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THE KLEIN SOLUTION TO PAINLEVÉ'S SIXTH EQUATION

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Abstract. We will describe a method for constructing explicit algebraic solutions to the sixth Painlevé equation. There are basically two steps: First we explain how to construct finite braid group orbits of triples of elements of $\mathrm{SL}_2(\mathbb{C})$ out of triples of generators of three-dimensional complex reflection groups. (This involves the Fourier–Laplace transform for certain irregular connections.) Then we adapt a result of Jimbo to produce the Painlevé VI solutions. (In particular this solves a Riemann–Hilbert problem explicitly.)

Each step will be illustrated using the complex reflection group associated to Klein's simple group of order 168. This leads to a new algebraic solution with seven branches. We will also prove that, unlike the algebraic solutions of Dubrovin–Mazzocco and Hitchin, this solution is not equivalent to any solution coming from a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$.

1. Introduction

Klein's quartic curve

$$X^3Y + Y^3Z + Z^3X = 0 \quad \subset \; \mathbb{P}^2(\mathbb{C})$$

is of genus three and has the maximum possible number $84(g-1) = 168$ of holomorphic automorphisms. Klein found these automorphisms explicitly (in terms of 3×3 matrices). They constitute Klein's simple group $K \subset \mathrm{PGL}_3(\mathbb{C})$ which is isomorphic to $\mathrm{PSL}_2(7)$.

Lifting to $\mathrm{GL}_3(\mathbb{C})$ there is a two-fold covering group $\widehat{K} \subset \mathrm{GL}_3(\mathbb{C})$ of order 336 which is a complex reflection group—there are complex reflections

$$r_1,\; r_2,\; r_3 \;\in \mathrm{GL}_3(\mathbb{C}) \tag{1}$$

which generate $\widehat{K}$. (Recall a pseudo-reflection is an automorphism of the form "one plus rank one", a complex reflection is a pseudo-reflection of finite order and a complex reflection group is a finite group generated by complex reflections. Here, each generator r_i has order two—as for real reflections.)

Using the general tools to be described in this paper we will construct, starting from the Klein complex reflection group $\widehat{K}$, another algebraic curve with affine equation

$$F(y,t) = 0 \tag{2}$$

given by a polynomial F with integer coefficients. This curve will be a seven-fold cover of the t-line branched only at $0, 1, \infty$ and such that the function $y(t)$, defined implicitly by (2), solves the Painlevé VI differential equation.

One upshot of this will be to construct an explicit rank three Fuchsian system of linear differential equations with four singularities (at $0, t, 1, \infty$, for some t) on $\mathbb{P}^1$, and with monodromy group equal to $\widehat{K}$ in its natural representation (so the monodromy around each of the finite singularities $0, t, 1$ is a generating reflection).

In general the construction of linear differential equations with finite monodromy group is reasonably straightforward provided one works with rigid representations of the monodromy groups. In our situation the representation is minimally non-rigid; it lives in a complex two-dimensional moduli space, and this is the basic reason the (second order) Painlevé VI equation arises.

Apart from the many physical applications, from a mathematical perspective our basic interest in the Painlevé VI equation is that it is the explicit form of the simplest isomonodromy (=non-abelian Gauss–Manin) connection. In brief, the isomonodromy connections arise by replacing the closed differential forms and periods appearing in the usual (abelian) Gauss–Manin picture, by flat connections and monodromy representations, respectively.

Indeed one may view the Painlevé VI equation as a natural nonlinear analogue of the Gauss hypergeometric equation. From this point of view the thrust of this paper is towards finding the analogue of Schwartz's famous list of hypergeometric equations with algebraic solutions.

Before carefully describing the contents of this paper we will briefly recall exactly how the sixth Painlevé equation arises.

Consider a Fuchsian system of differential equations (with four singularities) of the form

$$\frac{d\Phi}{dz} = A(z)\Phi; \qquad A(z) = \sum_{i=1}^{3} \frac{A_i}{z-a_i} \tag{3}$$

where the A_i's are 2×2 traceless matrices. We wish to deform (3) isomonodromically—i.e. when the pole positions (a_1, a_2, a_3) are moved in $\mathbb{C}^3 \setminus$ diagonals we wish to vary the coefficients A_i such that the conjugacy class of the corresponding monodromy representation is preserved. Such isomonodromic deformations are governed by Schlesinger's equations:

$$\frac{\partial A_i}{\partial a_j} = \frac{[A_i, A_j]}{a_i - a_j} \quad \text{if } i \neq j, \text{ and } \quad \frac{\partial A_i}{\partial a_i} = -\sum_{j\neq i} \frac{[A_i, A_j]}{a_i - a_j}. \tag{4}$$

Let us view these more geometrically as a nonlinear connection on a fibre-bundle. First observe that Schlesinger's equations preserve the adjoint orbit O_i containing each A_i and are invariant under overall conjugation of (A_1, A_2, A_3, A_4), where $A_4 = -A_1 - A_2 - A_3$ is the residue of (3) at infinity. Thus one sees that Schlesinger's equations amount to a flat connection, the *isomonodromy connection*, on the trivial fibre bundle

$$\mathcal{M}^* := (O_1 \times O_2 \times O_3 \times O_4)/\!\!/G \times B \longrightarrow B \tag{5}$$

over $B := \mathbb{C}^3 \setminus$ diagonals, where the fibre $(O_1 \times \cdots \times O_4)/\!\!/G$ is the quotient of

$$\left\{ (A_1, A_2, A_3, A_4) \in O_1 \times O_2 \times O_3 \times O_4 \;\middle|\; \sum A_i = 0 \right\}$$

by overall conjugation by $G = \mathrm{SL}_2(\mathbb{C})$. (Generically this fibre is two dimensional and has a natural complex symplectic structure.)

Now for each point (a_1, a_2, a_3) of the base B one can also consider the set

$$\mathrm{Hom}_{\mathcal{C}}(\pi_1(\mathbb{C} \setminus \{a_i\}), G)/G \tag{6}$$

of conjugacy classes of representations of the fundamental group of the four-punctured sphere, where the representations are restricted to take the simple loop around a_i into the conjugacy class $\mathcal{C}_i := \exp(2\pi\sqrt{-1}O_i) \subset G$ $(i = 1, \ldots, 4, a_4 = \infty)$. These spaces of representations are also generically two dimensional (and complex symplectic) and fit together into a fibre bundle

$$M \longrightarrow B.$$

Moreover this bundle M has a *complete* flat connection defined locally by identifying representations taking the same values on a fixed set of fundamental group generators. The isomonodromy connection is the pullback of this complete connection along the natural bundle map

$$\nu : \mathcal{M}^* \longrightarrow M$$

defined by taking the systems (3) to their monodromy representations (cf. [18, 5]).

To obtain Painlevé VI one considers the double cover $\widehat{\mathcal{M}}^*$ of $\mathcal{M}^*$ (defined by choosing an ordering of the eigenvalues of A_4) and lifts the isomonodromy connection to $\widehat{\mathcal{M}}^*$. Upon choosing specific local coordinates x, y on the fibres of $\widehat{\mathcal{M}}^*$ and restricting the pole positions to $(a_1, a_2, a_3) = (0, t, 1)$, the isomonodromy connection then amounts to two first order coupled nonlinear equations for $x(t), y(t)$. Eliminating x yields (cf. e.g. [23]) the sixth Painlevé equation[1] (PVI):

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}$$
$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{(t-1)}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right).$$

The four parameters $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ here are directly related to the choice of the adjoint orbits O_i. From another viewpoint, we will see the monodromy spaces (6) are affine cubic surfaces and Iwasaki [21] has recently pointed out that the four parameters correspond to the moduli of such surfaces, appearing in the Cayley normal form.

The sixth Painlevé equation has critical singularities at $0, 1, \infty$ and is remarkable in that any of its solutions have wonderful analytic continuation properties: any locally-defined solution $y(t)$ may be analytically continued to a meromorphic function on the universal cover of the three-punctured sphere $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. (This is the so-called Painlevé property.)

From the geometric viewpoint, the monodromy of PVI (i.e. the analytic continuation of solutions around $\mathbb{P}^1 \setminus \{0, 1, \infty\}$) corresponds to the monodromy of the nonlinear connection on $\widehat{\mathcal{M}}^*$. In turn this connection is the pullback of the complete connection on the bundle M. Being complete, the monodromy of the connection on M amounts to an action of the fundamental group of the base B on the standard fibre (6). In other words: the monodromy of solutions to PVI is governed by the standard action of the pure three-string braid group $P_3 = \pi_1(B)$ on the space (6) of monodromy data.

Our main concern in this paper is to construct algebraic solutions to PVI. One knows that, for generic values of the four parameters, any solution of PVI is a 'new transcendental function' on the universal cover of the three-punctured sphere. However, for special values of the parameters it is possible that there are solutions expressible in terms of standard transcendental functions, or even solutions which are algebraic—i.e. are defined by polynomial equations. For example there are the algebraic solutions of Hitchin [19, 20],

[1] The general PVI equation was first written down by R. Fuchs (son of L. Fuchs) and it was added to the list of Painlevé equations by Painlevé's student B. Gambier.

Dubrovin [11] and Dubrovin–Mazzocco [13] related to the dihedral, tetrahedral, octahedral and icosahedral groups.

The problem of constructing algebraic solutions may be broken into two steps. First, the algebraic solutions will have a finite number of branches and so one may start by looking for finite orbits of the P_3 action on the space of monodromy data (6).

Clearly if we start with a linear system (3) whose monodromy is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, then the corresponding braid group orbit will be finite. The solutions of Hitchin, Dubrovin and Mazzocco mentioned above are equivalent to solutions arising in this way.

The basic idea underlying the present paper is that PVI also arises as the equation for isomonodromic deformations of certain rank three Fuchsian systems. Namely we replace A_1, A_2, A_3 in (3) by 3×3 matrices B_1, B_2, B_3 each of rank one. Then the corresponding moduli spaces are still of dimension two, and one finds again that PVI governs the isomonodromic deformations (and that any PVI equation arises in this way). Note that the rank one condition implies the corresponding monodromy group will be generated by a triple of pseudo-reflections in $\mathrm{GL}_3(\mathbb{C})$.

Rather than work throughout with this equivalent 3×3 representation of PVI, we will pass between the two pictures in order to use existing machinery developed in the 2×2 framework (in particular the work of Jimbo [22]).

Our starting point will be to describe a method of constructing finite braid group orbits of triples of elements of $\mathrm{SL}_2(\mathbb{C})$ starting from any triple of complex reflections generating a complex reflection group in $\mathrm{GL}_3(\mathbb{C})$. In general this will yield more exotic finite braid group orbits than those from finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. The key idea behind this construction is to use the Fourier–Laplace transformation to convert the rank three Fuchsian system into a rank three system with an irregular singularity, then to apply a simple scalar shift and transform back, so that the resulting Fuchsian system is reducible, and we take the irreducible rank two quotient or subsystem. Of crucial importance here is Balser–Jurkat–Lutz's computation [1] of the action of the Fourier–Laplace transformation on monodromy data, relating the monodromy data of the Fuchsian system to the Stokes data $u_\pm$ of the irregular system. This correspondence may be described by the explicit formula

$$r_3 r_2 r_1 = u_-^{-1} t^2 u_+$$

(dating back at least to Killing) for the Birkhoff factorisation of the product of generating reflections, and enables us to compute the action of the scalar shift on the reflections.

Lots of finite braid group orbits of $\mathrm{SL}_2(\mathbb{C})$ triples are obtained in this way: We recall that Shephard–Todd [30] have classified all the complex reflection groups and showed that in three-dimensions, apart from the real reflection groups, there are four irreducible complex reflection groups generated by triples of reflections, of orders 336, 648, 1296 and 2160 respectively, as well as two infinite families $G(m, p, 3), m \geq 3, p = 1, m$ of groups of orders $6m^3/p$. For $m = 2$ and $p = 1, 2$ these would be the symmetry groups of the octahedron and tetrahedron respectively. (In general, for other p dividing m, $G(m, p, 3)$ is not generated by a triple of reflections.) The main example we will focus on, the Klein group, is thus the smallest non-real exceptional complex reflection group. This leads to a P_3 orbit of size seven which we will prove is not isomorphic to any orbit coming from a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$.

The second step in the construction of algebraic solutions is to pass from the finite braid group orbit to the explicit solution. For this we adapt (and correct) a result of Jimbo [22] giving an explicit formula for the leading term in the asymptotic expansion at zero of the solution $y(t)$ on each branch. By using the PVI equation this is sufficient to determine the solution curve precisely.

The general strategy of this paper is the same as the paper [13] of Dubrovin–Mazzocco. Indeed part of our motivation was to extend their work to (a dense open subset of) the full four parameter family of PVI equations. Recall that [13] dealt with the real three-dimensional reflection groups and for this it was sufficient to only consider a one-parameter family of PVI equations (corresponding to fixing each of A_1, A_2, A_3 to be nilpotent, so the remaining parameter is the choice of orbit of A_4).

In relation to [13] the key results of the present paper are firstly to see how to extend their method of passing from generating triples of real reflections to finite P_3 orbits of (unipotent) $\mathrm{SL}_2(\mathbb{C})$ triples. (Reading the earlier papers [11, 12] of Dubrovin was helpful to fully understand this aspect of [13].)

Secondly we were able to fix Jimbo's asymptotic formula. (Dubrovin–Mazzocco did not use Jimbo's asymptotic result, but adapted Jimbo's argument to prove a version of it for their nilpotent situation). The key point here was to find a sign error hidden in the depths of Jimbo's asymptotic formula—perhaps we should emphasize that without the correction the construction of this paper will not work at all. (Namely at some point we need to obtain precise rational numbers out of the transcendental formulae.) This sign is also important because it is needed to obtain the correct connection formulae for solutions of the Painlevé VI equation.

The two main tools of this paper (construction of finite P_3 orbits of $\mathrm{SL}_2(\mathbb{C})$ triples, and Jimbo's formula) are independent and will have separate applications. For example one may take any triple of elements of a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ and try to apply Jimbo's formula to find solutions to PVI. (E.g. in [2] we have classified the inequivalent P_3 orbits of generators of the binary icosahedral group and, as a further test of Jimbo's formula, constructed an algebraic solution to PVI with 12 branches, involving 105 twenty digit integers—this is the largest genus zero icosahedral solution and is interesting since its parameters lie on none of the reflecting hyperplanes of Okamoto's affine F_4 action.)

The layout of this paper is as follows. In section 2 we explain in a direct algebraic fashion how to obtain finite braid group orbits of (conjugacy classes of) triples of elements of $\mathrm{SL}_2(\mathbb{C})$ from triples of generators of three-dimensional complex reflection groups. Section 3 (which could be skipped on a first reading) then explains how the formulae of section 2 were found. This is somewhat more technical, involving the action of the Fourier–Laplace transform on monodromy data, but is necessary to understand the origin of the procedure of section 2. We also mention in passing (Remark 23) the relation with the $\mathrm{GL}_n(\mathbb{C})$ quantum Weyl group actions. Next, in section 4, we give Jimbo's formula for the leading term in the asymptotic expansion at zero of the PVI solution $y(t)$ on the branch specified by a given $\mathrm{SL}_2(\mathbb{C})$ triple. This is applied in section 5 to find the Klein solution explicitly. Section 6 then proves that the Klein solution is not equivalent (under Okamoto's affine F_4 action) to any solution coming from a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. Finally in section 7 we explain how to reconstruct, from such a PVI solution, an explicit rank three Fuchsian system with monodromy group generated by the triple of complex reflections we started with in section 2.

It should be mentioned that, in the short paper [3], we previously showed by a different method that the equations for isomonodromic deformations of the 3×3 Fuchsian systems mentioned above are equivalent to PVI—this was written before Jimbo's formula was fixed and also does not give the relation between the rank two and three monodromy data.

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2. Braid group orbits

In this section we will explain how to obtain some interesting finite braid group orbits of triples of elements of $\mathrm{SL}_2(\mathbb{C})$ from triples of generators of three-dimensional complex reflection groups.

The motivation is simply the fact that branches of a solution to PVI are parameterised by pure braid group orbits of conjugacy classes of triples of elements of $\mathrm{SL}_2(\mathbb{C})$. Clearly any algebraic solution of PVI has a finite number of branches and so the first step towards finding an algebraic solution is to find a finite braid group orbit, which is what we will do here.

2×2 case.
Let $G=\mathrm{SL}_2(\mathbb{C})$ and consider the standard action of the three-string braid group B_3 on G^3 generated by

$$\beta_1(M_3,M_2,M_1)=(M_2,M_2^{-1}M_3M_2,M_1)\tag{7}$$
$$\beta_2(M_3,M_2,M_1)=(M_3,M_1,M_1^{-1}M_2M_1)$$

where $M_i\in G$. We are interested in the induced action of B_3 on the set of conjugacy classes of such triples.

First we recall some basic facts (cf. e.g. [26]). To begin with note that the seven functions

$$m_1:=\mathrm{Tr}(M_1),\qquad m_2:=\mathrm{Tr}(M_2),\qquad m_3:=\mathrm{Tr}(M_3),$$
$$m_{12}:=\mathrm{Tr}(M_1M_2),\qquad m_{23}:=\mathrm{Tr}(M_2M_3),\qquad m_{13}:=\mathrm{Tr}(M_1M_3)\tag{8}$$
$$m_{321}:=\mathrm{Tr}(M_3M_2M_1).$$

on G^3 are invariant under the diagonal conjugation action of G and in fact generate the ring of invariant polynomials. Indeed the ring of invariants is isomorphic to the quotient of $\mathbb{C}[m_1,m_2,m_3,m_{12},m_{23},m_{13},m_{321}]$ by the (ideal generated by the) so-called Fricke relation:

$$m_{321}^2-Pm_{321}+Q=4\tag{9}$$

where P, Q are the following polynomials in the first six variables:

$$P = m_1m_{23} + m_2m_{13} + m_3m_{12} - m_1m_2m_3$$

$$Q = m_1^2 + m_2^2 + m_3^2 + m_{12}^2 + m_{23}^2 + m_{13}^2 + m_{12}m_{23}m_{13} - m_1m_2m_{12} - m_2m_3m_{23} - m_1m_3m_{13}.$$

(This relation appears in the book [14] of Fricke and Klein.) That there is precisely one relation fits nicely with the rough dimension count of $3 \cdot 3 - 3 = 6$ for the space of conjugacy classes of triples. Viewed as a quadratic equation for m_{321} the other root of (9) is $\mathrm{Tr}(M_1M_2M_3)$ so in particular we have

$$\mathrm{Tr}(M_1M_2M_3) = P - m_{321}. \tag{10}$$

Note that, upon fixing m_1, m_2, m_3, m_{321}, the Fricke relation is a cubic equation in the remaining three variables; the six dimensional variety we are studying is essentially a universal family of affine cubic surfaces [21].

Now, the induced B_3 action on conjugacy classes of triples induces an action on the invariant functions, and we will describe this action in terms of the seven chosen generators. Clearly m_{321} is fixed by both β_i, and the m_i are just permuted:

$$\beta_1(m_1, m_2, m_3) = (m_1, m_3, m_2), \qquad \beta_2(m_1, m_2, m_3) = (m_2, m_1, m_3).$$

Lemma 1. *The induced B_3 action on the quadratic functions is*

$$\beta_1(m_{12}, m_{23}, m_{13}) = (m_2m_{321} + m_1m_3 - m_{13} - m_{12}m_{23}, m_{23}, m_{12})$$

$$\beta_2(m_{12}, m_{23}, m_{13}) = (m_{12}, m_{13}, m_1m_{321} + m_2m_3 - m_{23} - m_{13}m_{12})$$

Proof. The second formula follows from the first by permuting indices. For the first formula the hard part is to establish

$$\mathrm{Tr}(M_2^{-1}M_3M_2M_1) = m_2m_{321} + m_1m_3 - m_{13} - m_{12}m_{23}.$$

One way to do this (which will extend to the 3×3 case below) is to write $M_i = \varepsilon_i(1+e_i\otimes\alpha_i)$ for some rank one matrix $e_i \otimes \alpha_i$ and number $\varepsilon_i \in \mathbb{C}^*$. Then $\mathrm{Tr}(M_2^{-1}M_3M_2M_1)$ can be expanded in terms of the numbers $\alpha_i(e_j)$ and the terms of the resulting expression can be identified as terms in the expansions of the seven invariant functions. □

Before moving on to the higher rank case we point out the evident fact that if (M_3, M_2, M_1) are a triple of generators of a finite subgroup of G then the corresponding braid group orbit is finite (and in turn the induced action on conjugacy classes of triples is also finite).

3×3 case.

Now we wish to find analogous formulae for the corresponding action of B_3 on conjugacy classes of triples of pseudo-reflections in $\mathrm{GL}_3(\mathbb{C})$.

Suppose r_1, r_2, r_3 are pseudo-reflections in $\mathrm{GL}_3(\mathbb{C})$, so that

$$r_i = 1 + e_i \otimes \alpha_i$$

for some $e_i \in V, \alpha_i \in V^*$ where $V = \mathbb{C}^3$. Choose six non-zero complex numbers $n_1, n_2, n_3, t_1, t_2, t_3$ such that t_i is a choice of square root of $\det(r_i)$ (i.e. $t_i^2 = 1 + \alpha_i(e_i)$), that the product $r_3r_2r_1$ has eigenvalues $\{n_1^2, n_2^2, n_3^2\}$ and that these square roots are chosen so that

$$t_1t_2t_3 = n_1n_2n_3 \tag{11}$$

(which is a square root of the equation $\prod(\det r_i) = \det r_3r_2r_1$). These square roots (and the choice of ordering of eigenvalues of $r_3r_2r_1$) will not be needed to describe the braid group actions here, but will be convenient later.

Now consider the following eight $\mathrm{GL}_3(\mathbb{C})$-invariant functions on the set of triples of pseudo-reflections:

$$t_1^2, \qquad t_2^2, \qquad t_3^2,$$

$$t_{12} := \mathrm{Tr}(r_1r_2) - 1, \qquad t_{23} := \mathrm{Tr}(r_2r_3) - 1, \qquad t_{13} := \mathrm{Tr}(r_1r_3) - 1, \tag{12}$$

$$t_{321} := n_1^2 + n_2^2 + n_3^2, \qquad t'_{321} := (n_1n_2)^2 + (n_2n_3)^2 + (n_1n_3)^2.$$

(Note that $t_i^2 = \mathrm{Tr}(r_i) - 2$, $t_{321} := \mathrm{Tr}(r_3r_2r_1)$ and $t'_{321} = \det(r_3r_2r_1)\mathrm{Tr}(r_3r_2r_1)^{-1}$.) The subtractions of 1 or 2 in this definition turn out to simplify the formulae below. The action of B_3 on triples of pseudo-reflections is generated by

$$\beta_1(r_3, r_2, r_1) = (r_2, r_2^{-1}r_3r_2, r_1), \tag{13}$$

$$\beta_2(r_3, r_2, r_1) = (r_3, r_1, r_1^{-1}r_2r_1).$$

Now consider the induced action on conjugacy classes of triples. First, it is clear that t_{321}, t'_{321} are B_3-invariant since $r_3r_2r_1$ is fixed. Also, as before, the functions t_i^2 are just permuted:

$$\beta_1(t_1^2, t_2^2, t_3^2) = (t_1^2, t_3^2, t_2^2), \qquad \beta_2(t_1^2, t_2^2, t_3^2) = (t_2^2, t_1^2, t_3^2).$$

Lemma 2. *The induced B_3 action on the functions (t_{12}, t_{23}, t_{13}) is as follows:*

$$\beta_1(t_{12}, t_{23}, t_{13}) = (t_{321} + t_1^2 + t_3^2 - t_{13} + (t'_{321} - t_{12}t_{23})/t_2^2, t_{23}, t_{12})$$

$$\beta_2(t_{12}, t_{23}, t_{13}) = (t_{13}, t_{23}, t_{321} + t_2^2 + t_3^2 - t_{23} + (t'_{321} - t_{13}t_{12})/t_1^2).$$

Proof. The non-obvious part is to establish

$$\mathrm{Tr}(r_2^{-1}r_3r_2r_1) = t_{321} + t_1^2 + t_3^2 - t_{13} + (t'_{321} - t_{12}t_{23})/t_2^2.$$

For this we first observe $r_i^{-1} = 1 - e_i \otimes \alpha_i/t_i^2$. Then expanding $\mathrm{Tr}(r_2^{-1}r_3r_2r_1)$ yields

$$t_{321} + 1 - t_2^2 - u_{12}u_{21} - u_{23}u_{32} - (u_{12}u_{23}u_{31} + u_{23}u_{32}u_{12}u_{21})/t_2^2$$

where $u_{ij} := \alpha_i(e_j)$. To simplify this we first use the following identities (obtained by expanding the traces t_{ij}):

$$u_{ij}u_{ji} = t_{ij} - t_i^2 - t_j^2 \qquad \text{if } i \neq j. \tag{14}$$

Then, to finish, we use the identity (analogous to (10)):

$$u_{12}u_{23}u_{31} = t_3^2t_{12} + t_2^2t_{13} + t_1^2t_{23} - (t_1t_2)^2 - (t_2t_3)^2 - (t_1t_3)^2 - t'_{321},$$

which is obtained by expanding $\mathrm{Tr}(r_1^{-1}r_2^{-1}r_3^{-1}) = \mathrm{Tr}\left((r_3r_2r_1)^{-1}\right) = n_1^{-2} + n_2^{-2} + n_3^{-2}$. □

Again we have the evident fact that if (r_3, r_2, r_1) are a triple of generators of a finite subgroup of $\mathrm{GL}_3(\mathbb{C})$, i.e. if they are generators of a three-dimensional complex reflection group, then the corresponding braid group orbit is finite (and in turn the induced action on conjugacy classes of triples is also finite).

Remark 3. A rough dimension count gives $3\cdot 5-8=7$ for the space of conjugacy classes of pseudo-reflections, so we expect there to be a relation amongst the eight invariant functions. This is the analogue of the Fricke relation and comes from the identity

$$(u_{12}u_{23}u_{31})(u_{32}u_{21}u_{13}) = (u_{23}u_{32})(u_{12}u_{21})(u_{13}u_{31}).$$

Rewriting each bracketed term in terms of the eight functions yields the desired relation:

$$\begin{aligned}\big(t_3^2t_{12}+t_2^2t_{13}+t_1^2t_{23}-(t_1t_2)^2-(t_2t_3)^2-(t_1t_3)^2-t'_{321}\big)&\\ \times(t_{321}+t_1^2+t_2^2+t_3^2-t_{12}-t_{13}-t_{23})&\\ =(t_{12}-t_1^2-t_2^2)(t_{13}-t_1^2-t_3^2)(t_{23}-t_2^2-t_3^2).&\end{aligned} \tag{15}$$

From 3×3 to 2×2.

Now we will define a B_3-equivariant map from the space of triples of pseudo-reflections to the space of $\mathrm{SL}_2(\mathbb{C})$ triples. The main application of this here is just the observation that we will then obtain more exotic finite $\mathrm{SL}_2(\mathbb{C})$ braid group orbits from any triple of generators of a complex reflection group.

Suppose we are given the data

$$\mathbf{t} := (t_1,t_2,t_3,n_1,n_2,n_3,t_{12},t_{23},t_{13})$$

associated to a triple of pseudo-reflections. (We extend the B_3-action to the set of such data—i.e. with square root choices etc.—in the obvious way, permuting the t_i and fixing the n_i.) Define a map φ taking $\mathbf{t}$ to the $\mathrm{SL}_2(\mathbb{C})$ data $\mathbf{m}$ given by:

$$\begin{gathered} m_1 := \frac{t_1}{n_1}+\frac{n_1}{t_1},\qquad m_2 := \frac{t_2}{n_1}+\frac{n_1}{t_2},\qquad m_3 := \frac{t_3}{n_1}+\frac{n_1}{t_3},\\ m_{12} := \frac{t_{12}}{t_1t_2},\qquad m_{23} := \frac{t_{23}}{t_2t_3},\qquad m_{13} := \frac{t_{13}}{t_1t_3},\\ m_{321} := \frac{n_2}{n_3}+\frac{n_3}{n_2}.\end{gathered} \tag{16}$$

Theorem 1. *The map φ is B_3-equivariant. In particular finite B_3-orbits of $\mathrm{SL}_2(\mathbb{C})$ triples are obtained from triples of generators of three-dimensional complex reflection groups.*

Proof. This may be proved by direct calculation (a less direct proof will be given in section 3, along with a description of the origins of the above formulae). For example if we write $\mathbf{m}'=\beta_1(\varphi(\mathbf{t}))$ and $\mathbf{m}''=\varphi(\beta_1(\mathbf{t}))$, then the tricky part is to see $m'_{12}=m''_{12}$. However it is straightforward to show that the expression obtained for $m'_{12}-m''_{12}$ (using the above formulae) has a factor of $t_1t_2t_3-n_1n_2n_3$ in its numerator, which is zero due to (11). (Similarly for β_2.) □

Remark 4. It is possible to check directly (using Maple) that the map φ is well-defined; i.e. that $\varphi(\mathbf{t})$ satisfies the Fricke relation (9) provided that $\mathbf{t}$ satisfies both (11) and (15).

Group	Degrees $x_i + 1$	$(\theta_1, \theta_2, \theta_3, \theta_4)$
$G(m, m, 3)$	$3, m, 2m$	$(m-2, m-2, m-2, m)/2m$
$G(m, 1, 3)$	$m, 2m, 3m$	$(m-2, m-2, 2m-4, 4m)/6m$
Icosahedral	$2, 6, 10$	$(0, 0, 0, 4/5)$
G_{336}	$4, 6, 14$	$(2, 2, 2, 4)/7$
G_{648}	$6, 9, 12$	$(0, 0, 0, 1/2)$
G_{1296}	$6, 12, 18$	$(4, 7, 7, 12)/18$
G_{2160}	$6, 12, 30$	$(5, 5, 5, 9)/15$

Table 1. Parameters for solutions from standard generating triples.

Painlevé parameters.

The Painlevé VI equation that arises by performing isomonodromic deformations of the rank 2 Fuchsian system with monodromy data M_1, M_2, M_3, M_4 (where $M_4M_3M_2M_1 = 1$) has parameters

$$\alpha = (\theta_4 - 1)^2/2, \ \beta = -\theta_1^2/2, \ \gamma = \theta_3^2/2, \ \delta = (1 - \theta_2^2)/2 \tag{17}$$

where the θ_j $(j = 1, 2, 3, 4)$ are such that M_j has eigenvalues $\exp(\pm\pi i\theta_j)$, i.e.

$$\mathrm{Tr}(M_j) = 2\cos(\pi\theta_j).$$

Now suppose the M_j arise under the map φ from some data $\mathbf{t}$ associated to a three-dimensional pseudo-reflection group. We can then relate the Painlevé parameters to the invariants of the pseudo-reflection group (cf. [3] Lemma 3). If we choose (for $j = 1, 2, 3$) λ_j, μ_j such that

$$t_j = \exp(\pi i\lambda_j), \qquad n_j = \exp(\pi i\mu_j), \qquad \sum \lambda_i = \sum \mu_i \tag{18}$$

then we have:

Lemma 5. *The Painlevé parameters corresponding to the data* $\mathbf{t}$ *under the map* φ *are*

$$\theta_i = \lambda_i - \mu_1 \ (i = 1, 2, 3), \quad \theta_4 = \mu_3 - \mu_2. \tag{19}$$

Proof. From the definition (16) of φ we have $\mathrm{Tr}(M_i) = m_i = 2\cos\pi(\lambda_i - \mu_1)$ and $\mathrm{Tr}(M_4) = m_{321} = 2\cos\pi(\mu_3 - \mu_2)$. □

In particular if we are considering a complex reflection group G that is generated by a triple of reflections, then, by 5.4 of [30], we may choose a generating triple (r_1, r_2, r_3) such that the μ_i are related to the exponents $x_1 \leqslant x_2 \leqslant x_3$ of the group G as follows:

$$\mu_i = x_i/h, \quad (i = 1, 2, 3) \qquad h := x_3 + 1.$$

This result enables us to compile Table 1 of parameters of the Painlevé equations corresponding to the standard generators of the three-dimensional complex reflection groups, where a suitable permutation of $\{\mu_i\}$ has been used in each case and we have taken each θ_i to be positive (since negating any θ_i leads to equivalent PVI parameters). (Not every three-dimensional complex reflection group may be generated by three reflections but those that can be are listed.)

Example. Let us consider the reflections generating the Klein complex reflection group. Explicitly the standard generators (from [30] 10.1) are:

$$r_1 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -\overline{a} \\ -1 & 1 & -\overline{a} \\ -a & -a & 0 \end{pmatrix}, \qquad r_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad r_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where $a := (1+i\sqrt{7})/2$. These are order two complex reflections so $\det(r_i) = -1$ and are ordered so that the exponents $\{3,5,13\}$ of the group appear in the eigenvalues of the product $r_3r_2r_1$. Namely $r_3r_2r_1$ has eigenvalues $\{\exp(2\pi i\frac{3}{14}), \exp(2\pi i\frac{5}{14}), \exp(2\pi i\frac{13}{14})\}$. Also we compute:

$$\mathrm{Tr}(r_1r_2) = 1, \qquad \mathrm{Tr}(r_2r_3) = 1, \qquad \mathrm{Tr}(r_1r_3) = 0.$$

Thus if we set $\lambda_1 = \lambda_2 = \lambda_3 = 1/2, \mu_1 = 3/14, \mu_2 = 5/14, \mu_3 = 13/14$ so that

$$t_1 = t_2 = t_3 = i, \qquad n_1 = \exp\frac{3\pi i}{14}, \qquad n_2 = \exp\frac{5\pi i}{14}, \qquad n_3 = \exp\frac{13\pi i}{14}$$

then the image of this data under φ is

$$m_1 = m_2 = m_3 = 2\cos(2\pi/7), \quad m_{321} = 2\cos(4\pi/7), \quad m_{12} = m_{23} = 0, \quad m_{13} = 1.$$

Clearly (cf. Lemma 5) the parameters of the corresponding Painlevé equation are:

$$\theta_1 = \theta_2 = \theta_3 = 2/7, \quad \theta_4 = 4/7 \qquad \text{and so} \qquad (\alpha,\beta,\gamma,\delta) = (9,-4,4,45)/98.$$

The corresponding braid group orbit is easy to calculate by hand; Observe each of m_1, m_2, m_3, m_{321} is fixed by B_3, and that, since $4\cos(2\pi/7)\cos(4\pi/7) + 4\cos^2(2\pi/7) = 1$, the formula for the action on the quadratic functions simplifies to

$$\beta_1(m_{12}, m_{23}, m_{13}) = (1 - m_{13} - m_{12}m_{23}, m_{23}, m_{12}),$$

$$\beta_2(m_{12}, m_{23}, m_{13}) = (m_{12}, m_{13}, 1 - m_{23} - m_{13}m_{12}).$$

In this way we find the B_3 orbit has size seven, with values

(20)

m_{12}	m_{23}	m_{13}
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0.

Upon restriction to the pure braid group $P_3 \subset B_3$ generated by β_1^2, β_2^2, we find the orbit still has size seven. Thus the corresponding solution of Painlevé VI has seven branches and, via (20), the branches of the solution are conveniently labeled by the (binary) numbers from zero to six. The pure braid group generators are represented by the following permutations of the seven branches:

$$\beta_1^2: \qquad (05)(14)(236)$$

$$\beta_2^2: \qquad (03)(12)(465)$$

whose product also has two 2-cycles and a 3-cycle. This should be the monodromy representation of the solution curve as a cover of $\mathbb{P}^1$ branched at $0, 1, \infty$. Using the Riemann–Hurwitz formula we thus see the solution curve has genus zero. Also, for example, one can calculate the monodromy group of the cover; the subgroup of the symmetric group generated by these (even) permutations is as large as possible, namely A_7. This gives a clear picture of the solution curve topologically as a cover of the Riemann sphere. Our next aim (after explaining how the formulae of this section were found) will be to find an explicit polynomial equation for the solution curve and for the function y on it solving Painlevé VI.

Remark 6. Given the data corresponding to any of these branches one can easily solve the seven equations (8) to find a corresponding $\mathrm{SL}_2(\mathbb{C})$ triple. For example for branch zero it is straightforward to find the triple:

$$M_1 = \begin{pmatrix} \phi & 0 \\ 0 & \phi^{-1} \end{pmatrix}, \qquad M_2 = \begin{pmatrix} w & x \\ -x & \overline{w} \end{pmatrix}, \qquad M_3 = \begin{pmatrix} w & \mu x \\ -x/\mu & \overline{w} \end{pmatrix}, \tag{21}$$

where $\phi = e^{2\pi i/7}$, $w = \frac{1+\phi^2}{\phi-\phi^3}$, $x = \sqrt{1-|w|^2} \in \mathbb{R}_{>0}$ and $\mu = (r + i\sqrt{4-r^2})/2$ where $r = \frac{(1+\phi^2)^2}{1-\phi^2} \in [0,1] \subset \mathbb{R}$.

Lemma 7. *The group generated by M_1, M_2, M_3 is an infinite subgroup of* SU_2*.*

Proof. By construction the M_i are in SU_2 (somewhat surprisingly). Since the group is nonabelian and contains elements of order seven we are still to check it is not some large dihedral group. Let ε be an eigenvalue of $M_1^4 M_2$, so that $\varepsilon + \varepsilon^{-1} = \tau$ where $\tau = -(1+\phi^2)(\phi+\phi^4+\phi^6)$ and so $\varepsilon = (\tau \pm \sqrt{\tau^2-4})/2$. Thus ε is some algebraic number of modulus one and we claim it is not a root of unity. To see this we take the product of $z - \varepsilon'$ over all the Galois conjugates of ε and find the minimal polynomial of ε is

$$p(z) = z^6 - 3z^5 - z^4 - 7z^3 - z^2 - 3z + 1.$$

Thus if $\varepsilon^N = 1$ for some integer N we would have that p divides $z^N - 1$ so all the roots of p are roots of unity. However clearly $p(1) = -13$ so p has a real root greater than 1. □

(Note also that triples in the same braid group orbit generate the same group.)

Some properties of φ.

Before moving on to the Fourier–Laplace transform we will describe some properties of the map φ; we will show that the different choices of square roots etc. give isomorphic P_3 orbits and also examine the fibres of φ. This motivates the definition of φ.

First of all, the conjugacy class of a triple of pseudo-reflections only determines the values of the functions $t_1^2, t_2^2, t_3^2, t_{12}, t_{23}, t_{13}$ and the symmetric functions in the n_i^2, whereas the map φ involves each t_i, n_i. Thus we must make a choice of ordering of the eigenvalues n_i^2 of $r_3r_2r_1$ and of the square roots t_i, n_i such that $t_1t_2t_3 = n_1n_2n_3$ in order to obtain the 2×2 data. In general different choices lead to different 2×2 data (cf. Remark 14). However the pure braid group orbits obtained via different choices are all isomorphic:

Lemma 8. *Let π be a permutation of $\{1,2,3\}$ and choose signs $\varepsilon_i, \delta_i \in \{\pm 1\}$ for $i = 1,2,3$ such that $\varepsilon_1\varepsilon_2\varepsilon_3 = \delta_1\delta_2\delta_3$. Consider the map σ on the set of data*

$$\mathbf{t} = (t_1, t_2, t_3, n_1, n_2, n_3, t_{12}, t_{23}, t_{13})$$

(satisfying (11) *and* (15)*) defined by*

$$\sigma(\mathbf{t}) := (\varepsilon_1 t_1, \varepsilon_2 t_2, \varepsilon_3 t_3, \delta_1 n_{\pi(1)}, \delta_2 n_{\pi(2)}, \delta_3 n_{\pi(3)}, t_{12}, t_{23}, t_{13}).$$

Then map σ commutes with the action of the pure braid group $P_3 \subset B_3$, and in particular the P_3 orbits through $\varphi(\mathbf{t})$ and $\varphi(\sigma(\mathbf{t}))$ are isomorphic.

Proof. The pure braid group action is generated by β_1^2, β_2^2 and so fixes the t_i, n_i pointwise. From Lemma 2 the action on the functions (t_{12}, t_{23}, t_{13}) is independent of the sign and ordering choices. □

We also wish to check that the different possible sign/ordering choices lead to Painlevé VI equations with parameters which are equivalent under the action of the affine F_4 Weyl group symmetries defined by Okamoto [29]. To this end we lift the map σ to act on the data $\mathbf{\Lambda} := (\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3)$ of (18) as

$$\sigma(\mathbf{\Lambda}) = (\lambda_1 + a_1, \lambda_2 + a_2, \lambda_3 + a_3, \mu_{\pi(1)} + b_1, \mu_{\pi(2)} + b_2, \mu_{\pi(3)} + b_3)$$

where a_i, b_i are integers such that $\sum a_i = \sum b_i$ and π is a permutation of $\{1, 2, 3\}$.

Lemma 9. *The Painlevé VI parameters associated to $\mathbf{\Lambda} = (\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3)$ in Lemma 5 are equivalent, under Okamoto's affine F_4 Weyl group action, to those associated to $\sigma(\mathbf{\Lambda})$.*

Proof. Since the set of such σ's forms a group it is sufficient to check the lemma on generators. The translations and the permutations may be dealt with separately since the group is a semi-direct product. First for the translations (fixing π to be the trivial permutation) this is straightforward; for example it is easy to express the corresponding translations of the θ's in terms of the translations of [28] (34). For the permutation just swapping μ_2 and μ_3 this amounts to negating θ_4 which is obtained by the transformation $T_{\varpi_2}^{-1} T_{\varpi_1}^2 s_1$ in the terminology of [28]. Finally the permutation just swapping μ_1 and μ_2 is obtained from the transformation $(s_0 s_3 s_4) s_2 (s_0 s_3 s_4)$ in the terminology of [28]. □

Remark 10. Perhaps it is helpful to recall that there are several symmetry groups of PVI considered in the literature, amongst which we have

$$\text{affine } D_4 \ < \ \text{extended affine } D_4 \ < \ \text{affine } F_4$$

the first two of which are for example considered in [28]. In brief[2] the first two differ by the Klein four-group and do not involve changing the time parameter t, whereas the full affine F_4 action of Okamoto involves changing t (by automorphisms of $\mathbb{P}^1$ permuting $0, 1, \infty$). In fact only the extended affine D_4 symmetries were used above, although the full F_4 action will be considered in section 6.

Next we will examine the fibres of φ. From our rough dimension counts we see these fibres should be one dimensional. The continuous part of the fibres arises as follows. Define an action of $\mathbb{C}^*$ on the pseudo-reflection data $\{\mathbf{t}\}$ by declaring $h \in \mathbb{C}^*$ to act as

$$t_i \mapsto h t_i, \qquad n_i \mapsto h n_i, \qquad t_{ij} \mapsto h^2 t_{ij}. \tag{22}$$

[2] I am grateful to M. Noumi for clarifying this to me.

Observe that this does indeed act within the fibres of φ, i.e. that $\varphi(h\mathbf{t}) = \varphi(\mathbf{t})$ for any $h \in \mathbb{C}^*$. Moreover a simple direct calculation shows this $\mathbb{C}^*$ action commutes with the B_3 action on $\{\mathbf{t}\}$. (The simplicity of this action is deceptive since we carefully chose the functions t_i, t_{ij}.)

Thus, for example, we can always use this action to move to the (B_3-invariant) subset of the pseudo-reflection data having $n_1 = 1$.

Lemma 11. *The map φ is surjective, and the restriction of φ to the subset of the pseudo-reflection data $\{\mathbf{t}\}$ having $n_1 = 1$ is a finite map.*

Proof. Given arbitrary $\mathrm{SL}_2(\mathbb{C})$ data $\mathbf{m}$ we just try to solve for $\mathbf{t}$ (having first set $n_1 = 1$). One finds a solution always exists and there are five sign choices, so a generic fibre has 32 points. □

Remark 12. One may check algebraically that if $\mathbf{t}$ has $n_1 = 1$, satisfies $n_2 n_3 = t_1 t_2 t_3$ and is such that $\varphi(\mathbf{t})$ satisfies the Fricke relation (9) then $\mathbf{t}$ satisfies the 3×3 analogue (15) of the Fricke relation.

Moving to the subset of the data on which $n_1 = 1$ implies we are forcing 1 into the spectrum of the product $r_3 r_2 r_1$. This implies that the representation (of the free group on three letters) defined by (r_3, r_2, r_1) is reducible: This is clear if $r_i = 1 + e_i \otimes \alpha_i$ for some e_i which are not a basis of $\mathbb{C}^3$ (since the span of the e_i is an invariant subspace). Otherwise we have

Lemma 13 (cf. [7] 10.5.6, [8] 3.7). *If $r_i := 1 + e_i \otimes \alpha_i$ for a basis e_1, e_2, e_3 of $V = \mathbb{C}^3$ and $v \in V$ satisfies $r_3 r_2 r_1 v = v$ then $r_i v = v$ for $i = 1, 2, 3$.*

Proof. If $r_3 r_2 r_1 v = v$ then

$$r_2 r_1 v - v = r_3^{-1} v - v,$$

the lefthand side of which is a linear combination of e_2, e_1, and the righthand side is a multiple of e_3, since $r_3^{-1} = 1 - e_3 \otimes \alpha_3 / t_3^2$. Thus both sides vanish so $r_3 v = v$ and $r_2 r_1 v = v$. Then similarly we see both sides $r_1 v - v = r_2^{-1} v - v$ vanish. □

Thus we can use the $\mathbb{C}^*$ action to move to a reducible triple. The map φ is defined simply by first moving to a reducible triple by setting $h := n_1^{-1}$ so that we can write the r_i in block upper triangular form. In general there will then be a size two and a size one block (with entry 1) on the diagonal. We then define $\widehat{M_i} \in \mathrm{GL}_2(\mathbb{C})$ to be the size two block, which will have eigenvalues $\{1, (t_i/n_1)^2\}$. Hence defining $M_i = n_1 \widehat{M_i} / t_i$ yields an $\mathrm{SL}_2(\mathbb{C})$ triple. Computing the various traces then gives the stated formulae for the map φ. (In more invariant language we take the projection to $\mathrm{SL}_2(\mathbb{C})$ of the rank two part of the 'semisimplification' of the reducible representation.)

Thus we have motivated the map φ in terms of the $\mathbb{C}^*$ action. In the next section we will motivate this action as the image under the Fourier–Laplace transform of a simple scalar shift.

Real reflection groups.

Let us check that the unipotent 2×2 monodromy data and the real three-dimensional reflection groups considered by Dubrovin–Mazzocco in [13] are related by the map φ defined in (16).

The monodromy data in [13] is parameterised by four numbers (x_1, x_2, x_3, μ) related by the condition

$$m = 2\cos(2\pi\mu) \qquad \text{where } m := 2 + x_1x_2x_3 - (x_1^2 + x_2^2 + x_3^2),$$

which is equivalent to [13] (1.21). The degenerate cases $m = \pm 2$ are ruled out. Although the main interest is in real reflection groups the formulae here make sense for complex values of the parameters; effectively we are restricting φ to a complex three-dimensional slice.

Without loss of generality one may assume $x_1 \neq 0$ and then the 2×2 monodromy data is given by the triple ([13] (1.20)):

$$M_1 = \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} 1 + x_2x_3/x_1 & -x_2^2/x_1 \\ x_3^2/x_1 & 1 - x_2x_3/x_1 \end{pmatrix}. \tag{23}$$

Note that, generically, each of these matrices is conjugate to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and in all cases $m_j := \mathrm{Tr}(M_j) = 2$. Also, straightforward computations give that

$$m_{12} = 2 - x_1^2, \qquad m_{23} = 2 - x_2^2, \qquad m_{13} = 2 - x_3^2, \qquad m_{321} = m.$$

(By nondegeneracy there is always some index j such that $x_j \neq 0$ and one may obtain the same values of the invariant functions from an analogous 2×2 triple.)

The corresponding 3×3 reflections considered in [13] are ([13] (1.51)):

$$r_1 = \begin{pmatrix} -1 & -x_1 & -x_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad r_2 = \begin{pmatrix} 1 & 0 & 0 \\ -x_1 & -1 & -x_2 \\ 0 & 0 & 1 \end{pmatrix}, \qquad r_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_3 & -x_2 & -1 \end{pmatrix}$$

which preserve the nondegenerate symmetric bilinear form given by the matrix

$$\begin{pmatrix} 2 & x_1 & x_3 \\ x_1 & 2 & x_2 \\ x_3 & x_2 & 2 \end{pmatrix}.$$

Immediate computation then gives that

$$t_1^2 = t_2^2 = t_3^2 = -1,$$
$$t_{12} = x_1^2 - 2, \qquad t_{23} = x_2^2 - 2, \qquad t_{13} = x_3^2 - 2$$
$$t_{321} = -m - 1 = -t'_{321}.$$

The characteristic polynomial of $r_3r_2r_1$ is

$$(\lambda + 1)(\lambda^2 + m\lambda + 1)$$

which has roots

$$n_1^2 = -1, \qquad n_2^2 = -\exp(2\pi i\mu), \qquad n_3^2 = -\exp(-2\pi i\mu).$$

Now we claim that if we choose the square roots appropriately then the map φ of (16) takes this 3×3 data onto the unipotent 2×2 data above. Indeed setting $n_1 = t_1 = t_2 = t_3 = i$ clearly gives the correct values $m_j = i/i + i/i = 2$ and $m_{jk} = t_{jk}/i^2 = -t_{jk}$. Also if we set $n_2 = i\exp(\pi i\mu)$ and $n_3 = i\exp(-\pi i\mu)$ then $n_2/n_3 = \exp(2\pi i\mu)$ and so $m_{321} = n_2/n_3 + n_3/n_2 = 2\cos 2\pi\mu = m$ as required. Thus the map φ does indeed extend the above correspondence used by Dubrovin–Mazzocco.

Remark 14. If instead we choose to order the eigenvalues of $r_3r_2r_1$ as

$$n_1^2 = -\exp(2\pi i\mu), \qquad n_2^2 = -1, \qquad n_3^2 = -\exp(-2\pi i\mu)$$

then we claim that, with appropriate square root choices, the corresponding 2×2 data (under φ) has the remarkable property that the four local monodromies $M_1, M_2, M_3, M_3M_2M_1$ all lie in the same conjugacy class: Namely if we choose $n_2 = t_1 = t_2 = t_3 = i$, let n_1 be any square root of $-\exp(2\pi i\mu)$ and define $n_3 := 1/n_1$ then we find

$$m_1 = m_2 = m_3 = m_{321} = i/n_1 + n_1/i = \pm 2\cos(\pi\mu).$$

Thus if this common value is not ± 2, the corresponding $\mathrm{SL}_2(\mathbb{C})$ matrices are regular semisimple and we have established the claim. However if this value is ± 2, it follows that $n_1^2 = -1$ and then that $m = 2$ contradicting the nondegeneracy assumption.

Such 'symmetric' $\mathrm{SL}_2(\mathbb{C})$ triples have been studied in this context by Hitchin (cf. [20]), and that they arise from real reflection groups was known to Dubrovin–Mazzocco (cf. [13] Remark 0.2). It is interesting to note that, from triples of generating reflections of the real three-dimensional tetrahedral, cubical, and icosahedral groups, one obtains in this way triples of generators of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. However, rather bizarrely, the tetrahedral and cubical groups are swapped in the process: the tetrahedral reflection group maps to the binary cubical/octahedral subgroup of $\mathrm{SL}_2(\mathbb{C})$ and vice-versa; the cubical reflection group maps to the binary tetrahedral subgroup of $\mathrm{SL}_2(\mathbb{C})$. The three inequivalent triples of generating reflections of the icosahedral reflection group do all map to triples of generators of the binary icosahedral group though.

3. Isomonodromic deformations

The main aim of this section is to see how the $\mathbb{C}^*$ action of (22) on the invariants of the pseudo-reflection data arises, and, on the other side of the Riemann–Hilbert correspondence, to describe the corresponding $\mathbb{C}$ action on the rank three Fuchsian systems.

This is the key ingredient needed to motivate the map φ of (16). As a corollary we will see why φ is B_3-equivariant.

We will work in a somewhat more general context in this section than the rest of the paper; the reader interesting mainly in the construction of algebraic solutions to PVI could skip straight to the next section.

Apart from the desire to explain how the procedure of section 2 was found, the motivation for this section is to enable us to see (in section 7) how one may work back from an explicit solution to PVI to an explicit rank three system of differential equations. This will give a mechanism for constructing new non-rigid systems of differential equations with finite monodromy group. (Except for this the proofs given in the other sections are independent of the results of this section.)

Let us begin with some generalities on isomonodromic deformations of Fuchsian systems and the Schlesinger equations. Let $V = \mathbb{C}^n$ and suppose we have matrices $B_1, \ldots, B_{m-1} \in \mathrm{End}(V)$ and distinct points $a_1, \ldots, a_{m-1} \in \mathbb{C}$. Then consider the following meromorphic connection on the trivial rank n holomorphic vector bundle over the Riemann sphere:

$$\nabla := d - \left(B_1\frac{dz}{z-a_1} + \cdots + B_{m-1}\frac{dz}{z-a_{m-1}}\right). \tag{24}$$

This has a simple pole at each a_i and at infinity. Write

$$B_m = B_\infty := -(B_1 + \cdots + B_{m-1})$$

for the residue matrix at infinity. Thus, on removing disjoint open discs $D_1, \ldots, D_m$ from around the poles and restricting ∇ to the m-holed sphere

$$S := \mathbb{P}^1 \setminus (D_1 \cup \cdots \cup D_m),$$

we obtain a (nonsingular) holomorphic connection. In particular it is flat and so, taking its monodromy, a representation of the fundamental group of the m-holed sphere is obtained. This procedure defines a holomorphic map, which we will call the monodromy map or Riemann–Hilbert map, from the set of such connection coefficients to the set of complex fundamental group representations:

$$\left\{(B_1, \ldots, B_m) \;\middle|\; \sum B_i = 0\right\} \xrightarrow{\mathrm{RH}} \left\{(M_1, \ldots, M_m) \;\middle|\; M_m \cdots M_1 = 1\right\}$$

where appropriate loops generating the fundamental group of S have been chosen and the matrix $M_i \in G := \mathrm{GL}_n(\mathbb{C})$ is the automorphism obtained by parallel translating a basis of solutions around the ith loop.

The Schlesinger equations are the equations for isomonodromic deformations of the connection (24). Suppose we move the pole positions $a_1, \ldots, a_{m-1}$. Then we wish to vary the coefficients B_i, as functions of the pole positions, such that the monodromy data $(M_1, \ldots, M_m)$ only changes by diagonal conjugation by G. This is the case if the B_i vary according to Schlesinger's equations:

$$\frac{\partial B_i}{\partial a_j} = \frac{[B_i, B_j]}{a_i - a_j} \quad \text{if } i \neq j, \quad \text{and} \quad \frac{\partial B_i}{\partial a_i} = -\sum_{j \neq i, m} \frac{[B_i, B_j]}{a_i - a_j} \tag{25}$$

where $i = 1, \ldots, m-1$. Observe that these equations imply that the residue at infinity B_m is held constant. Also note that the Schlesinger equations are equivalent to the flatness of the connection

$$d - \left(B_1 \frac{dz - da_1}{z - a_1} + \cdots + B_{m-1} \frac{dz - da_{m-1}}{z - a_{m-1}}\right). \tag{26}$$

In terms of differential forms Schlesinger's equations may be rewritten as

$$dB_i = -\sum_{j \neq i, m} [B_i, B_j] d_{ij} \tag{27}$$

where d is the exterior derivative on $\{a_i\}$ and $d_{ij} := d\log(a_i - a_j) = (da_i - da_j)/(a_i - a_j)$. In turn it will be convenient to rewrite this as

$$dB_i = [L_i, B_i] \quad \text{where} \quad L_i := \sum_{j \neq i, m} B_j d_{ij}. \tag{28}$$

Note that if we have a local solution of Schlesinger's equations and we construct the L_i from the formula (28) then firstly we have that $\nabla_i := d - L_i$ is a flat connection and secondly that B_i is a horizontal section of ∇_i (in the adjoint representation).

Now let us specialise to the case where the dimension n equals the number $m-1$ of finite singularities, and where each of the finite residues $B_1, \ldots, B_n$ is a rank one matrix. Thus

$$B_i = f_i \otimes \beta_i \quad \text{for some} \quad f_i \in V, \ \beta_i \in V^*.$$

Then we may lift the Schlesinger equations from the space of residues B_i to the space of f_i's and β_i's. Namely, suppose we have a local solution of the Schlesinger equations on some polydisc D. Then we can write $B_i = f_i \otimes \beta_i$ for $i = 1, \ldots, n$ at some base-point and then evolve f_i, β_i over D, as solutions to the equations:

$$df_i = L_i f_i \qquad d\beta_i = -\beta_i L_i \tag{29}$$

where the L_i are defined in terms of the given Schlesinger solution. Then one finds immediately that the $f_i \otimes \beta_i$ solve (28), and so $f_i \otimes \beta_i = B_i$ throughout D (since they agree at the basepoint and solve Schlesinger's equations). Alternatively one can view (29) as a coupled system of nonlinear equations for $\{f_i, \beta_i\}$, by defining L_i in terms of the $B_j := f_j \otimes \beta_j$ as in (28). We will refer to these as the *lifted equations* (they were introduced in [24] and further studied in [16]). The above considerations show:

Proposition 15. *Any solution of the Schlesinger equations* (25) *may be lifted to a solution of the lifted equations* (29) *by only solving linear equations. Conversely any solution of the lifted equations projects to a solution of* (25) *by setting* $B_i = f_i \otimes \beta_i$.

Now we wish to define an action of $\mathbb{C}$ which will be the additive analogue of the $\mathbb{C}^*$ action of (22).

Suppose we have a local solution $\{B_1(\mathbf{a}), \ldots, B_n(\mathbf{a})\}$ of Schlesinger's equations on some polydisc D, where $\mathbf{a} = (a_1, \ldots, a_n)$, such that the images of the B_i are linearly independent (i.e. for any choice of f_i, β_i such that $B_i = f_i \otimes \beta_i$, the f_i make up a basis of V). Then we can define the following action of the complex numbers on the set of such solutions:

Proposition 16. *For any complex number* $\lambda \in \mathbb{C}$ *the matrices*

$$\widetilde{B}_i := B_i + \lambda f_i \otimes \widehat{f}_i$$

constitute another solution to Schlesinger's equations on D*, where* $\widehat{f}_1, \ldots, \widehat{f}_n \in V^*$ *are the dual basis defined by* $\widehat{f}_i(f_j) = \delta_{ij}$.

Proof. First note that this is well-defined since the projectors $f_i \otimes \widehat{f}_i$ are independent of the choice of f_i's. Then lift the B_i arbitrarily to a solution $\{f_i(\mathbf{a}), \beta_i(\mathbf{a})\}$ of (29) over D. Straightforward computations then give that $\widehat{f}_i$ satisfies $d\widehat{f}_i = -\sum_{j\neq i} \beta_i(f_j)\widehat{f}_j d_{ij}$ and using this one easily confirms $d\widetilde{B}_i = [\widetilde{L}_i, \widetilde{B}_i]$ where $\widetilde{L}_i = L_i + \lambda \sum_{j\neq i} f_j \otimes \widehat{f}_j d_{ij}$. □

Remark 17. One arrives at this action as follows. Given a local solution $\{f_i(\mathbf{a}), \beta_i(\mathbf{a})\}$ of the lifted equations one may check that the matrix $B \in \mathrm{End}(V)$ defined by $(B)_{ij} = \beta_i(f_j)$ satisfies the nonlinear differential equation

$$dB = \left[B, \mathrm{ad}_{A_0}^{-1}([dA_0, B])\right] \tag{30}$$

where $A_0 := \mathrm{diag}(a_1, \ldots, a_n)$. (Note that $\mathrm{ad}_{A_0} : \mathrm{End}(V) \to \mathrm{End}(V)$ is invertible when restricted to the matrices with zero diagonal part and that $[dA_0, B]$ has zero diagonal part.) This is the 'dual' equation to the Schlesinger equations in the present context (in the sense of Harnad [16]) and arises as the equation for isomonodromic deformations of the irregular connection

$$d - \left(\frac{A_0}{w^2} + \frac{B}{w}\right) dw, \tag{31}$$

which, after an appropriate coordinate change, appears as the (twisted) Fourier–Laplace transform of the original Fuchsian system (cf. [1] and references therein). Equation (30) appears in the theory of Frobenius manifolds [11] for skew-symmetric B and is related to quantum Weyl groups [6]. Note that equation (30) is equivalent to the Schlesinger equations in that its solutions may also be lifted to solutions of (29) by only solving the linear equations

$$df_i = \sum_{j\neq i}(B)_{ji} f_j d_{ij} \qquad d\beta_i = -\sum_{j\neq i}(B)_{ij}\beta_j d_{ij}$$

where $B(\mathbf{a})$ solves (30).

Now from the form of (30) it is transparent that replacing B by $B+\lambda$ maps solutions to solutions (where $\lambda\in\mathbb{C}$ is constant). (Observe this corresponds to tensoring the irregular connection (30) by the meromorphic connection $d-\lambda\frac{dw}{w}$ on the trivial line bundle.) If B is translated in this way, then (provided the f_i are a basis) we can see how to change the corresponding Schlesinger solutions as follows. First note:

Lemma 18. *Suppose $f_1,\ldots,f_n\in V$ is an arbitrary basis, $\beta_i\in V^*$ is arbitrary, $B_i = f_i\otimes\beta_i\in \operatorname{End}(V)$ and $(B)_{ij}=\beta_i(f_j)$. Then*

$$(B_1,\ldots,B_n) \text{ is conjugate to } (E_1B,\ldots,E_nB)$$

where $E_i\in\operatorname{End}(V)$ has (i,i) entry 1 and is otherwise zero.

Proof. Define $g\in \mathrm{GL}(V)$ to have ith column f_i. Then observe that $g^{-1}B_ig = E_iB$. □

Now replacing B by $B+\lambda$ changes $B_i = gE_iBg^{-1}$ to $B_i+\lambda gE_ig^{-1} = B_i+\lambda f_i\otimes\widehat{f_i}$ and so we deduce the action of Proposition 16.

The next step is to find the action on monodromy data corresponding to the $\mathbb{C}$ action above. Suppose $B_j = f_j\otimes\beta_j$ for $j=1,\ldots,n$ and each

$$\lambda_j := \operatorname{Tr}(B_j) = \beta_j(f_j)$$

is not an integer. Then one knows that the monodromy matrix M_j around a_j is conjugate to $\exp(2\pi i B_j)$ and so is a (diagonalisable) pseudo-reflection. We will write

$$r_j = M_j = 1+e_j\otimes\alpha_j \qquad \text{where } e_j\in V, \alpha_j\in V^*$$

for this pseudo-reflection. Clearly the non-identity eigenvalue of r_j is $\exp(2\pi i\lambda_j)$ so setting $t_j=\exp(\pi i\lambda_j)$ (as in (18)) implies $t_j^2=\det(r_j)$ agreeing with the definition (12).

Now from Lemma 18 we see that the residue at infinity

$$B_\infty = -(B_1+\cdots+B_n) \quad \text{is conjugate to} \quad -(E_1B+\cdots+E_nB) = -B.$$

Thus translating B by λ implies that the translated residue at infinity $\widetilde{B}_\infty$ is conjugate to $B_\infty-\lambda$. Thus the monodromy around a large positive loop is just scaled:

$$\widetilde{r}_n\cdots\widetilde{r}_2\widetilde{r}_1 \quad \text{is conjugate to} \quad r_n\cdots r_2r_1h^2 \tag{32}$$

where

$$h := \exp(\pi i\lambda)\in\mathbb{C}^*,$$

at least if B_∞ is sufficiently generic (no distinct eigenvalues differing by integers). (Here $\widetilde{r}_1,\ldots,\widetilde{r}_n$ are the monodromy data of the connection obtained by replacing each each B_i by $\widetilde{B}_i$ in (24).)

In brief the additive action was determined by the fact that sum $\sum B_i$ was just translated by λ (assuming the f_i make up a basis, which is held fixed). We will see below that the multiplicative action (i.e. the action on monodromy data) is determined by the fact that the product $r_n\cdots r_1$ is just scaled by h^2 (assuming the e_i make up a basis, which is held fixed).

First let us recall a basic algebraic fact about pseudo-reflections.

Suppose $e_1,\ldots,e_n$ are a basis of V and $\alpha_1,\ldots,\alpha_n\in V^*$ are such that $r_i := 1+e_i\otimes\alpha_i\in \mathrm{GL}(V)$, i.e. $1+\alpha_i(e_i)\neq 0$. Define two $n\times n$ matrices t^2, u by

$$t^2 := \mathrm{diag}(1+\alpha_1(e_1),\ldots,1+\alpha_n(e_n)),\qquad (u)_{ij}=\alpha_i(e_j).$$

(We do not need to choose a square root t of t^2 at this stage, but it is convenient to keep the notation consistent with other sections of the paper.) Then let u_+, u_- be the two unipotent matrices determined by the equation

$$t^2u_+ - u_- = u \tag{33}$$

where u_+ is upper triangular with one's on the diagonal and u_- is lower triangular with one's on the diagonal.

Theorem 2 (Killing [25], Coxeter [9], Coleman [8])**.** *The matrix representing the product $r_n\cdots r_1$ (in the e_i basis) is in the big-cell of $\mathrm{GL}_n(\mathbb{C})$, and so may be written uniquely as the product of a lower triangular, a diagonal and an upper triangular matrix. Moreover this factorisation is given explicitly by $u_-^{-1}t^2u_+$:*

$$r_n\cdots r_2r_1 = u_-^{-1}t^2u_+. \tag{34}$$

Remark 19. The history of this result is discussed by Coleman [8] (cf. Corollary 3.4). Coxeter proves this for genuine reflections—i.e. coming from a symmetric bilinear form in [9]. The starting point of this paper was the simple observation that Coxeter's argument may be extended to the pseudo-reflection case. Dubrovin had used Coxeter's version in relation to Frobenius manifolds (cf. [12]) and the author was interested in extending Dubrovin's picture to the general case (cf. [4]). Despite asking various complex reflection group experts the author only found Coleman's paper (and hence the link to Killing) since [8] is in the same volume as a well-known paper of Beukers–Heckman.

It is worth clarifying the fact that generically the matrix u determines $(r_1,\ldots,r_n)$ up to conjugacy:

Lemma 20. *If $\det(u)\neq 0$ then there is a matrix $g\in\mathrm{GL}(V)$ such that, for $i=1,\ldots,n$ we have*

$$r_i = g(1+e_i^o\otimes\gamma_i)g^{-1}$$

where $\gamma_i\in V^*$ is the ith row of the matrix u and e_i^o is the standard basis of V.

Proof. By definition of u, if $\det(u)\neq 0$ then the e_i are a basis of V. Then the result follows since we know the action on a basis: $r_i(e_j)=e_j+u_{ij}e_i$. □

Note that if we define $u_\pm, t^2$ by the equation $u=t^2u_+-u_-$ then the condition that $\det(u)\neq 0$ is equivalent to saying 1 is not an eigenvalue of $u_-^{-1}t^2u_+$, since $\det(u) = \det(t^2u_+-u_-)=\det(u_-^{-1}t^2u_+-1)$.

Thus generically the n-tuple $(r_1,\ldots,r_n)$ is determined up to overall conjugacy by the matrix u, and in turn by the product $r_n\cdots r_1$, by Theorem 2. From (32) the obvious guess is therefore that that $\widetilde{r}_n \cdots \widetilde{r}_1 = u_-^{-1} t^2 u_+ h^2 = u_-^{-1} h^2 t^2 u_+$ so that $\widetilde{\alpha}_i(\widetilde{e}_j) = h^2 t^2 u_+ - u_-$, which should determine $(\widetilde{r}_1, \ldots, \widetilde{r}_n)$ up to overall conjugacy. The following theorem says that this is indeed the case, at least generically. Suppose each λ_i and each eigenvalue of $\sum B_i$ is not an integer (and that the same holds after translation by λ). Then we have:

Theorem 3 (Balser–Jurkat–Lutz [1]). *Let $\widetilde{u}$ be the the matrix*

$$\widetilde{u} = h^2 t^2 u_+ - u_-$$

where $u_\pm, t^2, h$ are as defined above. Then there is a basis $\widetilde{e}_i$ of $V = \mathbb{C}^n$ and $\widetilde{\alpha}_1, \ldots \widetilde{\alpha}_n \in V^*$ such that $\widetilde{r}_i = 1 + \widetilde{e}_i \otimes \widetilde{\alpha}_i$ and $(\widetilde{u})_{ij} = \widetilde{\alpha}_i(\widetilde{e}_j)$ for all i, j.

Remark 21. This is not written down in precisely this way in [1] so we will describe how to extract it in Appendix A. The key point is that the matrices $u_\pm$ are essentially the Stokes matrices of the irregular connection (31) and are easily seen to be preserved under the scalar shift. Then one computes bases of solutions of the Fuchsian connection as Laplace transforms of standard bases of solutions of (31) and this enables the Stokes matrices to be related to the pseudo-reflection data u as in equation (33). The observation that this implies the Fuchsian monodromy data and the Stokes data are then related by the beautiful equation (34) in Theorem 2 does not seem to appear in [1]. In summary we see that equation (34) is the manifestation of the Fourier–Laplace transformation on monodromy data, relating the monodromy data of the Fuchsian connection to the monodromy/Stokes data of the corresponding irregular connection.

In other words: in general the matrix u determines $(r_1, \ldots, r_n)$ up to overall conjugation and Theorem 3 explains how the matrix u varies: the lower triangular part is fixed, the upper triangular part is scaled by h^2, and the diagonal part $t^2 - 1$ becomes $h^2 t^2 - 1$. Let us make this more explicit in the $n = 3$ case. We start with a connection

$$d - \sum_1^3 \frac{B_i}{z - a_i} dz \tag{35}$$

where, up to overall conjugation:

$$B_1 = \begin{pmatrix} \lambda_1 & b_{12} & b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & \lambda_3 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{31} & b_{32} & \lambda_3 \end{pmatrix}$$

for some numbers b_{ij} with $i \neq j$. Then we take the monodromy data of this and obtain pseudo-reflections r_1, r_2, r_3 which, up to overall conjugation, are of the form

$$r_1 = \begin{pmatrix} t_1^2 & u_{12} & u_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad r_2 = \begin{pmatrix} 1 & 0 & 0 \\ u_{21} & t_2^2 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}, \qquad r_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u_{31} & u_{32} & t_3^2 \end{pmatrix}$$

where $t_j = \exp(\pi i \lambda_j)$. Then we replace λ_i by $\lambda_i + \lambda$ in (35) for each i, and Theorem 3 says that the monodromy of the resulting connection is conjugate to

$$\widetilde{r}_1 = \begin{pmatrix} h^2 t_1^2 & h^2 u_{12} & h^2 u_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \widetilde{r}_2 = \begin{pmatrix} 1 & 0 & 0 \\ u_{21} & h^2 t_2^2 & h^2 u_{23} \\ 0 & 0 & 1 \end{pmatrix}, \ \widetilde{r}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u_{31} & u_{32} & h^2 t_3^2 \end{pmatrix} \tag{36}$$

where $h := \exp(\pi i \lambda)$.

Taking the various traces yields the fact that the invariant functions of the monodromy matrices are related as:

$$\widetilde{t}_i^2 = h^2 t_i^2, \qquad \widetilde{t}_{ij} = h^2 t_{ij}, \qquad \widetilde{t}_{321} = h^2 t_{321}, \qquad \widetilde{t}'_{321} = h^4 t'_{321}.$$

These equations hold for any λ since the invariants are analytic functions of the coefficients B_i and so vary holomorphically with the parameter λ. This motivates the definition of the $\mathbb{C}^*$ action in (22), and in turn this yields the definition of the map φ as explained just after Lemma 13, by taking the projection to $\mathrm{SL}_2(\mathbb{C})$ of the rank two part of the semisimplification of (36) when $\lambda = -\mu_1$.

Braid group actions.

Let us check that the $\mathbb{C}^*$ action commutes with the braid group action on the level of the matrices u. (One suspects this is the case since the braid group actions are obtained by integrating the isomonodromy equations, and we have seen in Proposition 16 that the corresponding $\mathbb{C}$ action commutes with the Schlesinger flows.)

The standard braid group action of the n-string braid group B_n on n-tuples of pseudo-reflections may be given by generators $\gamma_1, \ldots, \gamma_{n-1}$ with γ_i acting as

$$\gamma_i(r_n, \ldots, r_1) = (\ldots, r_{i+2}, r_i, r_i^{-1} r_{i+1} r_i, r_{i-1}, \ldots)$$

only affecting r_i, r_{i+1} and preserving the product $r_n \cdots r_1$. (For $n = 3$ we previously labeled the generators differently: $\beta_1 = \gamma_2, \beta_2 = \gamma_1$.) Now suppose we write $r_i = 1 + e_i \otimes \alpha_i$ with $e_i \in V, \alpha_i \in V^*$, where $V = \mathbb{C}^n$. Let us restrict to the case where the r_i are linearly independent (in the sense that any such e_i form a basis of V). Then it is easy to lift the above B_n action to an action on the $2n^2$-dimensional space

$$W := \{\ (e_n, \ldots, e_1, \alpha_n, \ldots \alpha_1) \ \big|\ \{e_i\} \text{ a basis of } V,\ \alpha_i \in V^*, \alpha_i(e_i) \neq -1\ \} \tag{37}$$

by letting γ_i fix all e_j, α_j except for $j = i, i+1$:

$$\gamma_i(\ldots, e_{i+1}, e_i, \ldots, \alpha_{i+1}, \alpha_i, \ldots) = (\ldots, e_i, r_i^{-1} e_{i+1}, \ldots, \alpha_i, \alpha_{i+1} \circ r_i, \ldots)$$

where $r_i := 1 + e_i \otimes \alpha_i \in \mathrm{GL}(V)$. (We think of W as the multiplicative analogue of the space on which the lifted equations (29) were defined.) It is simple to check this action is well-defined on W.

Now we may project this lifted B_n-action to the space of the matrices u. Recall the $n \times n$ matrix u was defined by setting $u_{ij} = \alpha_i(e_j)$. By a straightforward computation we find that, if we set $u' = \gamma_i(u)$ then $u'_{jk} = u_{jk}$ unless one of j or k equals i or $i+1$, and

$$\begin{aligned} u'_{ii} &= u_{i+1i+1}, & u'_{i+1i+1} &= u_{ii} \\ u'_{ii+1} &= t_i^2 u_{i+1i}, & u'_{i+1i} &= u_{ii+1}/t_i^2 \\ u'_{ij} &= u_{i+1j} + u_{i+1i} u_{ij}, & u'_{ji} &= u_{ji+1} - u_{ji} u_{ii+1}/t_i^2 \\ u'_{i+1j} &= u_{ij}, & u'_{ji+1} &= u_{ji} \end{aligned}$$

for any $j \notin \{i, i+1\}$ where $t_i^2 := 1 + u_{ii}$.

In turn u contains precisely the same data as the matrix

$$u_-^{-1} t^2 u_+ \in G^0 \subset \mathrm{GL}_n(\mathbb{C})$$

where $u_\pm, t^2$ are determined by the equation $u = t^2 u_+ - u_-$, and G^0 denotes the big-cell, consisting of the invertible matrices that may be factorised as the product of a lower triangular and an upper triangular matrix. Thus the B_n-action on $\{u\}$ is equivalent to a B_n-action on G^0. Let us describe this. First let $P_i \in \mathrm{GL}_n(\mathbb{C})$ denote the permutation matrix corresponding to the permutation swapping i and $i+1$. Thus P_i equals the identity matrix except in the 2×2 block in the $i, i+1$ position on the diagonal, where it equals $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Also for any unipotent upper triangular matrix u_+, let $\xi_i(u_+)$ denote the matrix which equals the identity matrix except in the 2×2 block in the $i, i+1$ position on the diagonal, where it equals

$$\begin{pmatrix} 1 & (u_+)_{ii+1} \\ 0 & 1 \end{pmatrix}.$$

(This map ξ_i defines a homomorphism from U_+ to the root group of $\mathrm{GL}_n(\mathbb{C})$ corresponding to the ith simple root—cf. e.g. (3.10) [6].)

Proposition 22. *The induced B_n-action on G^0 is given by the formula*

$$\gamma_i(a) = P_i \xi_i(u_+) a \xi_i(u_+)^{-1} P_i.$$

where $a = u_-^{-1} t^2 u_+ \in G^0$.

Proof. Lifting back up to W, write $r_j = 1 + e_j \otimes \alpha_j$ and denote

$$(\ldots, e'_{i+1}, e'_i, \ldots, \alpha'_{i+1}, \alpha'_i, \ldots) = \gamma_i(\ldots, e_{i+1}, e_i, \ldots, \alpha_{i+1}, \alpha_i, \ldots).$$

Note that the product $R := r_n \cdots r_1 \in \mathrm{GL}(V)$ is fixed by the B_n action. By Theorem 2 the matrix for R in the e_j basis of V is $a = u_-^{-1} t^2 u_+$, and in the e'_j basis the matrix for R is $\gamma_i(a)$. Thus $\gamma_i(a) = S^{-1} a S$ where S is the matrix for the change of basis from $\{e_j\}$ to $\{e'_j\}$. From the formula for the action on the e_j, S equals the identity matrix except in the 2×2 block in the $i, i+1$ position on the diagonal, where it equals

$$\begin{pmatrix} -u_{ii+1}/t_i^2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & u_{ii+1}/t_i^2 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Finally from the equation $u = t^2 u_+ - u_-$ we see $u_{ii+1}/t_i^2 = (u_+)_{ii+1}$ so $S = \xi_i(u_+)^{-1} P_i$. □

Remark 23. This B_n action on the big-cell also appears as the classical limit of the so-called quantum Weyl group actions (cf. [10] and [6] Remark 3.8), provided we use the permutation matrices rather than Tits' extended Weyl group. Thus we have shown, for $\mathrm{GL}_n(\mathbb{C})$, how the classical action of the quantum Weyl group is related to the standard action of B_n on n-tuples of pseudo-reflections. Presumably this is related to Toledano Laredo's proof [31], for $\mathrm{GL}_n(\mathbb{C})$ of the Kohno–Drinfeld theorem for quantum Weyl groups.

Corollary 24. *The $\mathbb{C}^*$ action on $\{u\}$ commutes with the B_n-action defined above.*

Proof. On passing to G^0, we recall that the $\mathbb{C}^*$ action just scales t^2 and leaves both $u_\pm$ fixed. However t^2 does not appear in the formula of Proposition 22 for the B_n action. □

One can now see directly why the map φ will be B_3 equivariant. Upon using the $\mathbb{C}^*$ action to make 1 an eigenvalue of $r_3 r_2 r_1$ we know that the r_i are all block triangular in some basis. Then we just note the obvious fact that the braid group action (13) on the pseudo-reflections restricts to the action (7) in the 2×2 block on the diagonal.

4. Jimbo's leading term formula

So far we have described how to find some $\mathrm{SL}_2(\mathbb{C})$ triples living in finite orbits of the braid group, and read off some properties of the corresponding solution to Painlevé VI (in particular we saw that the set of branches of the solution correspond to the orbit under the pure braid group of the conjugacy classes of such triples). In this section and the next we will describe a method to find the corresponding solution explicitly. This method is quite general and should work with any sufficiently generic SL_2 triple in a finite braid group orbit—in particular it is not a priori restricted to any one-parameter family of Painlevé VI equations. (One just needs to check conditions b),c),d) below for each branch of the solution.)

The crucial step is the following formula:

Theorem 4. (M. Jimbo [22]) *Suppose we have four matrices $M_j \in \mathrm{SL}_2(\mathbb{C})$, $j = 0, t, 1, \infty$ satisfying*

a) $M_\infty M_1 M_t M_0 = 1$,

b) M_j *has eigenvalues* $\{\exp(\pm\pi i\theta_j)\}$ *with* $\theta_j \notin \mathbb{Z}$,

c) $\mathrm{Tr}(M_0 M_t) = 2\cos(\pi\sigma)$ *for some nonzero* $\sigma \in \mathbb{C}$ *with* $0 \le \mathrm{Re}(\sigma) < 1$,

d) None of the eight numbers

$$\theta_0 \pm \theta_t \pm \sigma, \quad \theta_0 \pm \theta_t \mp \sigma, \quad \theta_\infty \pm \theta_1 \pm \sigma, \quad \theta_\infty \pm \theta_1 \mp \sigma$$

is an even integer.

Then the leading term in the asymptotic expansion at zero of the corresponding Painlevé VI solution $y(t)$ on the branch corresponding to $[(M_0, M_t, M_1)]$ is

$$\frac{(\theta_0+\theta_t+\sigma)(-\theta_0+\theta_t+\sigma)(\theta_\infty+\theta_1+\sigma)}{4\sigma^2(\theta_\infty+\theta_1-\sigma)\widehat{s}}t^{1-\sigma} \tag{38}$$

where

$$\widehat{s} = c \times s, \qquad s = \frac{a+b}{d}$$

$$a = e^{\pi i\sigma}(i\sin(\pi\sigma)\cos(\pi\sigma_{1t}) - \cos(\pi\theta_t)\cos(\pi\theta_\infty) - \cos(\pi\theta_0)\cos(\pi\theta_1))$$

$$b = i\sin(\pi\sigma)\cos(\pi\sigma_{01}) + \cos(\pi\theta_t)\cos(\pi\theta_1) + \cos(\pi\theta_\infty)\cos(\pi\theta_0)$$

$$d = 4\sin\left(\frac{\pi}{2}(\theta_0+\theta_t-\sigma)\right)\sin\left(\frac{\pi}{2}(\theta_0-\theta_t+\sigma)\right)\sin\left(\frac{\pi}{2}(\theta_\infty+\theta_1-\sigma)\right)\sin\left(\frac{\pi}{2}(\theta_\infty-\theta_1+\sigma)\right)$$

$$c = \frac{(\Gamma(1-\sigma))^2\,\widehat{\Gamma}(\theta_0+\theta_t+\sigma)\,\widehat{\Gamma}(-\theta_0+\theta_t+\sigma)\,\widehat{\Gamma}(\theta_\infty+\theta_1+\sigma)\,\widehat{\Gamma}(-\theta_\infty+\theta_1+\sigma)}{(\Gamma(1+\sigma))^2\,\widehat{\Gamma}(\theta_0+\theta_t-\sigma)\,\widehat{\Gamma}(-\theta_0+\theta_t-\sigma)\,\widehat{\Gamma}(\theta_\infty+\theta_1-\sigma)\,\widehat{\Gamma}(-\theta_\infty+\theta_1-\sigma)}$$

where $\widehat{\Gamma}(x) := \Gamma(\frac{1}{2}x+1)$ *(with Γ being the usual gamma function) and where $\sigma_{jk} \in \mathbb{C}$ (for $j, k \in \{0, t, 1\}$) is determined by* $\mathrm{Tr}(M_j M_k) = 2\cos(\pi\sigma_{jk}), 0 \le \mathrm{Re}(\sigma_{jk}) \le 1$, *so* $\sigma = \sigma_{0t}$.

Remark 25. The formula (38) is computed directly from the formula [22] (2.15) for the asymptotics as $t \to 0$ for the coefficients of the isomonodromic family of rank two systems. The formulae for $\widehat{s}$ and s are as in [22] except for a sign difference in s. Since this formula is crucial for us and since s is not derived in [22] we will give a derivation in the appendix.

Remark 26. D. Guzzetti repeated Jimbo's computations in [15] Section 8.3 and Appendix, but did not reduce the formula to as short a form; see [15] (A.6) and (A.30) (but note (A.30) is not quite correct but is easily corrected by examining (A.28) and (A.29)). However we can state that the corrected version of Jimbo's formula agrees numerically with the corrected version of Guzzetti's (at least for the values of the parameters used in this paper, and for those of several hundred randomly chosen $\mathrm{SL}_2(\mathbb{C})$ triples). It is puzzling that Guzzetti does not state that his formula does not agree with Jimbo's.[3]

Remark 27. To agree with Jimbo's notation we are thus relabeling the triples (M_1, M_2, M_3) as (M_0, M_t, M_1) as well as the θ parameters $(\theta_1, \theta_2, \theta_3, \theta_4) \mapsto (\theta_0, \theta_t, \theta_1, \theta_\infty)$. To keep track of this it is perhaps simplest to bear in mind the corresponding monodromy relations

$$M_4 M_3 M_2 M_1 = 1 \qquad \text{and} \qquad M_\infty M_1 M_t M_0 = 1.$$

5. The Klein solution

For the Klein solution σ is either $1/2$ or $1/3$ depending on the branch. If the solution is to be algebraic then Jimbo's formula will give the leading term in the *Puiseux* expansion at 0 of each branch of the solution. Thus we find the leading term on the jth branch is of the form $C_j t^{1-\sigma_j}$ where

$$C_j = \frac{57}{28\widehat{s}_j}$$

on the four branches with $\sigma = 1/2$ $(j = 0, 1, 2, 3)$ and

$$C_j = \frac{475}{308\widehat{s}_j}$$

on the other three branches, having $\sigma = 1/3$ $(j = 4, 5, 6)$. Now we would like to evaluate these precisely on each branch and identify them as algebraic numbers. A simple numerical inspection shows that C_0 has argument $\pi/4$, C_6 is real and negative,

$$C_1 = -iC_0, \qquad C_2 = iC_0, \qquad C_3 = -C_0$$

and

$$C_4 = \exp(-2\pi i/3)C_6, \qquad C_5 = \exp(2\pi i/3)C_6.$$

Thus we would hope that C_0^4 and C_6^3 are rational numbers. Using Maple we calculate the various $\widehat{s}_j$'s numerically and then deduce:

$$C_0^4 = -7/3^4, \qquad \text{so that} \qquad C_0 = \frac{(1+i)7^{1/4}}{3\sqrt{2}}$$

$$C_6^3 = -5^3/14, \qquad \text{so that} \qquad C_6 = \frac{-5}{14^{1/3}}.$$

Thus we now know precisely the leading coefficient C_j of the Puiseux expansion at 0 of each branch of the solution $y(t)$. By substituting back into the Painlevé VI equation these leading terms determine, algebraically, any desired term in the Puiseux expansion. If the solution is to be algebraic it should satisfy an equation of the form

$$F(t, y(t)) = 0$$

[3] Also there is some confusion as to the range of validity of Jimbo's work: namely the restriction $0 \le \mathrm{Re}(\sigma) < 1$, is equivalent to $\mathrm{Tr}(M_0 M_t) \notin \mathbb{R}_{\le -2}$ rather than the much stronger condition $|\mathrm{Tr}(M_0 M_t)| \le 2$ and $\mathrm{Re}\mathrm{Tr}(M_0 M_t) \neq -2$ appearing in [15] (1.30).

for some polynomial $F(t,y)$ in two variables. Since the solution has 7 branches F should have degree 7 in y. Let us write F in the form

$$F = q(t)y^7 + p_6(t)y^6 + \cdots + p_1(t)y + p_0(t)$$

for polynomials p_i, q in t and define rational functions $r_i(t) := p_i/q$ for $i = 0,\dots,6$. If $y_0,\dots,y_6$ denote the (locally defined) solutions on the branches then for each t we have that $y_0(t),\dots,y_6(t)$ are the roots of $F(t,y)=0$ and it follows that

$$y^7 + r_6(t)y^6 + \cdots + r_1(t)y + r_0(t) = (y-y_0(t))(y-y_1(t))\cdots(y-y_6(t)).$$

Thus, expanding the product on the right, the rational functions r_i are obtained as symmetric polynomials in the y_i:

$$r_0 = -y_0\cdots y_6,\ \dots\ , r_6 = -y_0 - \cdots - y_6.$$

Since the r_i are global rational functions, the Puiseux expansions of the y_i give the Laurent expansions at 0 of the r_i. Clearly only a finite number of terms of each Laurent expansion are required to determine each r_i, and indeed it is simple to convert these truncated Laurent expansions into global rational functions. Clearing the denominators then yields the solution curve:

$$F(t,y) =$$

$$\left(162\,t^3 - 243\,t^2 - 243\,t + 162\right)y^7 + \left(-567\,t^3 + 2268\,t^2 - 567\,t\right)y^6 +$$

$$\left(-1701\,t^3 - 1701\,t^2\right)y^5 + \left(1407\,t^4 + 2856\,t^3 + 1407\,t^2\right)y^4 +$$

$$\left(14\,t^5 - 2849\,t^4 - 2849\,t^3 + 14\,t^2\right)y^3 + \left(-21\,t^5 + 3444\,t^4 - 21\,t^3\right)y^2 +$$

$$\left(-567\,t^5 - 567\,t^4\right)y + (125\,t^6 - 88\,t^5 + 125\,t^4).$$

One may easily check on a computer that this curve has precisely the right monodromy over the t-line (and in particular is genus zero, and has monodromy group A_7). Also one finds that it has 10 singular points; 6 double points over $\mathbb{C}\setminus\{0,1\}$ and 4 more serious singularities over the branch points. Finally since it is a genus zero curve we can look for a rational parameterisation. Using the CASA package [17] we find the solution may be parameterised quite simply as:

$$y = -\frac{(8\,s^2 - 18\,s + 81)\,(8\,s^2 - 36\,s + 81)\,(s-9)^2}{27\,s\,(2\,s-9)\,(2\,s+9)\,(4\,s^2 - 27\,s + 81)},$$

$$t = -2\,\frac{(8\,s^2 - 18\,s + 81)^2\,(s-9)^3}{(4\,s^2 - 27\,s + 81)^2\,(2\,s+9)^3}.$$

Note that the polynomial F defining the solution curve is quite canonical but there are many possible parameterisations. Using the parameterisation it is easy to carry out the ultimate test and substitute back into the Painlevé VI equation (with parameters $(\alpha,\beta,\gamma,\delta) = (9,-4,4,45)/98$) finding that we do indeed have a solution.

6. Inequivalence Theorem

We know (cf. Remark 14 above and [13] Remark 0.2) that the five 'platonic' solutions of [13] are equivalent (via Okamoto transformations) to solutions associated to finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. In other words, even though the unipotent matrices (23) generate an infinite group, there is an equivalent solution with finite 2×2 monodromy group.

This raises the following question: Even though the 2×2 monodromy data (21) associated to the Klein solution generates an infinite group, is there an equivalent solution with finite 2×2 monodromy? We will prove this is not the case:

Theorem 5. *Suppose there is an algebraic solution of some Painlevé VI equation which is equivalent to the Klein solution under Okamoto's affine F_4 action. Then the corresponding 2×2 monodromy data (M_1, M_2, M_3) also generate an infinite subgroup of* $\mathrm{SL}_2(\mathbb{C})$.

Proof. First the parameters $(\theta_1, \theta_2, \theta_3, \theta_4)$ should be equivalent to the corresponding parameters $(2, 2, 2, 4)/7$ of the Klein solution. If (M_1, M_2, M_3) generate a binary tetrahedral, octahedral or icosahedral group then we will not be able to get any sevens in the denominators (since these groups have no elements of order seven) so any solution associated to these groups is inequivalent to the Klein solution. (This uses the simple observation that Okamoto's transformations act within the ring $\mathbb{Z}[\frac{1}{2}, \theta_1, \theta_2, \theta_3, \theta_4]$ so that if the θ_i are rational numbers with no sevens in the denominators, then no equivalent set of parameters has a seven in any denominator.)

Next, suppose $F(y, t) = 0$ is the curve defining the Klein solution and $F_1(y_1, t_1) = 0$ is the curve defining the equivalent solution. Then we know ([29] p.361) that t, t_1 are related by a Mobius transformation permuting $0, 1, \infty$.

Lemma 28. *There is an isomorphism of the curves $F(y, t) = 0$ and $F_1(y_1, t_1) = 0$ covering the automorphism of the projective line mapping t to t_1.*

Proof. Let us recall some facts about Okamoto's transformations from [29, 28, 27]. First write $q := y, q_1 := y_1$. Then, from the formulae for the action of the Okamoto transformations [28] Table 1, [27] (7.14), we see that q_1 is a rational function of q, p, t, where p is the conjugate variable to q in the Hamiltonian formulation of Painlevé VI (cf. [29] (0.6)). The first of Hamilton's equations says $\frac{dq}{dt} = \frac{\partial H}{\partial p}$, where $H = H_{VI}$ is the Hamiltonian [29] p.348. By observing that H is a quadratic polynomial in p (and rational in t and polynomial in q) we deduce immediately that p is a rational function of $\frac{dq}{dt}, q$ and t. Moreover since $q = y$ satisfies the polynomial equation $F(y, t) = 0$ implicit differentiation enables us to express $\frac{dq}{dt}$ as a rational function of q, t. Thus p is a rational function of just q, t and so in turn q_1 is a rational function of just q, t.

Now by the symmetry of the situation the same argument also shows q is a rational function of q_1, t_1. This sets up an isomorphism between the fields $\mathbb{C}(q, t) \cong \mathbb{C}(q_1, t_1)$ extending the isomorphism $\mathbb{C}(t) \cong \mathbb{C}(t_1)$ given by mapping t to t_1. Dualising this gives the desired isomorphism of the corresponding curves. □

In particular we see that F_1 must have degree seven in y_1, since the curves have the same number of branches. This implies (M_1, M_2, M_3) cannot generate a cyclic group, since cyclic groups are abelian and so the pure braid group acts trivially; all such solutions have just one branch.

Finally we need to rule out the binary dihedral groups which will need more work. Write the elements of the binary dihedral group of order $4d$ as

$$\widetilde{I_2(d)} = \{1, \zeta, \dots, \zeta^{2d-1}, \tau, \tau\zeta, \dots, \tau\zeta^{2d-1}\}$$

where $\zeta := \left(\begin{smallmatrix} \varepsilon & \\ & \varepsilon^{-1} \end{smallmatrix}\right)$, $\tau := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\varepsilon = \exp(\pi i/d)$.

Below we will abbreviate $\tau\zeta^a$ as just τa and ζ^a as a.

The basic strategy is to go through all possible triples of elements and show in each case that, on conjugacy classes, the generators $p_1 := \beta_1^2, p_2 := \beta_2^2$ of the pure braid group action cannot have two two-cycles and a three-cycle.

The basic relations we will use repeatedly are:

$$\tau\zeta^k = \zeta^{-k}\tau, \qquad \zeta^{2d} = 1.$$

First let us record the formulae for the action of p_1 on all possible pairs of elements. (Here β_1 acts by mapping a pair (x, y) of elements to $(y, y^{-1}xy)$, and p_1 is the square of β_1.)

Lemma 29. *Suppose a, b are arbitrary integers. Then, in abbreviated form:*

$$\begin{aligned} p_1(a, b) &= (a, b), \\ p_1(\tau a, b) &= (\tau(a + 2b), -b), \\ p_1(a, \tau b) &= (-a, \tau(b - 2a)), \\ p_1(\tau a, \tau b) &= (\tau(2b - a), \tau(3b - 2a)). \end{aligned}$$

Proof. Straightforward computation. □

Now, on triples, β_1 (respectively β_2) maps (x, y, z) to $(y, y^{-1}xy, z)$ (resp. $(x, z, z^{-1}yz)$). Immediately we see that any triple of the form

$$(a, b, c), \qquad (\tau a, b, c), \quad \text{or} \quad (a, b, \tau c)$$

will be fixed by one or both of p_1, p_2. Thus the corresponding permutation representation will have a one-cycle, which is not permitted.

In general the triples of elements of $\widetilde{I_2(d)}$ fall into eight 'types' depending on if each element contains a τ or not. From Lemma 29 the pure braid group clearly takes triples to triples of the same type. After the three types already dealt with the next four are:

$$(a, \tau b, c), \qquad (\tau a, \tau b, c), \qquad (a, \tau b, \tau c), \qquad (\tau a, b, \tau c).$$

For each of these one finds, from Lemma 29, that either p_1^2 or p_2^2 (or both) act trivially. This implies that there will be no three-cycles in the permutation representation of one or both of p_1 or p_2 on conjugacy classes of such triples.

Finally we need to rule out the triples of type $(\tau a, \tau b, \tau c)$. First let us note that the conjugacy class of τ has size d and contains the elements $\tau(2a)$ for any integer a, and the conjugacy class of $\tau\zeta$ has size d and contains the elements $\tau(2a+1)$. It follows that, upto overall conjugacy, we have:

$$p_1(\tau a, \tau b, \tau c) \cong (\tau a, \tau b, \tau(c - k)) \quad \text{where } k := 2(b - a), \text{ and} \tag{39}$$

$$p_2(\tau a, \tau b, \tau c) \cong (\tau(a - l), \tau b, \tau c) \quad \text{where } l := 2(c - b).$$

Moreover the only (possibly distinct) triple of the form $(\tau p, \tau b, \tau q)$ that is conjugate to $(\tau a, \tau b, \tau c)$ is $(\tau(2b - a), \tau b, \tau(2b - c))$, which is obtained by conjugating by τb.

Lemma 30. *Let $o(k)$ be the order of the element ζ^k where $k = 2(b-a)$. Then in the permutation representaion of p_1 the conjugacy classes of the triple through $(\tau a, \tau b, \tau c)$ lies in a cycle of length $o(k)$.*

Proof. First if $\tau a = \tau(2b-a)$ then $\zeta^k = 1$ so $o(k) = 1$ and (39) says the conjugacy class of $(\tau a, \tau b, \tau c)$ is fixed by p_1.

Secondly if $\tau a \neq \tau(2b-a)$, i.e. $o(k) > 1$ then by (39) we see $p_1^r(\tau a, \tau b, \tau c) \cong (\tau a, \tau b, \tau(c - rk))$. This is conjugate to $(\tau a, \tau b, \tau c)$ if and only if $\zeta^{-rk} = 1$ (using the fact that $\tau(2b-a) \neq \tau a$) i.e. if and only if r is divisible by $o(k)$. Thus we are in a cycle of length $o(k)$. □

Similarly if $o(l)$ is the order of ζ^l where $l = 2(c-b)$ then the conjugacy class of $(\tau a, \tau b, \tau c)$ is an a cycle of p_2 of length $o(l)$.

Thus in order to be equivalent to the Klein solution we need $o(k), o(l) \in \{2, 3\}$ for all the possible k's and l's that occur in the orbit. It is straightforward to check this is not possible: First from (39) note that p_1 maps the pair of integers $[k, l]$ to $[k, l-2k]$ and similarly $p_2[k, l] = [k + 2l, l]$. Thus:

i) If $o(k) = o(l) = 2$ then $o(l-2k) = o(k+2l) = 2$ and, repeating, we see only two-cycles appear in the orbit, whereas we need a three-cycle.

ii) If $o(k) = 2, o(l) = 3$ then $o(k + 2l) = 6$ and so we get an unwanted six-cycle. (Similarly if $o(k) = 3, o(l) = 2$.)

iii) If $o(k) = o(l) = 3$ then $\zeta^{k+2l}, \zeta^{l-2k}$ each have order either one or three. Thus either an unwanted one-cycle appears or we only get three cycles; no two-cycles appear.

Thus we conclude that the Klein solution is not equivalent to any solution coming from a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. □

7. Reconstruction

Given a triple r_1, r_2, r_3 of generators of a three-dimensional complex reflection group, we have explained how to obtain an $\mathrm{SL}_2(\mathbb{C})$ triple M_1, M_2, M_3 (in an isomorphic braid group orbit) and then how, if Jimbo's formula is applicable, to obtain an algebraic solution $y(t)$ to the sixth Painlevé equation.

In this section we will explain how to obtain from $y(t)$ a rank three Fuchsian system with four poles on $\mathbb{P}^1$ and monodromy conjugate to the original complex reflection group (generated by three reflections).

First we recall (from [23]) that the solution $y(t)$ and its derivative determine algebraically an $\mathfrak{sl}_2$ two system

$$\frac{d\Phi}{dz} = A(z)\Phi; \qquad A(z) = \sum_{i=1}^{3} \frac{A_i}{z - a_i} \tag{40}$$

with monodromy (M_1, M_2, M_3), where $(a_1, a_2, a_3) = (0, t, 1)$, with respect to some choice of loops generating the fundamental group of the four-punctured sphere. (The exact formulae will be given below.) Now define

$$\widehat{A}_i = A_i + \theta_i/2$$

for $i=1,2,3$, so that $\widehat{A}_i$ has rank one (and eigenvalues $\{0,\theta_i\}$). Then the system

$$\frac{d}{dz}-\sum_{i=1}^{3}\frac{\widehat{A}_i}{z-a_i} \tag{41}$$

has monodromy $(\widehat{M_1},\widehat{M_2},\widehat{M_3})$ where $\widehat{M_i}=M_i\exp(\pi\sqrt{-1}\theta_i)$, which are pseudo-reflections in $\mathrm{GL}_2(\mathbb{C})$. Write these rank one matrices as

$$\widehat{A}_i = h_i\otimes\gamma_i \quad \text{for some } h_i\in\mathbb{C}^2, \gamma_i\in(\mathbb{C}^2)^*,\ \ i=1,2,3.$$

In general the span of the h_i will be two dimensional and without loss of generality we will suppose that h_1,h_2 are linearly independent (otherwise we can relabel below). Now consider the three 3×3 rank one matrices given by

$$B_i := \begin{pmatrix} \widehat{A}_i & \begin{matrix}0\\0\end{matrix} \\ c_i\gamma_i & 0\end{pmatrix} \qquad i=1,2,3$$

for some constants c_1,c_2,c_3 and the corresponding rank three system

$$\frac{d}{dz}-\sum_{i=1}^{3}\frac{B_i}{z-a_i}. \tag{42}$$

By overall conjugation, since h_1,h_2 are linearly independent, we can always assume $c_1=c_2=0$. Now if $c_3=0$ then (42) is block diagonal and reduces to (41). However if $c_3\neq 0$ we obtain a rank three system with

$$B_i = f_i\otimes\beta_i$$

for a basis f_i of $V:=\mathbb{C}^3$—namely:

$$f_1=\begin{pmatrix}h_1\\0\end{pmatrix},\ f_2=\begin{pmatrix}h_2\\0\end{pmatrix},\ f_3=\begin{pmatrix}h_2\\c_3\end{pmatrix},\quad \beta_i=\begin{pmatrix}\gamma_i & 0\end{pmatrix}.$$

Moreover, up to overall conjugation this system is independent of the choice of nonzero c_3 (since conjugating by $\mathrm{diag}(1,1,c)$ scales c_3 arbitrarily).

In particular the invariant functions of the monodromy of the system (42) are independent of c_3 and are equal to the invariants of the monodromy of the limiting system with $c_3=0$, since the invariants are holomorphic functions of any parameters.

Now we can perform the scalar shift of section 3 in reverse. Namely, in the f_i basis (B_1,B_2,B_3) have the form

$$B_1=\begin{pmatrix}\widetilde{\lambda}_1 & b_{12} & b_{13}\\ 0&0&0\\0&0&0\end{pmatrix},\qquad B_2=\begin{pmatrix}0&0&0\\ b_{21} & \widetilde{\lambda}_3 & b_{23}\\ 0&0&0\end{pmatrix},\qquad B_3=\begin{pmatrix}0&0&0\\0&0&0\\ b_{31}&b_{32}&\widetilde{\lambda}_3\end{pmatrix} \tag{43}$$

for some numbers $b_{ij},\widetilde{\lambda}_i$. Then the scalar shift just translates each $\widetilde{\lambda}_i$ by the same scalar.

If (as we are assuming) we started with a solution $y(t)$ as constructed with the procedure of this paper then $\widetilde{\lambda}_i=\lambda_i-\mu_1$, where λ_i,μ_i are related as in (18) to the original complex reflections r_1,r_2,r_3.

Theorem 6. *The system obtained by replacing each* $\widetilde{\lambda}_i$ *by* λ_i *in* (43) *has monodromy conjugate to* (r_1, r_2, r_3). *In other words there is a choice of fundamental solution* Φ *and of simple positive loops* l_i *around* a_i *for* $i = 1, 2, 3$ *generating* $\pi_1(\mathbb{P}^1 \setminus \{a_1, a_2, a_3, \infty\})$ *such that* Φ *has monodromy* r_i *around* l_i.

Proof. Consider the system obtained by replacing $\widetilde{\lambda}_i$ by $\lambda_i + \lambda$ for each i, for varying λ (so $\lambda = -\mu_1$ is the original system). Write

$$\widehat{\mathbf{t}}(\lambda) = (t_i^2(\lambda), t_{ij}(\lambda), t_{321}(\lambda), t'_{321}(\lambda))$$

for the invariant functions of the monodromy of the corresponding system. These functions vary holomorphically with λ for any $\lambda \in \mathbb{C}$.

Write $r'_i = 1 + e_i \otimes \alpha_i, u_{ij} = \alpha_i(e_j)$ for the monodromy data at $\lambda = 0$. By construction the eigenvalues of r'_1, r'_2, r'_3 and of the product $r'_3 r'_2 r'_1$ are the same as those of the r_i (since they are determined by the residues of the Fuchsian system).

Now the invariants $\widehat{\mathbf{t}}(0)$ are easily expressed in terms of u (cf. proof of Lemma 2) and we know how u varies with λ (Theorem 3). It follows that

$$\widehat{\mathbf{t}}(\lambda) = (h^2 t_i^2, h^2 t_{ij}, h^2 t_{321}, h^4 t'_{321}) \tag{44}$$

where $(t_i^2, t_{ij}, t_{321}, t'_{321}) = \widehat{\mathbf{t}}(0)$ and $h = \exp(\pi i \lambda)$.

By construction we know the invariants $\widehat{\mathbf{t}}(-\mu_1)$ of the original system, namely they equal the invariants of the block diagonal monodromy data

$$\begin{pmatrix} \widehat{M_1} & \\ & 1 \end{pmatrix}, \qquad \begin{pmatrix} \widehat{M_2} & \\ & 1 \end{pmatrix}, \qquad \begin{pmatrix} \widehat{M_3} & \\ & 1 \end{pmatrix},$$

which is the monodromy of the limiting system with $c_3 = 0$.

But this was set up precisely so that $\widehat{\mathbf{t}}(0)$ (obtained by inverting (44) when $\lambda = -\mu_1$) are the invariants of the original complex reflection group generators.

Finally we remark that the conjugacy class of (r_1, r_2, r_3) is uniquely determined by the value of the invariants $\widehat{\mathbf{t}}$. This will be clear in the example below and follows in general from the fact that the invariants $\widehat{\mathbf{t}}$ generate the ring of conjugation invariant functions on triples of pseudo-reflections, and that the triple (r_1, r_2, r_3) is irreducible. □

Remark 31. Having established that the resulting system has the correct monodromy, let us record a more direct way to go from the $\widehat{A}_i$ to the B_i of (43). The key point is that the pairwise, and three-fold, traces of distinct $B'_i s$ are independent of the scalar shift λ (and that the constant c_3 does not contribute). Thus if $i \neq j$ then we find

$$b_{ij} b_{ji} = \mathrm{Tr}(B_i B_j) = \mathrm{Tr}(\widehat{A}_i \widehat{A}_j),$$

$$b_{32} b_{21} b_{13} = \mathrm{Tr}(B_3 B_2 B_1) = \mathrm{Tr}(\widehat{A}_3 \widehat{A}_2 \widehat{A}_1).$$

In general these are sufficient to determine (B_1, B_2, B_3) uniquely up to conjugacy.

Now we will recall (from [23]) the formulae for the $\widehat{A}_i$ in terms of y, y'. Let us first go in the other direction, and then invert. Consider the following rank one matrices

$$\widehat{A}_1 := \begin{pmatrix} z_1 + \theta_1 & -u z_1 \\ (z_1 + \theta_1)/u & -z_1 \end{pmatrix}, \ \widehat{A}_2 := \begin{pmatrix} z_2 + \theta_2 & -w z_2 \\ (z_2 + \theta_2)/w & -z_2 \end{pmatrix}, \ \widehat{A}_3 := \begin{pmatrix} z_3 + \theta_3 & -v z_3 \\ (z_3 + \theta_3)/v & -z_3 \end{pmatrix}$$

so that $\widehat{A}_i$ has eigenvalues $\{0, \theta_i\}$ for $i = 1, 2, 3$. Now if we define

$$k_1 := (\theta_4 - \theta_1 - \theta_2 - \theta_3)/2, \qquad k_2 := (-\theta_4 - \theta_1 - \theta_2 - \theta_3)/2$$

and impose the equations

$$z_1 + z_2 + z_3 = k_2, \quad uz_1 + vz_3 + wz_2 = 0, \quad (z_1 + \theta_1)/u + (z_3 + \theta_3)/v + (z_2 + \theta_2)/w = 0 \tag{45}$$

then $\widehat{A}_1 + \widehat{A}_2 + \widehat{A}_3 = -\mathrm{diag}(k_1, k_2)$, and the corresponding $\mathfrak{sl}_2$ matrices satisfy $A_1 + A_2 + A_3 = -\mathrm{diag}(\theta_4, -\theta_4)$.

Now we wish to define two T-invariant functions x, y on the set of such triples $(\widehat{A}_1, \widehat{A}_2, \widehat{A}_3)$, where the one-dimensional torus $T \subset \mathrm{SL}_2(\mathbb{C})$ acts by diagonal conjugation. (The function x is denoted $\widetilde{z}$ in [23].) First note that the $(1, 2)$ matrix entry of

$$\widehat{A} := \frac{\widehat{A}_1}{z} + \frac{\widehat{A}_2}{z - t} + \frac{\widehat{A}_3}{z - 1}$$

is of the form $\frac{p(z)}{z(z-1)(z-t)}$ for some *linear* polynomial $p(z)$. Thus $\widehat{A}_{12}$ has a unique zero on the complex plane and we define y to be the position of this zero. Explicitly one finds:

$$y = \frac{tuz_1}{(t + 1)uz_1 + tvz_3 + wz_2}. \tag{46}$$

Then we define

$$x = \frac{z_1}{y} + \frac{z_2}{y - t} + \frac{z_1}{y - 1}. \tag{47}$$

To motivate x we note that if we set $z = y$ then $\widehat{A}$ is lower triangular and its first eigenvalue (i.e. its top-left entry) is $x + \frac{\theta_1}{y} + \frac{\theta_2}{y-t} + \frac{\theta_3}{y-1}$. so x is clearly T-invariant.

Now the fact is that we can go backwards and express the six variables $\{z_1, z_2, z_3, u, v, w\}$ in terms of y, x. That is, given y, x we wish to solve the five equations (45), (46), (47) in the six unknowns $\{z_1, z_2, z_3, u, v, w\}$. To fix up the expected one degree of freedom we impose a sixth equation

$$(t + 1)uz_1 + tvz_3 + wz_2 = 1$$

so that (46) now says $y = tuz_1$. (This degree of freedom corresponds to the torus action mentioned above.) One then finds, algebraically, that these six equations in the six unknowns admit the unique solution:

$$z_1 = y\frac{E - k_2^2(t + 1)}{t\theta_4}, \qquad z_2 = (y - t)\frac{E + t\theta_4(y - 1)xk_2^2 - tk_1k_2}{t(t - 1)\theta_4}$$

$$z_3 = -(y - 1)\frac{E + \theta_4(y - t)x - k_2^2 t - k_1k_2}{(t - 1)\theta_4},$$

$$u = \frac{y}{tz_1}, \qquad v = -\frac{y - 1}{(t - 1)z_3}, \qquad w = \frac{y - t}{t(t - 1)z_2}$$

where

$$E = y(y - 1)(y - t)x^2 + \big(\theta_3(y - t) + t\theta_2(y - 1) - 2k_2(y - 1)(y - t)\big)x + k_2^2 y - k_2(\theta_3 + t\theta_2).$$

Finally we recall that if $y(t)$ solves PVI then the variable x is directly expressible in terms of the derivative of y (cf. [23] above C55):

$$x = \frac{1}{2}\left(\frac{t(t-1)y'}{y(y-1)(y-t)} - \frac{\theta_1}{y} - \frac{\theta_3}{y-1} - \frac{\theta_2+1}{y-t}\right).$$

Thus given a solution $y(t)$ to PVI we may use these formulae to reconstruct the matrices $\widehat{A}_1, \widehat{A}_2, \widehat{A}_3$ upto overall conjugation by the diagonal torus.

Example. For the Klein solution, suppose we set the parameter $s = 3$ (this is chosen to give reasonably simple numbers below). Then $t = 121/125$ and the above formulae yield (cf. Remark 31):

$$b_{12}b_{21} = \frac{3}{224}, \qquad b_{23}b_{32} = \frac{249}{2464}, \qquad b_{13}b_{31} = \frac{5}{176},$$
$$b_{32}b_{21}b_{13} = \frac{21}{1408}.$$

These values determine (B_1, B_2, B_3) uniquely upto conjugacy, and it is easy to find a representative triple:

Corollary 32. *Let*

$$B_1 = \begin{pmatrix} \frac{1}{2} & \frac{3}{224} & \frac{21}{1408} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{5}{176} \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{332}{49} & 1 & \frac{1}{2} \end{pmatrix}.$$

Then the Fuchsian system

$$\frac{d}{dz} - \left(\frac{B_1}{z} + \frac{B_2}{z - \frac{121}{125}} + \frac{B_3}{z-1}\right)$$

has monodromy equal to the Klein complex reflection group, generated by reflections.

Appendix A

We wish to explain how to extract Theorem 3 from the paper [1] of Balser, Jurkat and Lutz.

Let us briefly recall the set-up of [1]. Given an $n \times n$ matrix A_1 and a diagonal matrix $B_0 = \mathrm{diag}(b_1, \dots, b_n)$ (with the b_i pairwise distinct) one considers the Fuchsian connection ([1] (0.2))

$$d - (B_0 - z)^{-1}(1 + A_1)dz = d - \sum_{i=1}^{n} \frac{-E_i(1+A_1)}{z - b_i} dz \tag{48}$$

where E_i is the $n \times n$ matrix with a one in its i, i entry and is otherwise zero. (To avoid confusion with other notation of the present paper we have relabelled $t \mapsto z, \Lambda \mapsto B_0, \lambda_i \mapsto b_i$ from [1].) Write $\Lambda' = \mathrm{diag}(\lambda'_1, \dots, \lambda'_n)$ for the diagonal part of A_1 and suppose that

i) No λ'_i is an integer, and that

ii) No eigenvalue of A_1 is an integer.

Condition *i*) implies that each residue of (48) is rank one and has non-integral trace.

Now one chooses an admissible branch cut direction η and cuts the complex z-plane from each b_i to ∞ along the direction η, leaving a simply connected domain $\mathcal{P}_\eta \subset \mathbb{C}$. (In fact ([1] p.694) one takes $\eta \in \mathbb{R}$ and uses η to give logarithm choices on $\mathcal{P}_\eta$ near each b_i.) The direction η is said to be 'admissible' if none of these cuts overlap, and the inadmissible

η in the interval $(-\pi/2, 3\pi/2]$ are labelled $\eta_0, \ldots, \eta_{m-1}$ (with $\eta_{i+1} < \eta_i$). This labelling is extended to all integral subscripts ν by setting $\eta_\nu = \eta_{\nu+km} + 2\pi k$ for any integer k.

Now, given an admissible η one may canonically construct a certain fundamental solution $Y^*(z)$ of (48) on $\mathcal{P}_\eta$ and define an $n \times n$ matrix $C = C(\eta)$, with one's on the diagonal, such that (by [1] Lemma 1): After continuing Y^* along a small positive loop around b_k (and crossing the kth cut), Y^* becomes

$$Y^*(1 + C_k^*)$$

where C_k^* is zero except for its kth column which equals the kth column of $C\widetilde{D}$, where $\widetilde{D} := \exp(-2\pi i \Lambda') - 1$.

If η varies through admissible values then $C(\eta)$ does not change. Thus we choose an integer ν and let $C = C_\nu$ be $C(\eta)$ for any $\eta \in (\eta_{\nu+1}, \eta_\nu)$, as in [1] Remark 3.1 p.699.

Thus if we suppose ([1] p.697) that, when looking along η towards infinity, that b_{k+1} lies to the right of b_k for $k = 1, \ldots, n-1$, then the monodromy of Y^* around a large positive loop encircling all the b_i is the product of pseudo-reflections

$$(1 + C_1^*) \cdots (1 + C_n^*).$$

The main facts we need from [1] now are:

1) That the Stokes matrices $C_\nu^\pm$ of the irregular connection

$$d - \left(B_0 + \frac{A_1}{x}\right) dx$$

are determined by C_ν by the equation

$$C_\nu D = C_\nu^+ - e^{2\pi i \Lambda'} C_\nu^- \tag{49}$$

where $D := 1 - \exp 2\pi i \Lambda'$. (This is equation (3.25) of [1], and that the $C_\nu^\pm$ are the Stokes matrices is the content of [1] Theorem 2, p.714.)

2) That the Stokes matrices are unchanged if A_1 undergoes a scalar shift $A_1 \mapsto A_1 - \lambda$ ([1] Remark 4.4, p.712).

This is sufficient to determine how the pseudo-reflections $1 + C_k^*$ vary under the scalar shift; one doesn't need to know how the Stokes matrices are defined, only that they are triangular matrices with one's on the diagonal. The main subtlety one needs to appreciate is that: *in the above convention (with b_{k+1} to the right of b_k) then*

$$C_\nu^+ \textit{ is lower triangular and } C_\nu^- \textit{ is upper triangular.} \tag{50}$$

Indeed ([1] p.701, paragraph before (3.16)) $C_\nu^{+/-}$ are upper/lower triangular respectively if $b_1, \ldots, b_n$ are ordered according to the dominance relation on $S'_{\nu+1}$. This dominance relation is defined (top of p.699) so that it coincides with the natural ordering of the indices if $b_1, \ldots, b_n$ are ordered so that the jth cut (along the direction $\eta \in (\eta_{\nu+1}, \eta_\nu)$) lies to the right of the kth cut whenever $j < k$ (again looking along η towards infinty). This is *opposite* to the previous ordering of the b_i. Thus sticking to our original ordering we deduce (50).

In summary if we set $V = \mathbb{C}^n$ and write $1 + C_k^* = 1 + v_k \otimes \beta_k$ (where $\{\beta_i\}$ is the standard basis of V^* and $v_k \in V$ is the kth column of $v := C_\nu \widetilde{D}$, so $v_{ij} = \beta_i(v_j)$) then

$$v = C_\nu \widetilde{D} = (C_\nu^+ - e^{2\pi i \Lambda'} C_\nu^-) D^{-1} \widetilde{D} \sim e^{-2\pi i \Lambda'} C_\nu^+ - C_\nu^-$$

where we note that $e^{2\pi i\Lambda'}\widetilde{D} = D$ and where $\sim$ is defined so that $A \sim B$ if there is an invertible diagonal matrix s such that $A = sBs^{-1}$. (This conjugation by s just corresponds to different choices of v_k, β_k such that $1 + C_k^* = 1 + v_k \otimes \beta_k$ and so clearly does not affect the corresponding pseudo-reflections.)

Thus under the scalar shift, the upper triangular part of v is fixed, the lower triangular part is scaled by $\exp(2\pi i\lambda)$ and the diagonal part $e^{-2\pi i\Lambda'} - 1$ is changed to $e^{-2\pi i(\Lambda'-\lambda)} - 1$.

Finally let us relate this back to our conventions in the body of the paper. Namely we have a connection

$$d - \sum \frac{B_i}{z - a_i} dz$$

with rank one residues, monodromy r_i around a_i and monodromy $r_n \cdots r_1$ around a large positive loop. The images of the B_i make up a basis of V so we may conjugate $(B_1, \dots, B_n)$ such that each B_i is zero except in row $(n - i + 1)$.

Then we set $b_i = a_{n-i+1}$ and define $A_1 = -1 - \sum B_i$ so that

$$\frac{-E_i(1 + A_1)}{z - b_i} = \frac{B_{n-i+1}}{z - a_{n-i+1}}$$

for each i and that

$$(1 + C_1^*, \dots, 1 + C_n^*) = (r_n, \dots, r_1)$$

upto overall conjugation. Thus if we write $r_i = 1 + e_i \otimes \alpha_i$ and define u by $u_{ij} = \alpha_i(e_j)$ we have that

$$u \sim \Omega v \Omega$$

where Ω is the order reversing permutation matrix ($\Omega_{ij} = \delta_{in-j+1}$). Note that we denoted the trace of B_i as λ_i so that the diagonal part Λ' of A_1 is

$$\Lambda' = -1 - \Omega\Lambda\Omega$$

where $\Lambda := \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Thus, if we write

$$u = t^2 u_+ - u_-$$

with t^2 diagonal and $u_{+/-}$ upper/lower triangular with one's on the diagonal, we have that, upto overall conjugation by a diagonal matrix:

$$u_+ = \Omega C_\nu^+ \Omega, \qquad u_- = \Omega C_\nu^- \Omega$$

and $t^2 = \exp(2\pi i\Lambda)$. Therefore under the scalar shift both $u_\pm$ are fixed and so u is changed to $h^2 t^2 u_+ - u_-$, establishing Theorem 3.

Remark 33. One can check independently that it is the upper triangular part of u that should be scaled by h^2, rather than the lower triangular part, since we know that the eigenvalues of $r_n \cdots r_1$ should be scaled by h^2. Indeed if we expand

$$\mathrm{Tr}(r_n \cdots r_1) = n + \sum_i u_{ii} + \sum_{i>j} u_{ij}u_{ji} + \sum_{i>j>k} u_{ij}u_{jk}u_{ki} + \cdots + u_{nn-1}u_{n-1n-2}\cdots u_{21}u_{n1}$$

$$= \sum_i t_i + \sum_{i>j} u_{ij}u_{ji} + \sum_{i>j>k} u_{ij}u_{jk}u_{ki} + \cdots + u_{nn-1}u_{n-1n-2}\cdots u_{21}u_{n1}$$

we see that scaling just the upper triangular part of u (and the t_i) scales each term here by h^2 as required, and otherwise one obtains higher powers of h.

Appendix B

We will explain how to derive the formula for the parameter s in Jimbo's formula (38). This formula is stated incorrectly, and not derived, in [22]. Since it is not immediately clear how to derive the formula we sketch the main steps here, and point out the (probably typographical) error. (We remark that the whole procedure described in the present paper does not work without this correction.)

Suppose we have four matrices $M_j \in \mathrm{SL}_2(\mathbb{C})$, $j = 0, t, 1, \infty$ satisfying

$$M_\infty M_1 M_t M_0 = 1, \tag{51}$$

and M_j has eigenvalues $\{\exp(\pm\pi i\theta_j)\}$ where $\theta_j \notin \mathbb{Z}$. Write $\varepsilon_\infty = \exp(\pi i\theta_\infty)$ and suppose M_∞ is actually diagonal ($M_\infty = \mathrm{diag}(\varepsilon_\infty, \varepsilon_\infty^{-1})$). Define $\sigma_{jk} \in \mathbb{C}$ with $0 \le \mathrm{Re}(\sigma_{jk}) \le 1$ (for $j, k \in \{0, t, 1, \infty\}$) by

$$\mathrm{Tr}(M_j M_k) = 2\cos(\pi\sigma_{jk}),$$

and let $\sigma := \sigma_{0t}$ and $\varepsilon := \exp(\pi i\sigma)$.

Under the further assumptions that σ is nonzero, that $0 \le \mathrm{Re}(\sigma) < 1$ and that none of the eight numbers

$$\theta_0 \pm \theta_t \pm \sigma, \quad \theta_0 \pm \theta_t \mp \sigma, \quad \theta_\infty \pm \theta_1 \pm \sigma, \quad \theta_\infty \pm \theta_1 \mp \sigma$$

is an even integer, Jimbo [22] p.1141 points out that, up to overall conjugacy by a diagonal matrix, M_0, M_t, M_1 are given, for some $s \in \mathbb{C}^*$, by:

$$(is_\sigma)M_0 = C^{-1}\begin{pmatrix} \varepsilon c_0 - c_t & 2s\alpha'\gamma' \\ -2s^{-1}\beta'\delta' & -\varepsilon^{-1}c_0 + c_t \end{pmatrix} C, \quad (is_\sigma)M_t = C^{-1}\begin{pmatrix} \varepsilon c_t - c_0 & -2s\varepsilon\alpha'\gamma' \\ 2s^{-1}\varepsilon^{-1}\beta'\delta' & -\varepsilon^{-1}c_t + c_0 \end{pmatrix} C$$

$$(is_\infty)M_1 = \begin{pmatrix} c_\sigma - \varepsilon_\infty^{-1}c_1 & -2\varepsilon_\infty^{-1}\beta\gamma \\ 2\varepsilon_\infty\alpha\delta & -c_\sigma + \varepsilon_\infty c_1 \end{pmatrix}, \qquad \text{where} \qquad C = \begin{pmatrix} \delta & \beta \\ \alpha & \gamma \end{pmatrix}$$

and we have used the temporary notation:

$$c_j := \cos(\pi\theta_j), \quad c_\sigma := \cos(\pi\sigma), \quad s_j := \sin(\pi\theta_j), \quad s_\sigma := \sin(\pi\sigma),$$

$$\alpha = \sin\frac{\pi}{2}(\theta_\infty - \theta_1 + \sigma), \beta = \sin\frac{\pi}{2}(\theta_\infty + \theta_1 + \sigma), \gamma = \sin\frac{\pi}{2}(\theta_\infty + \theta_1 - \sigma), \delta = \sin\frac{\pi}{2}(\theta_\infty - \theta_1 - \sigma),$$

$$\alpha' = \sin\frac{\pi}{2}(\theta_0 - \theta_t + \sigma), \beta' = \sin\frac{\pi}{2}(\theta_0 + \theta_t + \sigma), \gamma' = \sin\frac{\pi}{2}(\theta_0 + \theta_t - \sigma), \delta' = \sin\frac{\pi}{2}(\theta_0 - \theta_t - \sigma).$$

Notice that σ_{01} and σ_{1t} do not appear in these formulae; The idea now is to express the parameter s in terms of σ_{01}, σ_{1t} (and the other parameters). Naively we can just calculate $\mathrm{Tr}(M_0 M_1)$ and $\mathrm{Tr}(M_1 M_t)$ from the above formulae, equate them with $2\cos(\pi\sigma_{01}), 2\cos(\pi\sigma_{1t})$ respectively and try to solve for s. However this yields a complicated expression in the trigonometric functions and a simple formula looks beyond reach.

The key observation to simplify the computation is that the above parameterisation of the matrices is such that $CM_tM_0C^{-1}$ is diagonal and equal to $\Delta := \mathrm{diag}(\varepsilon, \varepsilon^{-1})$ (and also equal to $CM_1^{-1}M_\infty^{-1}C^{-1}$ by (51)). Thus we find

$$2\cos(\pi\sigma_{01}) = \mathrm{Tr}(M_0 M_1) = \mathrm{Tr}\big(CM_\infty^{-1}C^{-1}\Delta^{-1}(CM_0C^{-1})\big) \tag{52}$$

$$2\cos(\pi\sigma_{1t}) = \mathrm{Tr}(M_1 M_t) = \mathrm{Tr}\big(CM_\infty^{-1}C^{-1}\Delta^{-1}(CM_tC^{-1})\big) \tag{53}$$

whose right-hand sides are more manageable expressions in the trigonometric functions, and are linear in $1, s, s^{-1}$. If we take the combination (52)+ε(53) then the s^{-1} terms cancel and upon rearranging we find:

$$\begin{aligned} 2i\det(C)s_\sigma(c_{01}+\varepsilon c_{1t}) =&(\gamma\delta\varepsilon_\infty^{-1}-\alpha\beta\varepsilon_\infty)(\varepsilon-\varepsilon^{-1})c_2+ \\ &(\gamma\delta\varepsilon_\infty-\alpha\beta\varepsilon_\infty^{-1})(\varepsilon^2-1)c_0+2s\alpha\gamma\alpha'\gamma'(\varepsilon_\infty-\varepsilon_\infty^{-1})(\varepsilon-\varepsilon^{-1}). \end{aligned} \tag{54}$$

where $c_{01}=\cos(\pi\sigma_{01})$ and $c_{1t}=\cos(\pi\sigma_{1t})$. To proceed we note:

Lemma 34.
a) $\gamma\delta-\alpha\beta := \det(C) = -s_\infty s_\sigma$
b) $\gamma\delta\varepsilon_\infty^{-1}-\alpha\beta\varepsilon_\infty = is_\infty(\varepsilon c_\infty - c_1)$
c) $\gamma\delta\varepsilon_\infty-\alpha\beta\varepsilon_\infty^{-1} = is_\infty(c_1-\varepsilon^{-1}c_\infty)$

Proof. A few applications of standard trigonometric formulae yield

$$\alpha\beta=(c_1-(c_\infty c_\sigma - s_\infty s_\sigma))/2, \qquad \gamma\delta=(c_1-(c_\infty c_\sigma+s_\infty s_\sigma))/2$$

which gives $a)$ immediately and also enable $b), c)$ to be easily deduced. □

Substituting these into (54) and cancelling a factor of $2s_\sigma s_\infty = -(\varepsilon-\varepsilon^{-1})(\varepsilon_\infty-\varepsilon_\infty^{-1})/2$ yields the desired formula:

$$s=\frac{\varepsilon(is_\sigma c_{1t}-c_tc_\infty-c_0c_1)+is_\sigma c_{01}+c_tc_1+c_\infty c_0}{4\alpha\gamma\alpha'\gamma'}.$$

This differs from formula (1.8) of [22] in a single sign, in α, which was crucial for us.

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