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GRADIENT QUATERNIONIC VECTOR FIELDS AND A CHARACTERIZATION OF THE QUATERNIONIC PROJECTIVE SPACE

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Abstract. On a quaternionic Kähler manifold (M^{4n}, g, Q), of positive reduced scalar curvature ν, a gradient vector field $Z = gradf$ which preserves the quaternionic structure Q is studied. The corresponding potential f is proved to be an eigenfunction of the Laplacian with the eigenvalue $\mu = 2\nu(n+1)$. A second order differential equation for the 1-form $\xi = df$ is established. We prove that this equation is equivalent to the Obata-Blair equation for the pull-back $\Psi = \pi^*\xi$ of the 1-form ξ on the total space of the Sasakian SO_3-principal bundle $\pi : F \to M$ associated with (M^{4n}, g, Q). Using the results of Obata, Blair and Ishihara we characterize the quaternionic projective space as the unique quaternionic Kähler manifold of positive scalar curvature which satisfies one of the following properties: i) there exists a non Killing vector field Z which preserves Q: ii) there exists an eigenfunction of the Laplacian with the eigenvalue μ

1. Quaternionic Kähler manifolds (basic definitions).

A quaternionic Kähler manifold is a $4n$-dimensional Riemannian manifold (M, g) with a 3-dimensional sub bundle Q of the bundle EndTM of endomorphisms, which satisfies the following conditions:

1) Q is spanned locally by three anticommuting almost complex structures J_1, J_2, $J_3 = J_1 J_2$. (A base $f = (J_{1|x}, J_{2|x}, J_{3|x})$ of a fiber Q_x of Q will be called an admissible base in a point $x \in M$).

2) The Riemannian metric g is Q-Hermitian, that is all endomorphisms from Q are skew-symmetric with respect to g;

3) The sub bundle Q is invariant with respect to the Levi-Civita connection ∇^g of g.

The condition 3) may be written as

$$\nabla^g_X J_\alpha = \omega_\gamma(X) J_\beta - \omega_\beta(X) J_\gamma \qquad X \in TM$$

Key words and phrases. Quaternionic Kähler manifold; Quaternionic vector field; eigenfunction of Laplacian.

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where $\omega_\alpha, \alpha = 1,2,3$, are local 1-forms and (α,β,γ) stand for a cyclic permutation of $(1,2,3)$.

We will assume $n > 1$. Then g is an Einstein metric:

$$Ric(g) = (K/4n)g$$

where $K = 4n(n+2)\nu$ is the scalar curvature and ν is a constant called the reduced scalar curvature, and the following structure equations hold:

$$d\omega_\alpha + \omega_\beta \wedge \omega_\gamma = -\nu g \circ J_\alpha (= -\nu g(J_\alpha \cdot, \cdot)) \tag{1}$$

$$[R(X,Y), J_\alpha] = \nu(g(X, J_\gamma Y)J_\beta - g(X, J_\beta Y)J_\gamma) \qquad X, Y \in TM \tag{2}$$

where R is the curvature tensor. (Here we use conventions: $d\omega_\alpha = 2Alt(\nabla\omega_\alpha)$ and $\omega_\beta \wedge \omega_\gamma = 2Alt(\omega_\beta \otimes \omega_\gamma)$).

We will assume that the scalar curvature K of the metric g is positive.

Then M is compact and simply connected [4]. Rescaling the metric g we may assume that $\nu = 1$.

We note that the case of non positive scalar curvature was studied in [14]; see also [3].

Definition. 1) a transformation φ of a quaternionic Kähler manifold (M, g, Q) is called to be quaternionic if it preserves the quaternionic structure $Q : \varphi^* Q = Q$.

2) A vector field $Z \in \chi(M)$ is called to be quaternionic if it generates a (local) one-parameter group of quaternionic transformations.

We denote by $aut(Q)$ (resp., $aut(g)$) the Lie algebra of all quaternionic (resp., Killing) vector fields on (M, g, Q). Since we assume that $\nu > 0$ and $n > 1$, the quaternionic structure Q is completely determined by the metric g (see [4]) and, hence $aut(g) \subset aut(Q)$.

2. Characterization of a gradient quaternionic vector field.

For any $Z \in aut(Q)$ we set

$$f_Z = -[1/2(n+1)\nu] divZ$$

where $divZ = \nabla^g_i Z^i$ is the divergence.

In [3] we proved the following

Proposition 1. *The mapping*

$$p : aut(Q) \ni Z \mapsto Z - gradf_Z$$

is the projection of the space $aut(Q)$ of quaternionic vector fields onto the space $aut(g)$ of Killing vector fields.

2) Denote by $\mathcal{P}$ the kernel of p. Then

$$aut(Q) = aut(g) + \mathcal{P}$$

is a reductive decomposition:

$$[aut(g), \mathcal{P}] \subset \mathcal{P}$$

and the space $\mathcal{P}$ consists of all gradient quaternionic vector fields. Moreover,

3) the map

$$\mathcal{P} \ni Z \mapsto f_Z$$

is the isomorphism of $\mathcal{P}$ onto the eigenspace $\mathcal{F}(2\nu(n+1))$ of the Laplacian with the eigenvalue $2\nu(n+1)$. The inverse mapping is

$$grad : \mathcal{F}(2\nu(n+1)) \ni f \mapsto Z = gradf.$$

Now we characterize a gradient quaternionic vector field $Z \in \mathcal{P}$ in terms of associated 1-form $\xi = g \circ Z$.

Proposition 2. *Let Z be a vector field and $\xi = g \circ Z$ is the associated 1-form. Then the following conditions are equivalent:*

i) $Z \in \mathcal{P}$

ii) ξ is closed and the bilinear form $\nabla^g \xi$ is the Q-Hermitian

iii) ξ satisfies the equation

$$(\nabla^g)^2_{X,Y}\xi = -(\nu/4)P(X,Y)\xi \qquad \forall X, Y \in TM \tag{3}$$

where P is the parallel (1,3) *tensor field on M defined by*

$$\begin{aligned} P(X,Y)\xi &= 2\xi(X) g \circ Y + \xi(Y) g \circ X + g(X,Y)\xi \\ &\quad - \sum_\alpha \xi(J_\alpha Y) g \circ J_\alpha X - \sum_\alpha g(J_\alpha X, Y)\xi \circ J_\alpha \\ &\quad \xi \in T^*M, \quad X, Y \in T^*M \end{aligned}$$

Proof. The equivalence of conditions i), ii) was proved in [3] (see also [2]). In [3] we proved also that i) implies iii); now we show that, conversely, iii) implies ii). We note that for a solution ξ of (3) one has $\nabla^g d\xi = 0$ and $\nabla^g \nabla^- \xi = 0$, where $\nabla^- \xi$ is the skew-Q-Hermitian part of bilinear form $\nabla^g \xi$; in particular $\delta d\xi = \delta \nabla^- \xi = 0$. Then statement follows by compactness of M and by observing that the restriction of the divergence δ to a skew-Q-Hermitian bilinear forms is the (formal) adjoint of ∇^-.

3. The main theorems.

We prove the following

Theorem 1. *Let (M, g, Q) be a 4n-dimensional $(n > 1)$ quaternionic Kähler manifold with positive scalar curvature $K = 4n(n+2)\nu$. If there exists a non zero solution $\xi \in \Lambda^1 M$ of the equation (3), then (M, g) is isometric to the quaternionic projective space $\mathbb{H}P^n$ with the standard metric of the same scalar curvature K.*

As a Corollary, we obtain

Theorem 2. *A quaternionic Kähler manifold* (M,g,Q) *with* $K>0$ *is isometric to* $\mathbb{H}P^n$ *if one of the following conditions holds:*
i) there exists a quaternionic non-Killing vector field on M*;*
ii) there exists an eigenfunction of the Laplacian with the eigenvalue $2(n+1)\nu$*.*

Other corollary of the Theorem 1 will be derived in [3].

For proving Theorem 1, we consider the pull-back Ψ of a solution ξ of (3) onto the principal Sasakian SO_3-bundle $\pi : F \to M$ associated with (M,g,Q) and show that Ψ satisfies some equation stated by M. Obata, [13]. Then the Theorem follows directly from the results of M. Obata and S. Ishihara, [10]

4. The Sasakian bundle associated with a quaternionic Kähler manifold.

Let (M,g,Q) be a quaternionic Kähler manifold. We will assume that the reduced scalar curvature $\nu = 1$. Denote by F the set of all admissible bases $f = (J_1, J_2, J_3 = J_1 J_2)$ of the quaternionic structure Q (in all point $x \in M$). The group SO_3 acts naturally on F with the orbit space M. We will refer the principal SO_3-bundle $\pi : F \to M$ as the Sasakian bundle associated with (M,g,Q) (see Remark below) and denote by E_α the vertical fundamental vector fields on f corresponding to the standard base $\mathbf{e}_\alpha, \alpha = 1,2,3$, of the Lie algebra SO_3

$$\mathbf{e}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\gamma.$$

(Here and below (α,β,γ) is a cyclic permutation of (1,2,3)).

Levi-Civita connection ∇^g of g preserves the bundle $F \subset End(TM)$ and hence, induces a connection ∇^F in π. We denote by $\eta : TF \to so_3$ the connection form of ∇^F and by $\Omega : \Lambda^2 TF \to so_3$ its curvature form defined by

$$\Omega = d\eta + (1/2)[\eta,\eta].$$

We may write

$$\eta = \sum\nolimits_\alpha \mathbf{e}_\alpha \otimes \eta_\alpha, \quad \Omega = \sum\nolimits_\alpha \mathbf{e}_\alpha \otimes \Omega_\alpha.$$

where $\eta_\alpha, \Omega_\alpha (\alpha = 1,2,3)$ are scalar forms. Then

$$\eta_\alpha(E_\sigma) = \delta_{\alpha\sigma}, \quad \Omega_\alpha = d\eta_\alpha + \eta_\beta \wedge \eta_\gamma.$$

Denote by $H = Ker\eta$ the horizontal distribution of ∇^F and by

$$\pi_H = \pi_{*|H} : H \to TM$$

the restriction to H of the natural projection $\pi_* : TF \to TM$.

We define the tautological operators I_α and horizontal 2-forms Φ_α on F by

$$I_{\alpha|f} = \pi_H^{-1} \circ J_\alpha \circ \pi_*, \quad \Phi_{\alpha|f} = -\pi_H^*(g \circ J_\alpha)$$

or

$$\Phi_\alpha(X, Y) = g(\pi_* X, J_\alpha \pi_* Y), \quad \forall X, Y \in T_f F.$$

Since the tensor fields $I_\alpha \Phi_\alpha$ are SO_3-equivalent

$$[A^* I_\alpha = \sum_p A^\rho_\alpha I_p, \quad A^* \Phi_\alpha = \sum_p A^\rho_\alpha \Phi_\rho, A \in SO_3]$$

their Lie derivatives with respect to the vector fields E_ρ are given by:

$$E_\alpha \cdot I_\alpha = 0, \quad E_\alpha \cdot I_\beta = I_\gamma, \quad E_\beta \cdot I_\alpha = -I_\gamma$$

The equation (1) implies

Lemma 1. *The following structure equations hold:*

$$\Phi_\alpha = \Omega_\alpha \equiv d\eta_\alpha + \eta_\beta \wedge \eta_\gamma \tag{4}$$

$$d\Phi_\alpha = \Omega_\beta \wedge \eta_\gamma - \eta_\beta \wedge \Omega_\gamma \tag{5}$$

Following J. Vilms (see [4], page 249), we define a Riemannian metric h on F by

$$h = \pi^* g + \sum_\alpha \eta_\alpha \otimes \eta_\alpha$$

and denote by ∇ the Levi-Civita connection of h. For any vector field $X \in \chi(M)$ we denote by $\underline{X} = \pi_H^{-1} X$ the horizontal lift of X. We define operators

$$L_\alpha := \nabla E_\alpha \quad \alpha = 1, 2, 3$$

The relations between $h, E_\alpha, I_\alpha, \Phi_\alpha$ are described by the following lemma

Lemma 2. *1)* $\Phi_\alpha = -h \circ I_\alpha$

2) $E_\alpha, \alpha = 1, 2, 3$, *are orthonormal Killing vector fields of the Riemannian manifold* (F, h) *and*

$$\eta_\alpha = h \circ E_\alpha (= h_{ij} E_\alpha j)$$

3) $L_\alpha \equiv \nabla E_\alpha = -(1/2)[I_\alpha + E_\gamma \otimes \eta_\beta - E_\beta \otimes \eta_\gamma]$, $h \circ L_\alpha = \nabla \eta_\alpha = (1/2) d\eta_\alpha = (1/2)[\Phi_\alpha - \eta_\beta \wedge \eta_\gamma]$

4) $\nabla_{E_\alpha} \underline{X} = \nabla_{\underline{X}} E_\alpha = -(1/2) L_\alpha \underline{X} = -(1/2) I_\alpha \underline{X} \quad \forall X \in \chi(M)$

5) $\nabla_{\underline{X}} \underline{Y} = \underline{\nabla^g_X Y} - (1/2) \sum_\alpha \Phi_\alpha(\underline{X}, \underline{Y}) E_\alpha \quad \forall X, Y \in \chi(M)$

Proof. 1), 2) are checked directly; 3) follows from the structure equations (1) and implies 4),5).

Remarks. 1) The projection $\pi : (F, h) \to (M, g)$ is a Riemannian submersion, with totally geodesic fibers; hence 3)-5) may be obtained also by using O'Neill formulas (see [4], page 240).

2) The triple (E_1, E_2, E_3) is strictly related to a canonical 3-Sasakian structure of F (See for example [10], [7]).

5. Calculation of the second covariant derivatives of 1-form Ψ on F.

To derive a differential equation for the pull back $\pi^*\xi$ of a solution $\xi \in \Lambda^1 M$ of the equation (3), we calculate now the second derivatives $\nabla^2\Psi$ of any 1-form $\Psi = \pi^*\xi$, where ξ is a 1-form on M as follows.

Proposition 3. *Let $\xi \in \Lambda^1 M$ be a 1-form on M and $\Psi = \pi^*\xi$ is its pull-back to F. Then*

1) $\nabla_{\underline{Y}}\Psi = \pi^*(\nabla^g_Y\xi) + (1/2)\sum_\alpha \Psi(I_\alpha\underline{Y})\eta_\alpha$

2) $\nabla_{E_\alpha}\Psi = (1/2)\Psi \circ I_\alpha$

3)

$$\nabla^2_{\underline{Y},\underline{X}}\Psi = \pi^*((\nabla^g)^2_{X,Y}\xi) + (1/2)\sum_\alpha [a_\xi(Y, \pi_* I_\alpha\underline{X}) + a_\xi(X, \pi_* I_\alpha\underline{Y})]\eta_\alpha$$
$$+ (1/4)\sum_\alpha \Psi(I_\alpha\underline{Y})\Phi_\alpha \circ \underline{X} + (1/4)\sum_\alpha \Phi_\alpha(\underline{X}, \underline{Y})\Psi \circ I_\alpha$$

4) $\nabla^2_{E_\alpha,\underline{Y}}\Psi = (1/2)[a_\xi(Y, \pi_* I_\alpha\cdot) + a_\xi(\pi_*\cdot, \pi_* I_\alpha\underline{Y})] - (1/4)\Psi(\underline{Y})\eta_\alpha$

5) $\nabla^2_{\underline{X},E_\alpha}\Psi = -[a_\xi(X, \pi_* I_\alpha\cdot) + a_\xi(\pi_*\cdot, \pi_* I_\alpha\underline{X})] - (1/2)\Psi(\underline{X})\eta_\alpha$

6) $\nabla^2_{E_\alpha,E_\beta}\Psi = 0$

7) $\nabla^2_{E_\alpha,E_\alpha}\Psi = -(1/4)\Psi$

where $a_\xi(X,Y) = (\nabla_X\xi)(Y)$ *and* $X, Y \in \chi(M)$.

Proof of Proposition. 1),2) follows from 5),4) of Lemma 2. Now we indicate the idea for proving 3)-7). Let $s : x \mapsto s(x) = (J^{(x)}_\alpha)$ be a local section of π. We define the horizontal lift $\underline{J_\alpha}$ of the almost complex structure $J^{(x)}_\alpha$, $\alpha = 1, 2, 3$, by

$$\underline{J_\alpha}|_f = \pi_H^{-1} \circ J^{(x)}_\alpha \circ \pi_* \qquad \forall f \in F, \pi f = x$$

The proof of 3)-7) comes straightforwardly by derivation of 1),2) and basing on the following

Lemma 3. *Let $\underline{J_\alpha}$ be the horizontal lift of a local section $s : x \mapsto s(x) = (J^{(x)}_\alpha)$. Then*

1) $\nabla_{\underline{Z}}\, \underline{J_\alpha} = \underline{\nabla^g_Z J^{(x)}_\alpha} - (1/2)h \circ \sum_\sigma(\underline{J_\alpha} I_\sigma \underline{Z}) \wedge E_\sigma \qquad \forall Z \in \chi(M)$

2) $\nabla_{E_\sigma}\underline{J_\alpha} = -(1/2)[I_\sigma, \underline{J_\alpha}]$

3) $\underline{J_\alpha}|_{s(x)} = I_\alpha|_{s(x)}$

4) If the section s is horizontal in a point $x \in M$, that is $T_{s(x)}s(M) = H_{s(x)}$ (the horizontal subspace in the point $s(x)$), then

$$\nabla_{\underline{X}}\underline{J_\alpha}|_{s(x)} = \nabla_{\underline{X}} I_\alpha|_{s(x)} \qquad \forall X \in T_x M$$

6. Proof of Theorem 1.

Assume now that 1-form $\xi \in \Lambda^1 M$ is a solution of the equation (3). Then the terms in the bracket [ ] in the formulas 4),5) of Proposition 3 vanish, since $\nabla\xi$ is Q-Hermitian, and the following result is checked directly

Proposition 4. *Let* $\xi \in \Lambda^1 M$ *be a solution of* (3). *Then 1-form* $\Psi = \pi^* \xi$ *on* F *satisfies the following equation*

$$\nabla^2_{X,Y} \Psi = -(1/4)[2\Psi(X) h \circ Y + \Psi(Y) h \circ X + h(X,Y)\Psi] \tag{6}$$

for any $Z, X \in TF$.

Now Theorem 1 follows from the following results of M. Obata and S.Ishihara.

Theorem (M. Obata [13]**).** *Let* (F,h) *be a simply connected Riemannian manifold. Then there exists a non trivial solution* Ψ *of* (6)*, where* Ψ *is a closed 1-form, if and only if* (F,h) *is isometric to the Euclidean sphere of radius 2.*

Theorem (Ishihara [10]**).** *Let* (M,g) *be a quaternionic Kähler manifold and* (F,h) *is the total space of the associate Sasakian bundle. Assume that* (F,h) *is isometric to the Euclidean sphere of radius 2. Then* (M,g) *is isometric to the quaternionic projective space with standard metric* g_{can} *of reduced scalar curvature* $\nu = 1$.

Note that to prove the last Theorem one computes the curvature tensor of (M,g) from curvature tensor of (F,h) by using O'Neill formula (see [4], page 241):

$$\begin{aligned} R^M(X,Y,Z,T) &= R^F(\underline{X},\underline{Y},\underline{Z},\underline{T}) - 2h(A_{\underline{X}}\underline{Y}, A_{\underline{Z}}\underline{T}) \\ &\quad + h(A_{\underline{Y}}\underline{Z}, A_{\underline{X}}\underline{T}) - h(A_{\underline{X}}\underline{Z}, A_{\underline{Y}}\underline{T}) \end{aligned}$$

where, by 5) of Lemma 1,

$$A_{\underline{X}}\underline{Y} = -(1/2)\sum_\alpha \Phi_\alpha(\underline{X}, \underline{Y} E_\alpha) \qquad \forall X, Y \in \xi(M).$$

Then

$$\begin{aligned} R^M(X,Y)Z &= (1/4)[g(Y,Z)X - g(X,Z)Y \\ &\quad + \sum_\alpha [g(X, J_\alpha Z) J_\alpha Y - g(Y, J_\alpha Z) J_\alpha X \\ &\quad + 2g(X, J_\alpha Y) J_\alpha Z] \end{aligned}$$

which characterizes $(\mathbf{H}P^n, g_{can})$.

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