\}$ which is a locally finite set in $\mathbb{R}$ since the orbit $G.x_0$ is discrete and closed in M.

Let $g \in G$ and assume that there exists a Dirichlet neighbors chain from x_0 to $h.x_0$ whenever $d_h < d_g$. Applying g_1 we then also conclude that there exists a Dirichlet neighbors chain from $g_1.x_0$ to $g_2.x_0$ whenever $d(g_1.x_0, g_2.x_0) < d_g$. Consider a minimal geodesic c from x_0 to $g.x_0$ of length d_g.

Case 1. Suppose that c meets $\bigcup_{e\neq k\in G} H_{x_0,k.x_0}$ in $x = c(t_1) \in H_{x_0,k.x_0}$ at distance $t_1 < \frac{1}{2}d_g$. See fig. 1. Then $k \neq g$. Consider a minimal geodesic c_1 from x_0 to $k.x_0$. Then c_1 meets $H_{x_0,k.x_0}$ in $c_1(\frac{1}{2}d_k)$. The minimal geodesic c_2 from x to $k.x_0$ has length t_1. Thus $d_k \leq 2t_1 < d_g$ by the triangle inequality. By induction there exists a Dirichlet neighbors chain from x_0 to $k.x_0$. The minimal geodesic c_3 from $k.x_0$ to $g.x_0$ has lenght $d_{k^{-1}.g} < t_1 + d(x, g.x_0) = d_g$ since otherwise $k.x_0 = x_0$. By induction there exists a Dirichlet neighbors chain from $k.x_0$ to $g.x_0$. So we get a Dirichlet neighbors chain from x_0 via $k.x_0$ to $g.x_0$, as required.

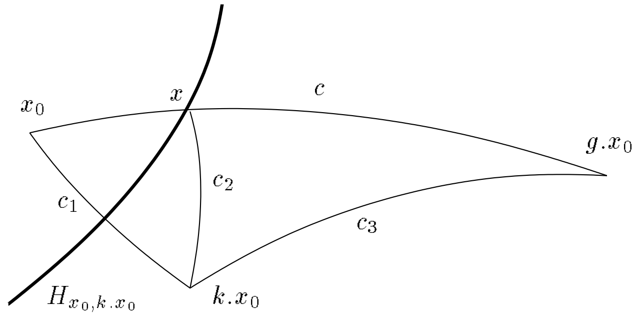


FIGURE 1.

Case 2. Suppose that c meets $\bigcup_{e\neq k\in G} H_{x_0,k.x_0}$ for the first time at $x = c(\frac{1}{2}d_g) \in H_{x_0,g.x_0}$ and that x lies in no other central hypersurface. Then there exists an open convex ball U with center x which meets only $H_{x_0,g.x_0}$ from $\bigcup_{e\neq k\in G} H_{x_0,k.x_0}$. By lemma (2.2) the central hypersurface $H_{x_0,g.x_0}$ cuts U in two connected components $\{y \in U : d(x_0,y) \gtrless d(g.x_0,y)\}$ and is the boundary of both. One of them is in $D(x_0)^o$ and the other is in the interior $D(h.x_0)^o$ of a neighbor of $D(x_0)$. So $y = c(\frac{1}{2}d_g + \varepsilon) \in D(h.x_0)^o$ for some $\varepsilon > 0$. Then $d(y,h.x_0) < d(y,x_0)$ and $d_{h^{-1}g} = d(h.x_0,g.x_0) \leq d(h.x_0,y) + d(y,g.x_0) < d_g$. By induction there is a Dirichlet neighbors chain from $h.x_0$ to $g.x_0$, thus also from x_0 to $g.x_0$.

Case 3. Suppose that c meets $\bigcup_{e\neq k\in G} H_{x_0,k.x_0}$ for the first time at $x = c(\frac{1}{2}d_g) \in H_{x_0,g.x_0} \cap H_{x_0,k.x_0}$ for $k \neq g$. We have $d(x,x_0) = d(x,k.x_0)$. Consider the minimal geodesic c_1 from x_0 to $k.x_0$ which meets $H_{x_0,k.x_0}$ in $c_1(\frac{1}{2}d_k)$. Then $d_k = d(x_0,k.x_0) < d(x_0,x) + d(x,k.x_0)$ since otherwise the curve following c from x_0 to x and then the minimal geodesic from x to $k.x_0$ would be a minimal geodesic and could not have an angle $\neq 0$ at x which implies that $k.x_0 = g.x_0$. By induction there is Dirichlet neighbors chain from x_0 to $k.x_0$. Moreover, $d_{k^{-1}g} = d(k.x_0,g.x_0) < d(k.x_0,x) + d(x,g.x_0) = d_g$ since otherwise the piecewise minimal geodesic from $k.x_0$ via x to $g.x_0$ would be a minimal geodesic and thus $k.x_0 = x_0$. By induction again there is a Dirichlet neighbor chain from $k.x_0$ to $g.x_0$ which together with the first chain gives a chain from x_0 to $g.x_0$, as required. $\square$

2.6. Reflections. Let (M,γ) be a connected complete Riemannian manifold. A *reflection* in M is an isometry $s \in \operatorname{Isom}(M)$ such that for some fixed point x_0 of s the tangent mapping $T_{x_0}s$ is a reflection in the Euclidean space $(T_{x_0}M, \gamma_{x_0})$, with repect to a hyperplane: For some vector $0 \neq X_{x_0} \in T_{x_0}M$ we have $T_{x_0}s.X_{x_0} = -X_{x_0}$, whereas $T_{x_0}s|X_{x_0}^\perp = \operatorname{Id}$.

Lemma. *Let s be a reflection on a complete connected Riemannian manifold M. Then we have:*

(1) *Every connected component N of the fixed point set M^s is a totally geodesic submanifold, and for each $x \in N$ the tangent mapping T_xs equals the identity on T_xN and $-\operatorname{Id}$ on $T_xN^\perp$.*
(2) *Every connected component N of M^s determines s completely as follows: For $y \in M$ there exists $x \in N$ such that $d(y,x) = \operatorname{dist}(y,N)$. Let $t \mapsto \exp(t.Y_x)$ be a minimal geodesic which reaches y at $t = 1$. Then $s(y) = \exp(-Y_x)$.*

(3) *At least one connected component of* M^s *is of codimension 1. Any such component is called a reflection hypersurface for* s.
(4) *For any* $y \in M \setminus M^s$ *we have* $M^s \subseteq H_{y,s.y}$.

Proof. (1) Let $x_0 \in M^s$ be a point such that $T_{x_0}s$ is a Euclidean reflection. Then $T_{x_0}s \circ T_{x_0}s = \mathrm{Id}_{T_{x_0}M}$, thus s is also an involution. Consequently $T_x s$ is an Euclidean involution for each fixed point x, thus it is diagonalizable with eigenvalues $+1$ on the eigenspace $T_x N$ and eigenvalue -1 on the eigenspace $T_x N^\perp$ where N is the connected component N of M^s containing x.

(2) Note that $Y_x \in T_x N^\perp$ and that $s(\exp(t.Y_x)) = \exp(t.T_x s.Y_x) = \exp(-t.Y_x)$.

(3) The connected component of M^s containing x_0 is of codimension 1.

(4) For $x \in M^s$ let c be a minimal geodesic from x to y. Then $s \circ c$ is a minimal geodesic from $s.x = x$ to $s.y$. Thus $d(x,y) = d(x,s.y)$. □

An example of a reflection s which is generated by two different reflection hypersurfaces H, H': Let $M = S^1$, $H = \{1\}$, $H' = \{-1\}$, and let s be complex conjugation. Another 2-dimensional example with three reflecting hypersurfaces is drawn in fig. 2. The 2-dimensional example in fig. 2 also shows that two different

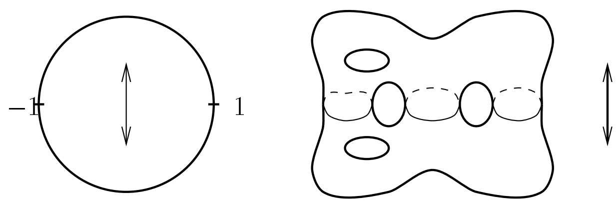


FIGURE 2. Reflections generated by different reflection hypersurfaces.

reflection hypersurfaces H, H' for the same reflection s need not be parallel, i.e., $\mathrm{dist}(x, H')$ is not constant in $x \in H$.

In [30] one finds the following theorem: If an irreducible Riemannian symmetric space M of noncompact type admits an involutory isometry whose fixed point set has codimension one, then M is a real hyperbolic space. This extends a result of Iwahori [20] concerning irreducible Riemannian symmetric spaces of compact type; however, the proofs of these two results are substantially different.

2.7. Disecting reflections. An isometry $s \in \mathrm{Isom}(M)$ is called *disecting* if the complement of the fixed point set M^s is not connected.

Lemma.

(1) *A disecting isometry* s *is a reflection.*
(2) *For a disecting reflection* s *the fixed point set* M^s *disects* M *into exactly 2 pieces. The reflection* s *permutes these two pieces.*
(3) *For a disecting reflection* s *the fixed point set* M^s *is a disjoint union of codimension 1 submanifolds.*
(4) *For a disecting reflection* s *and any* $y \in M \setminus M^s$ *we have* $M^s = H_{y,s.y}$.

Proof. (1) Since $M \setminus M^s$ is not empty and disconnected, the fixed point set M^s which is a disjoint union of closed totally geodesic submanifolds contains at least one connected component of codimesion 1. For any x in a codimension 1 component

H the tangent mapping $T_x s$ equals $\mathrm{Id}_{T_x H}$ on $T_x H$ and is a nontrivial isometry on the 1-dimensional subspace $T_x H^\perp$, thus equals multiplication by -1 there.

(2) Let $x_0 \in M \setminus M^s$. By (2.6.4) we have $M^s \subseteq H_{x_0,s.x_0}$. By (2.3.2) for the group $\{\mathrm{Id}, s\}$, removing the set $H_{x_0,s.x_0}$ decomposes M into exactly two connected pieces. Thus the subset $M^s \subseteq H_{x_0,s.x_0}$ cannot decompose it into more than two pieces.

(3) The union M^s_1 of all codimension 1 connected components of M^s also disects M into two connected components, since removing also the components of higher codimension does not change connectedness any more. Let N be a connected component of codimension ≥ 2 of M^s. Then N is contained in one component of $M \setminus M^s_1$ and s thus has to map it into the other component, by (2). Thus N is empty.

(4) By (2.6.4) we have $M^s \subseteq H_{y,s.y}$. Let $z \in H_{y,s.y}$. If $z \notin M^s$ then z and $s(z)$ lie in different components of $M \setminus M^s$. Let c_1 be a minimal geodesic from y to z, and let c_2 be a minimal geodesic (of the same length) from $s(y)$ to z. Then the broken geodesic $c_1 c_2^{-1}$ from y to $s(y)$ has to meet M^s in some point $x \in M^s$ since y and $s(y)$ lie in different components of $M \setminus M^s$. If x is an inner point on c_2, say, then the broken geodesic following $s(c_2)$ from y to $s(x) = x$ and then c_2 from x to z has the same length as c_2 and hence c_1. It has an angle at x (otherwise $z = s(z)$ and we are done), thus there is a geodesic from y to z shorter than c_1, a contradiction. □

2.8. Theorem. *Let M be a simply connected complete Riemannian manifold.*

Then any reflection σ on M is disecting, and its fixed point set M^σ is a connected orientable totally geodesic closed hypersurface.

Proof. Let $x \in M \setminus M^\sigma$ and let H be a connected component of M^σ of codimension 1. Choose a minimal geodesic c^+ from x to H. It hits H orthogonally by minimality, and thus we may continue it by $c^- = \sigma.c^+$ to obtain a geodesic c_0 from x to $\sigma(x)$ which hits H in exactly one point.

Suppose that $M \setminus H$ is connected. Then there exists a smooth curve c_1 from x to $\sigma(x)$ in $M \setminus H$. Since M is simply connected, there exists a smooth homotopy $h : [0,1] \times [0,1] \to M$ with $h(0,s) = x$, $h(1,s) = \sigma.x$, $h(t,0) = c_0(t)$, and $h(t,1) = c_1(t)$. We can also assume that h is transversal to H. But then $h^{-1}(H)$ is a closed 1-dimensional submanifold in $[0,1]^2$ which hits the boundary exactly once in $[0,1] \times \{0\}$ and never in $\{0\} \times [0,1]$ or $\{1\} \times [0,1]$. So the connected component hitting once must hit again in $[0,1] \times \{1\}$. Thus c_1 hits H, a contradiction.

Thus $M \setminus H$ is not connected, and H cuts M into two components, M^+ and M^-. Moreover, $M^\sigma = H$ since σ interchanges M^+ and M^-. □

2.9. Reflection groups and chambers. Let $G \subset \mathrm{Isom}(M)$ be a discrete subgroup of isometries of a connected complete Riemannian manifold M which is generated by all reflections contained in G. We shall call any such group G a *reflection group* of M. By a *(Weyl) chamber* we mean the closure in M of a connected component of the (open) complement of the union of all reflection hypersurfaces of all reflections in G. By an *open (Weyl) chamber* we mean the open interior C^o of a Weyl chamber C. For a chamber C a *wall* is a connected component of $C \cap M^s$ for a reflection s if it contains a non-empty open subset of M^s of codimension 1 in M. Two walls F_i, F_j are called neighbours is the intersection $F_i \cap F_j$ has a connected component of codimension 2.

Lemma. *For a chamber C and any regular point $x_0 \in C^o$ the Dirichlet domain $D(x_0)$ is contained in C. Thus C is a union of Dirichlet domains of the form $D(g.x_0)$, for all $g \in N_G(C)$.*

Moreover, G acts transitively on the set of all chambers.

Proof. Since the set M_{reg} of regular points is open and dense in M, we may choose a regular $x_0 \in C$. We claim that $C \supseteq D(x_0)$.

For $x \in M$ consider a minimal geodesic c from $x_0 = c(0)$ to $x = c(1)$. If c hits a reflection hypersurface H in $c(t)$ for $t < 1$, we may consider the minimal geodesic from x_0 to $c(t)$, followed by the minimal geodesic from $c(t)$ to $s_H(x)$, which is a broken geodesic from x_0 to $s_H(x)$ of length $d(x_0, c(t)) + d(c(t), s_H(x)) = d(x_0, c(t)) + d(c(t), x) = d(x_0, x)$. Since it has a proper angle at $c(t)$, a minimal geodesic from x_0 to $s_H(x)$ has length $d(x_0, s_H(x)) < d(x_0, x)$. Thus we see: *Whenever the minimal geodesic from x_0 to a point x hits a reflection hypersurface H in an intermediate point, $d(x_0, s_H(x)) < d(x_0, x)$.*

Now let $y \in D(x_0)$. Then $d(x_0, y) = \operatorname{dist}(x_0, G.y)$. By the statement above, any minimal geodesic from x_0 to y can hit a reflection hypersurface at most in y. Thus $y \in C$.

That G acts transitively on the set of all chambers follows from (2.3). $\square$

2.10. Examples of non-disecting reflections. We consider the real projective plane $\mathbb{RP}^2$, with the metric induced from S^2, and a reflection $s = s_H$ on one line H in it. Look at fig. 3 where the line at infinity L is chosen orthogonal to H so that L is invariant under s. On the line L one still has to identify antipodically. The fixed point set of s consists of H and the single point x on L farthest from H. There is only one chamber $C = \mathbb{RP}^2 \setminus (H \cup \{x\})$ which is a punctured disk and is dense in $\mathbb{RP}^2$. But there are two Dirichlet domains $D(z)$ and $D(sz)$ depending on z which for z on the line through x othogonal to L meet in $H \cup L$. As another

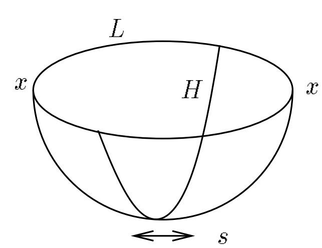


FIGURE 3. A reflection on $\mathbb{R}P^2$.

example, consider $M = SO(3) = \mathbb{R}P^3$ with the biinvariant metric. Then $g \mapsto g^{-1}$ is a non-disecting reflection whose fixed point set is the disjoint union of $\{e\}$ and some $\mathbb{R}P^2$. This reflection generates a Coxeter group.

2.11. Theorem. *Let M be a complete Riemannian manifold and let $G \subseteq \operatorname{Isom}(M)$ be a discrete group of isometries which is generated by all its reflections. Let C be a Weyl chamber in M for G. Let $F_1, F_2, \dots$ be the walls of C and let s_i be the reflection with respect to the wall F_i.*

Then the reflections $s_1, s_2, \dots$ generate G, and they satisfy the following relations:

(1) $(s_i)^2 = 1$

(2) *If two walls F_i, F_j are neighbors then $(s_i s_j)^{n_{ij}} = 1$ for some natural number n_{ij}.*

See fig. 4 for a Weyl chamber in the Poincaré upper halfplane with infinitely many walls.

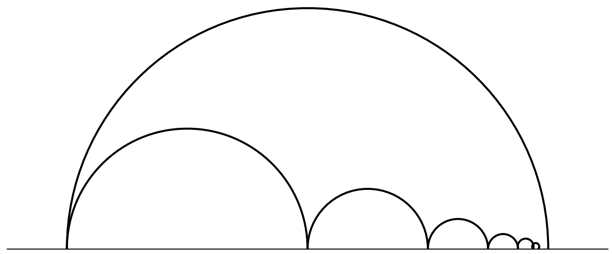


FIGURE 4.
A Weyl chamber in the Poincaré upper halfplane with infinitely many walls.

Proof. We prove that the reflections $s_1, s_2, \dots$ generate G. Let C' be any other Weyl chamber in M. Then we choose a smooth curve $c : [0,1] \to M$ from a regular point $x_0 \in C$ to a regular point $x' \in C'$ which changes Weyl chambers only transversally through open interiors of walls. First the curve passes from C through the interior of a wall F_{i_1} to a neighbor $s_{i_1}(C)$, and then through a wall F of this chamber to the next. For the reflection s_F in F we have $s_F = s_{i_1}.s_{i_2}.s_{i_1}$ for some wall F_{i_2} of C. If we now follow the curve c through all interiors of walls we see that C' is of the form $C' = g(C)$ for g in the subgroup generated by $s_1, \dots, s_l$. Any reflection in G is of the form s_F for some wall of some chamber C'. But then the argument above shows that $s_F = g.s_{i_k}.g^{-1}$, so G is generated by $s_1, s_2, \dots$ as claimed.

Relations (1) and (2) follow, since if x is an interior point of the face $f = F_i \cap F_j$ (i.e., there are no other walls through x) then the stabilizer G_x is faithfully and orthogonally represented in the two-dimensional space $T_x(f)^\perp$, and any finite subgroup of $O(2)$ which contains a reflection is a dihedral group. $\square$

2.12. Remark. In the setting of theorem (2.11) there might be more relations than specified in (2.11.1) and (2.11.2), see fig. 5. The left part of fig. 5 is a flat 2-

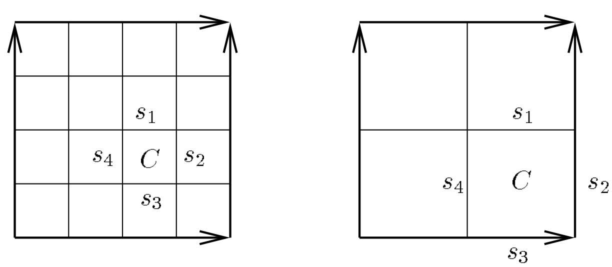


FIGURE 5. 2-tori with $\mathbb{Z}_2^4$ and $\mathbb{Z}_2^2$ as reflection groups.

torus with a chamber C specified, with reflections s_1, s_2, s_3, s_4 and angular relations

$(s_i s_{i+1})^2 = 1$ for $i = 1, \dots, 4 \mod (4)$ as decribed in (2.11.2). But moreover the relations $(s_2 s_4)^2 = 1$ and $(s_1 s_3)^2 = 1$ hold which are not described by (2.11.2).

In right hand part of fig. 5 we even have $s_1 = s_3$ and $s_2 = s_4$.

2.13. Lifting reflection groups to the universal covering. Let $\pi : \tilde{M} \to M$ be the universal covering of a Riemannian manifold M with a reflection group G, and let $\pi_1(M) = \Gamma \subset \mathrm{Isom}(\tilde{M})$ be the group of deck transformations of π. Any isometry of M can be lifted to an isometry of $\tilde{M}$. A lift $\tilde{s}$ of a reflection s in G is a reflection on $\tilde{M}$ if and only if it has a fixed point $\tilde{x} \in \tilde{M}$ with $\pi(\tilde{x})$ in a reflection hypersurface of s in M. The group $\tilde{G}$ generated by all reflections which are lifts of reflections in G, is a reflection group in $\tilde{M}$ which is normalized by Γ in $\mathrm{Isom}(\tilde{M})$. Then $\tilde{G}\Gamma$ is the group of all lifts of transformations in G, and $G = (\tilde{G}\Gamma)/\Gamma = \tilde{G}/(\tilde{G} \cap \Gamma)$. If $\tilde{C}$ is a chamber for $\tilde{G}$ in $\tilde{M}$ then $\pi(\tilde{C})$ is a chamber for G in M, since the union of all reflections hypersurfaces of $\tilde{G}$ equals the inverse image under π of the union of all reflection hypersurfaces of G.

Let s be a reflection in G, and let $\tilde{s}$ be a reflection covering s in $\tilde{G}$. According to (2.8) each reflection $\tilde{s}$ in $\tilde{G}$ is disecting, $\tilde{M}^{\tilde{s}}$ is one reflection hypersurface, and $\tilde{M} \setminus \tilde{M}^{\tilde{s}}$ consists of exactly two connected components $\tilde{M}^{\tilde{s}}_{+}$ and $\tilde{M}^{\tilde{s}}_{-}$.

If G is generated by disecting reflections then G acts simply transitively on the set of all chambers, see (3.5) below. The converse is not true, even if G is a Coxeter group, see fig. 7 in (2.15).

Suppose that one (equivalently any) chamber is simply connected. Then G acts simply transitively on the set of all chambers if and only if $\Gamma \subseteq \tilde{G}$. To see this, note that the universal cover $\pi : \tilde{M} \to M$ restricts to a diffeomorphism for each chamber $\tilde{C}$ in $\tilde{M}$ onto a chamber $C = \pi(\tilde{C})$ in M. If Γ contains a nontrivial deck transformation γ, then for a chamber $\tilde{C}$ covering C the set $\gamma(\tilde{C})$ is another chamber covering C. By (2.10) and (3.5) there exists a unique $\tilde{g} \in \tilde{G}$ with $\tilde{g}(\tilde{C}) = \gamma(\tilde{C})$. But then $\tilde{g} = \gamma$ if and only if $\tilde{g}$ covers Id_M in G.

2.14. Proposition. *Let G be a reflection group on a simply connected complete Riemannian manifold M. Then each chamber C is simply connected.*

Proof. Suppose for contradiction that some chamber C is not simply connected: Let $c : [0,1] \to C$ be a closed smooth curve through a regular point $x_0 \in C$ which is not contractible to the constant curve through x_0 in C with fixed ends at x_0.

Since M is simply connected there exists a smooth homotopy $h : [0,1] \times [0,1] \to M$ with $h(0,t) = c(t)$, $h(s,0) = x_0$, $h(s,1) = x_0$, and $h(1,t) = x_0$. We may assume that h is transversal to each reflection hypersurface and to each intersection of such hypersurfaces, since these form a locally finite family by the discreteness of G. Thus for each intersection hypersurface H_i the set $h^{-1}(H_i)$ is a 1 dimensional embedded submanifold of $[0,1]^2$ which does not meet the boundary, so it is a disjoint set of embedded circles in C which may touch only the bottom boundary $\{0\} \times [0,1]$. Moreover, the sets $h^{-1}(H_i)$ are all pairwise transversal 1-dimensional submanifolds in $(0,1)^2$, or empty, since this is the case for the (geodesically closed) H_i in M. Fig. 6 is an illustration. See [25], section 6, for transversality theorems on manifolds with corners. Now $h_0 = c$ is completely contained in $\bar{c}$ and we consider the curve $h_s = h|(\{s\} \times [0,1])$ for s moving from 0 to 1. So we move $\{s\} \times [0,1]$ upwards inside $[0,1]^2$. If this line hits $h^{-1}(H_i)$ we start reflecting back into C the point $h_s(t)$ for those t which lie inside $h^{-1}(H_i)$. If we meet another $h^{-1}(H_j)$ we add the reflection s_{H_j} at the right, etc. Since the different $h^{-1}(H_i)$ are transversal to each other

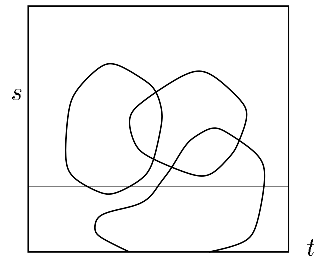


FIGURE 6. The sets $h^{-1}(H_i)$ in $[0,1]^2$.

this is welldefined, in particular at $s=0$, where the sets $(\{0\}\times[0,1])\cap h^{-1}(H_i)$ are disjoint by transversality. This proceedure transforms the smooth homotopy $h:[0,1]^2\to M$ to a continuous homotopy $\bar h:[0,1]^2\to C$ which contracts c to x_0. Thus C is simply connected which contradicts our assumption. □

2.15. Maximal torus of a compact Lie group as manifold with reflections. Let G be a semisimple compact Lie group with Lie algebra $\mathfrak{g}_0$ and let T be a maximal torus in G. The Lie subalgebra $\mathfrak{t}_0$ to T is then a Cartan subalgebra. Let $\Delta\subset\mathfrak{t}^*$ be the set of roots where $\mathfrak{t}=\mathfrak{t}_0\otimes\mathbb{C}$ is the complexification of $\mathfrak{t}_0$ and where $\mathfrak{t}^*$ is the dual space of $\mathfrak{t}$. Each root is purely imaginary on $\mathfrak{t}_0$. We have the following inclusion of lattices in $\mathfrak{t}^*$:

$$\mathbb{Z}\Delta\subseteq\Lambda_{\text{anal}}\subseteq\Lambda_{\text{alg}},\qquad\text{where}$$

$\mathbb{Z}\Delta$ is the root lattice, generated by Δ,

Λ_{anal} is the lattice of analytically integral forms $\lambda\in L(\mathfrak{t}_0,i\mathbb{R})$; they are characterized by the following property: whenever $H\in\mathfrak{t}_0$ satisfies $\exp(H)=1$ then $\lambda(H)\in 2\pi i\mathbb{Z}$; equivalently: there exists a multiplicative character $\xi_\lambda:T\to S^1$ such that $e^{\lambda(H)}=\xi_\lambda(\exp(H))$ for all $H\in\mathfrak{t}_0$.

Λ_{alg} is the weight lattice consisting of all $\lambda\in L(\mathfrak{t}_0,i\mathbb{R})$ such that that $2\langle\alpha,\lambda\rangle/|\alpha|^2\in\mathbb{Z}$ for all roots $\alpha\in\Delta$.

Now $\exp:\mathfrak{t}_0\to T$ induces an isomorphism $\mathfrak{t}_0/\Lambda^*_{\text{anal}}=T$, where Λ^*_{anal} is the dual lattice $\{X\in\mathfrak{t}_0:\lambda(X)\in\mathbb{Z}\text{ for all }\lambda\in\Lambda_{\text{anal}}\}$. Recall that G has trivial center if and only if $\Lambda_{\text{anal}}=\mathbb{Z}\Delta$, that G is simply connected if and only if $\Lambda_{\text{anal}}=\Lambda_{\text{alg}}$, that in general $\Lambda_{\text{anal}}/\mathbb{Z}\Delta$ is the center of G, and that the order of $\Lambda_{\text{alg}}/\mathbb{Z}\Delta$ equals the determinant of the Cartan matrix of $\mathfrak{g}$. The reflections on T are induced by the reflections in the Weyl group in $\mathfrak{t}_0$; to visualize it we consider the reflections hyperplanes and the lattice Λ^*_{anal} which consists of vectors orthogonal to the reflection hyperplanes. Then we consider a standard fundamental domain of the additive action of Λ^*_{anal}. We see that for $A_1^k=SU(2)^k$ all reflections in T are disecting, but that for semisimple nonabelian G we always get nondisecting reflections.

See fig. 7 for an example: It shows for $A_2=SU(3)$ the Cartan algebra $\mathfrak{t}_0$ as the universal covering of T with the reflection hyperplanes (bold) for $W\rtimes\Lambda^*_{\text{anal}}$, the lattice Λ^*_{anal}, and the fundamental domain (dashed). The reflections on T are not disecting, and the reflection group acts freely on the set of chambers in T, which are numbered.

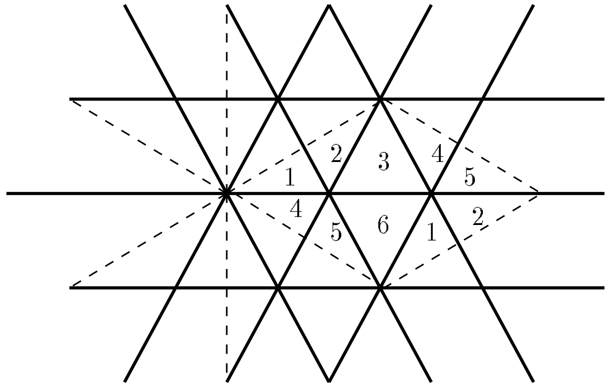


FIGURE 7. $\mathfrak{t}_0$ as universal covering of T for $A_2 = SU(3)$.

3. Coxeter Riemannian manifolds

3.1. Coxeter groups. [4] Recall that a *Coxeter group* is a group G which is a quotient of a free group $G(S)$ with a set S of generators by the subgroup generated by the relations $s^2 = 1$ and $(ss')^{n_{s,s'}} = 1$ for all $s, s' \in S$, where $n_{s,s'} \in \{1, 2, \dots, \infty\}$ indicates the order of ss' in G.

The set S is called a set of *standard generators* of G, and (G, S) is called a *Coxeter system* for G. Any subset $S' \subset S$ generates a subgroup $G(S') \subset G$ such that $(G(S'), S')$ is again a Coxeter system. $G(S')$ is called a *Coxeter subgroup*. The set of all Coxeter subgroups is a partially ordered set with respect to inclusion. A Coxeter system is described by a Coxeter diagram with vertices corresponding to the elements of S, where s and s' are connected by $n_{s,s'} - 2$ edges if $(ss')^{n_{ss'}} = 1$ and $1 < n_{ss'} < \infty$. The Coxeter diagram of a Coxeter subgroup $(G(S'), S')$ for $S' \subset S$ is obtained from the Coxeter diagram of (G, S) by deleting all vertices in $S \setminus S'$ and all edges leading to such vertices.

The length $\ell(g)$ of an element $g \in G$ is the minimum number l such that $g = s_{i_1} \dots s_{i_l}$ for $s_{i_k} \in S$. It satisfies $\ell(gg') \le \ell(g) + \ell(g')$, $\ell(g^{-1}) = \ell(g)$, and $|\ell(g') - \ell(g)| \le \ell(g'g^{-1})$.

In a Coxeter group (G, S) let $P_s^+ := \{g \in G : \ell(sg) > \ell(g)\}$ and $P_s^- := sP_s^+$. Then we have [4], iv, 1, 7:

(1) $\bigcap_{s\in S} P_s^+ = \{e\}$.
(2) $G = P_s^+ \sqcup P_s^-$ (disjoint union) for each $s \in S$.
(3) Let $s, s' \in S$ and $g \in G$. If $g \in P_s^+$ and $gs' \notin P_s^+$ then $s = gs'g^{-1}$.

Conversely, let G be a group with a generating set S of idempotents. Let $(P_s)_{s\in S}$ be a family of subset of G which satisfies

(4) $e \in P_s$ for all $s \in S$.
(5) $P_s \cap sP_s = \emptyset$ for all $s \in S$.
(6) Let $s, s' \in S$ and $g \in G$. If $g \in P_s$ and $gs' \notin P_s$ then $s = gs'g^{-1}$.

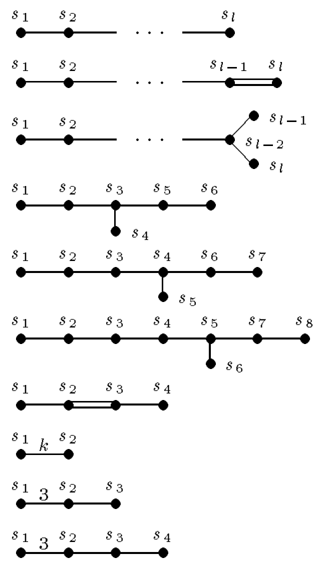
Then (G, S) is a Coxeter system and $P_s = P_s^+$.

3.2. Riemannian Coxeter manifold. Let $G \subset \mathrm{Isom}(M)$ be a discrete subgroup of isometries of a complete Riemannian manifold M which is generated by disecting reflections. Then (M, G) is called a *Riemannian Coxeter manifold.*

3.3. Coxeter manifolds of constant curvature. We recall some classical results.

Let (G, S) be a Coxeter system such that G is a finite group and let $S = \{s_1, \dots, s_n\}$. Then there exists a unique orthogonal representation of G as a linear reflection group on an Euclidean space $\mathbb{R}^n$ such that the s_i are reflections. The Weyl chamber associated to S is a simplicial cone with walls $F_1, \dots, F_n$ such that s_i is the reflection in F_i. Then the angle α_{ij} between F_i and F_j is given by $\alpha_{ij} = \pi/n_{ij}$ where $(s_i s_j)^{n_{ij}} = 1$ and n_{ij} is minimal. In the following table we give the list of all finite Coxeter systems which are irreducible in the sense that they are not a direct product of two (commuting) Coxeter subsystems.

A_l	$l \geq 1$	
B_l	$l \geq 2$	
D_l	$l \geq 4$	
E_6		
E_7		
E_8		
F_4		
$\mathrm{Di}(k+2)$	$k \geq 4$	$G_2 = \mathrm{Di}(6)$
H_3	$m \geq 3$	
H_4	$m \geq 3$	



If the Coxeter group has no dihedral group $\mathrm{Di}(k+2)$ as direct factor, then the angle between two walls may only take the values $\alpha = \pi/n$ for $n = 2, 3, 4, 5, 6$.

Conversely any simplicial cone with walls $F_1, \dots, F_n$ having angles $\alpha_{ij} = \pi/n_{ij}$ between F_i and F_j where $n_{i,j} \in \mathbb{N}$, is the Weyl chamber of a uniquely given Coxeter system with finite Coxeter group, by [32], theorem 1. The Coxeter diagram of (G, S) contains also all information about the Weyl chamber. The angle between the walls F_i and F_j is $\alpha_{ij} = \pi/n_{i,j}$ where $n_{ij} - 2$ is the number of edges connecting the vertices s_i and s_j.

If $g \in G$ preserves a codimension k face (an intersection of k walls) $F = F_{i_1} \cap \dots \cap F_{i_k}$ which does not contain a line through 0, then it it preserves it pointwise. Namely, g has a fixed point x in the interior of F since F is convex. By the lemma of Chevalley, g is contained in the Coxeter subgroup generated by all reflections s_i fixing x which correspond to all walls through x. Since x is an inner point of F, these walls also contain F. Thus g fixes F pointwise.

The angle in $F_{i_1} \cap \dots \cap F_{i_k}$ between $F_{i_1} \cap \dots \cap F_{i_k} \cap F_{i_{k+1}}$ and $F_{i_1} \cap \dots \cap F_{i_k} \cap F_{i_{k+2}}$ is in general not of the form π/n; nevertheless it is uniquely determined by the Coxeter system.

3.4. Example of Coxeter manifolds of non-constant curvature. Let G be a linear reflection group on $\mathbb{R}^n$.

(1) Let S be the unit sphere of $\mathbb{R}^n$. Then G acts on S and is generated by reflections. Choose a chamber C in $\mathbb{R}^n$ and a $(n-1)$-ball B in $C \cap S$. By surgery one may glue any compact $(n-1)$-dimensional manifold M to ∂B and do this in each chamber via the transformations of the group G. Obviously one can also put a G-invariant Riemannian metric on the resulting manifold, which then has complicated topology but carries a finite subgroup of the group of isometries which is generated by disecting reflections.

(2) Choose a chamber C in $\mathbb{R}^n$ and within C a regular point. Connect this point by a smooth curve to some point in each interior of each wall of C. Distributing this by G into all chambers of $\mathbb{R}^n$ yields a graph on which G acts. Now replace each point in the walls by a S^1 which lies completely in the interior of the wall, and replace the piece of the graph in the chamber C by a smooth compact surface which all the S^1's as boundary components, meeting the walls orthogonally. Distribute this to all chambers by the G-action and obtain a smooth compact surface with induced Riemannian metric on which G acts as a group of isometries generated by reflections.

3.5. Theorem. *Let (M, G) be a Riemannian Coxeter manifold. Then G is a Coxeter group and (G, S) is a Coxeter system for G, where S is the set of reflections with respect to the walls of C. Moreover, G acts simply transitively on the set of chambers.*

Proof. We follow arguments from [12]. Let Q be a chamber. For a reflection s with respect to a wall F of Q we set

$$P_s := \{g \in G : gQ \subset M^s_+\}$$

where M^s_+ is the connected componet of $M \setminus M^s$ which contains Q.

Lemma. $P_s = P_s^+ =: \{g \in G : \ell(sg) > \ell(g)\}$.

Proof. It is sufficent to check the properties (3.1.4), (3.1.5), and (3.1.6). The first two properies are obvious. We check (3.1.6). Let s, s' be reflections with respect to walls F, F' of the chamber Q and $g \in P_s$ but $gs' \notin P_s$. The chambers $Q, s'Q$ have a common wall W and the chambers gQ, $gs'Q$ have a common wall gW. Since they are on different sides of the hypersurface M^s, the wall gW belongs to M^s, see Fig. 8. Then $s(gQ) = gs'Q$ and s', $g^{-1}sg$ are two reflections which map

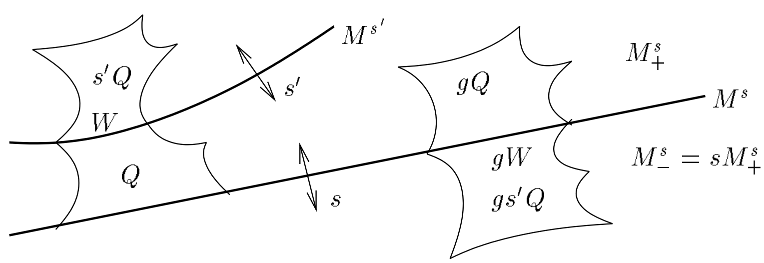


FIGURE 8.

Q to $s'Q$. Moreover $W \subset M^{s'}$, and $gW \subset M^s$ implies $W \subset g^{-1}M^s = M^{g^{-1}sg}$, so that $M^{s'} = M^{g^{-1}sg}$. Thus $s' = g^{-1}sg$. $sg = gs'$. ∎

Now the theorem follows from (3.1). Indeed, by (2.9) the group G acts transitively on the set of chambers. Assume that $gQ = Q$ for some $g \in G$. Then $g \in \bigcap_{s\in S} P_s = \bigcap_{s\in S} P^+_{s_i} = 1$ by property (3.1.1). $\square$

3.6. Corollary. *A discrete group G of isometries on a Riemannian manifold M generated by reflections is a quotient of a Coxeter group.*

Proof. This follows by (2.11), or by (2.13) and (3.5). $\square$

3.7. Question. Does there exist a discrete group of isometries which is generated by reflections but is not a Coxeter group? If so, can one characterize those which are Coxeter groups?

3.8. Corollary. *Let G be a reflection group on a complete connected Riemannian manifold M such that G acts freely and transitively on the set of all chambers, e.g., a Coxeter manifold. Let C be a chamber. Then we have:*

(1) *C is the Dirichlet domain associated with an interior point of C.*
(2) *Each chamber is convex and its interior consists of regular points.*
(3) *Any central hypersurface $H_{x_0,g.x_0}$ of a regular point x_0 and $1 \neq g \in G$ is a reflection hypersurface.*
(4) *$M_{\ reg} = \bigcup_{g\in G} gC^o$.*
(5) *Let F_1 and F_2 be two walls of the chamber C such that $F_1 = g.F_2$ for some $g \in G$. Then $F_1 = F_2$.*
(6) *The natural projection $\pi : M \to M/G$ induces a homeomorphism $C \to M/G$.*

Proof. (1) By lemma (2.9) the chamber C is a union of Dirichlet domains; but by (3.5) G acts simply transitively on the set of chambers, thus C is just one Dirichlet domain, by (2.3).

(2) By (1) and (2.3.2) each chamber consists of regular points. For convexity we have to show that any minimal geodesic arc between two points in C is contained in C. This follows from [2], 3.5.

(3) By (1), the union of all open chambers equals the union of all open Dirichlet domains $D(x)$ for all regular points x. Thus also their complements in M are the same: The union of all reflection hypersurfaces for G in M equals the union of all central hypersurfaces with respect to some (each) regular point. Thus the reflection hypersurfaces are exactly the central hypersurfaces $H_{g.x_0,g'.x_0}$.

(4) If $x \in H_{x_0,g.x_0}$ then by (3) the isotropy group of x is not trivial, so x is not regular. Thus by (2.3.2) we have $M_{\text{reg}} = \bigcup_{g\in G} g.C^o$.

(5) Let F_1^o be the open interior of F_1 in some central hypersurface H. F_1 is contained in the intersection of exactly two chambers, namely $F_1 \subseteq C \cap h.C$, where h is the reflection in the hypersurface H. Also $F_2 = g.F_1 = g.C \cap g.h.C$, but one of the two chambers must be C. Thus $g = h^{-1}$ is the reflection at H and so $F_1 = F_2$.

(6) follows from (5) and from the fact that G acts simply transitively on the set of all chambers. $\square$

3.9. Let (M, G) be a connected Riemannian Coxeter manifold and let C be a chamber. We denote by W the set of walls of C and by $G(W)$ the free group, generated by involutive generators r_F corresponding to all walls $F \in W$. Since G is generated by reflections with respect to walls in W, there is a natural homomorphism

$G(W) \to G$. We denote its kernel by R. We define the normal subgroup R_a of angular relations of $G(W)$as follows:

> Let $F_i, F_j \in W$ be neighboring walls with non empty intersection f containing a codimension 2 submanifold, and let F_i and F_j have angle π/n for a natural number n along some codimension 2 connected component of f, then $(r_{F_i} r_{F_j})^n$ is a generator of R_a in $G(W)$.

We denote by M_i, $i = 2, 3$ the complement in M of the union of codimension $\geq i$ intersections of reflection hypersurfaces. Note that these intersections are totally geodesic submanifolds as fixed point sets of finitely many isometries.

Theorem. *In this situation, the group R_a of angular relations is a normal subgroup of the group R of all relations in G. Moreover, $\pi_1(M_3, x_0) = \pi_1(M, x_0)$ and $\pi_1(C^o, x_0) = \pi_1(C, x_0)$, and we have the following exact sequences of groups:*

$$\{1\} \to \pi_1(C^o, x_0) *_e G(W) \to \pi_1(M_2, x_0) \to G(W) \to G \to \{1\}$$
$$\{1\} \to \pi_1(C, x_0) *_e G(W)/R_a \to \pi_1(M, x_0) \to G(W)/R_a \to G \to \{1\}$$

where for groups H and G the group $H *_e G$ is the kernel of the projection $p_G : H * G \to G$ from the free product to G. In particular,

$$\pi_1(M_2, x_0)/(\pi_1(C^o, x_0) *_e G(W)) = R,$$
$$\pi_1(M, x_0)/(\pi_1(C, x_0) *_e G(W)/R_a) = R/R_a.$$

Proof. By (3.8.6) the composition $C \to M \to M/G$ is a homeomorphism thus $\pi_1(C, x_0) \to \pi_1(M, x_0)$ is injective. By restriction $C \cap M_2 \to M_2 \to M_2/G$ is also a homeomorphism thus $\pi_1(C \cap M_2, x_0) \to \pi_1(M_2, x_0)$ is injective. By (3.8.5) we have $\pi_1(C^o, x_0) = \pi_1(C, x_0)$ since a closed curve in C may be deformed into C^o.

Any element in $\pi_1(M, x_0)$ can be represented by a closed smooth curve c through x_0 in M which we may assume to be transversal to all intersections of walls. By dimension, c lies in M_2 and first meets a wall F_1 of C transversally. Next it meets a wall $s_{F_1}(F_2)$ of $s_{F_1}(C)$ transversally. And so on until it comes back to x_0. We assign to c the expression (word) $r_{F_1} r_{F_2} \dots r_{F_k}$ in $G(W)$. A homotopy moving c in M_2 just allows cancellations in this expression using $r_F^2 = 1$. Replacing the r_F in this expression by the corresponding s_F we get an element in the reflection group G which maps C to C and thus is the identity, by theorem (3.5).

Let f_i be a fixed curve from x_0 to $s_{F_i}(x_0) \in s_{F_i}(C)$ hitting F_i once transversally. Any expression $r_{F_1} \dots r_{F_k}$ in $G(W)$ which maps to the identity in G, is assigned to the closed curve in M_2 which first follows f_1 from x_0 to $s_{F_1}(x_0)$, then $s_{F_1} \circ f_2$ from $s_{F_1}(x_0)$ to $s_{F_1} s_{F_2}(x_0)$, etc., until it ends again in x_0. Thus the sequence is exact at $G(W)$.

A curve representing an element in $\pi_1(M_2, x_0)$ which is transversal to walls can be described, up to 'transversal' homotopy, by a word $c_0 r_{F_1} c_1 r_{F_2} c_2 \dots r_{F_k} c_k$ where:

- $c_i \in \pi_1(s_{F_1} s_{F_2} \dots s_{F_i}(C^o), s_{F_1} s_{F_2} \dots s_{F_i}(x_0)) \cong \pi_1(C^o, x_0)$,
- r_{F_i} stands for the curve $s_{F_1} s_{F_2} \dots s_{F_{i-1}}(f_i)$.
- $s_{F_1} s_{F_2} \dots s_{F_k} = e$ in G since the curve is closed.

Thus the word describes a unique element of the free product $\pi_1(C^o, x_0) * G(W)$ which is in the kernel of $\pi_1(C^o, x_0) * G(W) \to G$. The curve in $\pi_1(M_2, x_0)$ maps to $e \in G(W)$ if and only if the word above also satisfies

- $r_{F_1} r_{F_2} \dots r_{F_k} = e$ in $G(W)$.

These are the elements of $\pi_1(C^o, x_0) *_e G(W)$.

So the first sequence is left exact, and surjectivity at G follows from (2.11).

The second exact sequence follows from the first one: any homotopy in M between smooth curves in M_2 may be assumed to be transversal to all intersections of reflection hypersurfaces of codimension ≥ 2. Then it avoids all intersection of codimension ≥ 3, so it lies in M_3. Thus $\pi_1(M, x_0) = \pi_1(M_3, x_0)$. If the homotopy meets an intersection $f = F_1 \cap F_2$ transversely, moving the curve through f means a cancellation in the expression assigned to the curve which is given by the corresponding generator $(r_{F_1} r_{F_2})^n$ of R_a. □

3.10. Theorem. *Let (M, G) be a simply connected Riemannian Coxeter manifold and let C be a chamber. Then we have:*

(1) *In terms of* (3.9) *we have $R_a = R$. In other words, the relations* (2.11.1) *and* (2.11.2) *generate all relations of the Coxeter system (G, S).*
(2) *The stabilizer G_x of a point $x \in C$ is a finite Coxeter group generated by reflections with respect to the walls $F_{i_1}, \dots, F_{i_k}$ through x. Moreover, if G_x has no factor isomorphic to the dihedral group $D(m)$ for $m = 5$ or > 6, then the angles between two walls through x take values π/n for $n = 2, 3, 4, 6$.*

For linear Coxeter groups this result was proved by Vinberg [32].

Proof. (1) This follows from $\pi_1(M, x_0) = R/R_a$ from (3.9).

(2) Let $g = s_{F_1} \dots s_{F_j} \in G_x$. Since any $h \in G$ preserves the union of all reflection hypersurfaces, g permutes the set of reflection hypersurfaces through x. Thus $g(f) = f$ where f is the connected component of $F_{i_1} \cap \dots \cap F_{i_k}$ containing x. Then $C \cap gC \supseteq f$.

We shall use the method of proof of theorem (3.9). Now choose a regular point $x_0 \in C$ near x and a curve c_1 in M_2 from x_0 to gx_0 which transverses the walls F_j, then $s_{F_1}(F_2)$, etc. Choose a second smooth curve c_2 in M_2 from x_0 to gx_0 in M_2 which is near x so that it intersects only walls through x. Then we choose a homotopy in M between c_1 and c_2 which we may assume to be transversal to all codimension ≥ 2 intersections of reflection hypersurfaces. Then it is in M_3 and cuts intersections of two reflection hypersurfaces transversely. Moving c_1 to c_2 via this homotopy amounts to do angular cancellations (in R_a) in the representation of g. Thus g is represented also as a word in reflections in hypersurfaces through x according to the transversing of c_2 of the corresponding walls. □

4. Riemannian manifolds with corners of Coxeter type

4.1. Manifolds with corners. For more details see [25], section 2. A *quadrant $Q \subset \mathbb{R}^n$ of index k* is a subset of the form $Q = \{x \in \mathbb{R}^n : l_1(x) \geq 0, \dots, l_k(x) \geq 0\}$ where $l_1, \dots, l_k$ are independent linear functionals on $\mathbb{R}^n$. If $x \in Q$ and exactly j of the l_i vanish on x then x is called a *corner of index j*. For an open subset $U \subset Q$ a mapping $f : U \to \mathbb{R}^p$ is called C^r $(0 \leq r \leq \infty)$ if all partial derivatives of f of order $\leq r$ exist and are continuous on U. By the Whitney extension theorem this is the case if and only if f can be extended to a C^r function $\tilde{f} : \tilde{U} \to \mathbb{R}^p$, where $\tilde{U} \subset \mathbb{R}^n$ is open and $U = \tilde{U} \cap Q$. If $f : U \to U'$ is a diffeomorphism between open subsets of quadrants in $\mathbb{R}^n$ then the index of $x \in U$ equals the index of $f(x) \in U'$.

A *smooth manifold with corners M* is defined in the usual way: it is modelled on open subsets of quadrants in $\mathbb{R}^n$; a chart on M is a diffeomorphism $u : U \to u(U)$ from an open subset $U \subset M$ onto an open subset $u(U)$ of a quadrant in $\mathbb{R}^n$, where

$n = \dim(M)$. The chart (U, u, Q) is called centered on x if $u(x) = 0$. A point $x \in M$ is called a *corner of index* j if there is a chart (U, u, Q) of M with $x \in U$ and and $u(x)$ a corner of index j in Q.

A subset $N \subset M$ is called a *submanifold with corners* of the manifold with corners M, if for any $y \in N$ there is a chart (U, u, Q) of M centered at y and there is a quadrant $Q' \subseteq \mathbb{R}^k \subseteq \mathbb{R}^n$ such that $Q' \subseteq Q$ and $u(N \cap U) = u(U) \cap Q'$. A submanifold with corners N of M is called *neat* if the index in N of each $y \in N$ coincides with its index in M. Only neat submanfolds have tubular neigborhoods.

Let us denote by $\partial^j M$ the set of all corners of index j of M. Note that $\partial^0 M = M$. Then each $\partial^j M$ is a submanifold without boundary of M. Let $\partial M := \bigcup_{j\geq 1} \partial^j M$.

Each closure (in M) of a connected component of $\partial^j M$ is a submanifold with cornes of M which is called a *codimension j face* of M; it is of dimension $n - j$. A codimension 1 face is also called a *wall*. A face is not neat. The set of all faces is a partially ordered set with respect to inclusion.

The *tangent bundle of a manifold with corners M* is constructed in the following way: Let $(U_\alpha, u_\alpha, Q_\alpha)$ be an atlas of M. Then TM is the quotient space of the disjoint union $\bigsqcup_\alpha (\{\alpha\} \times U_\alpha \times \mathbb{R}^n)/\sim$ by the following equivalence relation: $(\alpha, x, v) \sim (\beta, y, w)$ if $x = y$ and $d(u_\beta \circ u_\alpha^{-1})(u_\alpha(x))v = w$. Then $\pi_M : TM \to M$ is a smooth vector bundle, and the total space TM is again a manifold with corners: the corners are all in the base.

A tangent vector X is called *inner* (short for: not outer) if there is a smooth curve $c : [0, 1) \to M$ with $\dot c(0) = X$. If $X \in T_x M$ and if (U, u, Q) is a chart with $x \in U$, and if the quadrant Q is given by the independent linear functionals $l_1, \dots, l_k$, and if $Tu(X) = (u(x), v) \in u(U) \times \mathbb{R}^n$, then X is inner if and only if the following holds: If $l_i(u(x)) = 0$ then $l_i(v) \geq 0$, for all i. Let us call the tangent vector *strictly inner* if $l_i(u(x)) = 0$ implies $l_i(v) > 0$, for all i. Let us denote the space of all inner vectors by ${}^iTM \subset TM$. It is not a manifold with corners any more. For example, ${}^iT[0, \infty) = \{(x, v) : x \geq 0, x = 0 \implies v \geq 0\}$.

An inner vector field on M is a smooth vector field $X : M \to TM$ whose values are all inner tangent vectors. By pasting local solutions one can show that there exists a smooth open semiflow of X in the following sense: There is a set $W \subset \mathbb{R} \times M$ containing $\{0\} \times M$ and $[0, \varepsilon_x) \times \{x\}$ for some $\varepsilon_x > 0$ for each $x \in M$ and a smooth mapping $\mathrm{Fl}^X : W \to M$ with $\mathrm{Fl}^X_0(x) = x$ and $\frac{d}{dt}\mathrm{Fl}^X_t(x) = X(\mathrm{Fl}^X_t(x))$. But Fl^X_t is not even a local diffeomorphism (it may map a corner to an interior point).

By a partition of unity argument one can show that there exists a smooth vector field Y on M which is strictly inner, and one may adapt it in such a way that its flow Fl^Y_t is defined everywhere on M for $0 \leq t \leq \varepsilon$ for $\varepsilon > 0$. Then $\mathrm{Fl}^Y_\varepsilon$ maps M into its interior $M \setminus \partial M$. Thus: *Each manifold with corners M is a submanifold with corners of a manifold without boundary of the same dimension.* See also [18].

Let X be a vector field on M which is tangential to the boundary: if $x \in \partial^j M$ then $X(x) \in T_x \partial^j M$ for all j. Then there exists a local flow for X for positive and for negative time; the set $W \subset \mathbb{R} \times M$ is open.

4.2. Equipment of a manifold with corners. Let M be an n-dimensional manifold with corners. Consider a surjective mapping s from the set W of all walls (codimension 1 faces) of M onto the set of generators S of a Coxeter system (G, S) (see (3.1)). Any face f of M of codimension k is the intersection of k many walls $W_1, \dots W_k$ (but not conversely). Then we extend the map s to a map s from the set of faces of M into the set of Coxeter subgroups of G as follows :

$$s : f = F_1 \cap \ldots \cap F_k \mapsto s(f) = G(s(F_1), \ldots s(F_k))$$

where $G(s(F_1), \ldots s(F_k)$ is the subgroup of G generated by $s(F_1), \ldots s(F_k)$.

The mapping s is called a *Coxeter equipment* of M by the Coxeter system (G, S), if $G(F)$ is a finite group for each face of codimension ≥ 1. It follows that s is an partial order reversing homomorphism of the poset of all faces of M into the poset of all Coxeter subgroups of the Coxeter system (G, S) if we also put $s(\emptyset) = G$. Note that $s(M) = \{1\}$.

4.3. Riemannian manifolds with corners. A *Riemannian metric on a manifold with corners* M is as usual a smooth section $\gamma : M \to S^2_+ T^*M$. So it can be smoothly extended to a Riemannian metric on a manifold without boundary of the same dimension which contains M as a submanifold with corners. If the Riemannian metric has the property that each closure of a face is a totally geodesic submanifold, then for each each inner tangent vector $X_x \in {}^iT_xM$ the geodesic $t \mapsto \exp_x(tX_x)$ is defined for small nonnegative t.

This can be expressed by the property of the geodesic spray to be 'inner' and 'tangential' to all boundary strata $\partial^j M$, see [25], section 2. In detail: A vector $\Xi \in TTM$ is called an *inner tangent vector* to ${}^iTM$ if there exists a smooth curve $c : [0, \varepsilon) \to TM$ with $\pi_{TM}(\Xi) = c(0)$, $c([0, \varepsilon)) \subset {}^iTM$, and $c'(0) = \Xi$. For example, let $Q = \{x \in \mathbb{R}^n : l_1(x) \geq 0, \ldots, l_k(x) \geq 0\}$ be a quadrant and let $(x, u) \in {}^iTQ$. A vector $(x, u; v, w) \in T^2Q$ then is inner to ${}^iTM$ if and only if:

(1) If x is inner, so u is arbitrary, then (v, w) is arbitrary.
(2) If $l_i(x) = 0$ and $l_i(u) > 0$ then $l_i(v) \geq 0$ and w is arbitrary.
(3) If $l_i(x) = 0$ and $l_i(u) = 0$ then $l_i(v) \geq 0$ and $l_i(w) \geq 0$.

Let us denote by ${}^iT^2M$ the set of all vectors which are inner to ${}^iTM$. A *spray* S on the manifold with corners M is a smooth mapping $S : TM \to T^2M$ such that

(4) $T(\pi_M) \circ S = \mathrm{Id}_{TM}$.
(5) $\pi_{TM} \circ S = \mathrm{Id}_{TM}$.
(6) $T(m_t).S(X) = \frac{1}{t}S(t.X)$ for $0 \neq t \in \mathbb{R}$, where $m_t : TM \to TM$ is scalar multiplication by t.

The spray is called *inner* if $S({}^iTM) \subset {}^iT^2M$ and it is called *tangential* if moreover S is tantent to each boundary stratum: $S(T\partial^j M) \subset T^2(\partial^j M)$.

If γ is a smooth Riemannian metric on the manifold with corners M, then we may extend γ to a Riemannian metric $\tilde{\gamma}$ on a suitable open manifold $\tilde{M}$ of the same dimension which contains M as submanifold with boundary. We may compute the geodesic (Levi-Civita) spray $\tilde{S}$ of $\tilde{\gamma}$ and restrict it again to TM. This spray is an inner tangential spray if and only if in (M, γ) all closures of faces are totally geodesic submanifolds, and we have $\exp = \pi_M \circ \mathrm{Fl}^S_1$.

Thus we conclude (see also [25], 2.10):

Lemma. [25], 2.10 *Let γ be a Riemannian metric on a manifold with corners M such that all faces are totally geodesic. Then there exists a suitable open neighborhood V of the zero section in T^iM such that the geodesic exponential mapping* $\exp : V \to M$ *is defined. If V is small enough then* $\exp$ *has the following properties:*

(1) $\exp(0_x) = x$ *for all* $x \in M$.
(2) $\exp_x : V_x := V \cap T^i_xM \to M$ *is a diffeomorphism of V_x onto an open neighborhood W_x of x in M.*

(3) *V_x is the intersection of an open ball $B_x \subset (T_xM, \gamma_x)$ with a quadrant $Q_x \subset T_xM$.*
(4) *The mapping $(\pi_M, \exp) : V \to M \times M$ is a diffeomorphism onto an open neighborhood of the diagonal in $M \times M$.*
(5) $\exp$ *restricts to the exponential mapping of the induced Riemannian metric on each closure of a face.*

4.4. Riemannian chambers and their Coxeter equipment. An *Riemannian chamber* is a manifold with corners C with a Riemannian metric γ such that each face is totally geodesic and such that the following two conditions (1) and (2) are satisfied.

(1) The angle between neighboring walls W_i and W_j is a constant of the form π/n_{ij} for $n_{ij} \in \mathbb{N}$ along any codimension 2 connected component of $W_i \cap W_j$.

Let $V \subset T^iC$ be small as in (4.3). Then $\exp_x : V_x = V \cap T^i_xC \to W_x \subset C$ is a diffeomorphism. Recall from (4.3) that V_x is the intersection of an open ball B_x in (T_xC, γ_x) with a quadrant Q_x whose walls contain the inverse images under $\exp_x$ of the closed walls of C containing x. The angles between the hyperplanes T_xW_i and T_xW_j in the Euclidean space (T_xC, γ_x) are exactly π/n_{ij}, by (1). By [32], theorem 1, this equivalent to the fact that the group $G_x \subset O(T_xC, g_x)$ generated by the reflections in the hyperplanes T_xW_i is a finite Coxeter group with fundamental Weyl chamber $\mathbb{R}_{\geq 0}.V_x$.

Consider the pullback Riemannian metric $(\exp_x |V_x)^*\gamma$ on V_x. Now we can formulate the second condition:

(2) If we extend the Riemannian metric $(\exp_x |V_x)^*\gamma$ on V_x to the ball $B_x = G_x.V_x$ by using the elements of G_x as isometries, then the resulting G_x-invariant Riemannian metric $\tilde\gamma_{B_x}$ on B_x is smooth.

If G is a discrete group of isometries of a complete Riemannian manifold (M, γ) which is generated by disecting reflections, and if a chamber C is also a Dirichlet domain, then obviously (C, γ) is a Riemannian chamber.

Proposition. *Any Riemannian chamber C carries a universal Coxeter equipment.*

Proof. Let $\{W_i\}$ be the set of all walls of C. For each wall W_i of C we take a generator s_i. Then let G be the group generated by all s_i, with relations $(s_is_j)^{n_{ij}} = 1$, whenever $W_i \cap W_j \neq \emptyset$ and where the angle between W_i and W_j is π/n_{ij}. Then G is a Coxeter group with Coxeter system $(G, \{s_i\})$. For each $x \in C$ we constructed in (4.4) a linear Coxeter group $G_x \subset O(T_xC, g_x)$ which is generated by those s_i for which $x \in W_i$. Obviously, G_x is a finite subgroup of G. Moreover, let $F = W_1 \cap \dots W_k$ be a nonempty face. Then $G(F)$ is generated by the reflections $s_1, \dots, s_k$ which satisfy pairwise $(s_is_j)^{n_{ij}} = 1$ for $2 \leq n_{ij} < \infty$. Thus $G(F)$ is finite for each nonempty face. $\square$

The Coxeter equipment constructed in this proposition is called universal since the mapping s is injective. Other Coxeter equipments are possible, if different walls are mapped to the same generator in such a way, that the isotropy group of each face F stay isomorphic to $G(F)$ as above, and the full group is still a Coxeter group.

Thus we say that a Coxeter equipment s of the Riemannian chamber C is admissible, if for any two different walls W_i and W_j with nonempty intersection the element $s(W_i)s(W_j)$ has order exactly n_{ij} in G, where the angle between W_i and

W_j is π/n_{ij}. The right hand side of fig. 2 gives an example of a not universal equipment.

4.5. The Coxeter Riemannian manifolds associated with a Riemannian chamber. Note that by (2.11) and (3.9) the Weyl chamber C of a Coxeter G-manifold M has the natural structure of a Riemannian chamber with the admissible equipment $s : F \mapsto s(F) = \langle s \in S : M^s \supset F\rangle$. In the non-disecting case this is not true: In (2.10) the chamber of the non-disecting reflection on $\mathbb{RP}^2$ equals $\mathbb{RP}^2$ and the generating reflection is not associated to a wall since $\mathbb{RP}^2$ has no boundary.

For Coxeter manifolds the converse statement is also true as the following theorem shows.

Theorem. *Let C be a Riemannian chamber.*

Then to each admissible Coxeter equipment G of C there exists a smooth Riemannian manifold $\mathcal{U}(G,C)$ without boundary and a discrete subgroup G of isometries which is generated by reflections such that C is isometric to a chamber of M which is also a Dirichlet domain.

If C is connected then also M is connected. If the equipment $G = G_{univ}$ is the universal one then G is generated by disecting reflections and $\pi_1(\mathcal{U}(G,C)) = \pi_1(C) *_e G_{univ}$. In general we have an exact sequence:

$$\{1\} \to \pi_1(C) *_e G_{univ} \to \pi_1(\mathcal{U}(G,C)) \to G_{univ} \to G \to \{1\}.$$

Proof. We use first the universal equipment. Let $\{F_i\}$ be the set of all closures of walls of C. We construct first the group G, as follows. For each wall F_i of C we take a generator s_i of G. Then $G = G_{\text{univ}}$ is the group generated by all s_i and with relations $(s_i s_j)^{n_{ij}} = 1$, when $F_i \cap F_j \neq \emptyset$ and where the angle between F_i and F_j is π/n_{ij}. For each $x \in C$ we constructed in (4.4) a linear Coxeter group $G_x \subset O(T_xC, g_x)$ which is generated by those s_i for which $x \in F_i$. Obviously, G_x is a subgroup of G.

Now we construct $M = \mathcal{U}(G,C)$ as topological space by putting $\mathcal{U}(G,C) := G \times C/\sim$ where

$$(g.s_i, x) \sim (g, s_i(x)) = (g, x) \text{ for } x \in F_i, \quad \text{or equivalently}$$
$$(g, x) \sim (h, y) \iff x = y \text{ and } g^{-1}h \in G_x.$$

So $\mathcal{U}(G,C)$ is a quotient of the disjoint union of $|G|$ copies of C which are glued together only along walls.

We construct an atlas for $\mathcal{U}(G,C)$ as follows, using the arguments from (4.4). For a corner x of C consider the Riemannian metric $\tilde\gamma_{B_x}$ on the open ball $B_x \subset T_xC$ which is smooth by condition (4.4.2), and the smooth exponential mapping $\exp_x : V_x = B_x \cap Q_x \to W_x \subset C$. We extend it to a G_x-equivariant homeomorphism $\tilde{\exp}_x$ from B_x to the open neighborhood $U_x = \bigcup_{g\in G_x}(\{g\} \times W_x)$ of x in M by putting $\tilde{\exp}_x(g.X) = (g, \exp_x(X))$ for $X \in V_x$ and $g \in G_x$. Then $(U_x, u_x := \tilde{\exp}_x{}^{-1} : U_x \to B_x \in T_xC)$ is a chart on M.

If $x \in C$ is a regular point we use the inverse of the exponential mapping on such a small neigborhood of 0 in T_xC that its image does not meet any wall. These charts we the distribute from $C = \{\mathrm{Id}\} \times C$ to the whole of M by using the transformations from g.

We claim that this gives a smooth atlas for $\mathcal{U}(G,C)$: Suppose that x and y are corners of C such that $W_x \cap W_y \neq \emptyset$. We have to show that $u_x \circ u_y^{-1}$ is smooth.

We may assume that $y \in W_x$ since we may connect x and y by finitely many chart changings with this property. But then this is a chart change of exponential mappings at different base points of the smooth Riemannian metric in $B_x \subset T_x C$.

Finally, G acts on the smooth manifold $\mathcal{U}(G, C)$ by construction: $g.(g_1, x) = (gg_1, x)$, and it consists of isometries. By construction G acts freely and transitively on the set of all chambers of $\mathcal{U}(C, G)$. We claim that the generators s_i of G are disecting. Suppose for contradiction that a generator s is not disecting. Choose regular point $x_0 \in C^o$ and a smooth curve c in $\mathcal{U}(C,G) \setminus \mathcal{U}(C,G)^s$ from x_0 to $s.x_0$ which is transversal to all intersections of reflection hypersurfaces. Then c passes from C to a neighbor $s_{i_1}C$, then to a neighbor $s_{i_1}s_{i_2}C$ of $s_{i_1}C$, and so on, till it reaches the chamber $s_{i_1}\dots s_{i_k}C = sC$ containing $s.x_0$. None of the s_{i_j} equals s since c does not meet $\mathcal{U}(C,G)^s$. Since G acts freely and transitively on the set of chambers we have $s = s_{i_1}\dots s_{i_k}$ in G, a contradiction. (WHY??)

Finally, for a general admissible equipment we have a normal subgroup $R \subset G$ of further relations which by the description of an admissible equipment acts freely and discretely on the universal M which thus is a covering of the resulting manifold. The statement on fundamental groups follows from (3.9). □

4.6. Remark. We can also consider manifold with corners C with a smooth Riemannian metric g which satisfies only condition (4.4.1). Then we can construct a topological manifold M which is smooth off the union of all reflection hypersurfaces, with a Riemannian metric which is only continuous along the the reflection hypersurfaces, in general. It might be worthwile to study this object.

4.7. Theorem. *Let C be a manifold with corners with a Coxeter equipment $s : W \mapsto s(W) \in S$ where (G, S) is a Coxeter system.*

Then there exists a Riemannian metric γ such that (C, γ) is a Riemannian chamber and s is an admissible equipment for it.

Proof. We construct the metric inductively starting from faces which are manifolds without boundary. On each such face F we put an arbitrary Riemannian metric γ_F.

Now let F be a face which contains corners of index (in F) at most 1, i.e., F is a manifold with boundary ∂F which is a disjoint union of faces F_1 without boundary. Along each boundary component F_1 of F we consider an open collar $F_1 \times [0, 1) \subset F$ and extend the metric by $\gamma(x,t) = dt^2 + \gamma_{F_1}(x)$ where $x \in F_1$ and t is the coordinate function on $[0, 1)$. With a partition of unity we may extend this metric to the whole of F in such a way that near each F_1 it is not changed. Note that F_1 is totally geodesic in F, and that the metric is constant in the direction t normal to F_1.

Now let F be a face which contains corners of index (in F) at most 2, i.e., ∂F contains walls F_1^i of F which are manifolds with boundary. We already defined Riemannian metrics $\gamma_{F_1^i}$ on F_1^i. If F_2 is a boundary component of $F_1^1 \cap F_1^2$ we consider an open tubular wedge neigbourhood $F_2 \times D$ of F_2 in F with the following property. Each fiber $\{x\} \times D$ intersects F_1^i exactly in the fiber $\{x\} \times [0, 1)$ of that collar of F_2 in F_1^i for each $x \in F_2$ which was used above to construct the Riemannian metric on F_1^i. The fiber $D \subset \mathbb{R}^2$ is an open 0-neighborhood in a quadrant with angle $\alpha_F(F_1^1, F_1^2)$ as in fig. 9. Here $\alpha_F(F_1^1, F_1^2)$ is determined by the Coxeter equipment: If in terms of walls W_i of C we have

$$F = W_{i_1} \cap \dots \cap W_{i_{n-2}},$$

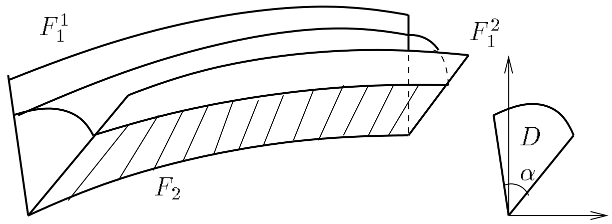


FIGURE 9. The tubular wedge neigborhood $F_2 \times D$ and its fiber D.

$$F_1^1 = W_{i_1} \cap \dots \cap W_{i_{n-2}} \cap W_{i_{n-1}}, \qquad F_1^2 = W_{i_1} \cap \dots \cap W_{i_{n-2}} \cap W_{i_n},$$
$$F_2 = W_{i_1} \cap \dots \cap W_{i_{n-2}} \cap W_{i_{n-1}} \cap W_{i_n},$$

then $\alpha_F(F_1^1, F_1^2)$ is determined by the (finite) Coxeter system $(G(F_2) = G(S'), S')$ where $S' = \{s_{i_1}, \dots, s_{i_n}\}$, by considering the angle between and in the corresponding faces in the Weyl chamber of $(G(S'), S')$, as decribed in (3.3).

We now put the product metric $\gamma_D(u) + \gamma_{F_2}(x)$ for $(x, u) \in F_2 \times D$ on the tubular wedge neigbourhood $F_2 \times D$, where γ_D is the standard Euclidean metric on $\mathbb{R}^2$ restricted to D. This gives a metric on $F_2 \times D$ which induces the already constructed metric $\gamma_{F_1^i}$ on the intersection with F_1^i since $F_2 \times D$ intersects F_1^i in the collar used to construct $\gamma_{F_1^i}$. Moreover F_2 and the parts of F_1^i are totally geodesic, and the metric is constant in directions normal to any relevant face, near that face.

We do this construction near any face of codimension 2 of F. Then we use a collar $(F_1^i \setminus \partial F_1^i) \times [0, 1)$ of the interior of the face F_1^i in F such that the fiber near any F_2 coincides with the normal geodesic in $F_2 \times D$ in the metric constructed there. Put the metric $dt^2 + \gamma_{F_1^i}(x)$ for $(x, t) \in (F_1^i \setminus \partial F_1^i) \times [0, 1)$ on this collar, and use a partition of unity on the union of all these collars and the wedge neighborhoods which is constant in the normals near any face to glue the metrics in such a way that the resulting metric is constant in the normal directions near any face and each face is totally geodesic. With another partition of unity we extend this metric into the interior of F and not changing it near any face.

We proceed inductively. We assume that we have already constructed in this way metrics on each face which consists of corners of index $\geq k$ in C and consider now a face F which consists of corners of index $\geq k - 1$ in C. Then the boundary ∂F is a union of faces where we alredy constructed the metric. Let F_k be a minimal face in ∂F, i.e., F_k does not contain any other face. Then F_k is a manifold without boundary where we already have a metric γ_{F_k}. Moreover F_k is the transversal intersection of k walls $F_1^1, \dots, F_1^k$ of F, where k is the codimension of F_k in F. We then choose a tubular wedge neighbourhood $F_k \times D^k$ of F_k in F which intersects fiber respectingly each intersection of $k - 1$ of the walls $F_1^1, \dots, F_1^k$ of F in the tubular wedge neighborhood which was used previously to construct the metric $\gamma_{F_1^i}$ on each of the walls. Here D^k is an open 0-neighborhood in a quadrant in $\mathbb{R}^k$ with walls whose angles $\alpha_F(F_1^i, F_1^j)$ are determined by the Coxeter equipment as described above. We now put the metric $\gamma_{D^k}(u) + \gamma_{F_k}(x)$ for $(x, u) \in F_k \times D^k$ on the tubular wedge neigbourhood $F_k \times D^k$, where γ_{D^k} is the standard Euclidean metric on $\mathbb{R}^k$ restricted to D^k. This gives a metric on $F_k \times D^k$ which induces the already constructed metric $\gamma_{F_1^i}$ on the intersection with F_1^i since $F_k \times D^k$ intersects F_1^i in the tubular wedge neighborhood used to construct $\gamma_{F_1^i}$. Moreover F_k and the parts of F_1^i are totally geodesic, and the metric is constant in directions normal to any face near that face.

We do this construction near any minimal face of F. Then we use a collar $(F_1^i \setminus \partial F_1^i) \times [0,1)$ of the interior of the face F_1^i in F such that the fiber near any minimal face F_l coincides with the normal geodesic in $F_l \times D^l$ in the metric constructed there. Put the metric $dt^2 + \gamma_{F_1^i}(x)$ for $(x,t) \in (F_1^i \setminus \partial F_1^i) \times [0,1)$ on this collar, and use a partition of unity on the union of all these collars and the wedge neighborhoods which is constant in the normals near any face, to glue the metrics in such a way that the resulting metric is constant in the normal directions near any face and each face is totally geodesic. With another partition of unity we extend this metric into the interior of F.

Eventually we exhaust each connected component of C. $\square$

4.8. Proposition. *Let C be a manifold with corners with a Coxeter equipment $s : W \mapsto s(W) \in S$ where (G,S) is a Coxeter system. Let γ and γ' be two Riemannian metrics on C such that (C,γ) and (C,γ') are both Riemannian chambers and s is an admissible equipment for both.*

Then the smooth manifolds $\mathcal{U}(G,C,\gamma)$ and $\mathcal{U}(G,C,\gamma')$ constructed via (4.5) are diffeomorphic.

Proof. Since the construction as a topological space described in the proof of (4.5) depends only on the equipment, the two manifolds are canonically homeomorphic. For a corner $x \in C$ let $u_x : U_x \to B_x \subset T_xC$ and $u'_x : U'_x \to B_x \subset T_xC$ be two charts as described in the proof of (4.5) for the two Riemannian metrics γ and γ'. But then the chart change $u'_x \circ u_x^{-1}$, considered in a manifold without boundary which contains C as a submanifold with corners (see (4.1)), consists of the exponential mapping of the extended Riemannian metric $\tilde{\gamma}$ followed by the inverse of the exponential mapping of $\tilde{\gamma}'$, which is obviously smooth. Thus the canonical homeomorphism between $\mathcal{U}(G,C,\gamma)$ and $\mathcal{U}(G,C,\gamma')$ is a diffeomorphism. $\square$

5. Orbifolds

5.1. Smooth orbifolds. We recall the definition of orbifold. Let X be a second countable Hausdorff space. An atlas of a smooth n-dimensional orbifold (or V-manifold) on X is a family $\{U_i\}_{i\in I}$ of open sets that satisfy:

(1) $\{U_i\}_{i\in I}$ is an open cover of X.
(2) For each $i \in I$ a local uniformizing system consisting of a triple $\{\tilde{U}_i, G_i, \varphi_i\}$, where $\tilde{U}_i$ is a connected open subset of $\mathbb{R}^n$ containing the origin, G_i is a finite group of diffeomorphisms acting effectively and properly on $\tilde{U}_i$, and $\varphi_i : \tilde{U}_i \to U_i$ is a continuous map of $\tilde{U}_i$ onto U_i such that $\varphi_i \circ g = \varphi_i$ for all $g \in G_i$ and the induced map of $\tilde{U}_i/G_i$ onto U_i is a homeomorphism. The finite group G_i is called a local uniformizing group.
(3) Given $\tilde{x}_i \in \tilde{U}_i$ and $\tilde{x}_j \in \tilde{U}_j$ such that $\varphi_i(\tilde{x}_i) = \varphi_j(\tilde{x}_j)$, there is a diffeomorphism $\varphi_{ij} : \tilde{V}_j \to \tilde{V}_i$ from a neighborhood $\tilde{V}_i \subset \tilde{U}_i$ of $\tilde{x}_i$ onto a neighborhood $\tilde{V}_j \subset \tilde{U}_j$ of $\tilde{x}_j$ such that $\varphi_i = \varphi_j \circ \varphi_{ji}$.

Two atlases are equivalent if their union is again an atlas of a smooth orbifold on X. An orbifold is the space X with an equivalence class of atlaces of smooth orbifolds on X.

Proposition. [31] *If M is an n-dimensional smooth manifold and G is a group acting smoothly and discretely on M, then $X = M/G$ has a structure of orbifold.*

Proof. Let $x \in X$. Choose $\tilde{x} \in M$ projecting to x, and denote by G_x the isotropy group of $\tilde{x}$. Choose a neigborhood of $\tilde{U}_x$ invariant by G_x and disjoint from $g(U_x)$ for all $g \in G \setminus G_x$ such that there is a local chart $k : U_x \to \tilde{U}_x \subset \mathbb{R}^n$ on M with $k(x) = 0$. We take $(\tilde{U}_x, G_x, \varphi_x)$, where φ_x is a composition of k^{-1} with the projection $\tilde{U} \to \tilde{U}/G_x$, for a local uniformizing system. It is easily checked such local uniformizing systems form an atlas of a smooth n-dimensional orbifold on M/G. $\square$

In the definition of atlas of a smooth orbifold on X we can always take the finite subgroups G_i to be subgroups of the orthogonal group $O(n)$ acting naturally on $\mathbb{R}^n$. Condition (3) implies that for each $g_i \in G_i$ there exists $g_j \in G_j$ such that $\varphi_{ji} \circ g_i = g_j \circ \varphi_{ji}$.

Let $\{\tilde{U}_i, G_i, \varphi_i\}$ be a unifomizing system such that $\tilde{U}_i$ contains the origin, the group G_i is a subgroup of $O(n)$, and $x = \varphi_i(0)$. Then the group $G_x = G_i$ is independent of the uniformizing system $\{\tilde{U}_i, G_i, \varphi_i\}$. More precisely, this group is defined up to isomorphism and its action on $\mathbb{R}^n$ is defined up to isomorphism as well. The point $x \in X$ is called regular if the corresponding group G_x is trivial and otherwise singular.

5.2. Reconstruction of the orbifold structure from the structure sheaf. Let again $\{\tilde{U}_i, G_i, \varphi_i\}$ be a unifomizing system such that $\tilde{U}_i$ contains the origin, the group G_i is a subgroup of $O(n)$, and $x = \varphi_i(0)$. Then there is a representation $\rho : G_i \to O(n)$, a ball B in $\mathbb{R}^n$ centered at the origin, and a map $\varphi : B \to X$ such that $\varphi(0) = x$ and $\{B, G_i, \varphi\}$ is a uniformizing system of the orbifold X.

A function $f : U_i \to \mathbb{R}$ is called smooth if $f \circ \varphi_i$ is a smooth function on $\tilde{U}$. The germs of smooth functions on X define a sheaf S_X on X.

5.3. Definition. *Let X and $\tilde{X}$ be two smooth orbifolds. The orbifold $\tilde{X}$ is called a covering orbifold for X with a projection $p : \tilde{X} \to X$ if p is a continuous map of underlying topological spaces and each point $x \in X$ has a neighborhood $U = \tilde{U}/G$ (where $\tilde{U}$ is an open subset of $\mathbb{R}^n$) for which each component V_i of $p^{-1}(U)$ is isomorphic to $\tilde{U}/G_i$, where $G_i \subset G$ is some subgroup. The above isomorphisms $U = \tilde{U}/G$ and $V_i = \tilde{U}/G_i$ must respect the projections.*

Note that the projection p in the above definition is not a cover of underlying topological spaces.

Hereafter we suppose that all orbifolds and their covering orbifolds are connected.

5.4. Theorem. [31] *An orbifold X has a universal covering orbifold $p : \tilde{X} \to X$. More precisely, if $x \in X$, $\tilde{x} \in \tilde{X}$ are regular points and $p(\tilde{x}) = x$, for any other covering orbifold $p' : \tilde{X}' \to X$ and $\tilde{x}' \in \tilde{X}'$ such that $p'(\tilde{x}') = x$ there is a cover $q : \tilde{X} \to \tilde{X}'$ such that $p = p' \circ q$ and $q(\tilde{x}) = \tilde{x}'$. For any points $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ there is a deck transformation of $\tilde{X}$ taking $\tilde{x}$ to $\tilde{x}'$.*

Suppose $\rho : G \to O(n)$ is a representation of a finite group G, $\mathbb{R}^n/G$ is the corresponding orbifold, and $S_{\mathbb{R}^n/G}$ is the corresponding sheaf. By the Hilbert theorem the ring $\mathbb{R}[\mathbb{R}^n]^G$ is finitely generated. Let $\sigma^1, \dots, \sigma^m$ be a system of homogeneous generators of $\mathbb{R}[\mathbb{R}^n]^G$ and $y^1, \dots, y^m$ the corresponding functions on $\mathbb{R}^n/G$. Consider the map $\sigma = (\sigma^1, \dots, \sigma^m) : \mathbb{R}^n \to \mathbb{R}^m$ called the orbit map. It is known

[2] that the map σ induces a homeomorphism between $\sigma(\mathbb{R}^n)$ and the orbit space $\mathbb{R}^n/G$ which establishes an isomorphism between the restriction of the sheaf C^∞_m of smooth functions on $\mathbb{R}^m$ to $\sigma(\mathbb{R}^n)$ and the sheaf $S_{\mathbb{R}^n/G}$.

It is clear that for each orbifold X and $x \in X$ there is a neghborhood U_x and a representation $\rho : G_x \to O(n)$ such that the restriction of S_X to U_x is isomorphic to the restriction of the sheaf $S_{\mathbb{R}^n/G_x}$ to some ball centered at the origin.

For a representation $\rho : G \to O(n)$ a diffeomorphism of the orbit space $\mathbb{R}^n/G$ is an automorphism of the sheaf $S_{\mathbb{R}^n/G}$ by definition. Let $f : \mathbb{R}^n/G \to \mathbb{R}^n/G$ be a diffeomorphism and $h^1, \dots, h^m$ a system of generators of $S_{\mathbb{R}^n/G}$. Then f is uniquely defined by the images of generators h^i and these images are the generators of $S_{\mathbb{R}^n/G}$ again. Denote by R the set of all reflections contained in G and by $A(G,R)$ the set of all automorphisms of the group G which preserves the set R.

5.5. Theorem. [23] *For each diffeomorphism f of the orbit space $\mathbb{R}^n/G$ there is a smooth lift $F : \mathbb{R}^n \to \mathbb{R}^n$. For each such lift F there is an automorphism $a \in A(G,R)$ such that for all $g \in G$ and $x \in \mathbb{R}^n/G$ we have $F(gx) = a(g)F(x)$.*

The local version of this theorem is also true, i.e. if B is a ball in $\mathbb{R}^n$ centered at the origin and f is a diffeomorphism of the sheaf $S_{B/G}$, then there is a smooth lift $F : B \to B$ with the same property as above.

5.6. Theorem. *An orbifold X is defined uniquely by its sheaf S_X.*

Proof. Note that for a regular point $x \in X$ the ring $S_X(x)$ of the germs of S_X at x is isomorphic to the ring of germs at 0 of smooth functions on $\mathbb{R}^n$. Then the dimension of the orbifold X is defined by the sheaf S_X. Next note that if $\rho : G \to O(n)$ is a representation of a finite group G, then the group preserving all smooth G-invariant functions on $\mathbb{R}^n$ coincides with $\rho(G)$. If this group is infinite there is a regular point with non trivial stabilizer, which is impossible. The result then follows from the fact that the order of G equals the cardinality of a regular orbit.

It is sufficient to prove that for each a finite group G, a representation $G \to O(n)$, a ball B in $\mathbb{R}^n$, and the map $\varphi : B \to X$ which induces an isomorphism of the sheaf $S_{B/G}$ and the restriction S_U of the sheaf S_X to some open subset U of X, $\{B, G, \varphi\}$ is a uniformizing system on X.

Let $\{B_1, G_1, \varphi_1\}$ be such a uniformizing system, corresponding to the representation $\rho_1 : G_1 \to O(n)$, $\varphi(0) = x$, and $\{B_2, G_2, \varphi_2\}$ a uniformizing system of the orbifold X which is induced by some representation $\rho_2 : G_2 \to O(n)$ such that $\varphi_2(0) = x$. We may assume that $B_1 = B_2 = B$ and $\varphi_1(B) = \varphi_2(B) = U$. Then the rings of functions on B which are compositions of φ_1 and φ_2 with the sections of S_X on U coincides. By the above remark $\rho_1(G_1) = \rho_2(G_2) = G$.

For $i = 1, 2$ denote by $\bar\varphi_i$ the diffeomorphism $B/G \to U$ induced by φ_i. Then $\bar\varphi_2^{-1} \circ \bar\varphi_1$ is a diffeomorphism of B/G. By Theorem (5.5) there is a smooth lift $B \to B$ of this diffeomorphism. But this means that $\{B_1, G_1, \varphi_1\}$ is a uniformizing system of the orbifold X. □

5.7. Corollary. *Let a group G acts discretely on a smooth simply connected manifold M and S_X the corresponding sheaf on $X = M/G$. Then M is a universal covering orbifold for X.*

Proof. Evidently manifold M is a covering orbifold for X. If $\tilde X$ is universal covering orbifold for X, then there is a cover $q : \tilde X \to M$. By the definition of cover $\tilde{X}$ should be a manifold and q a cover of manifilds. Therefore q is a diffeomorphism. $\square$

Theorems (5.6) and (5.7) imply the following statement.

5.8. Corollary. *Let a group G act discretely on a smooth simply connected manifold M and S_X the corresponding sheaf on $X = M/G$. Then each diffeomorphism of the orbit space X, i.e. an automorphism of the sheaf S_X has a smooth lift to M.*

6. Coxeter orbifold

6.1. Coxeter orbifolds. A smooth orbifold X is called a *Coxeter orbifold* if for each local uniformizing system $(\tilde{U}_i \subset \mathbb{R}^n, G_i \subset O(n), \varphi_i)$ the group G_i is a finite linear Coxeter group.

6.2. Example. Let M be a Coxeter Riemannian manifolds with reflection group G. Then any Weyl chamber is a Coxeter orbifold. This follows from proposition (5.1) and (2.11).

6.3. Coxeter orbifold as a manifold with corners and its universal Coxeter equipment. Let X be a Coxeter orbifold. Let $(\tilde{U}_i, G_i, \varphi_i)_{i\in I}$ be an atlas of local uniformizing systems on X such that (U_i) is an open cover of X. Then $\tilde{U}_i \subset \mathbb{R}^n$ is an open neighborhood of 0 which is invariant under the Coxeter group G_i. Thus the orbit space $\tilde{U}_i/G_i$ is an open neighborhood of 0 in a linear Weyl chamber of the group G_i. The (equivariant) chart changings φ_{ij} induce smooth chart changings between open subsets of $\tilde{U}_j/G_j$ and $\tilde{U}_i/G_i$. These respect the indices of corners (see (4.1)). Thus they describe a smooth atlas for the structure of a manifold with corners on X. So walls and faces are defined and to each wall W one can associate a generator $s(W)$ of the Coxeter system with the following property: If $W \cap U_i \neq \emptyset$ for a local uniformizing system $(\tilde{U}_i, G_i, \varphi_i)$, then $s(W)$ equals the generator of G_i which is given by the reflection in the wall $\varphi_i^{-1}(W) \subset \tilde{U}_i$. Then $(s(W)s(W'))^{n(W,W')} = 1$ if $\varphi_i^{-1}(W), \varphi_i^{-1}(W') \neq \emptyset$ the generators corresponding to them in G_i satisfy the same relation.

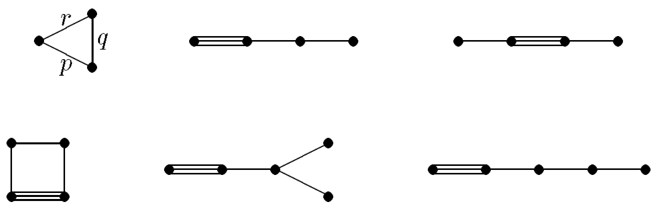
6.4. Theorem. *Any Coxeter orbifold is the Weyl chamber of a Riemannian Coxeter manifold.*

Proof. This follows from (6.3) and (4.7). $\square$

6.5. Corollary. *Any Coxeter orbifold is good in the sense of Thurston* [31].

6.6. Coxeter orbifold structures on a simplex. Let Δ_n be the standard n-simplex with vertices $0, 1, \ldots, n$. If s is a Coxeter (G, S)-equipment of Δ_n, then there exist a Riemannian metric γ on Δ_n, such that (Δ_n, γ) is a Riemannian chamber and the equipment s is admissible. We denote by $M = \mathcal{U}(G, \Delta_n, s, \gamma)$ the associated Coxeter G-manifold. It is simply connected. The homeomorphism $M/G \cong \Delta_n$ define on Δ_n a structure of Coxeter orbifold, with the universal covering manifold M, which depends only on the equipment s, by (4.8). Hence, a description of Coxeter orbifold structures on Δ_n and also Coxeter G-manifolds with the orbit space Δ_n up to a G-diffeomorphism reduces to a description of Coxeter equipments of Δ.

For any finite Coxeter group G with the generators $S = \{s_0, \ldots, s_n\}$ there exist a unique natural equipment such that the wall $W_i = (0, 1, \ldots, \widehat{i}, \ldots, n)$ corresponds to s_i for $i = 0, \ldots, n$. The corresponding Coxeter manifold M is the sphere S^n with the natural action of G induced by the standard representation of G in $\mathbb{R}^{n+1}$. Let now G be an infinite Coxeter group with system of generators S. There exists a Coxeter (G,S)-equipment of Δ_n if and only if $|S| = n+1$ and the Coxeter subgroup generated by $S \setminus \{s\}$ is finite for any $s \in S$. In term of the Coxeter diagram Γ of the group G, this means that all connected components of Γ with exception of one component Γ_0 correspond to finite Coxeter groups, and the component Γ_0 corresponds to an infinite Coxeter group, but after deleting any node it become a Coxeter diagram of a finite Coxeter group. One can easily check that such a Coxeter diagram Γ_0 is either a connected parabolic Coxeter diagram (i.e., extended Dynkin diagram of a simple Lie algebra) or one of the following diagrams:



An interesting question is to classify such equipments for other polyhedra, e.g., a cube, prism etc.

May be, it is possible to construct a non trivial example of compact Coxeter manifolds, e.g. simply connected 3-manifolds.

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