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Frames and MV–Algebras

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FRAMES AND MV-ALGEBRAS

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Abstract

This is a preliminary study of a class of MV-algebras which is a natural generalization of the class of "algebras of continuous functions". More specifically, we're interested in the algebra of frame maps $Hom_{\mathcal{F}}(\Omega(A),\ \mathbf{K})$ in the category $\mathcal{F}$ of frames, where A is a topological MV-algebra, $\Omega(A)$ the lattice of open sets of A, and $\mathbf{K}$ an arbitrary frame.

Given a topological space X and a topological MV-algebra A, we have the algebra $C(X, A)$ of continuous functions from X to A. We can look at this from a frame point of view. Among others we have the result: if $\mathbf{K}$ is spatial, then $\mathcal{C}(pt(\mathbf{K}), A)$, $pt(\mathbf{K})$ the points of $\mathbf{K}$, embeds into $Hom_{\mathcal{F}}(\Omega(A), \mathbf{K})$ analogous to the case of $\mathcal{C}(X, A)$ embedding into $Hom_{\mathcal{F}}(\Omega(A), \Omega(X))$.

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0. Introduction and Preliminaries

The theory of MV-algebras has its origin in the study of the system of infinite-valued logic originated by Łukasiewicz. The completeness of the propositional Łukasiewicz logic was first published by Rose and Rosser in 1958 [9]. An earlier proof by Wajsberg was never published.

In 1958 C. C. Chang [2,3] developed an algebraic version of Łukasiewicz propositional logic and provided an algebraic proof of the completeness. The resulting algebraic system became known as an MV-algebra.

MV-algebras, therefore, stand in relation to the Łukasiewicz infinite-valued logic as Boolean algebras stand in relation to classical 2-valued logic. Boolean algebras, of course, have not stayed glued to their origin in logic, their uses showing up in other areas of mathematics. Moreover there has been extensive investigations concerning their structure.

The same can be said about MV-algebras, that is their connecting to other areas of mathematics [1,5,8] and investigations of their intrinsic structure. The latter has shown the existence of a rich variety of various types of MV-algebras and there is continuing research into the classification problems.

This is a preliminary study of a class of MV-algebras which, on the surface, is a natural generalization of the class of "algebras of continuous functions". More specifically, we're interested in the algebra of frame maps $Hom_{\mathcal{F}}(\Omega(A),\ \mathbf{K})$ in the category $\mathcal{F}$ of frames, where A is a topological MV-algebra, $\Omega(A)$ the lattice of open sets of A, and $\mathbf{K}$ an arbitrary frame.

We shall, in general, confine ourselves to the cases where A is linearly ordered and locally compact in the interval topology.

Given a topological space X and a topological MV-algebra A, we have the algebra $C(X,A)$ of continuous functions from X to A. We can look at this from a frame point of view.

Let $\Omega(X),\ \Omega(A)$ be the frames of open sets of $X,\ A$ respectively. Consider the set $Hom_{\mathcal{F}}(\Omega(A),\Omega(X))$ of frame maps. Under certain conditions this is an MV-algebra. Moreover, $C(X,A)$ embeds as a subalgebra into $Hom_{\mathcal{F}}(\Omega(A),\Omega(X))$ by sending a continuous function f to its inverse image map f^{-1}. This is an isomorphism.

We can look at a more general setting by replacing $\Omega(X)$ by an arbitrary frame $\mathbf{K}$. If A is also locally compact and Hausdorff in its topology, then $Hom_{\mathcal{F}}(\Omega(A),\mathbf{K})$ will be an MV-algebra. We note that all atomic linearly ordered algebras are locally compact and Hausdorff.

We can ask to what extent does $Hom_{\mathcal{F}}(\Omega(A),\mathbf{K})$ relate to an algebra of continuous functions. We have the result: if $\mathbf{K}$ is spatial, then $\mathcal{C}(pt(\mathbf{K}),A)$, $pt(\mathbf{K})$ the points of $\mathbf{K}$, embeds into $Hom_{\mathcal{F}}(\Omega(A),\mathbf{K})$ analogous to the case of $\mathcal{C}(X,A)$ embedding into $Hom_{\mathcal{F}}(\Omega(A),\Omega(X))$. We may ask when this embedding is an isomorphism of $\mathcal{C}(pt(\mathbf{K}),A)$ into $Hom_{\mathcal{F}}(\Omega(A),\mathbf{K})$.

If A is not locally finite, then $\mathcal{C}(X,A)$ is never semi-simple. The same is true in the case of $Hom_{\mathcal{F}}(\Omega(A), \mathbf{K})$. So we can ask, when is $Hom_{\mathcal{F}}(\Omega(A), \mathbf{K})$ semisimple? It should certainly be some kind of generalization of a semisimple algebra.

We say that a structure $A = (A, 0, 1, *, \odot, \oplus)$ is an MV-algebra iff A satisfies the following equations:

1. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $x \oplus y = x \oplus y$;
3. $x \oplus 0 = x$;
4. $x \oplus 1 = 1$;
5. $0^* = 1$;
6. $1^* = 0$;
7. $x \odot y = (x^* \oplus y^*)^*$;
8. $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

¿From (8), with $y = 0$ it follows $(x^{**}) = x$ and with $y = 1$, $x^* \oplus x = 1$. On A two new operations $\vee$ and $\wedge$ are defined as follows: $x \vee y = (x^* \oplus y)^* \oplus y$ and $x \wedge y = (x^* \odot y)^* \odot y$. The structure $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice. We shall write $x \leq y$ iff $x \wedge y = x$. We say that an MV-algebra A is an MV-chain when, as a lattice, A is linearly ordered. Boolean algebras are just the MV-algebras obeying the additional equation $x \odot x = x$.

The basic MV-algebraic operations on the unit interval of real numbers, $[0, 1]$, are

$$x \oplus y = min(1, x + y);$$

$$x \odot y = max(0, x + y - 1);$$

$$x^* = 1 - x.$$

We refer to this MV-algebra by $[0, 1]$. An *ideal* of an MV-algebra A is a non-empty subset I of A which is closed under $\oplus$ and $x \leq y, \quad y \in I$ imply $x \in I$. A *prime* ideal P of A is an ideal of A such that $x \wedge y$ implies $x \in P$ or $y \in P$. An ideal M of A is called *maximal* if $M \subseteq I$ implies $I = A$ or $I = M$, I an ideal of A. As MV-algebras form an equational class, the notions of MV-isomorphism, quotient, subalgebra, product, etc., are just the particular cases of the corresponding universal algebraic notions. Let A

be an MV-algebra and B a subset of A, then the subalgebra of A, generated by B, will be denoted by $< B >$. The intersection of all maximal ideals, the *radical* of A, will be denoted by $RadA$. We write nx instead of $x \oplus ... \oplus x$ (n-times) and x^n instead of $x \odot ... \odot x$ (n-times). The least integer for which $nx = 1$ is called the *order* of x. When such an integer exists, we denote it by $ordx$ and say that x has *finite order*, otherwise we say that x has *infinite order* and write $ordx = \infty$. An MV-algebra A such that $RadA = 0$ is called *semisimple*. If $\mathrm{ord}x$ is finite for every $x \in A \setminus \{0\}$, then we say that A is *locally finite*. If for every element x of the MV-algebra A there is an integer n such that $nx = (n+1)x$ then A will be called *quasi-boolean*. A *quasi-boolean algebra of index* n is a quasi boolean algebra A such that there is an integer n with nx an idempotent for all $x \in A$. For all unexplained MV-algebraic notions we refer to [4] and for frames to [7].

Let X be a non-empty set. Then the set $B = [0,1]^X$ of all $[0,1]$-valued functions over X, equipped with pointwise operations, is an MV-algebra. Up to isomorphism, subalgebras of B provide the most general possible examples of semisimple MV-algebras; locally finite MV-algebras are subalgebras of the MV-algebra $[0,1]$.

Let A be an MV-algebra and S, T subsets of A. Then we define $S \oplus T$, $S \odot T$ and S^*, respectively as follows:

$$S \oplus T = \{x \oplus y \mid x \in S,\ y \in T\}.$$

$$S \odot T = \{x \odot y \mid x \in S,\ y \in T\}.$$

and

$$S^* = \{x^* \mid x \in S\}.$$

In case, say $S = \{x\}$ we write $x \odot T$. Also we put $(a,b) = \{x \in A \mid a < x < b\}$, $[a,b) = \{x \in A \mid a \le x < b\}$, $(a,b] = \{x \in A \mid a < x \le b\}$ and $[a,b] = \{x \in A \mid a \le x \le b\}$, with $a \le b$ and $a, b \in A$.

Proposition 0.1. Let A be an atomless MV-algebra. Then A is densely ordered.

Proof.

Suppose that for some $a,\ b \in A$ that $a < b$ and $(a,b) = \emptyset$. Let $c = b \odot a^* > 0$. A is atomless so for some $x \in A,\ 0 < x < c$. Then $b \odot x^* \ge b \odot c^* = b \odot (b^* \oplus a) = a$. Therefore, $a \oplus x \le b$. This yields $a \oplus x = a$ or $a \oplus x = b$. The former implies that $x = 0$ so $a \oplus x = b$. So $c = a^* \wedge x \le x$ and this is absurd.

Lemma 0.1. Let A be an MV-chain. Let $x, y, a, b \in A$ such that: $x < a,\ y < b$ and $x \oplus y = a \oplus b$. Then $a \oplus b = 1$.

Proof.

If not we would have $x \oplus y = x \oplus (x^* \odot a) \oplus y \oplus (y^* \odot b) < 1$. Thus, by cancellation, $(x^* \odot a) \oplus (y^* \odot b) = 0$. It follows that $a \leq x,\ b \leq y$.

Lemma 0.2. Let A be an atomless MV-chain. Let $a, b \in A$ such that: $a < b$. Then there is an $r \in A$ such that $0 < r,\ a < a \oplus r < b$.

Proof.

As $b \odot a^* \neq 0$ there is an $r \in A,\ 0 < r < b \odot a^*$. Then $a \oplus r \leq a \oplus b \odot a^* = b$. If we have equality here, then when $b \neq 1$ we would have $r = b \odot a^*$ contrary to assumption. In the case that $b = 1$ we can choose a $c,\ a < c < b = 1$. Then there is an r with $0 < r < a \oplus r < c < b$.

Lemma 0.3. Let A be an atomless MV-chain. Let $0 < a \in A$. Then for some $r \in A$ we have $0 < 2r < a$.

Proof.

Suppose that for all non-zero $r < a$ that $a \leq 2r$. Then for all non-zero $r, s < a$ we must have $a \leq r \oplus s$. But by Lemma 0.2, if $0 < r < a$, then for some non-zero s we have $r \oplus s < a$.

Corollary 0.1. Let A be an atomless MV-chain. Let $a, b \in A$ such that $a < b$. Then for some non-zero $e \in A$ we have $a < a \oplus 2e < b$.

Proof.

By Lemma 0.2 there is an $r \neq 0,\ a < a \oplus r < b$. By Lemma 0.3, there is an $0 < e$ with $2e < r$. Therefore $a \oplus 2e < a \oplus r < b$.

Call A *divisible* if for each $a \in A,\ n > 0$ there is a $b \in A$ such that $nb = a,\ (n-1)b \neq a$.

Lemma 0.4. Let A be a divisible MV-chain. Then for $n > 0$, if $nx = ny \neq 1$, we have $x = y$.

Proof. Clear for $n = 1$. Suppose true for $k < n$. Assume that $nx = ny \neq 1$. We also may assume that $x < y$. Now we have $x \oplus (n-1)x = y \oplus (n-1)y \neq 1$. Also, as $x < y,\ (n-1)x \leq (n-1)y$. Equality of the latter implies $x = y$ by the induction assumption. Thus $(n-1)x < (n-1)y$. Applying Lemma 0.1 we obtain $nx = ny = 1$ which is absurd. Thus it's false that $x < y$. By symmetry we may infer that $x = y$.

We see that if A satisfies the divisibility hypothesis and if $nb = a < 1$, then b is unique. We denote b by $\frac{a}{n}$.

Lemma 0.5. Let A be a divisible and locally finite MV-algebra. For $a \in A, n > 0$, if $na < 1$, then $\frac{na}{n} = a$.

Proof

By definition, $n(\frac{na}{n}) = na$. From Lemma 0.4 we may infer that $n(\frac{na}{n}) = a$.

Lemma 0.6. Let A be an MV-algebra, $a,\ b \in A$, $a < b \neq 1$, $0 < y < b \odot a^*$. Then the following conditions hold:

(i) $a < b \odot y^*$

(ii) $[0,\ y) \oplus (a,\ b \odot y^*) \subseteq (a,\ b)$.

Proof.

(i): From $0 < y < b \odot a^*$, by adding b^* we get $b^* < y \oplus b^* \leq a^* \vee b^* = a^*$. Hence $a \leq y^* \odot b < b$. Let us show that it cannot be $a = y^* \odot b$. Indeed, assuming true the above equality, via negation, we get $a^* = y \oplus b^*$ and then $a^* \odot b = (y \oplus b^*) \odot b = y \wedge b \leq y$. Hence, $a^* \odot b \leq y$, in contradiction with $y < b \odot a^*$. So we get, as claimed $a < y^* \odot b$.

(ii): Let $z \in [0,\ y)$, $w \in (a,\ b \odot y^*)$. Then $a < z \oplus w \leq b$. Suppose $z \oplus w = b$. Then $y \odot z^* \oplus z \oplus w = b \oplus y \odot z^*$. Hence, $y \oplus w = b \oplus y \odot z^* \geq b$. But $y \oplus w \leq b$; therefore $y \oplus w = b = y \oplus b \odot y^* \neq 1$. It follows that $w \wedge y^* = b \odot y^* \leq w$ which is in contradiction with $w < b \odot y^*$.

Lemma 0.7. Let A be an atomless MV-chain. Let $x \in (a,\ b)$, $b \neq 1$. Then for some $y < b \odot a^*$ we have $x \in [0,\ y) \oplus (a,\ b \odot y^*)$.

Proof.

As $0 < b \odot x^*$ we can choose a y_x, $0 < y_x < b \odot x^*$. Then $x \oplus y_x \leq x \oplus b \odot x^* = b$. Since $b \neq 1$, equality here would imply $y_x = b \odot x^*$. Thus $x \oplus y_x < b$. Now let $0 < y < b \odot a^* \wedge y_x$. From Lemma 0.6 we have that $[0,\ y) \oplus (a,\ b \odot y^*) \subseteq (a,\ b)$. Assume that $b \odot y^* \leq x$. Then $y \oplus b \odot y^* \leq y \oplus \leq y_x \oplus b < b$ which is absurd. So, $x < b \odot y^*$, that is $x \in (a,\ b \odot y^*)$. As $x = 0 \oplus x$, the result follows.

Lemma 0.8. Let A be an atomless MV-chain. Let $u, a, b \in A$ such that $u \in [0, a \oplus b)$. Then there are $u_1, u_2 \in A$ such that $u_1 \leq a$, $u_2 \leq b$ and $u_1 \oplus u_2 = u$.

Proof.

If $u < a$, then $u = u \oplus 0 \in [0,\ a) \oplus [0,\ b)$; similarly, if $u < b$, then $u = 0 \oplus u \in [0,\ a) \oplus [0,\ b)$. Assume, then, that $a \leq b \leq u$. If $u \odot a^* \geq b$, then $u = a \oplus u \odot a^* \geq a \oplus b$. Hence, $u \odot a^* < b$. Since the algebra is assumed to be atomless, there exists an x, $u \odot a^* < x < b$. Thus $a \oplus u \odot a^* \leq a \oplus x$. Equality here would give, $u = a \oplus u \odot a^* = a \oplus x$. As $u \neq 1$, cancellation gives $u \odot a^* = x$. Thus $u < a \oplus x$. If $u \odot x^* \geq a$, then $u = x \oplus u \odot x^* \geq a \oplus x$. Hence, $u \odot x^* < a$. Therefore $u = u \odot x^* \oplus x$.

Proposition 0.2. Let A be an atomless MV-chain. Let $a, b, c, d \in A$

such that $a < b$ and $c < d$. Then $(a \oplus c,\ b \oplus d) \subseteq (a,\ b) \oplus (c,\ d)$.

Proof.

Let $u \in (a \oplus c,\ b \oplus d)$. Then $u \in [0,\ b \oplus d)$. Thus by Lemma 0.8, there are $u_1 < b,\ u_2 < d$ such that $u_1 \oplus u_2 = u$. If $u_1 \leq a,\ u_2 \leq c$, then $u \leq a \oplus c$. Hence, $a < u_1$ or $c < u_2$. We may assume that $a < u_1$, so that $u_1 \in (a,\ b)$. If $c < u_2$, we're done as $u_2 < d$.

Assume then that $u_2 \leq c$. Let $x = c \odot u_2^*$; if $u_1 \leq x$, then $u = u_1 \oplus u_2 \leq c$ which is false. So, $x < u_1$.

Suppose now that $u_1 \odot x^* \leq a$. Then $x \oplus u_1 \odot x^* = u_1 \leq x \oplus a$. Thus $u = u_1 \oplus u_2 \leq u_2 \oplus x \oplus a = c \oplus a$ which is false. Hence, $a < u_1 \odot x^*$.

Now, $u_2 \oplus x = c$; therefore, $u_1 \odot x^* \oplus c = u_1 \odot x^* \oplus u_2 \oplus x = u_1 \odot x^* \oplus$ $\oplus x \oplus u_2 = u_1 \oplus u_2 = u$. Let $v = u_1 \odot x^*$; then $v \in (a,\ b)$ and $v \oplus c = u$.

We know $d \odot c^* \wedge v \odot a^* > 0$. By Proposition 0.1 and Lemma 0.2, we can choose a $y,\ 0 < y < d \odot c^* \wedge v \odot a^*,\ c < c \oplus y < d,\ a < a \oplus y < v$. Thus, $v \odot y^* \oplus c \oplus y = (y \vee v) \oplus c = v \oplus c = u$. Noting that $y^* > v^* \oplus a$ we see that $v \odot y^* \geq v \wedge a = a$. If $v \odot y^* = a$, then $v = y \oplus a$. It follows then that $a < v \odot y^* < b$. As $c < c \oplus y < d$ we conclude that $u = v \odot y^* \oplus (c \oplus y) \in (a,\ b) \oplus (c,\ d)$ and the proposition is proved.

1.MV-algebras of frame morphisms

$\mathcal{F}$ will denote the category of frames and frame morphisms. Thus an object in $\mathcal{F}$ is a complete lattice $\mathbf{K}$ that satisfies the infinite distributive law, $a \wedge \Sigma_i a_i = \Sigma_i (a \wedge a_i)$. A frame morphism $f : \mathbf{K}_1 \to \mathbf{K}_2$ is a lattice homomorphism such that $f(\Sigma_i a_i) = \Sigma_i f(a_i)$.

Call a frame $\mathbf{K}$ *connected* if the only complemented elements are $0,\ 1$.

If X is any topological space, then $\Omega(X)$, the lattice of opens sets of X, is a frame. If $X_1,\ X_2$ are topological spaces and $\phi : X_1 \to X_2$ is a continuous map, then $f : \Omega(X_2) \to \Omega(X_1)$ is a frame morphism where f is the inverse image map of ϕ.

Let A be an MV-algebra. Then we say that A is a *topological* MV-algebra if on the set A is defined a topology with respect to which the MV-algebraic operations $\oplus$ and * are continuous. The continuity of the MV-algebraic operations $\oplus$ and * is equivalent to having the following properties be satisfied:

(a): Let O be an open set and $a, b \in A$ such that $a \oplus b \in O$. Then there are open sets O_1 and O_2 such that $a \in O_1$, $b \in O_2$ and $O_1 \oplus O_2 \subseteq O$;

(b): Let O be an open set and $a \in A$ such that $a^* \in O$. Then there is an open set O_1 such that $a \in O_1$ and $(O_1)^* \subseteq O$.

We observe that all atomic MV-chains are locally compact Hausdorff topological MV-algebras with respect to the interval topology generated by the sets $\{x \mid a < x < b\}$. Also, of course, the MV-algebra $[0,1]$ is of the same type. There exist non-atomic MV-chains endowed with its interval topology which are locally compact Hausdorff topological and not locally finite.

One example is the algebra A, where $A = \{(0,x) \mid x \in R+\} \cup \{(1,-x) \mid x \in R+\}$ ordered lexicographically and where $R+$ is the set of non-negative reals, and where $\oplus$, * are defined by:

$(0,x)^* = (1,\ -x),\ (1,\ -x)^* = (0,x)$

$(0,x) \oplus (0,\ y) = (0,\ x+y)$

$(1,-x) \oplus (1,\ -y) = (1,\ 0)$

$(0,x) \oplus (1,\ -y) = (1,\ 0)$ if $y \leq x$

$(0,\ x) \oplus (1,\ -y) = (1,\ x-y)$ if $x < y$.

Lemma 1.1. Let A be a locally compact Hausdorff topological MV-algebra. Let $\{O_i\}_I,\ \{O'_j\}_j$ be families of open sets of A. Assume that $O = \bigcup_{i \in I} O_i,\ O' = \bigcup_{j \in J} O'_j$. Then $O \oplus O' = \bigcup_{i,\ j}(O_i \oplus O'_j)$.

Proof.

Let $x \in O,\ y \in O'$. Then for $i \in I,\ j \in J$ we have $x \in O_i$ and $y \in O'_j$. Then $x \oplus y \in O_i \oplus O'_j)$. Conversely, suppose that $u \in \bigcup_{i,\ j}(O_i \oplus O'_j)$ for some $i \in I,\ j \in J$. Then there are $x \in O_i,\ y \in O'j$ with $u = x \oplus y$. But $x \in \bigcup_{i \in I} O_i = O$ and $y \in \bigcup_{j \in J} O'_j = O'$. Hence we get $u = x \oplus y \in O \oplus O'$.

Let $\mathcal{L}_{ch}$ be the category of locally compact Hausdorff spaces. Consider the functor $\Omega : \mathcal{L} \to \mathcal{F}$ defined as above.

Let A be a locally compact Hausdorff topological MV-algebra. Then as an object, $A \in \mathcal{L}$ and $\Omega(A) \in \mathcal{F}$. Now A has an addition, $\oplus : A \times A \to A$.

Consequently, applying the (contravariant) functor Ω we have

$$\oplus^{-1} : \Omega(A) \to \Omega(A \times A).$$

Let $\mathbf{K}$ be an arbitrary, but fixed frame. We have $Hom_{\mathcal{F}}(\ -\ ,\ \mathbf{K}) : \mathcal{F} \to \mathcal{S}et$ which sends any object L from the category $\mathcal{F}$ to the set of all frame morphisms from L to the frame K.

Thus we have an arrow,

$$Hom(\oplus^{-1}) : Hom_{\mathcal{F}}(\Omega(A \times A),\ \mathbf{K}) \to Hom_{\mathcal{F}}(\Omega(A),\ \mathbf{K}).$$

Coproducts exist in $\mathcal{F}$ so let $\Omega(A) \otimes \Omega(A)$ denote the to coproduct of $\Omega(A)$ with itself. Since A is locally compact we have $\Omega(A) \otimes \Omega(A) \cong \Omega(A \times A)$.

Let, $f,\ g : \Omega(A) \to \mathbf{K}$. We have the diagram

$$\begin{tikzcd}
\Omega(A) \arrow[r, "-\otimes 1"] \arrow[dr, "f"'] & \Omega(A) \otimes \Omega(A) \arrow[d, "f \otimes g"] & \Omega(A) \arrow[l, "1\otimes -"'] \arrow[dl, "g"] \\
 & \mathbf{K} &
\end{tikzcd}$$

where $f \otimes g$ is the unique frame map determined by the coproduct.

This gives rise to an isomorphism

$$Hom_{\mathcal{F}}(\Omega(A),\ \mathbf{K}) \times Hom_{\mathcal{F}}(\Omega(A),\ \mathbf{K}) \cong Hom_{\mathcal{F}}(\Omega(A) \otimes \Omega(A),\ \mathbf{K})$$

given by $(f,\ g) \to f \otimes g$.

Thus we have the diagram

$$\begin{tikzcd}
Hom_{\mathcal{F}}(\Omega(A),\ \mathbf{K}) \times Hom_{\mathcal{F}}(\Omega(A),\ \mathbf{K}) \arrow[r, phantom, "\cong"] & Hom_{\mathcal{F}}(\Omega(A) \otimes \Omega(A),\ \mathbf{K}) \arrow[d, "Hom(\oplus^{-1})"'] \\
 & Hom_{\mathcal{F}}(\Omega(A),\ \mathbf{K})
\end{tikzcd}$$

Hence for any frame morphism $d : \Omega(A) \otimes \Omega(A) \to \mathbf{K}$ we have $Hom(\oplus^{-1})(d) = d \circ \oplus^{-1}$.

In particular, for any $f,\ g \in Hom_{\mathcal{F}}(\Omega(A),\ \mathbf{K})$, we have the composition map , $(f,\ g) \to f \otimes g \to (f \otimes g) \circ \oplus^{-1}$.

Consider now the composition functor $H_{\mathbf{K}} : \mathcal{L} \to \mathcal{S}et$, where $H_{\mathbf{K}}(-) = Hom_{\mathcal{F}}(\Omega(-),\ \mathbf{K})$.

Again, for a topological MV-algebra A we have $\oplus : A \times A \to A$, and hence the map $H_{\mathbf{K}}((A \times A),\ \mathbf{K}) \to_{Hom(\oplus)} H_{\mathbf{K}}(A)$, where for $f,\ g \in H_{\mathbf{K}}(A)$, $Hom(\oplus)(f,\ g) = (f \otimes g) \circ \oplus^{-1}$ which we write as $f \oplus g$.

Now $H_{\mathbf{K}}((A \times A),\ \mathbf{K}) \cong Hom_{\mathcal{F}}(\Omega(A) \otimes \Omega(A),\ \mathbf{K})$, and $H_{\mathbf{K}} = Hom_{\mathcal{F}}(\Omega(A),\ \mathbf{K})$ so from above we have $H_{\mathbf{K}}(\oplus)(-) = (-) \circ \oplus^{-1}$.

Similarly, we have a map $* : A \to A$, hence a map $*^{-1} : \Omega(A) \to \Omega(A)$, thus $H_{\mathbf{K}} \to_{Hom(*)} H_{\mathbf{K}}$, where $Hom(*)(f) = f \circ *^{-1}$ which we write as f^*.

The constant functions on A are continuous, so they have inverse image maps. Let $a \in A$ and let $\hat{a} : A \to A$ be given by $\hat{a}(x) = a$. Then $\hat{a}^{-1} : \Omega(A) \to \Omega(A)$ is given by $\hat{a}^{-1}(O) = A$ if $a \in O$ and $\hat{a}^{-1}(O) = \emptyset$ if $a \notin O$.

Let now $f \in H_{\mathbf{K}}(A)$; then $H_{\mathbf{K}}(\hat{a})(f) = f \circ \hat{a}^{-1} \in H_{\mathbf{K}}(A)$.

If $O \in \Omega(A)$, then $f \circ \hat{a}^{-1}(O) = 1$, for $a \in O$, and $f \circ \hat{a}^{-1}(O) = 0$, for $a \notin O$.

We note that the resulting map is independent of the choice of f. Denote this map by $\mathbf{c}_a$.

It's easy to check that $H_{\mathbf{K}}(\hat{0})(f) = \mathbf{c}_0$ for any $f \in H_{\mathbf{K}}(A)$.

Theorem 1.1. Let A be a locally compact Hausdorff topological MV-algebra. Then, with the above notations we have that the structure

$$\mathcal{H}_{\mathbf{K}}(A) = (H_{\mathbf{K}}(A),\ Hom(\oplus),\ Hom(*),\ \mathbf{c}_0,\ \mathbf{c}_1)$$

is an MV-algebra.

Proof.

By [6,10 (Thm. 11.3.4)] we have that the functor $H_{\mathbf{K}}$ preserves commutative diagrams, thus preserves any identity expressible as a commutative diagram. Since A is an MV-algebra (whose axioms can be characterized by commutative diagrams) we may infer that $\mathcal{H}_{\mathbf{K}}(A)$ is an MV-algebra, where $\mathbf{c}_0$ and $\mathbf{c}_1$ are the '0' and "1" for $H_{\mathbf{K}}(A)$, respectively.

Another way to look at this is the following.

Let $\mathbf{K}_1,\ \mathbf{K}_2 \in \mathcal{F}$. Then we shall denote their coproduct by $\mathbf{K}_1 \otimes \mathbf{K}_2$. A typical element of $\mathbf{K}_1 \otimes \mathbf{K}_2$ will be of the form, $\Sigma_i(a_i \otimes b_i),\ a_i \in \mathbf{K}_1,\ b_i \in \mathbf{K}_2$.

The following identities hold in $\mathbf{K}_1 \otimes \mathbf{K}_2$.

1) $(a \otimes b) \wedge (c \otimes d) = (a \wedge c) \otimes (b \wedge d)$;
2) $1 \otimes 1 = 1 \in \mathbf{K}_1 \otimes \mathbf{K}_2$;
3) $a \otimes 0 = 0 \in \mathbf{K}_1 \otimes \mathbf{K}_2$;
4) $\Sigma_i(a_i \otimes b) = (\Sigma_i a_i) \otimes b$;
5) $\Sigma_i(a \otimes b_i) = a \otimes \Sigma_i b_i$.

Let $A = (A,\ \oplus,\ \odot,\ *, 0,\ 1)$ be a locally compact Hausdorff topological MV-algebra. Observe that the remaining operations on A, $\odot$, $\wedge$, $\vee$, are also continuous.

Let $\mathbf{F}$ be the frame $\Omega(A)$. Since we are assuming the topology on A to be locally compact, we have that $\mathbf{F} \otimes \mathbf{F} = \Omega(A \times A)$ where $A \times A$ has the product topology.

Let α be the inverse image map of $\oplus$ and ν that of *. Then $\alpha : \mathbf{F} \to \mathbf{F} \otimes \mathbf{F}$, $\nu : \mathbf{F} \to \mathbf{F}$ are frame maps. Thus, if $O \in \mathbf{F}$, we have $\nu(O) = O^* = \{x^* \mid x \in O\}$ and $\alpha(O) = \Sigma\{O_1 \otimes O_2 \mid O_1 \oplus O_2 \subseteq O\}$ where $O_1 \otimes O_2 = \{x \otimes y \mid x \in O_1,\ y \in O_2\}$.

Given a frame $\mathbf{K}$, let $\mathbf{A}_{\mathbf{K}} = Hom_{\mathcal{F}}(\mathbf{F},\ \mathbf{K})$. Let $f,\ g : \mathbf{F} \to \mathbf{K}$ be frame maps. We have the commutative diagram,

$$\mathbf{F}$$

$$\downarrow \alpha$$

$$\begin{tikzcd}
\mathbf{F} \arrow[r, "-\otimes 1"] \arrow[dr, "f"'] & \mathbf{F}\otimes\mathbf{F} \arrow[d, "h"] & \mathbf{F} \arrow[l, "1\otimes -"'] \arrow[dl, "g"] \\
& \mathbf{K} &
\end{tikzcd}$$

where h is the unique frame map determined by the coproduct.

Define $f \oplus g = h \circ \alpha$. Also define $f^* = f \circ \nu$. We can then define $f \odot g = (f^* \oplus g^*)^*$. Similarly we define $f \dot{\vee} g = f \oplus (f \oplus g^*)^*$, $f \dot{\wedge} g = (f^* \dot{\vee}\, g^*)^*$.

Consider then a typical element $\Sigma_i(O_i \otimes O'_i) \in \mathbf{F}\otimes\mathbf{F}$. Let $f,\ g,\ h$ be as in the diagram above. h is a frame map, so $h(\Sigma_i(O_i \otimes O'_i)) = \Sigma_i h((O_i \otimes O'_i) = \Sigma_i h((O_i \otimes 1) \wedge (1 \otimes O'_i)) = \Sigma_i(f(O_i \wedge g(O'_i))$.

Hence, if $O \in \mathbf{F}$, then $(f \oplus g) = \Sigma_{O_i \oplus O'_i \subseteq O} f(O_i) \wedge g(O'_i)$ where for $O_1,\ O_2 \in \mathbf{F},\ O_1 \oplus O_2 = \{x \oplus y \mid x \in O_1,\ y \in O_2\}$. Furthermore, consider now the following class of maps in $\mathbf{A_K}$. For $a \in A$, let $\mathbf{c}_a : \mathbf{F} \to \mathbf{K}$ be given by:

$$\mathbf{c}_a(O) = 1,\ \text{if } a \in O$$

$$\mathbf{c}_a(O) = 0,\ \text{if } a \notin O.$$

Proposition 1.1. Let A be a locally compact Hausdorff topological MV-algebra and $\mathbf{K}$ a frame. Then, with the above notation, for each $a \in A$, $\mathbf{c}_a$ is a frame homomorphism.

Proof.

The unit of $\mathbf{F}$ is A, so $\mathbf{c}_A(A) = 1$ for all $a \in A$. Similarly, $\mathbf{c}_a(\emptyset) = 0$ for all $a \in A$. Now let $O_1,\ O_2 \in \mathbf{F}$. Set $O = O_1 \cap O_2$. If $a \in O$, then $a \in O_1,\ a \in O_2$; hence $1 = \mathbf{c}_a(O) = \mathbf{c}_a(O_1) \wedge \mathbf{c}_a(O_2)$, If $a \notin O$, then $a \notin O_1$ or $a \notin O_2$. In either case, $0 = \mathbf{c}_a(O_1) \wedge \mathbf{c}_a(O_2)$. Finally, let $O = \bigcup_i O_i$. Clearly $\mathbf{c}_a(O) = \Sigma_i \mathbf{c}_a(O_i)$.

Then we have:

Theorem 1.2. Let A be a locally compact Hausdorff topological MV-algebra and $\mathbf{K}$ a frame. Then with the above notations we have that the structure $\mathbf{A_K} = (\mathbf{A_K}, \oplus, {}^*, \mathbf{c}_0,\ \mathbf{c}_1)$ is an MV-algebra.

Proof.

One can show directly, though tediously, that all the appropriate axioms for an MV-algebra will hold on $\mathbf{A_K}$, where $\mathbf{c}_0,\ \mathbf{c}_1$ act respectively as the "0", and "1" element of $\mathbf{A_K}$.

Corollary 1.1. Let A be locally compact Hausdorff topological MV-algebra and $\mathbf{K}$ a frame.Then the MV-algebras $\mathcal{H}_{\mathbf{K}}(A)$ and $\mathbf{A}_{\mathbf{K}}$ are isomorphic.

Proof.

From Theorems 1.1 and 1.2.

Theorem 1.3. Let A be a locally compact Hausdorff topological MV-algebra and $\mathbf{K}$ a frame. Then, up to isomorphism, the MV-algebra A is a subalgebra of the MV-algebra $\mathbf{A}_{\mathbf{K}}$.

Proof.

Since A is Hausdorff, the mapping $A \to \mathbf{A} : a \to \mathbf{c}_a$ is injective. Let $O \in \mathbf{F}$. Then, ${\mathbf{c}_a}^*(O) = \mathbf{c}_a(\nu(O)) = 1$ iff $a \in \nu(O)$ iff $a^* \in O$ iff $\mathbf{c}_{a^*}(O) = 1$. Therefore ${\mathbf{c}_a}^* = \mathbf{c}_{a^*}$. Now, $(\mathbf{c}_a \oplus \mathbf{c}_b)(O) = \Sigma_{O_1 \oplus O_2 \subseteq O} \mathbf{c}_a(O_1) \wedge \mathbf{c}_b(O_2)$. If $a \oplus b \in O$, then by property (b) there are open sets $O_1,\ O_2,\ a \in O_1,\ b \in O_2$ with $O_1 \oplus O_2 \subseteq O$. Then $(\mathbf{c}_a \oplus \mathbf{c}_b)(O) = 1 = \mathbf{c}_{a\oplus b}(O)$. If $a \oplus b \notin O$, then for all $O_1,\ O_2$ with $O_1 \oplus O_2 \subseteq O$, either $a \notin O_1$ or $b \notin O_2$. So $\mathbf{c}_a(O_1) \wedge \mathbf{c}_b(O_2) = 0$; that is, $(\mathbf{c}_a \oplus \mathbf{c}_b)(O) = 0 = \mathbf{c}_{a\oplus b}(O)$. We may infer, then, that $\mathbf{c}_a \oplus \mathbf{c}_b = \mathbf{c}_{a\oplus b}$.

Define for $O_1,\ O_2 \in \mathbf{F}$, $O_1 \vee O_2 = \{x \vee y \mid x \in O_1,\ y \in O_2\} = \{x \oplus (x \oplus y^*)^* \mid x \in O_1,\ y \in O_2\}$. Then for $f,\ g \in \mathbf{A}_{\mathbf{K}},\ O \in \mathbf{F}$ we will have $(f \dot{\vee} g)(O) = \Sigma_{O_1 \vee O_2 \subseteq O} f(O_1) \wedge g(O_2)$. From above we know $f \dot{\vee} g = g \dot{\vee} f$.

2. Spatial Frames and MV-algebras

Again we assume that A is a locally compact Hausdorff topological MV-algebra and $\mathbf{K}$ a frame and Spec($\mathbf{K}$) the spectrum of prime ideals of $\mathbf{K}$. Let $\mathrm{pt}(\mathbf{K}) = \{x \in \mathbf{K} \mid \mathrm{id}(x) \in \mathrm{Spec}(\mathbf{K})\}$. For $a \in \mathbf{K}$ take $\{x \in \mathrm{pt}(\mathbf{K}) \mid a \notin \mathrm{id}(x)\}$. It's easy to verify that the collection of these sets forms a topology for the set pt($\mathbf{K}$).

We also have a frame map $\phi : \mathbf{K} \to \Omega(\mathrm{pt}(\mathbf{K}))$ given by $a \to \{x \in \mathrm{pt}(\mathbf{K}) \mid a \notin \mathrm{id}(x)\}$. We say $\mathbf{K}$ is *spatial* or *has enough points* if ϕ is a frame isomorphism.

Proposition 2.1. Let A be a locally compact Hausdorff topological MV-algebra and $\mathbf{K}$ a spatial frame. Then $\mathcal{C}(\mathrm{pt}(\mathbf{K}),\ A)$ is a subalgebra of $\mathbf{A}_{\mathbf{K}}$.

Proof.

Consider the algebra $\mathcal{C}(\mathrm{pt}(\mathbf{K}),\ A)$ of continuous functions from pt($\mathbf{K}$) to A. Let $u \in \mathcal{C}(\mathrm{pt}(\mathbf{K}),\ A)$. Then the inverse image map $u^{-1} : \Omega(A) \to \Omega(\mathrm{pt}(\mathbf{K}))$ is a frame morphism. Hence, $\phi^{-1} \circ u^{-1} \in \mathbf{A}_{\mathbf{K}}$. Consider now the mapping, $u \to \phi^{-1} \circ u^{-1} : \mathcal{C}(\mathrm{pt}(\mathbf{K}),\ A) \to \mathbf{A}_{\mathbf{K}}$.

Suppose that $u,\ v \in \mathcal{C}(\mathrm{pt}(\mathbf{K}),\ A)$. Then $(u \oplus v) \to \phi^{=1} \circ (u \oplus v)^{-1}$. Let $O \in \Omega(A)$. Then $(u \oplus v)^{-1}(O) = \{x \in \mathrm{pt}(\mathbf{K}) \mid u(x) \oplus v(x) \in O\}$. Consider now, $\bigcup_{O_1 \oplus O_2 \subseteq O}(u^{-1}(O_1) \cap v^{-1}(O_2))$. Clearly, if $x \in u^{-1}(O_1) \cap v^{-1}(O_2))$ for some $O_1 \oplus O_2 \subseteq O$, then $u(x) \oplus v(x) \in O$. Hence, $\bigcup_{O_1 \oplus O_2 \subseteq O}(u^{-1}(O_1) \cap v^{-1}(O_2)) \subseteq (u \oplus v)^{-1}(O)$. Conversely, suppose that $x \in (u \oplus v)^{-1}(O)$ so that $u(x) \oplus v(x) \in O$. By, the continuity of the operation $\oplus$, there are $O_1,\ O_2$ open with $u(x) \in O_1,\ v(x) \in O_2$ and $O_1 \oplus O_2 \subseteq O$. But then $x \in u^{-1}(O_1) \cap v^{-1}(O_2)$ and so we may infer that $(u \oplus v)^{-1}(O) = \bigcup_{O_1 \oplus O_2 \subseteq O}(u^{-1}(O_1) \cap v^{-1}(O_2))$. Therefore, if $\mathbf{K}$ is spatial, $\phi^{-1}(u \oplus v)^{-1}(O) = \phi(\bigcup_{O_1 \oplus O_2 \subseteq O}(u^{-1}(O_1) \cap v^{-1}(O_2)))$. As ϕ^{-1} is a frame morphism, we then have $\phi^{-1}(u+v)^{-1}(O) = \bigcup_{O_1 \oplus O_2 \subseteq O}(\phi \circ u^{-1}(O_1) \cap \phi \circ v^{-1}(O_2))$. It follows that the map $u \to \phi^{-1} \circ u^{-1}$ preserves "$\oplus$".

Let $u \in \mathcal{C}(\mathrm{pt}(\mathbf{K}),\ A)$. Then $u^* \to \phi^{-1} \circ u^{*-1}$. For $O \in \Omega(A)$ we have $(\phi^{-1} \circ u^{-1})^*(O) = (\phi \circ u^{-1})(\nu(O))$. On the other hand, $(\phi^{-1} \circ u^{*-1})(O) = \phi^{-1}\{x \in \mathrm{pt}(\mathbf{K}) \mid u^*(x) \in O\} = \phi^{-1}\{x \in \mathrm{pt}(\mathbf{K}) \mid (u(x))^* \in O\} = \phi^{-1}\{x \in \mathrm{pt}(\mathbf{K}) \mid u(x) \in \nu(O)\} = (\phi^{-1} \circ u^{-1})(\nu(O))$. It follows that the map $u \to \phi^{-1} \circ u^{-1}$ preserves "$*$", and hence is an MV-homorphism.

Finally, supposes that $u,\ v \in \mathcal{C}(\mathrm{pt}(\mathbf{K}),\ A)$ and that $\phi^{-1} \circ u^{-1} = \phi^{-1} \circ v^{-1}$. Since ϕ^{-1} is an isomorphism, we have $u^{-1} = v^{-1}$. Assume that $u \neq v$. Then for $x \in \mathrm{pt}(\mathbf{K})$ we have $u(x) \neq v(x)$. As the topology of A is Hausdorff there is an open $O \in \Omega(A)$ with, say, $u(x) \in O,\ v(x) \notin O$. But then $x \in u^{-1}(O) - v^{-1}(O)$. We see, therefore, that the map $u \to \phi^{-1} \circ u^{-1}$ is an injection.

3. The interval topology of an MV-algebra and Frames

Let A be an MV-algebra endowed with its interval topology. For an element $a \in A$ we define a^- to be the open set $[0,\ a)$. Thus $0^- = \emptyset$. We also define a^+ to be the interior of $A \setminus (a^-)$, if $a \neq 0$ and $0^+ = (0,\ 1]$. For $a > 0$, then, a^+ is the largest open set disjoint from a^-.

Proposition 3.1. Let A be an MV-algebra and I an interval in A. Then we have one of the following cases:

(j) $I = a^-$ for some $a \in A$

(jj) $I = \bigcup\{(b_i)^+ \mid i \in J\}$ for some index set J

(jjj) $I = \bigcup\{(a_i)^+ \cap\ (b_i)^- \mid i \in J\}$ for some index set J.

Proof.

If $I = [0,\ a)$ for some a, then $I = a^-$. Suppose that $I = (b,\ 1]$. If A is atomic with atom e, then $I = (b \oplus e)^+$. Otherwise choose $x \in I$; for some $y \in I$ we have $b < x < y$. So $y^+ \subseteq I,\ x \in y^+$. Hence $I = \bigcup\{y^+ \mid y \in I\}$. Finally, suppose that $I = (a,\ b)$. So $I = (a,\ 1] \cap [0,\ b)$; thus $I = \bigcup\{y^+ \mid y \in$

$(a, 1]\} \cap b^- = \bigcup\{y^+ \cap\ b^- \mid y \in (a,\ 1]\}$.

Proposition 3.2. Let A be an MV-algebra endowed with its interval topology. Let $a \in A$, and O be an open set. Then one of the following hold:

(i): The set $O \odot a^-$ is open

(ii): $O \odot a^- = \{0\}$.

Proof. First, we claim that for $x \in A$ that if $x \odot a \neq 0$, then the set $x \odot a^- = (x \odot a)^-$. Clearly $x \odot a^- \subseteq (x \odot a)^-$. Suppose then that $0 \leq u < x \odot a$. Consider $x \odot (u \oplus x^*) = x \wedge u = u$. If $a \leq u \oplus x^*$, then $x \odot a \leq u$. Hence $u \oplus x^* < a$ and so $u \oplus x^* \in a^-$. Therefore $x \odot (u \oplus x^*) = u \in x \odot a^-$ and we see that $x \odot a^- = (x \odot a)^-$. Now, $O \odot a^- = \bigcup_{x \in O} x \odot a^- = \bigcup_{x \in O}(x \odot a)^-$ which is open or $O \odot a^- = \{0\}$ if $x \odot a = 0$ for all $x \in O$.

Proposition 3.3. Let A be a locally compact Hausdorff topological MV-algebra, with respect to its interval topology, and $\mathbf{K}$ an arbitrary frame. Then, for $a \in A,\ f,\ g \in \mathbf{A_K}$, the following identities hold:

(i) $(f\dot{\vee}g)(a^-) = f(a^-) \wedge g(a^-)$.

(ii)$(f\dot{\vee}g)(a^+) = f(a^+) \vee g(a^+)$.

(iii)$(f\dot{\wedge}g)(r^+) = f(r^-) \vee g(r^-)$.

(iv) If $f(a^-) = g(a^-),\ f(a^+) = g(a^+)$ for all $a \in A$ then $f = g$.

Proof.

(i): Let $O_1,\ O_2 \subseteq a^-$. Then, $O_1,\ O_2 \subseteq a^-$. Thus, $f(O_1) \leq f(a^-),\ g(O_2) \leq g(a^-)$ so $f(O_1) \wedge g(O_2) \leq f(a^-)$. Thus, $(f\dot{\vee}g)(a^-) \leq f(a^-) \wedge g(a^-)$. On the other hand, $(a^-) \vee (a^-) \subseteq a^-$ and so $f(a^-) \wedge g(a^-) \leq (f\dot{\vee}g)(a^-)$.

(ii): Suppose that $O_1 \vee O_2 \subseteq a^+$ and $O_1 \not\subseteq a^+$. Choose $x \in O_1 - (a^+)$. For $b \in O_2,\ x \vee b \in a^+$. But $x \vee b = x$ or $x \vee b = b$; $x \notin a^+$ implies $x \vee b = b$ and so $O_2 \subseteq a^+$. Hence we have that $O_1 \subseteq a^+$ or $O_2 \subseteq a+$. Therefore $f(O_1) \leq f(a^+)$ or $g(O_2) \leq g(a^+)$. In either case, $f(O_1) \wedge g(O_2) \leq f(a^+) \vee g(a^+)$, and so $(f\dot{\vee}g)(a^+) \leq f(a^+) \vee g(a^+)$. Now $a^+ \vee 0^+ \subseteq a^+$ so $f(a^+) \wedge g(0^+) = f(a^+) \leq f(a^+) \vee g(a^+)$. Similarly, $g(a^+) \leq (f\dot{\vee}g)(a^+)$ and we see that $f(a^+) \vee g(a^+) \leq (f\dot{\vee}g)(a^+)$.

(iii): $(f^*\dot{\vee}g^*)^*(r^-) = (f^*\dot{\vee}g^*)(\nu(r^-))$. Now $x \in \nu(r^-)$ iff $x^* \in r^-$ iff $x^* < r$ iff $r^* < x$ iff $x \in (r^*,\ 1]$. Applying part i) we see that $(f^*\dot{\wedge}g^*)((r^*,\ 1]) = f^*((r^*,\ 1]) \vee g^*((r^*,\ 1])$. Now $\nu((r^*,\ 1]) = r^-$ so $f^*((r^*,\ 1]) \vee g^*((r^*,\ 1]) = f(r^-) \vee g(r^-)$.

(iv): Let $O \in \Omega(A)$. Then $O = \bigcup_j I_j$ where the I_j are intervals. So $f(O) = \Sigma_j f(I_j)$, and $g(O) = \Sigma_j I_j$. Thus it suffices to show that $f(I) = g(I)$ for any interval I. We apply Proposition 3.1. If $I = a^-$ for some a, then by assumption $f(I) = g(I)$. If $I = \bigcup_j b_j^+$, then $f(I) = \Sigma_j f(b_j^+) = \Sigma_j g(b_j^+) = g(I)$ again by assumption. Finally, if $I = \bigcup_j((a_j)^-) \cap ((b_j)^+)$, then $f(I) = \Sigma_j f((a_j)^-) \cap f((b_j)^+) = \Sigma_j g((a_j)^-) \cap g((b_j)^+) = g(I)$.

Proposition 3.4. Let A be a locally compact topological MV-algebra endowed with its interval topology. Let $O_1,\ O_2$ be open sets of A. Then $O_1 \oplus O_2$ is open or $O_1 \oplus O_2 = O \cup \{1\}$ where O is open and $1 \notin O$.
Proof.

We can write $O_1 = \bigcup_i O_{1i},\ O_2 = \bigcup_j O_{2j}$ where the $O_{1i},\ O_{2j}$ are open intervals. By Lemma 1.1, $O_1 \oplus O_2 = \bigcup_{i,\ j}(O_{1i} \oplus O_{2j}) = \bigcup\{O_{1i} \oplus O_{2j} \mid O_{1i} \oplus O_{2j} \text{ open}\} \cup \bigcup\{O_{1i} \oplus O_{2j} \mid O_{1i} \oplus O_{2j} = \{1\}\}$. If $\bigcup\{O_{1i} \oplus O_{2j} \mid O_{1i} \oplus O_{2j} = \{1\}\} = \emptyset$ then $O_1 \oplus O_2$ is open. If $1 \in \bigcup\{O_{1i} \oplus O_{2j} \mid O_{1i} \oplus O_{2j} \text{ open}\}$, then $O_1 \oplus O_2 = \bigcup\{O_{1i} \oplus O_{2j} \mid O_{1i} \oplus O_{2j} \text{ open}\}$ and so is open. If $1 \notin \bigcup\{O_{1i} \oplus O_{2j} \mid O_{1i} \oplus O_{2j} \text{ open}\}$, and $\bigcup\{O_{1i} \oplus O_{2j} \mid O_{1i} \oplus O_{2j} = \{1\}\} \neq \emptyset$, then $O_1 \oplus O_2 = \bigcup\{O_{1i} \oplus O_{2j} \mid O_{1i} \oplus O_{2j} \text{ open}\} \cup \{1\} = O \cup \{1\}$ where O is open and $1 \notin O$.

We extend the above to,

Proposition 3.5. Let A be a locally compact topological MV-algebra endowed with its interval topology. Let $O_1,\ O_2, \ldots,\ O_n$ be open. Then $O_1 \oplus O_2 \oplus \ldots \oplus O_n$ is open or equals $O \cup \{1\}$ where O is open and $1 \notin O$.
Proof.

For $k = 2$ we have the above proposition. Suppose the claim of the proposition true for k. Then $O_1 \oplus O_2 \oplus \ldots \oplus O_{k+1} = O_1 \oplus O_2 \oplus \ldots \oplus O_k \oplus O_{k+1}$ which is of the form $O \oplus O_{k+1}$ where O is open or of the form $(O \cup \{1\}) \oplus O_{k+1}$ with O open and $1 \notin O$.

Now $O \oplus O_{k+1}$ is open or of the form $O' \cup \{1\}$ with $1 \notin O'$ and $(O \cup \{1\}) \oplus O_{k+1} = (O \oplus O_{k+1}) \cup \{1\}$ which is open or of the form $O' \cup \{1\}$ with $1 \notin O'$. By induction the result holds.

Proposition 3.6. Let A be a locally compact topological MV-algebra endowed with its interval topology. Let $\mathbf{K}$ be a frame, $f \in \mathbf{A_K}$ and $O \in \Omega(A)$. Then,

$$(nf)(O) = \Sigma_{O_1 \oplus \cdots \oplus O_n \subseteq O} f(O_1 \cap \cdots \cap O_n).$$

Proof. For $n = 1$ the statement is clear. Suppose true for all $k < n$. Let $O \in \Omega(A)$. Then $(nf)(O) = (f \oplus (n-1)f)(O) = \Sigma_{O_1 \oplus O_2 \subseteq O} f(O_1) \wedge (n-1)f(O_2) = \Sigma_{O_1 \oplus O_2} f(O_1) \wedge \Sigma_{O'_2 \oplus \cdots \oplus O'_n \subseteq O_2}(n-1)f(O'_2 \cap \cdots \cap O'_n) = \Sigma_{O_1 \oplus O_2 \subseteq O} \Sigma_{O'_2 \oplus \cdots \oplus O'_n \subseteq O_2} f(O_1 \cap O'_2 \cap \cdots \cap O'_n)$.

Now $O'_2 \oplus \cdots \oplus O'_n \subseteq O_2$ and $O_1 \oplus O_2 \subseteq O$ implies $O_1 \oplus O'_2 \oplus \cdots \oplus O'_n \subseteq O$. Thus $f(O_1 \cap O'_2 \cap \cdots \cap O'_n)$ is a summand of $\Sigma_{O_1 \oplus \cdots \oplus O_n \subseteq O} f(O_1 \cap \cdots \cap O_n)$ and we have $(nf)(O) \leq \Sigma_{O_1 \oplus \cdots \oplus O_n \subseteq O} f(O_1 \cap \cdots \cap O_n)$

3.1 The linear case

Proposition 3.1.1. If A is an atomic MV-chain, then the interval topology on A is discrete.

Proof.

Let $x \in A$. If $x = 0$, $\{x\} = [0,\ a)$ where a is the atom of A. By duality, $\{1\} = (a^*,\ 1]$. So assume that $0 < x < 1$. Suppose that $x < y < x \oplus a$. Then $0 \leq y \odot x^* \leq a \wedge x^*$. Now $y \odot x^* = 0$ implies $y = x$. As $a \wedge x^* \leq a$ we know that $a \wedge x^* = 0$ or $a \wedge x^* = a$. The former implies $y \odot x^* = 0$. Hence we have $y \odot x^* = a$ and so, $y = x \oplus a$ which is absurd. Thus, $(x,\ x \oplus a) \setminus \emptyset$. By duality, $(x \odot a^*,\ x) = \emptyset$. Thus, $\{x\} = (x \odot a^*,\ x \oplus a)$. Therefore points are open.

Proposition 3.1.2. Let A be an MV-chain such that some point, $\{a\}$, is open in the interval topology of A. Then A is atomic and all points are open in the interval topology.

Proof.

Suppose that $a = 0$. As $\{0\}$ is open there must be an r such that $[0,\ r) \subseteq \{0\}$. Then, $[0,\ r) = \{0\}$ and it follows that r is an atom. Similarly if $a = 1$. Thus assume that $a \neq 0,\ 1$. Let $c,\ d \in A$ be such that $c < a < d$, $(c,\ d) \subseteq \{a\}$. Then the interval $(c,\ a) = \emptyset$. By Proposition 3.1.1 A must be atomic, and so all points are open.

Proposition 3.1.3. Let A be a topological MV-chain which is connected in its interval topology. Then A is locally finite.

Proof.

Assume we have an $x \in A,\ x \neq 0$ of infinite order. Let $\mathrm{id}(x)$ be the ideal generated by x. Then $\mathrm{id}(x) = \bigcup_n [0,\ nx)$ which is an open set. Call it, say, O. Let $O' = \bigcup_{y \notin O} (y,\ 1]$. Now let $a \in A$. Assume that $a \notin O$. If $a = 1$, then $a \in (x^*,\ 1]$. As $a \notin O$ we see that $nx < a$ for all n. Let $y = a \odot x^*$. Suppose that $y \in O$, so that $y \leq nx$ for some n. Then $y' = a \odot (x^*)^n \leq y \leq nx$. Thus, $nx \oplus y' = nx \vee a = a \in O$ which is false. Hence $y \notin O$. Now $y \leq a$; if $y = a$, then $a^* \oplus x = a^* \oplus 0$. As $a \neq 1$ we have $x = 0$ which is also false. Thus $y < a$ and so $a \in (y,\ 1] \subseteq O'$. Therefore $A = O \cup O'$. But clearly $O \cap O' = \emptyset$ and as O' is also open, we have that A is not connected. This contradiction proves the proposition.

In the case that A is an MV-chain, the interval topology on A has as a basis of open sets, the sets, A, $\emptyset$, all sets of the form $[0,\ a)$, $a \neq 0$, $(b,\ 1]$, $b \neq 1$ and the sets $(a,\ b)$, $a < b$ where $[0, x)$, $(x,\ 1]$, $(a,\ b)$ are intervals of A.

We point out that if A is atomic, the intervals reduce to singletons of points $\{a\}$, or closed intervals $[a,\ b]$.

Lemma 3.1.1. Let A be an MV-chain endowed with its interval topology. Let O be open, $a \oplus b \in O$. Then there are $O_1,\ O_2$ open with $a \in O_1,\ b \in O_2$ and $O_1 \oplus O_2 \subseteq O$.
Proof.

If A is atomic we can pick $O_1 = \{a\},\ O_2 = \{b\}$. Thus we can assume that A is atomless. If $a \oplus b = 0$, then $a = 0 = b$. For some $c > 0$ we have $[0,\ c) \subseteq O$. We can choose an $e,\ 0 < 2e < c$ so that $[0,\ e) \oplus [0,\ e) \subseteq O$. Hence we can let $O_1 = O_2 = [0,\ e)$.

Suppose now that, say, $a = 0,\ b \neq 0$; then $b \in O$ so there are $u,\ v$ with $b \in (u,\ v) \subseteq O$. We can choose an $e > 0, b \oplus 2e < v$. Then $[0,\ e) \oplus (u,\ b \oplus e) \subseteq (u,\ v) \subseteq O$ and we can let $O_1 = [0,\ 1),\ O_2 = (u,\ b \oplus e)$.

If $a = 1 = b$ let $O_1 = O_2 = (u,\ 1]$ where we know that for some $u < 1$ we have $(u,\ 1] \subseteq O$

If $a = 1,\ b \neq 1$, we have a $u > 0,\ (u,\ 1] \subseteq O$. Thus set $O_1 = (u,\ 1],\ O_2 = [0,\ 1]$. Similarly if $a \neq 1,\ b = 1$.

Suppose now that $0 < a,\ b < 1$. We know there is an interval $I \subseteq O$ with $a \oplus b \in I$. I will have the form $(u,\ v)$ or $(u,\ v],\ v = 1$. In either case let v denote the upper end point of I. Choose an $e > 0,\ u \oplus 2e < a \oplus b$ and $e < a \wedge b$. Consider the intervals $(a \odot e^*,\ a \oplus b),\ (b \odot e^*,\ b \oplus e)$. If $a \leq a \odot e^*$, then $a \oplus e \leq a$ which is false. Similarly if $b \leq b \odot e^*$. Hence $a \in (a \odot e^*,\ a \oplus b),\ b \in (b \odot e^*,\ b \oplus e)$. Now let $x \in (a \odot e^*,\ a \oplus b),\ y \in (b \odot e^*,\ b \oplus e)$. Then, $a \odot e^* \oplus b \odot e^* \leq x \oplus y$. If $a \odot e^* \oplus b \odot e^* \leq u$, then $a \oplus b \leq u \oplus 2e < a \oplus b$. Therefore $u < x \oplus y$. If $I = (u,\ 1]$, then $x \oplus y \in I \subseteq O$. If $I = (u,\ v)$ we can modify e so that we also have $a \oplus 2e < v$. In this case $u < x \oplus y \leq a \oplus b \oplus 2e < v$. Ergo, choose $O_1 = (a \odot e^*,\ a \oplus b),\ O_2 = (b \odot e^*,\ b \oplus e)$.

Lemma 3.1.1, together with the obvious fact that * is always continuous on a linearly ordered algebra, imply that all linearly ordered MV-algebras are topological.

Proposition 3.1.4. Let A be an MV-chain endowed with its interval topology. For $a \in A, O$ open, the set $O \odot a^-$ is open or $O \odot a^- = \{0\}$.
Proof.

First, we claim that for $x \in A$ that if $x \odot a \neq 0$, then the set $x \odot a^- = (x \odot a)^-$. Clearly $x \odot a^- \subseteq (x \odot a)^-$. Suppose then that $0 \leq u < x \odot a$. Consider $x \odot (u \oplus x^*) = x \wedge u = u$. If $a \leq u \oplus x^*$, then $x \odot a \leq u$. Hence $u \oplus x^* < a$ and so $u \oplus x^* \in a^-$. Therefore $x \odot (u \oplus x^*) = u \in x \odot a^-$ and we see that $x \odot a^- = (x \odot a)^-$. Now, $O \odot a^- = \bigcup_{x \in O} x \odot a^- = \bigcup_{x \in O} (x \odot a)^-$

which is open or $O \odot a^{-} = \{0\}$ if $x \odot a = 0$ for all $x \in O$.

Lemma 3.1.2. Let A be an MV-chain endowed with its interval topology. Let $a,\ b,\ r \in A$. Then, $a \oplus b \geq r$ iff $a^{+} \oplus\ b^{+} \subseteq r^{+}$.
Proof.

Suppose first that $a \oplus b \geq r$. Let $x \in a^{+},\ y \in b^{+}$. Then $x \oplus y \geq a \oplus b \geq r$. Equality implies that $a \oplus b = 1$. But then $a^{+} \oplus\ b^{+} = \{1\} \subseteq r^{+}$.

Conversely, suppose that $a^{+} \oplus\ b^{+} \subseteq r^{+}$. If A is atomic, then $a^{+} = [a,\ 1],\ b^{+} = [b,\ 1]$ and so $a \oplus b \in r^{+}$; that is, $a \oplus b \geq r$. Assume, then, that A is atomless. If $a \oplus b < r$ we can find an $e > 0,\ a \oplus b \oplus 2e < r$. But then $(a \oplus e) \oplus (b \oplus e) < r$ which violates the assumption $a^{+} \oplus\ b^{+} \subseteq r^{+}$

Proposition 3.1.5. Let A be an MV-chain endowed with its interval topology. Let $I,\ I'$ be intervals with $I \oplus I' \subseteq r^{+}$ for some $r,\ 0 < r$. Then there exist $a,\ b \in A$ such that $I \subseteq a^{+},\ I' \subseteq b^{+}$ and $a \oplus b \geq r$.
Proof.

Let $0 < r < 1$. Suppose first that $0 \in I$. Then $I = u-$ for some u. Now $I' = \{0\} \oplus I' \subseteq r^{+}$. As $I \subseteq 0^{+} = [0,\ 1]$ and $0 \oplus r \geq r$ we have what's needed. Similarly if $0 \in I'$. Suppose then that $0 \notin I \cup I'$. Assume $I = a^{+},\ I' = b^{+}$. By Lemma 3.1.2 we have $a \oplus b \geq r$. Assume then that $I = a^{+},\ I' = (b,\ c),\ b < c$. Then $I \oplus b^{+} \subseteq r^{+}$; as $I' \subseteq b^{+}$ and the Lemma 3.1.2 gives $a \oplus b \geq r$ we have what we need. Finally suppose that $I = (a,\ a'),\ I' = (b,\ b')$. Then $I \subseteq a^{+},\ I' \subseteq b^{+}$ and $a^{+} \oplus b^{+} \subseteq r$. Again, invoking the Lemma 3.1.2 we have $a \oplus b \geq r$.

Suppose now that $r = 1$. If A is atomless this doesn't apply since $r^{+} = \emptyset$. If A is atomic, $r^{+} = 1^{+} = \{1\}$. Thus $0 \notin I \cup I'$. Therefore $I,\ I'$ are closed intervals, say, $I = [a,\ c],\ I' = [b,\ d]$. But then, as $I \oplus I' \subseteq r^{+}$ we have $a \oplus b = 1$; moreover, $I \subseteq a^{+},\ I' \subseteq b^{+}$ and the proof is complete.

Proposition 3.1.6. Let A be a topological MV-chain which is locally compact with respect its interval topology. Let $\mathbf{K}$ be a frame and $f, g \in \mathbf{A_K}$. Then the following equality holds:

$(f \oplus g)(r^{+}) = \Sigma_{s \oplus t \geq r}(f(s^{+}) \wedge g(t^{+}))$
Proof.

Suppose first that $s \oplus t \geq r$. Then, $s^{+} \oplus t^{+} \subseteq r^{+}$. Therefore $(f(s^{+}) \wedge g(t^{+})$ is a summand of $(f \oplus g)(r^{+})$. Conversely, assume that $O_1 \oplus O_2 \subseteq r^{+},\ O_1,\ O_2 \in \mathbf{F}$. Now $O_1 = \bigcup_j I_j,\ O_2 = \bigcup_m I'_m,\ I_j,\ I'_m$ intervals, so we see that $O_1 \oplus O_2 = \bigcup_{j,\ m}(I_j \oplus I'_m)$; thus, $I_j \oplus I'_m \subseteq r^{+}$ for all $j,\ m$. As, $f(O_1) = \Sigma_j f(I_j),\ g(O_2) = \Sigma_m f(I'_m)$. We have, therefore, $f(O_1) \wedge g(O_2) = \Sigma_{j,\ m}(f(I_j) \wedge g(I'_m))$. Thus it suffices to show that $f(I) \wedge g(I') \leq \Sigma_{s \oplus t \geq r} f(s^{+}) \wedge g(t^{+}),\ I \oplus I' \subseteq r^{+}$.

By Proposition 3.1.5, if $I,\ I'$ are intervals with $I \oplus I' \subseteq r^+$, then there are $s,\ t \in A$ such that $I \subseteq s^+,\ I' \subseteq t^+$ and $s \oplus t \geq r$. But then $f(I) \wedge g(I') \leq (f(s^+)) \wedge g(t^+) \leq \Sigma_{s\oplus t\geq r}(f(s^+) \wedge g(t^+))$ and the proof is complete.

3.2. Some ideal theory in $\mathbf{A_K}$

Again we assume A to be a topological MV-chain which is locally compact in its interval topology, $\mathbf{F} = \Omega(A)$ the frame of open subsets of A and $\mathbf{K}$ a frame.

Let $\mathbf{S}$ be a subset of $\mathbf{A_K}$. Let $\mathbf{F}_0 = \{O \in \mathbf{F} \mid 0 \in O,\ 1 \notin O\}$. Set $X_{\mathbf{S}} = \{f(O) \mid f \in \mathbf{S},\ O \in \mathbf{F}_0\} \subseteq \mathbf{K}$ and $\Phi(\mathbf{S})$ be the filter of $\mathbf{K}$ generated by $X_{\mathbf{S}}$. So we can map a subset $\mathbf{S}$ of $\mathbf{A_K}$ on a filter $\Phi(\mathbf{S})$ of $\mathbf{K}$. Moreover, let T be a subset of $\mathbf{K}$, we set $\mathbf{I}(T) = \{f \in \mathbf{A_K} \mid f(O) \in T \text{ for all } O \in \mathbf{F}_0\}$.

Proposition 3.2.1. Let A be a topological MV-chain which is locally compact in its interval topology, $\mathbf{K}$ a frame, $\mathbf{J}$ a subset of $\mathbf{A_K}$ and F a subset of $\mathbf{K}$. Then, with the above notations, the following statements hold:

(i) If F is a filter of $\mathbf{K}$ then $\mathbf{I}(F)$ is an ideal of $\mathbf{A_K}$;

(ii) If F is a proper filter of $\mathbf{K}$ then $\mathbf{I}(F)$ is a proper ideal of $\mathbf{A_K}$.

(iii) If F is a filter of $\mathbf{K}$ and $\mathbf{I}(F)$ is proper ideal of $\mathbf{A_K}$, then F is proper.

(iv) If $\mathbf{J}$ is an ideal of $\mathbf{A_K}$ then $X_{\mathbf{J}}$ is a filter base of $\mathbf{K}$;

(v) If $\mathbf{J}$ is an ideal of $\mathbf{A_K}$, then $\Phi(\mathbf{J})$ is proper iff $\mathbf{J}$ is proper.

(vi) If $\mathbf{J}$ is an ideal of $\mathbf{A_K}$, then $\mathbf{J} \subseteq \mathbf{I}(\Phi(\mathbf{J}))$.

(vii) If $\mathbf{M}$ is a maximal ideal of $\mathbf{A_K}$, then $\mathbf{M} = \mathbf{I}(\Phi(\mathbf{M}))$.

(viii) If F is a filter of $\mathbf{K}$, then $\Phi(\mathbf{I}(F)) \subseteq F$.

(ix) If F a minimal filter of $\mathbf{K}$ then $\Phi(\mathbf{I}(F)) = F$.

(x) Suppose that $F_1 \subseteq F_2$, where each F_i is a filter of $\mathbf{K}$. Then $\mathbf{I}(F_1) \subseteq \mathbf{I}(F_2)$.

(xi) If $\mathbf{M}$ is a maximal ideal of $\mathbf{A_K}$, then there is a maximal filter $F \subseteq \mathbf{K}$ such that $\mathbf{M} = \mathbf{I}(F)$.

Proof.

(i):Let F be a filter in $\mathbf{K}$. Then $\mathbf{I}(F) \neq \emptyset$ since $\mathbf{c}_0 \in \mathbf{I}(F)$. Let $f,\ g \in \mathbf{I}(F)$, $O \in \mathbf{F}_0$. Then $(f \oplus g)(O) = \Sigma_{O_1\oplus O_2\subseteq O}(f(O_1) \wedge g(O_2))$. If A is atomless it is easy to find $O_1,\ O_2 \in \mathbf{F}_0$ with $O_1 \oplus O_2 \subseteq O$. Then $f(O_1) \wedge g(O_2) \in F$. But $f(O_1) \wedge g(O_2) \leq (f \oplus g)(O)$. Hence $f \oplus g \in \mathbf{I}(F)$. On the other hand, if A is atomic let $O_1 = O_2 = \{0\}$. Again $O_1,\ O_2 \in \mathbf{F}_0,\ O_1 \oplus O_2 \subseteq O$ and as above we can infer $f \oplus g \in \mathbf{I}(F)$.

Suppose now that $g \leq f \in \mathbf{I}(F)$. Let $O \in \mathbf{F}_0$. For some $r > 0$, let $r^- \subseteq O$. Thus, $f(r^-) = (f \dot{\vee} g)(r^-) = f(r^-) \wedge g(r^-) \leq g(r^-) \leq g(O)$. As $r^- \in \mathbf{F}_0$ and $f(r^-) \in F$ we see that $g(O) \in F$ and so $g \in \mathbf{I}(F)$. We have then that $\mathbf{I}(F)$ is an ideal of $\mathbf{A_K}$.

(ii): Assume is a proper filter of $\mathbf{K}$. By (i), $\mathbf{I}(F)$ is ideal of $\mathbf{A_K}$. Since, by hypothesis $O \in \mathbf{F}_0$, then $\mathbf{c}_1 \notin \mathbf{I}(F)$.

(iii): Assume that F is a filter of $\mathbf{K}$ and that $\mathbf{I}(F)$ is proper ideal of $\mathbf{A_K}$. If $0 \in F$, then for all $O \in \mathbf{F}_0$ we have $\mathbf{c}_1(O) = 0 \in F$. Hence $\mathbf{c}_1 \in \mathbf{I}(F)$ which is absurd.

(iv): Let $\mathbf{J}$ be ideal of $\mathbf{A_K}$. Suppose that $f(O_1),\ g(O_2) \in X_{\mathbf{J}},\ f,\ g \in \mathbf{J},\ O_1,\ O_2 \in \mathbf{F}_0$. Now $0 \in O_i$ so for some $r_i > 0,\ (r_i)^- \subseteq O_i$. Let $r = r_1 \wedge r_2$. Then $r^- \subseteq O_1 \cap O_2$. As $(f \dot{\vee} g) \in \mathbf{J}$ and $r^- \in \mathbf{F}_0$ we have $(f \dot{\vee} g)(r^-) \in X_{\mathbf{J}}$. That is, $(f \dot{\vee} g)(r^-) = f(r^-) \wedge g(r^-) \leq f(O_1) \wedge g(O_2)$. Therefore $X_{\mathbf{J}}$ is a filter base.

(v): Assuming $\Phi(\mathbf{J})$ proper means that $0 \notin X_{\mathbf{J}}$. Now $1 = \mathbf{c}_1 \in \mathbf{J}$ implies $\mathbf{c}_1(1^-) = 0 \in X_{\mathbf{J}}$. Thus $\Phi(\mathbf{J})$ proper implies $\mathbf{J}$ proper. Conversely, suppose that $\mathbf{J}$ is proper ideal. Then for $f \in \mathbf{J}$ we have $\mathrm{ord} f = \infty$ and so $f(r^-) > 0$ for all $r > 0$. Consequently, if $O \in \mathbf{F}_0$ we have $f(O) > 0$ and therefore $0 \notin \Phi(\mathbf{J})$.

(vi) Let $\mathbf{J}$ be an ideal of $\mathbf{A_K}$. Let $f \in \mathbf{J}$; then for all $O \in \mathbf{F}_0$ we have $f(O) \in \Phi(\mathbf{J})$. Consequently, $f \in \mathbf{I}(\Phi(\mathbf{J}))$.

(vii):If $\mathbf{M}$ is a maximal ideal of $\mathbf{A_K}$, then $\mathbf{M} = \mathbf{I}(\Phi(\mathbf{M}))$, from the above statement.

(viii):Let $x \in \Phi(\mathbf{I}(F))$ Then for some $f \in \mathbf{I}(F),\ O \in \mathbf{F}_0$ we have $f(O) \leq x$. But $f \in \mathbf{I}(F)$. So $O \in \mathbf{F}_0$ means that $f(O) \in F$. Hence $x \in F$.

(ix):For a minimal filter F of $\mathbf{K}$ we have, by (viii) that $\Phi(\mathbf{I}(F)) = F$

(x): Let $f \in \mathbf{I}(F_1)$. Then for all $O \in \mathbf{F}_0,\ f(O) \in F_1$. Since $F_1 \subseteq F_2$ we see that $f \in \mathbf{I}(F_2)$.

(xi): We have a proper filter $\Phi(\mathbf{M}) \subseteq \mathbf{K}$. We know that $\mathbf{M} = \mathbf{I}(\Phi(\mathbf{M}))$. Let F be a maximal extension of $\Phi(\mathbf{M})$. By (x) we may infer $\mathbf{M} = \mathbf{I}(F)$.

Proposition 3.2.2. Let A be an MV-chain which is locally compact in its interval topology, $\mathbf{K}$ a frame and J a prime filter of $\mathbf{K}$. Then $\mathbf{I}(J)$ is a prime ideal of $\mathbf{A_K}$.

Proof.

Let J be a prime filter of $\mathbf{K}$, $f \wedge g \in \mathbf{I}(J)$ and assume that $f \notin \mathbf{I}(J)$. Then for some $O' \in \mathbf{F}_0$, we have $f(O') \notin J$. Since $0 \in O'$ we have an $r_0 > 0$ such that $(r_0)^- \subseteq O'$. Thus $f(r_0) \notin J$. Let $O \in \mathbf{F}_0$. Again, there is an $r > 0$ with $r^- \subseteq O$. Since $r_0^- \in \mathbf{F}_0$, we know that $(f \dot{\wedge} g)((r_0)^-) = f((r_0)^-) \wedge g((r_0)^-) \in J$. Hence, as J is prime, $g(r_0^-) \in J$. If $r_0 \leq r$, then $g((r_0)^-) \leq g(r^-) \leq g(O)$, and so $g(O) \in J$. If $r < r_0$, then $f(r^-) \notin J$, therefore $g(r^-) \in J$ and again it follows that $g(O) \in J$. We may infer, then, that $g \in \mathbf{I}(J)$.

Proposition 3.2.3. Let A be an MV-chain which is topological and locally compact in its interval topology and $\mathbf{K}$ a linearly ordered frame. Then, $\mathbf{A_K}$ is an MV-chain.

Proof.

If $\mathbf{K}$ is linearly ordered, then $\{1\}$ is a prime filter of $\mathbf{K}$. Hence by (5) of Proposition 3.2.2, $\mathbf{I}(\{1\})$ is a prime ideal of $\mathbf{A_K}$. Let $f \in \mathbf{I}_(\{1\})$. Then for every $O \in \mathbf{F}_0$ we have $f(O) \in \{1\}$; that is $f(O) = 1$. Therefore for all $r > 0$, $f(r^-) = 1 = \mathbf{c}_0(r^-)$. Also, for $r > 0$, $r^- \cap r^+ = \emptyset$. Therefore $0 = f(r^-) \wedge f(r^+) = f(r^+) = \mathbf{c}_0(r^+)$. For $r = 0$, $r^+ = A$ so $1 = f(r^+) = \mathbf{c}_0(r^+)$. Hence by Proposition 3.3(iv), $f = \mathbf{c}_0$. Thus $\{0\}$ is a prime ideal of $\mathbf{A_K}$ and so $\mathbf{A_K}$ is linearly ordered.

3.3 Spatial Frames and MV-chains

Throughout this section we assume A to be an MV-chain which is topological and locally compact in its interval topology and $\mathbf{K}$ to be a frame.

If L is a prime ideal of $\mathbf{K}$, then $F = \mathbf{K} - L$ is a prime filter. Thus if $x \in \mathrm{pt}(\mathbf{K})$ we have an associated prime filter $F_x = \mathbf{K} - \mathrm{id}(x)$, and consequently a prime ideal $\mathbf{I}(F_x)$. Thus we have a map $\theta : \mathrm{pt}(\mathbf{K}) \to \mathrm{Spec}(\mathbf{A_K})$ given by $x \to \mathbf{I}(F_x)$.

Proposition 3.3.1. Let A be an MV-chain which is topological and locally compact in its interval topology. Let $\mathbf{K}$ be a frame. Then, with the above notations we have that the following hold:

(1) The map θ is continuous.

(2) If $\mathbf{K}$ is spatial, then $\theta(\mathrm{pt}(\mathbf{K}))$ is a dense subset of $\mathrm{Spec}(\mathbf{A_K})$

(3) If $\mathbf{K}$ is spatial we have an embedding $\mathcal{C}(\theta(\mathrm{pt}(\mathbf{K}), A) \to \mathcal{C}(\mathrm{pt}(\mathbf{K}),\ A)$.

Proof.

(1): Firstly we have the following:

Claim: For $x \in \mathrm{pt}(\mathbf{K})$, $f \notin \theta(x)$ iff $f(0^+) \notin \mathrm{id}(x)$.

Indeed, assuming that $x \in \mathrm{pt}(\mathbf{K})$ we can make the following observation: $f \notin \theta(x)$ iff $f \notin \mathbf{I}(F_x)$ iff $f(O) \notin F_x$ for some $O \in \mathbf{F}_0$ iff $f(O) \in \mathrm{id}(x)$ for some $O \in \mathbf{F}_0$.

If $O \in \mathbf{F}_0$ we know that for some $r > 0$, $r^- \subseteq O$. Thus if $f(O) \in \mathrm{id}(x)$ so is $f(r^-)$. But then $f(0^+) \notin \mathrm{id}(x)$. For $A = r^- \cup 0^+$ so $1 = f(r^-) \vee f(0^+)$.

Viceversa, suppose that $f(0^+) \notin \mathrm{id}(x)$. If A is atomless we have that $0^+ = \bigcup_{0<r} r^+$ and $f(0^+) = \Sigma_{0<r} f(r^+)$. As $\mathrm{id}(x)$ is complete, for some $r > 0$, $f(r^+) \notin \mathrm{id}(x)$. Now $\emptyset = r^- \cap r^+$ so $0 = f(r^-) \wedge f(r^+) \in \mathrm{id}(x)$. Since $\mathrm{id}(x)$ is prime we have that $f(r^-) \in \mathrm{id}(x)$; as $r- \in \mathbf{F}_0$ we see that $f \notin \theta(x)$.

If A is atomic, with atom a, we have $0^+ = [a,\ 1]$. As above we have

$f(a^-) \in \mathrm{id}(x),\ a^- = \{0\}$. But a^- in this case is open and therefore in $\mathbf{F}_0$.

In both cases then, $f \notin \theta(x)$ iff $f(0+) \notin \mathrm{id}(x)$. So the Claim is proved.

Let $U_f = \{P \in \mathrm{Spec}(\mathbf{A_K}) \mid f \notin P\}$ be a basic open set in $\mathrm{Spec}(\mathbf{A_K})$. Then, by the Claim we have: $\theta^{-1}(U_f) = \{x \in \mathrm{pt}(\mathbf{K}) \mid f \notin \theta(x)\} = \{x \in \mathrm{pt}(\mathbf{K}) \mid f(0^+) \notin \mathrm{id}(x)\}$. But the latter set is an open set in $\mathrm{pt}(\mathbf{K})$.

(2): Suppose $\mathbf{K}$ is spatial. Since $\mathbf{K}$ is a complete lattice we have an $x_0 = \bigcap\{x \mid x \in \mathrm{pt}(\mathbf{K})\}$. As $x \leq y$ implies $\mathrm{id}(x) \subseteq \mathrm{id}(y)$ it follows that $x_0 \in \mathrm{id}(x)$ for all $x \in \mathrm{pt}(\mathbf{K})$. Therefore $\phi(x_0) = \emptyset = \phi(0)$. Since ϕ is an isomorphism we infer that $x_0 = 0$.

Let $f \in \theta(x)$; then $f(O) \in \mathbf{K} - \mathrm{id}(x)$ for all $O \in \mathbf{F}_0$, in particular for $O = r^-,\ r > 0$. Thus for all $r > 0$ we have $f(r^-) \notin \mathrm{id}(x)$. Since $r^- \cap r^+ = \emptyset$ and $\mathrm{id}(x)$ is prime, we see that $f(r^+) \in \mathrm{id}(x)$ for all $r > 0$.

Suppose now that $h \in \bigcap\{\theta(x) \mid x \in \mathrm{pt}(\mathbf{K})\}$. Then for all $x \in \mathrm{pt}(\mathbf{K})$ we have $h(r^+) \in \mathrm{id}(x)$ for all $r > 0$. Thus for all $x \in \mathrm{pt}(\mathbf{K})$, $h(r^+) \leq x$ for all $r > 0$. Hence, $h(r^+) = 0$ for all $r > 0$. Now let $O \in \mathbf{F}$. Suppose that $0 \in O$; then for some $s > 0,\ s^- \subseteq O$. If there is an $r > 0,\ r < s$ we have $1 = h(r^+) \vee h(s^-) = h(s^-) \leq h(O)$. If s is an atom we have $A = \{0\} \cup s^+$ and again $1 = h(s^+) \vee h(\{0\}) = h(\{0\}) \leq h(O)$. If $0 \notin O$, then $O \subseteq 0^+$ if A is atomless, and $0+ = \bigcup_{0<r} r^+$; thus $h(0^+) = \Sigma_{0<r} h(r^+) = 0$. If A is atomic with atom a we have $O \subseteq a^+$ so $h(O) \leq h(a^+) = 0$. It follows that $h = \mathbf{c}_0$, that is $\bigcap\{\theta(x) \mid x \in \mathrm{pt}(\mathbf{K})\} = \{\mathbf{c}_0\}$.

(3): By (1), above, we have a continuous map $\theta : \mathrm{pt}(\mathbf{K}) \to \mathrm{Spec}(\mathbf{A_K})$. We know if $\mathbf{K}$ is spatial, then $\theta(\mathrm{pt}(\mathbf{K}))$ is a dense subset of $\mathrm{Spec}(\mathbf{A_K})$. Let $f : \theta(\mathrm{pt}(\mathbf{K})) \to A$ be a continuous map, $\theta(\mathrm{pt}(\mathbf{K}))$ with the subspace topology. Then $f \circ \theta : \mathrm{pt}(\mathbf{K}) \to A$ is continuous. Thus we obtain a map $f \to f \circ \theta$ from the algebra of continuous functions from $\theta(\mathrm{pt}(\mathbf{K}))$ to A to the algebra of continuous functions from $\mathrm{pt}(\mathbf{K})$ to A. That is from $\mathcal{C}(\theta(\mathrm{pt}(\mathbf{K})),\ A) \to \mathcal{C}(\mathrm{pt}(\mathbf{K}),\ A)$.

Suppose we have $f,\ g \in \mathcal{C}(\theta(\mathrm{pt}(\mathbf{K})),\ A)$ and $f \circ \theta = g \circ \theta$. Then for all $x \in \mathrm{pt}(\mathbf{K})$ we have $f \circ \theta(x) = g \circ \theta(x)$, hence $f(\theta(x)) = g(\theta(x))$. So $(g^* \circ f)(\theta(x)) = 0 = (f^* \circ g)(\theta(x))$ for all $x \in \mathrm{pt}(\mathbf{K})$. Hence for all such x, $\mathrm{d}(f,\ g) \in \theta(x)$ where $\mathrm{d}(f,\ g) = (f^* \circ g) \oplus (g^* \circ f)$. Therefore, by (2), if $\mathbf{K}$ is spatial, we may infer that $f = g$. So the mapping $f \to f \circ \theta$ is injective.

Now given $f \in \mathcal{C}(\theta(\mathrm{pt}(\mathbf{K})),\ A)$ we have $f^* \to f^* \circ \theta$. Let $x \in \mathrm{pt}(\mathbf{K})$; then $f^* \circ \theta(x) = (f(\theta(x))^* = (f \circ \theta(x))^* = (f \circ \theta)^*(x)$. If $g \in \mathcal{C}(\theta(\mathrm{pt}(\mathbf{K})),\ A),\ x \in \mathrm{pt}(\mathbf{K})$,then $((f \oplus g) \circ \theta)(x) = (f \circ \theta)(x) \oplus (g \circ \theta)(x) = (f \circ \theta \oplus g \circ \theta)(x)$.

3.3.1. Semisimplicity of $\mathbf{A_K}$

Proposition 3.3.1.1 Let A be a locally finite and divisible MV-algebra and $\mathbf{K}$ a frame. If A is topological and locally compact in its interval topology, then $\mathbf{A_K}$ is semisimple.

Proof.

First we observe that for any $f \in \mathbf{A_K},\ n \geq 1,\ r \in A,\ r < 1$ we have $f(\frac{r}{n}^+) \leq (nf)(r^+)$. Indeed, we know that $n(\frac{r}{n}) = r$, thus if $O_i = (\frac{r}{n})^+$ for $i = 1,\ 2,\ \ldots,\ n$, then $O_1 \oplus O_2 \oplus \cdots \oplus O_n \subseteq r^+$. Hence $f(\frac{r}{n}^+) = f(O_1 \cap \cdots \cap O_n) \leq (nf)(r^+)$ by Proposition 3.1.9. Now let F be a prime filter in $\mathbf{K}$. By Proposition 3.2.2, we know that $M = I(F)$ is a prime ideal of $\mathbf{A_K}$. Suppose that $f \in \mathbf{A_K},\ f \notin M$. Then for some $O' \in \mathbf{F}_0$ we have $f(O') \notin F$. Since $0 \in O'$ there is an $r_0 \in A,\ r_0 > 0,\ r_0^- \subseteq O$. As $f(r_0^-) \leq f(O')$ we also have $f(r_0^-) \notin F$. Let $n > \mathrm{ord}(r_0)$. Let $r \in A,\ r > 0$. Then $r^* < 1$. Therefore $\frac{r^*}{n} < r_0$ since $n(\frac{r^*}{n}) = r < 1$. Moreover, $f(\frac{r^*}{n}^+) \leq (nf)((r^*)^+)$. Now $(r^*)^+ = \nu(r^-)$ so we obtain $f(\frac{r^*}{n}^+) \leq (nf)(\nu(r^-)) = (nf)^*(r^-) = (f^*)^n(r^-)$. Also, $A = \frac{r^*}{n}^+ \cup (r_0)^-$ and so $1 = f(\frac{r^*}{n}^+) \vee f(r_0)^-) \in F$. As F is prime and $f(r_0) \notin F$ we see that $f((\frac{r^*}{n})^-) \in F$. But then, $(f^*)^n(r^-) \in F$. Now if $O \in \mathbf{F}_0$ there is an $r \in A,\ r > 0$ with $r^- \subseteq O$. Therefore since $(f^*)^n(r^-) \in F$ we also have $f(O) \in F$. It follows that $(f^*)^n \in M$ and so M is a maximal ideal.

Now let $\mathbf{R} = \mathrm{Rad}\mathbf{A}$ and again let F be a prime filter of $\mathbf{K}$. Then $\mathbf{R} \subseteq M = I(F)$. Consider the filter $\Phi(\mathbf{R}) \subseteq \mathbf{K}$. Clearly $\Phi(\mathbf{R}) \subseteq \Phi(M)$. By Proposition 3.2.1 (viii), $\Phi(\mathbf{R}) \subseteq F$. Therefore $\Phi(\mathbf{R}) \subseteq \bigcap\{F \mid F \text{ a prime filter in } \mathbf{K}\}$. Hence, $\Phi(\mathbf{R}) = \{1\}$.

Now let $f \in \mathbf{R}$. then for all $r \in A,\ r > 0$ we have $f(r^-) = 1 = \mathbf{c}_0(r^-)$. Now let $s \in A,\ s > 0$; choose an $r,\ 0 < r < s$. Then $\emptyset = r^- \cap s^+$. Thus $0 = f(r^-) \wedge f(s^+) = f(s^+)$. Therefore $f(s^+) = \mathbf{c}_0(s^+),\ s > 0$. Now $0^+ = \bigcup_{s>0} s^+$ and so $f(0^+) = \Sigma_{s>0} f(s^+) = 0 = \mathbf{c}_0(0^+)$. By Proposition 3.3 (iv) we infer that $f = \mathbf{c}_0$. Hence $\mathbf{R} = \{\mathbf{c}_0\} = 0$.

Now, the condition of A being locally finite, divisible together with locally compact determines A completely. In fact A must be $[0,\ 1]$ as we now show.

Proposition 3.3.1.2. Let A be an MV-algebra such that

1) A is locally finite

2) A is divisible

3) A is locally compact in the interval topology

Then, $A = [0,\ 1]$.

We shall prove this in a series of results. As A is locally finite we can consider A as a subalgebra of $[0,\ 1]$.

First we show,

Lemma 3.3.1.1. A is dense in $[0,\ 1]$.

Proof.

Suppose there is a $(u,\ v) \subseteq [0,\ 1]$ such that $(u,\ v) \cap A = \emptyset,\ u < v$. If for all $x \in A,\ x \neq 0$ we have $u \leq x$, then there is an m such that $mx = 1$ for all $0 \neq x \in A$, just choose $m =$ord$u$. But since A is divisible there is a $y \in A$ such that $(m + 1)y = x$; this forces $A = \{0,\ 1\}$ which is not divisible. Thus we can find an $x \in A,\ 0 < x < u$. There is an n such that $(n - 1)x \leq u < v \leq nx$. Therefore $nx \cdot (x^*)^{n-1} \geq v \cdot u^* > 0$. Now $nx \cdot (x^*)^{n-1} = x \wedge (x^*)^{n-1} \geq v \cdot u^*$. Hence $v \cdot u^* \leq x$. It follows then that for all $x \in A,\ x \neq 0$ that $0 \neq v \cdot u^* \leq x$. As above there is an integer m such that $mx = 1$ for all $x \in A,\ x \neq 0$ and this violates divisibility.

Lemma 3.3.1.2. Suppose that A contains a closed interval $[u,\ v]$ of the algebra $[0,\ 1]$. Then $A = [0,\ 1]$.

Proof.

First note that $[0,\ v \cdot u^*] \subseteq A$. for let $x \in [0,\ v \cdot u^*]$, then $u < u \oplus x < v$ so $u \oplus x \in A$. But then $x = x \wedge u^* = u^* \cdot (u \oplus x) \in A$.

Secondly, assume for some $a \in A,\ a \neq 0$, that $[0,\ a] \subseteq A$. Let n be an integer and consider $x \in [0,\ na]$, and assume that $[0,\ (n-1)a] \subseteq A$. Then $x \cdot (a^*)^{n-1} \leq na \cdot (a^*)^{n-1} = a \wedge (a^*)^{n-1} \leq a$, So $x \cdot (a^*)^{n-1} \in [0,\ a] \subseteq A$. Thus, $(n-1)a \oplus x \cdot (a^*)^{n-1} = x \vee (n-1)a \in A$. If $x \geq (n-1)a \in A$, then $x = x \vee (n-1)a \in A$; if $x \leq (n-1)a$, then $x \in [0,\ (n-1)a] \subseteq A$ by the induction assumption. Hence $[0,\ na] \subseteq A$.

Applying this result to $[0,\ v \cdot u^*]$ we see that for any n, $[0,\ n(v \cdot u^*)] \subseteq A$. As $v \cdot u^*$ has finite order we infer that $[0,\ 1] \subseteq A$.

Now let $a \in A$; we know there is a compact neighborhood I_A of A with $a \in I_A$. Thus there are $u,\ v \in A$ with $a \in (u,\ v)_A \subseteq I_A$. Suppose that $A \neq [0.\ 1]$. Then $[u,\ v] \not\subseteq A$; thus there is a $w \in (u,\ v) - A$. By the density of A in $[0,\ 1]$ we can find an increasing sequence of elements $a_n \in A$ such that $u < u_n$ and $\sup\{u_n|n\} = w$. Similarly, there is a decreasing sequence v_n of elements of A such that $v_n < v$ and $\inf\{v_n|n\} = w$. Now $[0,\ u_1)_A \cup \bigcup_n (u,\ u_n)_A \bigcup_n (v_n,\ v)_A \cup (v_1.\ 1]_A$ covers I_A. But no finite subcover can contain all of the a_n or b_n.

It now follows that $A = [0,\ 1]$.

Corollary 3.3.1.1 Under the same hypotheses as in Proposition 3.3.1.1, if $\mathbf{K}$ is linearly ordered, then $\mathbf{A_K} = [0,\ 1]$.

Proof.

We know from Proposition 3.2.3 that $\mathbf{A_K}$ is linearly ordered. Since it's semisimple we must have $\mathbf{A} \subseteq [0,\ 1]$. But $[0,\ 1] \subseteq \mathbf{A_K}$ so $\mathbf{A_K} = A$.

4. The atomic chain case

Proposition 4.1. Let A be a topological atomic MV-chain endowed with its interval topology and $\mathbf{K}$ a connected frame. Then $\mathbf{A_K}$ is isomorphic to $\mathbf{A}$.

Proof.

Since A is an atomic chain, then $\Omega(A)$ is just the set of all subsets of A. For any $a \in A$ we have $\emptyset = \{a\} \cap \mathrm{co}(a)$, $A = \{a\} \cup \mathrm{co}(a)$ where $\mathrm{co}(a) = A - \{a\}$. Hence for any $f \in \mathbf{A_K}$, we have $0 = f(\{a\}) \wedge f(\mathrm{co}(a))$, $1 = f(\{a\}) \vee f(\mathrm{co}(a))$.

We see then that $f(\{a\})$ is a complemented element of $\mathbf{K}$ with complement $f(\mathrm{co}(a))$. Suppose now that for some $a,\ b \in A$ that $f(\{a\}) = f(\{b\}) = 1$. Then $0 = f(\{a\}) \wedge f(\mathrm{co}(a)) = f(\mathrm{co}(a))$. If $b \neq a$, then $b \in co(a)$ and so $f(\{b\}) = 0$. Thus, $f(\{a\}) = f(\{b\}) = 1$ implies $a = b$. Now, let $a \in A$, then, from above, we see that $f(\{a\})$ is complemented with complement $f(\mathrm{co}(a))$. As $\mathbf{K}$ is connected we have $f(\{a\}) = 0$ or $f(\{a\}) = 1$.

Suppose then that $f(\{a\}) = 1$, Then $f(\mathrm{co}(a)) = 0$. In this case we see that $f = \mathbf{c}_a$. Suppose then that $f(\{a\}) = 0$. Then $f(\mathrm{co}(a)) = 1$. Now $\mathrm{co}(a) = \bigcup\{\{r\} \mid r \in \mathrm{co}(a)\}$ and therefore $f(\mathrm{co}(a)) = \Sigma\{f(\{r\}) \mid r \in \mathrm{co}(a)\} = 1$. Since each $f(\{r\}) \in \{0,\ 1\}$ we must have a $w \in \mathrm{co}(a)$ with $f(\{w\}) = 1$. From the discussion above, w is unique, and $f = \mathbf{c}_w$. Thus we have that $\mathbf{A_K} = \{\mathbf{c}_a \mid a \in A\} \cong A$.

Proposition 4.2. Let n be a positive integer, A the finite topological MV-chain with $n+1$ elements, $\mathbf{K}$ a frame and F a prime filter of $\mathbf{K}$. Then the following statements hold:

(1) $f \in \mathbf{I}(F)$ iff $f(\{0\}) \in F$.

(2) $\mathbf{I}(F)$ is a maximal ideal of $\mathbf{A_K}$.

Proof.

We have, for $a \in A$, that $\emptyset = \{a\} \cap \mathrm{co}(a)$, $A = \{a\} \cup \mathrm{co}(a)$ and so, for $f \in \mathbf{A_K}$ we get $0 = f(\{a\}) \wedge f(\mathrm{co}(a))$, $1 = f(\{a\}) \vee f(\mathrm{co}(a))$. That is, $f(\{a\})$ is complemented with complement $f(\mathrm{co}(a))$.

Now if $O \in \Omega(A)$, then $O = \{r_1, \ldots, r_k\}$ and consequently $f(O) = f(\{r_1\} \cup \cdots \cup \{r_k\} = \Sigma_i f(\{r_i\})$. As each $f(\{r_i\})$ is complemented so is their sum. That is $f(O)$ is complemented.

As $\Omega(A)$ is finite, we see that for each $f \in \mathbf{A_K}$ the range of f lies in the boolean subalgebra of $\mathbf{K}$.

So, assume that $F \subseteq \mathbf{K}$ is a prime filter and that $f \in \mathbf{A_K}$. If $f \notin \mathbf{I}(F)$, then for some $o \in \mathbf{F}_0$ we have that $f(O) \notin F$. As $0 \in O$ and $\{0\} \subseteq O$ we see that $f(\{0\}) \notin F$.

Conversely, suppose that $f(\{0\}) \notin F$. Since $\{0\} \in \mathbf{F}_0$,we see that $f \notin \mathbf{I}(F)$. Therefore, we get (1).

Suppose then that $f \notin \mathbf{I}(F)$ and that for all $k > 0,\ (f^*)^k \notin \mathbf{I}(F)$. Then for all $k > 0,\ (f^*)^k(\{0\}) \notin \mathbf{I}(F)$.

Now $(f^*)^k(\{0\}) = (kf)^*(\{0\}) = (kf)(\nu(\{0\}))$. But $\nu(\{0\}) = \{x^* \mid x = 0\} = \{1\}$. Consequently we have that $(kf)(\{1\}) \notin F$. Hence $(kf)(1^-) \in F$ for all $k > 0$.

Using Proposition 3.6, we know that $(kf)(1^-) = \Sigma_{O_1 \oplus \cdots \oplus O_k \subseteq 1^-} f(O_1 \cap \cdots \cap O_k)$. Now let $k > n$. If $O_1 \oplus \cdots \oplus O_k \subseteq 1^-$, then some $O_i \subseteq \{0\}$, that is some $O_i = \{0\}$. Hence $O_1 \cap \cdots \cap O_k \subseteq \{0\}$ and therefore $f(O_1 \cap \cdots \cap O_k) \leq f(\{0\})$. It follows that $(kf)(1^-) \leq f(\{0\})$. But since $(kf)(1^-) \in F$ we see that $f(\{0\}) \in F$. This contradiction shows that for some $k > 0,\ (f^*)^k \in \mathbf{I}(F)$. So we have proved (2).

Proposition 4.3. Let n be a positive integer, A the finite topological MV-chain with $n+1$ elements and $\mathbf{K}$ a frame. Then $\mathbf{A_K}$ is semisimple.
Proof.

Let $\mathbf{R} = \mathrm{Rad}\mathbf{A_K}$ and let F be a prime filter of $\mathbf{K}$. Then $\mathbf{R} \subseteq M = \mathbf{I}(F)$. Consider the filter $\Phi(\mathbf{R}) \subseteq \mathbf{K}$, $\Phi(\mathbf{R})$ the filter generated by $\{f(O) \mid O \in \mathbf{F}_0,\ f \in \mathbf{R}\}$. Clearly $\Phi(\mathbf{R}) \subseteq \Phi(\mathbf{M})$. By Proposition 3.2.1 (viii), $\Phi(\mathbf{R}) \subseteq F$. Therefore $\Phi(\mathbf{R}) \subseteq \bigcap\{F \mid F \text{ a prime filter in } \mathbf{K}\}$. Hence, $\Phi(\mathbf{R}) = \{1\}$.

Now let $f \in \mathbf{R}$. then for all $r \in A_n,\ r > 0$ we have $f(r^-) = 1 = \mathbf{c}_0(r^-)$.

Now let $r \in A,\ r > 0$. Then $\emptyset = r^- \ \cap r^+$. Thus $0 = f(r^-) \wedge f(r^+) = f(r^+)$. Therefore $f(r^+) = \mathbf{c}_0(r^+),\ r > 0$. Now $0+ = \bigcup_{r>0} r+$ and so $f(0+) = \Sigma_{r>0} f(r+) = 0 = \mathbf{c}_0(0^+)$. By Proposition 3.3 (iv) we infer that $f = \mathbf{c}_0$. Hence $\mathbf{R} = \{\mathbf{c}_0\} = 0$.

Proposition 4.4. Let n be a positive integer, A the finite topological MV-chain with $n+1$ elements and $\mathbf{K}$ a frame. Then $f \in \mathbf{A_K}$ is an idempotent iff $f(\{0\})$ and $f(\{1\})$ are complements in $\mathbf{K}$.
Proof.

Let $f \in \mathbf{A_K}$, $O \in \Omega(A)$. We know that $(f \oplus f)(O) = \Sigma_{O_1 \oplus O_2 \subseteq O} f(O_1 \cap O_2)$. In particular, $(f \oplus f)(\{a\}) = \Sigma_{O_1 \oplus O_2 \subseteq \{a\}} f(O_1 \cap O_2)$.

Now $O_1 \oplus O_2 \subseteq \{a\}$ iff $O_1 = \{0\},\ O_2 = \{a\}$ or vice-versa. In either case, $O_1 \cap O_2 = \emptyset$ and we may infer that $(f \oplus f)(\{a\}) = 0$.

Suppose then that $f \oplus f = f$. From the above we see that $f(\{a\}) = 0$. Assume now that $0 < ka \neq 1$ and that $f(\{la\}) = 0$ for $l < k$. Then,

$(f \oplus f)(\{ka\}) = \Sigma_{O_1 \oplus O_2 \subseteq \{ka\}} f(O_1 \cap O_2)$. Now $O_1 \oplus O_2 \subseteq \{ka\}$ iff $O_i = \{0\}$, $O_j = \{ka\}$ or $O_i = \{ma\}$, $O_j = \{la\}$, $m + l = k$, where $i,\ j \in \{1,\ 2\}$, $i \neq j$.

Except in the case that $m = l$ we have $O_1 \cap O_2 = \emptyset$. Hence, $(f \oplus f)(\{ka\}) = f(ma)$, and as $m < k$, $f(ma) = 0$. Therefore for $0 < ka < 1$ we have $f(\{ka\}) = 0$. It now follows that $f(\{0\}) \vee f(\{1\}) = 1$. Since $f(\{0\}) \wedge f(\{1\}) = 0$ we may conclude, that if $f \in \mathbf{A_K}$ is an idempotent, then $f(\{0\})$, $f(\{1\})$ are complements in $\mathbf{K}$.

Suppose now that $f(\{0\})$, $f(\{1\})$ are complements, so that $f(\{0\}) \vee f(\{1\}) = 1$. Let $0 < ka < 1$; then we have $f(\{ka\}) = f(\{ka\}) \wedge f(\{1\}) \vee f(\{ka\}) \wedge f(\{1\}) = f(\{ka\} \cap \{0\}) \vee f(\{ka\} \cap \{1\}) = 0$.

Let $O \in \Omega(A)$. Then $f(O) = f(O \cap \{0\}) \vee f(O \cap \{1\})$. Thus,

1) if $0,\ 1 \in O$ then $f(O) = 1$;

2) if neither of 0, 1 are in O, then $f(O) = 0$;

3) if $0 \in O,\ 1 \notin O$, then $f(O) = f(\{0\})$;

4) if $0 \notin O,\ 1 \in O$, then $f(O) = f\{1\})$.

Consider then, $(f \oplus f)(O) = \Sigma_{O_1 \oplus O_2 \subseteq O} f(O_1 \cap O_2)$.

1') if $0,\ 1 \in O$ then there's a summand for the index $\{0,\ 1\} \oplus \{0,\ 1\}$ and the summand $f(\{0,\ 1\}) = 1$, so $(f \oplus f)(O) = 1$;

2') if neither of 0, 1 are in O, then 0, 1 are not in at least one of the O_i, $i = 1$or 2 so each summand $f(O_1 \cap O_2) = 0$ by 2) above;

3') if $0 \in O,\ 1 \notin O$, then $1 \notin O_1 \cup O_2$ and $0 \in O_1 \cap O_2$ and so $f(O_1 \cap O_2) = f(\{0\})$ or $0 \notin O_1 \cap O_2$ and $f(O_1 \cap O_2) = 0$ by 2) above. In either case $(f \oplus f)(O) = f(\{0\})$;

4') if $0 \notin O,\ 1 \in O$, then $0 \notin O_1 \cap O_2$. If $1 \in O_1 \cap O_2$, then $f(O_1 \cap O_2) = 1$ by 4) above, and if $1 \notin O_1 \cap O_2$ then $f(O_1 \cap O_2) = 0$ by 1) above. In any case the result is $(f \oplus f)(O) = f(\{1\})$.

Comparing 1) to 4) with 1') to 4') we have proved the proposition.

Since $f(\{0\}) \wedge f(\{1\}) = 0$ always, with the above notations, we have,

Corollary 4.1. $f \in \mathbf{A_K}$ is idempotent iff $f(\{0\}) \vee f(\{1\}) = 1$.

For a finite MV-chain A with $n+1$ elements $\mathcal{C}(X, A)$ it's the case that nf is an idempotent for every $f \in \mathcal{C}(X,\ A)$. That is $\mathcal{C}(X,\ A)$ is quasi-boolean of index n. The same is true for $\mathbf{A_K}$.

Proposition 4.5. Let n be a positive integer, A the finite topological MV-chain with $n + 1$ elements and $\mathbf{K}$ a frame. Then for every $f \in \mathbf{A_K}$, nf is an idempotent.

Proof.

Let $0 < ka < 1$. Suppose we have $O_1 \oplus O_2 \cdots \oplus O_n \subseteq \{ka\}$. Then some O_i must equal $\{0\}$. For if not, we would have $0 < x_i \in O_i$ for each i and so $\Sigma_i x_i = 1 \in \{ka\}$ which is absurd. Also, for some j, we must have $0 \notin O_j$ as otherwise $0 \in \{ka\}$. Consequently $O_1 \cap O_2 \cdots \cap O_n = \emptyset$. Therefore $(nf)(\{ka\}) = 0$. But $1 = \Sigma_{k=0}{}^n f(\{ka\}) = f(\{0\}) \vee f(\{1\})$. By the corollary above we see that nf is idempotent.

With the above notations, we have the following:

Corollary 4.2. $\mathbf{A_K}$ is quasi-boolean of finte index n.

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