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Nonholonomic Systems: Cartan's equivalence and Hamiltonization

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Dedicated to Alan Weinstein on his 60th Birthday

Abstract

A *nonholonomic mechanical system* is given by a configuration space Q^n endowed with a riemannian metric $\langle\, , \rangle$, a potential V, and a totally nonholonomic constraint distribution $\mathcal{H} \subset TQ$. Passing to the Jacobi metric, we may assume that $V = 0$, so the trajectories are geodesics of a (non metric) connection ∇_{NH} which mimics the Levi-Civita connection. The dynamical system can also be described in terms of an almost Poisson tensor $\{\, , \}_{NH}$ with non-zero Jacobinizer. The paper is divided into two parts, related by the use of *moving frames* in Q. In the first part we explore the *connection viewpoint*, and is devoted to the *local geometric invariants* obtained via Cartan's method of equivalence; as an specific example, we analyze Engel's (2-4) distribution. In the second part, we explore the *almost Hamiltonian description*. When a Lie symmetry group G is present, the dynamics can be either *reduced* (in the case of internal symmetries) or *compressed* (transversal symmetries). Important special cases are *G-Chaplygin systems*, in which the constraints are given by a connection on a principal bundle, with total space Q and base $S = Q/G$. These systems "compress" to the cotangent bundle T^*S of the base; in favorable cases, the compressed system is hamiltonizable. *Hamiltonization of a nH system with internal symmetry can also occur on a reduced stage.* Chaplygin's homogeneous sphere, a perfect ball rolling without slipping on the plane, is non hamiltonizable in $T^*SO(3)$, but it is hamiltonizable when *reduced* to T^*S^2. We conjecture that the same could be true for the general case of unequal inertia coefficients.

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1 Background, outline, results, and queries

Since Cartan [1926] moving frames have been an important tool in Riemannian geometry[1]; as it is well known, one of the classic applications was Chern's construction of characteristic classes. But in Analytical Mechanics, the moving frames method goes back to Poincaré [1901], actually much earlier, to Euler's rigid body equations.

Moving frames. What is the advantage of a moving frame based "operational system" for geometric mechanics? The philosophy in Koiller, Rios and Ehlers [2002] is to abolish Darboux's dictatorship, at least in a cotangent bundle T^*Q. Take a local coframe ϵ_i $i = 1, ..., n$ on Q^n, so that any element $p_q \in T^*Q$ can be written as $p_q = \sum m_i\, \epsilon_i(q)$. The natural 1-form α in T^*Q keeps the familiar expression

$$\alpha := pdq = m\epsilon \ . \tag{1.1}$$

Consequently, the canonical symplectic form $\Omega := d\alpha$ can be written as (for short a summation is implicitly understood):

$$\Omega := dp \wedge dq = dm \wedge \epsilon + m\, d\epsilon \ . \tag{1.2}$$

[1] There is a recent English translation of the Russian translation (Cartan [2001]).

The second term $m\,d\epsilon$, which deviates from Darboux's format, is *not* a nuisance; actually it carries most valuable information. For instance, the Kostant-Arnold-Kirillov-Sourier bracket can be instantly visualized. It sufficies to use a (left or right) invariant coframe and apply H. Cartan's "magic formula" on $d\epsilon$.

Moving frames are ideally suited when Lie symmetries are present in a mechanical system[2].

Nonholonomic systems. The aim of this paper is to discuss two aspects of *nonholonomic systems* (heretofore abreviated nH), using the moving frames method[3]:

- Geometric equivalence, using Cartan's description of nH systems in terms of affine connections (Cartan [1928]), and
- Compression, hamiltonization and reduction of Chaplygin systems, extending Chaplygin's method (Chaplygin [1911]) of the "reducing factor" via *affine symplectic structures.*

A nH system is given by a configuration space Q^n endowed with a riemannian metric g, a potential V, and a totally nonholonomic constraint distribution $\mathcal{H} \subset TQ$ (or equivalently by the co-distribution $\mathcal{H}^o \subset T^*Q$ anihilating $\mathcal{H}$). By *totally nonholonomic* we mean that the filtration

$$\mathcal{H} \subset \mathcal{H}_1 \subset \mathcal{H}_2 \subset \ldots$$

(where the next sub-bundle is obtained from the previous by adding combinations of all possible Lie brackets of vectorfields) ends in TQ. To avoid interesting complications we assume that all have constant rank. As usual, the metric defines a kinetic energy T, and the Lagrangian is of natural type, $L = T - V$ (in passing to the Jacobi metric, there is no loss in generality in assuming there is no potential energy). According to Sommerfeld [1952], the *d'Alembert-Lagrange principle* is the most natural foundation for Mechanics:

$$[L] := \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \in \mathcal{H}^o \ , \ \dot{q} \in \mathcal{H} \ . \tag{1.3}$$

It simply states that constraint forces exert no work on admissible motions $\dot{q} \in \mathcal{H}$.

For a survey of recent developments in geometric mechanics (with emphasis in nH systems) see Cendra, Marsden and Ratiu [2001] and Koiller [2003]. For a historical account on nH systems, from a somewhat "anti-reducionist" perspective, see Borisov and Mamaev [2002], and from the engineering viewpoint, Papastavridis [2002]. Recent books of interest, both from the mathematical and from the applied side, are Cushman and Bates [1997], Bloch [2003], Cortés [2002] and Oliva [2002]. As it is clear from the bibliography, *Reports on Mathematical Physics* has been publishing papers on nH mechanics on a regular basis, and *Regular and Chaotic Dynamics* devoted large parts of vols. 1/2 (2002) to the theme. For older eastern European literature see *PMM USSR, J. Applied Math. and Mechanics.*

Cartan equivalence. In the first part of the paper, we analyze nH systems under the *affine connection* perspective. We pursue the (local) classification programme for nH systems proposed by Cartan [1928] using his equivalence method[4]. Take adapted *orthonormal* coframes and write the structure equations. The

[2] As we learned from Alan at the banquet, the etymology for *symplectic* is "capable to join", themes and people. The latter is an important aspect of the sympectic "creed". *Question*: taking moving frames adapted for a given mathematical structure, would the non-Darboux term provide a *local* symplectic invariant?

[3] In Engineering, e.g. Hamel [1949], Papastavridis [2002], the keyword "moving frame" also disguises under the keyword "quasicoordinates".

[4] See Koiller, Rodrigues and Pitanga [2001] and Tavares [2002], for a rewrite of Cartan's paper in modern language.

method of equivalence is an iterative process, whereby taking exterior derivatives and suitably absorbing terms (in a more or less systematic way invented by Cartan) one ends up with the *invariants* of the theory. In Ehlers [2002] nH systems in a 3-manifold with a contact distribution were classified. Here we go one step further, looking at Engel's distribution in 4-manifolds. *This is the first such study for a distribution that is not strongly nonholonomic.* Next, we plan to study the famous Cartan's 2-3-5 distribution from the nH viewpoint.

Changing tack, in the second part of the paper we look at nH systems from the *almost Hamiltonian viewpoint*. If there is a Lie symmetry group G, it is natural to take adapted frames which are invariant under the group action.

External symmetries, G-Chaplygin systems, Compression. *External (or transversal) symmetries* appear frequently as a principal bundle action $G^r \hookrightarrow Q^n \rightarrow S^m$, $m+r=n$. The infinitesimal generators ξ_Q, $\xi \in \mathcal{G}$, which span the tangent space of the fibers $G.q$, are assumed to be Killing for the metric. The constraint distribution $\mathcal{H}$ is formed by the horizontal spaces of a connection in the bundle. In general, horizontal and vertical spaces are not orthogonal. These systems are called $G-Chaplygin$; Chaplygin considered the abelian case[5].

It is clear from symmetry that the dynamics can be *compressed* to T^*S. It takes the form of a Lagrangian with a gyroscopic force; alternatively, of a geodesic spray of a non metric connection, see Koiller [1992]. For our purposes it is more convenient use an *almost Hamiltonian* description (Koiller, Rios and Ehlers [2002]):

$$i_X \Omega_{NH} = dH\,,\ H = H^\phi : T^*S \rightarrow \Re\,,\ \Omega_{NH} = \Omega_{\rm can} + ({\rm J.K})\ . \tag{1.4}$$

Here (J.K) is a semi-basic 2-form on T^*S which in general is not closed. To define the compressed Hamiltonian H^ϕ we use the connection 1-form ϕ. As one may guess, J is the momentum map, and K is the curvature of the connection. Ambiguities cancel, since J is Ad^* while K is Ad-equivariant. Moreover, the construction is independent of the point q on the fiber over s.

Internal symmetries and Noether's theorem. An *internal symmetry* occurs when a Killing vectorfield ξ_Q for the metric g belongs to the distribution $\mathcal{H}$. It is easy so see that Noether's theorem from unconstrained mechanics still holds. The argument (cf. Arnold, Kozlov, and Neishtadt [1988]) goes as follows: denote by $\phi_\xi(s)$ the 1-parameter group generated by ξ and let $\phi(s,t) = \phi_\xi(s).q(t)$, so $\phi' = \frac{d}{ds}\phi = \xi_Q(\phi)$. Here $q(t)$ is chosen as a trajectory of the nonholonomic system. Differentiating with respect to s the identitly $L(\phi(s,t), \frac{d}{dt}\phi(s,t)) = $ const., after a standard integration by parts we get

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot q}\phi') = [L]\phi'\ .$$

This vanishes precisely when $\phi' = \xi_Q \in \mathcal{H}$ and in this case

$$I_\xi := \frac{\partial L}{\partial \dot q} \cdot \xi \tag{1.5}$$

is an integral of motion. $\diamondsuit$ Reduction of internal symmetries is described in Śniatycki [1998].

[5] A "historical" remark (by JK). Near the end of my post-doctoral year in Berkeley, way back in 1982, I became interested in nH systems, in particular with symmetries. Alan directed me to two wonderful books: Hertz [1899] *Foundation of Mechanics* and the treatise Neimark and Fufaev [1972]. In the latter I learned about Chaplygin systems, presented in coordinates. When I mentioned to Alan that I would like to examine the more general case of non-abelian group symmetries, Alan made a picture on his blackboard, and told me: "well, then, the constraints are given by a connection on a pricipal bundle". This was the starting point of Koiller [1992].

Terminology. Since Bates and Śniatycki [1993], and Bloch, Krishnaprasad, Marsden and Murray [1996], several authors have called attention on these two types of symmetries. Using them for *reduction by stages* has also been (at least implicitly) proposed. To stress the difference, reduction of external symmetries will be called here *compression.* The word *reduction* will be used only for internal symmetries. For G-Chaplygin systems we advocate that this reduction should be done *after* compression.

Hamiltonization. Affine symplectic structures. Since the first contact with symplectic geometry one is struck by the power of the closedness condition of the symplectic form (equivalently, the vanishing of the Poisson tensor Jacobinizer). Among many other things, this implies that the Poisson bracket of two integrals of a Hamiltonian vectorfield is again an integral of motion (Abraham and Marsden [1994], Arnold [1989]). Virtually all of the wonderful properties of symplectic geometry depend on closedness.

Nh systems have a reputation of having peculiar (even rebellious) dynamic behaviour (Arnold, Kozlov, and Neishtadt [1988]), and the general theory is way behind the theory of holonomic systems. Although the groundwork for an Hamilton-Jacobi theory for nH systems has been set up in Weber [1986], not much has been achieved since then. This undoubtly has to do with non-closedness. Nonetheless, in some special G-Chaplygin systems, ω_{NH} is *conformally symplectic* (Koiller and Rios [2001]); in such cases *we say that the compressed system is Hamiltonizable* and it becomes possible to use all of the standard symplectic techniques. Indeed, there is a well established theory of conformally symplectic forms, where one can rescue the symplectic machinery (Vaisman [1985], Wade [2000], Haller and Rybicki [1999], Haller and Rybicki [2001]).

Distilling a construction in Stanchenko [1985] we suggest considering *affine symplectic* structures, where one can add a 2-form Ω_o which is not "seen" by X, that is, $i_X\Omega_o = 0$. Thus $i_{X/f}(f\Omega_{NH} - \Omega_o) = dH$, and we must seek for an $f > 0$ and an Ω_o such that $d(f\Omega_{NH} - \Omega_o) = 0$. Assuming an educated guess for f, we give a simple criterium:

$$X = X_{NH} \text{ is hamiltonizable} \iff i_X d(f\Omega_{NH}) = 0 \ . \tag{1.6}$$

Question: It would be interesting to tie the "hamiltonizable" question with the invariants from the Cartan equivalence viewpoint.

Invariant measures and density functions for G-Chaplygin systems. In spite of many similarities, there are striking differences between nH and holonomic systems. For instance, nH systems do not have (in general) a smooth invariant measure. Necessary and sufficient conditions were given (explicitly in coordinates) by Blackall [1941]. In Kupka and Oliva [2001] and Kobayashi and Oliva [2003] a special context is considered (but in an invariant formulation). Namely, they find conditions insuring that the Riemann measure in TQ induced by the metric g is an invariant measure for the nH system. In the case of Chaplygin systems, a necessary and sufficient condition for the existence of an invariant measure for the compressed system is given in Cantrijn, Cortés, de Léon, and de Diego [2002], see Theorem 7.5. This is a very useful result for our purposes, because *the density function produces a natural candidate for the conformal factor alluded above.*

On Chaplygin's sphere: to be or not to be (Hamiltonizable), that's the question. It was very shocking for us to realize that nH Chaplygin systems possessing an invariant measure may not be Hamiltonizable. Consider a round sphere (possibly with different moments of inertia) rolling without slipping on a horizontal plane; it is assumed that the center of mass coincides with the geometric center[6]. A detailed account of the algebraic integrability is given in Duistermaat [2000]. The configuration space is $Q = R^2 \times SO(3)$. This problem has two external and three internal symmetries[7]. This would immediately imply integrability of the compressed system in $T^*SO(3)$ system, if it were Hamiltonizable. This dynamical system has an invariant measure, which gives us an educated guess f to use in the criterion $i_X d(f\Omega_{NH}) = 0$. However, it is NOT verified even in the homogeneous case, where $f \equiv 1$. These observations are in accordance with the opening statement in Duistermaat's paper:

> "Although the system is integrable in every sense of the word, it neither arises as a Hamiltonian system, nor is the integrability an immediate consequence of the symmetries".

However, there is still hope to Hamiltonize Chaplygin's sphere, when reduced to T^*S^2. We show that *is Hamiltonizable in the homogeneous case, where it reduces to an isotropic oscillator plus a Lorenz force.* Higher dimensional Chaplygin spheres, $Q = \Re^n \times SO(n)$ $n \geq 3$, also have invariant measures, but it is not known if they are integrable.

Question: Is the reduced system to T^*S^{n-1}, $S^{n-1} = SO(n)/SO(n-1)$ hamiltonizable? If the answer is positive, the integrability question may be addressed from a fresh perspective.

Jacobi's last multiplier and Hamiltonization on a reduced level. Integrability of Chaplygin's sphere stems from the existence of the invariant measure, which allows integration by *Jacobi's last multiplier method*. Suppose a dynamical system on a phase space M^n has an invariant measure and $n-2$ integrals. Take a coordinate system (x_1, x_2) on a given invariant surface. The restricted system has the form $\dot{x} = f_1(x,y)$, $\dot{y} = f_2(x,y)$, with an induced measure $\mu = F(x,y)\,dx \wedge dy$. As we learn in elementary calculus, the density F is the "integrating factor" and the system is declared integrable[8]. There is (at least locally) a function G such that $Ff_1 = \partial G/\partial x_2$, $Ff_2 = -\partial G/\partial x_1$. This is true because the condition for F to be an invariant measure is precisely $d\alpha = 0$, where $\alpha = -Ff_2 dx_1 + Ff_1 dx_2$.

One could introduce a Poisson structure $\{\, ,\}$ on the phase space M, in which the original integrals are all Casimirs, and the symplectic leaves are two dimensional. The system is manifestly Hamiltonian, $\dot{m} = \{m, \alpha\}$. The only proviso is that α may not be exact (it will be exact, though, on a suitable covering space). *In this sense, all systems integrable by Jacobi's last multiplier technique are Hamiltonizable.* However, for several reasons we want a more restricted definition of Hamiltonizable. We do not want a drastic change of Poisson structure as to change the Hamiltonian H to a totally different one G. It is also preferable to have symplectic leaves with dimension as high as possible; we would like to apply symplectic reduction *after* Hamiltonization, as a way to integrate the system. At any rate, we can reach a compromise and relax a bit the Hamiltonization question. We say that a nH system is Hamiltonizable when *some reduction of it* has the form

$$i_{\tilde{X}/f}\,\tilde{\Omega} = d\tilde{H} \;\; , \;\; d\,\tilde{\Omega} = 0 \; . \tag{1.7}$$

Question: find sufficient (ideally also necessary) conditions in terms of the original system.

[6] Cartan [1928] considered only the honest case (i.e., homogeneous), unaware perhaps that Chaplygin had shown in 1903 that even the dishonest case is integrable (see the reprint of the original paper in Chaplygin [2002]).

[7] Chaplygin's sphere is an example of a LR (left-right) nH system on a Lie group, where the metric is left invariant, but the constraints are right invariant (see Fedorov and Jovanovic [2003] for a very recent paper).

[8] More precisely, a recent trend in computer algebra is to find criteria for explicit solutions in given classes of functions.

2 Equivalence problem of nonholonomic geometry on Engel manifolds

Outline. A *nonholonomic structure* is a triple $(Q, ds^2 = \langle \cdot\, , \cdot \rangle, \mathcal{H})$ where M is an n-dimensional manifold endowed with a Riemannian metric ds^2 and a rank r, totally nonholonomic distribution $\mathcal{H}$. The motivation for studying such a structure is a free particle moving in Q, nonholonomically constrained to $\mathcal{H}$, with kinetic energy $T = \frac{1}{2}\langle \cdot\, , \cdot \rangle$. The nonholonomic geodesic equations are obtained by computing accelerations using the Levi-Civita connection associated with ds^2 and orthogonally projecting the result onto $\mathcal{H}$. This is called a *nonholonomic connection* (Lewis [1998]), and was introduced by Cartan [1928]. A programme to classify nH structures using the Cartan equivalence was outlined in Koiller, Rodrigues and Pitanga [2001], and applied in Ehlers [2002] to the simplest case of a contact distribution. Here we describe the next case, Engel manifolds.

The main question we address is the following. Given two nonholonomic structures $(Q, ds^2, \mathcal{H})$ and $(\bar{Q}, \bar{ds}^2, \bar{\mathcal{H}})$, is there a (local) diffeomorphism $f : U \subset Q \to \bar{U} \subset \bar{Q}$ carrying nonholonomic geodesics to nonholonomic geodesics? In Cartan's approach, this question is recast as an equivalence problem. The nonholonomic structure is encoded into a subbundle of the frame bundle over Q, called a G-structure. *The diffeomorphism f exists if the two corresponding G-structures are locally equivalent.* Necessary and sufficient conditions for the G-structures to be equivalent are given in terms of differential invariants found using the *method of equivalence.*

Our main example is the equivalence problem for nonholonomic geometry on an Engel manifold. An Engel distribution is a rank two distribution with growth vector $(2, 3, 4)$. By an Engel manifold, we simply mean a four dimensional manifold endowed with an Engel distribution. We derive all local invariants and apply the framing lemma, (lemma 2.4) to show that the symmetry group of such a structure has dimension at most four.

We begin by describing the nonholonomic geodesic equations. In the spirit of Cartan's program, we express them in terms of connection one-forms and (co)frames adapted to the distribution. This formulation is particularly well suited to the problem at hand; the nonholonomic geodesic equations are obtained by computing the Levi-Civita connection form and crossing out terms corresponding to directions complimentary to $\mathcal{H}$. We set up the equivalence problem for nonholonomic geometry, give a brief description of the equivalence method and apply it to the case of nonholonomic geometry on Engel manifolds.

Historical remarks. (Cartan [1928]) introduced the equivalence problem for nonholonomic geometry and studied the case of manifolds endowed with *strongly nonholonomic distributions.* A distribution $\mathcal{H}$ is strongly nonholonomic if any basis of vector fields spanning $\mathcal{H}$ on $U \subset Q$, together with their Lie brackets, span the entire tangent space over U. In his address, Cartan warned against attempts to study other cases because of the "plus compliqués" computations involved. In the meantime strides have been made in the equivalence method by Robert Gardner and his students that allow computations to be made at the Lie algebra level rather than at the group level (Gardner [1989]). This together with symbolic computation packages make equivalence problems tractable in many important cases. The equivalence problem for nonholonomic geometry was revisited in Koiller, Rodrigues and Pitanga [2001] and the generalization to arbitrary nonholonomic distributions was discussed. *Engel manifolds provide the simplest example involving distributions that are not strongly nonholonomic.*

2.1 Nonholonomic geometry

In this section we review some basic facts concerning nonholonomic distributions and the nonholonomic geodesic equations.

Totally nonholonomic distributions. A distribution $\mathcal{H}$ is a vector subbundle of the tangent bundle $T(Q)$ over Q. A path $c : \mathbf{R} \to Q$ is *horizontal* if $\dot{c}(t) \in \mathcal{H}_{c(t)}$ for all t. $\mathcal{H}$ is totally nonholonomic if, for any basis of tangent vector fields $\{X_1, ...X_r\}$ spanning $\mathcal{H}$, we have $\mathrm{span}\{X_i, [X_i, X_j], [X_i, [X_j, X_k]], ...\} = T(Q)$. Chow's theorem implies that if $\mathcal{H}$ is totally nonholonomic then any two points in Q can be joined by a horizontal path (see Montgomery [2002]). At the other extreme, the classical theorem of Frobenius implies that $\mathcal{H}$ is *integrable*, which is to say that Q is foliated by submanifolds whose tangent space coincides with $\mathcal{H}$ at each point, if and only if $[X_i, X_j] \in \mathcal{H}$ for all i and j (Warner [1971]).

As a specific example, consider the Engel distribution $\mathcal{H}$ on $\mathbf{R}^4$ with coordinates (x, y, z, w), spanned by $\{X_1 = \frac{\partial}{\partial w}, X_2 = \frac{\partial}{\partial x} + w\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\}$. There are, in fact, local coordinates on any Engel manifold so that the distribution is given by this normal form, see Montgomery [2002]. Then $\{X_1, X_2, X_3 = [X_1, X_2]\}$ spans a three-dimensional distribution $\mathcal{H}^{(1)}$, and $\{X_1, X_2, X_3, X_4 = [X_2, X_3]\}$ spans the entire tangent space of $\mathbf{R}^4$.

In what follows we will need a description of distributions in terms of coframes and differential ideals. Details can be found in Warner [1971] or Montgomery [2002]. Let $\mathcal{I}$ be the ideal in $\Lambda^*(Q)$ consisting of the differential forms annihilating $\mathcal{H}$. If $\mathcal{H}$ is rank r, then $\mathcal{I}$ is generated by $n - r$ independent one-forms. The *first derived ideal* of $\mathcal{I}$ is the ideal

$$(\mathcal{I})' := \{\theta \in \mathcal{I} \mid d\theta \equiv 0 \bmod (\mathcal{I})\}. \tag{2.1}$$

The first derived ideal $(\mathcal{I})'$ annihilates the distribution $\mathcal{H}^{(1)} = \mathrm{span}\{X_i, [X_i, X_j]\}$. If we set $\mathcal{I}^{(0)} = \mathcal{I}$ and $\mathcal{I}^{(n+1)} = (\mathcal{I}^{(n)})'$ we obtain a decreasing filtration

$$\mathcal{I} = \mathcal{I}^{(0)} \supset \mathcal{I}^{(1)} \supset \cdots \supset 0.$$

The filtration terminating with the 0 ideal is equivalent to the assumption that the distribution is completely nonholonomic. At the other extreme, the differential ideal version of the Frobenius theorem implies that $\mathcal{H}$ is integrable if and only if $(\mathcal{I})' \subset \mathcal{I}$ (Warner [1971]).

For the Engel example, the one forms $\eta^1 = dy - w dx$ and $\eta^2 = dz - y dx$ generate the ideal $\mathcal{I}^{(0)}$. Notice that $d\eta^2 = \eta^1 \wedge dx$ so $\eta^2 \in \mathcal{I}^{(1)}$ but $d\eta^1$ cannot be written in terms of η^1 or η^2 therefore $\eta^1 \notin \mathcal{I}^{(1)}$.

The nonholonomic geodesic equations. There are two different geometries commonly defined on a nonholonomic structure $(Q, ds^2 = \langle\ \cdot\ ,\ \cdot\ \rangle, \mathcal{H})$: *subriemannian geometry vs. nonholonomic geometry.* In subriemannian geometry one is interested in the *shortest paths.* The length of a path $c : [a, b] \to Q$ joining points x and y is $\ell(c) = \int \sqrt{\langle \dot{c}, \dot{c}\rangle} dt$. The distance from x to y is $d(x, y) = \inf(\ell(c))$ taken over all paths joining x to y that are tangent to $\mathcal{H}$. In nonholonomic geometry one is interested in *straightest paths*, which are solutions to the nonholonomic geodesic equations. Hertz [1899] was the first to notice that shortest $\neq$ straightest unless the constraints are holonomic. The nonholonomic geodesic equations are motivated by D'Alembert's principle:

> *Let $c : \mathbf{R} \to Q$ represent the position of a particle. Consider the mechanical system with kinetic energy $T = \frac{1}{2}ds^2(\dot{c}, \dot{c})$ and applied forces F, subject to the constraint $\dot{c} \in \mathcal{H}_{c(t)}$. Let ∇ be the Levi-Civita connection associated with ds^2. Then the constraining force $\nabla_{\dot{c}}\dot{c} - F$ is ds^2−perpendicular to the constraint subspace $\mathcal{H}_{c(t)}$ since it does not produce work.*

Notations. We will assume that there are no applied forces so the equations of motion reduce to the nonholonomic geodesic equations. They are obtained by computing the acceleration using the Levi-Civita connection and orthogonally projecting the result onto $\mathcal{H}$. It is convenient to adopt the following indicial conventions:

$$\begin{array}{rcl} 1 \leq & I, J, K & \leq n \\ 1 \leq & i, j, k & \leq r \quad (= \mathrm{rank}(\mathcal{H})) \\ r+1 \leq & \nu & \leq n \end{array} \tag{2.2}$$

Let $e = \{e_I\}$ be a local orthonormal frame for which the e_i span $\mathcal{H}$, and let $\eta = \{\eta_I\}$ be the dual coframe defined by $\eta_I(e_J) = \delta_{IJ}$, the Kronecker delta function. We note that the η^ν annihilate $\mathcal{H}$ and the metric, restricted to $\mathcal{H}$ is $ds^2|_{\mathcal{H}} = \eta^1 \otimes \eta^1 + \cdots + \eta^r \otimes \eta^r$. The Levi-Civita connection ∇ can be expressed in terms of local one-forms $\omega_{IJ} = -\omega_{JI}$ satisfying Cartan's structure equation $d\eta = -\omega \wedge \eta$.

A path $c : \mathbf{R} \to M$ is a *nonholonomic geodesic* if it satisfies the nonholonomic geodesic equations

$$\left[\frac{d}{dt}(v_i) + \sum_j v_j \omega_{ij}(\dot{c}) \right] e_i = 0, \tag{2.3}$$

where $1 \leq i, j, \leq r$ and $v_i = \eta^i(\dot{c})$ are the quasivelocities (Koiller, Rodrigues and Pitanga [2001]).

Example: the vertical rolling penny. A standard example of a mechanical system modeled by a nonholonomic Engel system is that of a penny rolling without slipping on the plane. Consider a coin of radius a rolling vertically on the xy-plane. The location of the coin is represented by the coordinates (x, y, θ, ϕ). The state space can be identified with the Lie group $SE(2) \times SO(2)$. The point of contact of the coin with the plane is (x, y), the angle made by the coin with respect to the positive x-axis is θ, and the angle made by the point of contact, the center of the coin, and a point marked on the outer edge of the coin is ϕ. The mass of the coin is m, the moment of inertia in the θ direction is J and the moment of inertia in the ϕ direction is I. The kinetic energy, which defines a Riemannian metric on the state space, is

$$ds^2 := \frac{1}{2}(dx \otimes dx + dy \otimes dy) + \frac{J}{2} d\theta \otimes d\theta + \frac{I}{2} d\phi \otimes d\phi. \tag{2.4}$$

The penny rolls without slipping giving rise to the constraints

$$\dot{x} = (a \cos\theta)\, \dot{\phi} \ \ , \ \ \dot{y} = (a \sin\theta)\, \dot{\phi} \ \ . \tag{2.5}$$

Consider the orthonormal frame (X_1, X_2, X_3, X_4) where

$$\begin{aligned} X_1 &:= \sqrt{\frac{2}{ma^2 + I}} \left(a\cos\theta \frac{\partial}{\partial x} + a \sin\theta \frac{\partial}{\partial y} + \frac{\partial}{d\phi} \right) \\ X_2 &:= \sqrt{\frac{2}{J}} \frac{\partial}{\partial \theta} \\ X_3 &:= \sqrt{\frac{2}{m}} \left(-\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y} \right) \\ X_4 &:= \sqrt{\frac{2}{m}} \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \right). \end{aligned} \tag{2.6}$$

Note that the constraint subspace $\mathcal{H} = \text{span}\{X_1, X_2\}$, and $\mathcal{H}^{(1)} = \text{span}\{X_1, X_2, X_3\}$). The dual coframe is $(\eta^1, \eta^2, \eta^3, \eta^4)$ where

$$\begin{aligned} \eta^1 &:= \sqrt{\frac{ma^2+I}{2}}d\phi \\ \eta^2 &:= \sqrt{\frac{J}{2}}d\theta \qquad (2.7) \\ \eta^3 &:= \sqrt{\frac{m}{2}}(-\sin\theta dx + \cos\theta dy) \\ \eta^4 &:= \sqrt{\frac{m}{2}}(\cos\theta dx + \sin\theta dy - d\phi). \qquad (2.8) \end{aligned}$$

To compute the Levi Civita connection form we determine $\omega = [\omega_{IJ}]$ such that $\omega_{IJ} = -\omega_{JI}$ and $d\eta = -\omega \wedge \eta$. Using simple linear algebra we find

$$\omega = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}\sqrt{\frac{m}{J(ma^2+I)}}\ \eta^3 & \frac{1}{\sqrt{2}}\sqrt{\frac{m}{J(ma^2+I)}}\ \eta^2 & 0 \\ -\frac{1}{\sqrt{2}}\sqrt{\frac{m}{J(ma^2+I)}}\ \eta^3 & 0 & -\frac{1}{\sqrt{2}}\sqrt{\frac{m}{J(ma^2+I)}}\ \eta^1 & 0 \\ -\frac{1}{\sqrt{2}}\sqrt{\frac{m}{J(ma^2+I)}}\ \eta^2 & \frac{1}{\sqrt{2}}\sqrt{\frac{m}{J(ma^2+I)}}\ \eta^1 & 0 & -\frac{\sqrt{2}}{\sqrt{J}}\eta^2 \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{J}}\eta^2 & 0 \end{pmatrix} \qquad (2.9)$$

so in particular

$$\omega_{12} = -\omega_{21} = \frac{1}{2}\sqrt{\frac{m}{J(ma^2+I})}\ \ \eta^3.$$

Let $c : \mathbf{R} \to Q$ be a nonholonomic geodesic given by $\dot{c}(t) = v_1(t)X_1 + v_2(t)X_2$. From the structure equations we see immediately that $\omega_{12}(\dot{c}(t)) = -\omega_{21}(\dot{c}(t)) = 0$ and the nonholonomic geodesic equations reduce to $\frac{d}{dt}(v_1) = \frac{d}{dt}(v_2) = 0$. The nonholonomic geodesics are solutions to $(\dot{x}, \dot{y}, \dot{\phi}, \dot{\theta}) = AX_1 + BX_2$. In particular,

$$\dot{x} = \frac{\sqrt{2}Aa\cos\theta(t)}{\sqrt{ma^2+I}}, \quad \dot{y} = \frac{\sqrt{2}Aa\sin\theta(t)}{\sqrt{ma^2+I}}, \quad \dot{\phi} = \frac{\sqrt{2}A}{\sqrt{ma^2+I}}, \quad \dot{\theta} = \frac{\sqrt{2}B}{\sqrt{J}}. \qquad (2.10)$$

The trajectories are spinning in place ($A = 0$), rolling along a line ($B = 0$), or circles ($A, B \neq 0$).

2.2 The equivalence problem for nonholonomic geometry.

Cartan's method of equivalence starts by encoding a geometric structure in terms of a subbundle of the coframe bundle called a G-structure. We begin this section by describing the G-structure for nonholonomic geometry. This G-structure was first presented by Cartan in his 1928 address to the International Congress of Mathematicians (Cartan [1928]). We then give a brief outline of some of the main ideas behind the method of equivalence as it is applied in our example of nonholonomic geometry on an Engel manifold. Details can be found in Gardner [1989], Montgomery [2002], or Bryant [1994]. We derive the invariants associated with a nonholonomic structure on a 4-dimensional manifold endowed with an Engel distribution.

Initial G-structure for nonholonomic geometry. A *coframe* $\eta(x)$ at $x \in Q^n$ is a basis for the cotangent space $T_x^*(Q)$. Alternatively, we can regard a coframe as a linear isomorphism $\eta(x) : T_x(Q) \to \mathbf{R}^n$ where $\mathbf{R}^n$ is represented by column vectors. The set of all coframes at x is denoted $F_x^*(Q)$ and has the canonical foot point mapping $\pi : F_x^*(Q) \to x$. The coframe bundle $F^*(Q)$ is the union of the $F_x^*(Q)$ as x varies over Q. A coframe is a smooth (local) section $\eta : Q \to F^*(Q)$ and is represented by a column vector of one-forms $(\eta^1, ..., \eta^n)^{\mathrm{tr}}$, where "tr" indicates transpose. A coframe can then be multiplied by a matrix on the left in the usual way. $F^*(Q)$ is a right $Gl(n)$-bundle with action $R_g\eta = g^{-1}\eta$ where g is a matrix in $Gl(n)$.

Definition 2.1. *If G is a matrix subgroup of $Gl(n)$, then a G-structure is simply a G-subbundle of $F^*(Q)$.*

We now describe the G-structure encoding the nonholonomic geometry associated with a nonholonomic structure $\{Q, ds^2, \mathcal{H}\}$. Given a nonholonomic structure $(Q, ds^2, \mathcal{H})$ we can choose an orthonormal coframe $\eta = (\eta^i, \eta^\nu)^{\mathrm{tr}}$ on $U \subset Q$ so that the η^ν annihilate $\mathcal{H}$ and use this coframe to write down the nonholonomic geodesic equations as described above. On the other hand, given a coframe $\bar\eta = (\bar\eta^i, \bar\eta^\nu)^{\mathrm{tr}}$ on Q we can construct a nonholonomic structure $(Q, \bar{ds}^2 = \Sigma\ \bar\eta^i \otimes \bar\eta^i + \bar\eta^\nu \otimes \bar\eta^\nu, \bar{\mathcal{H}})$ where $\bar{\mathcal{H}}$ is annihilated by the $\bar\eta^\nu$. How is $\bar\eta$ related to η if it is to lead to the same nonholonomic geodesic equations as η? In order to preserve $\mathcal{H}$ we must have $\eta^\nu - \bar\eta^\nu = 0 \pmod{I}$. In matrix notation, any modified coframe $\bar\eta$ must be related to η by

$$\begin{pmatrix} \bar\eta^i \\ \bar\eta^\nu \end{pmatrix} = \begin{pmatrix} A & b \\ 0 & a \end{pmatrix} \begin{pmatrix} \eta^i \\ \eta^\nu \end{pmatrix}. \tag{2.11}$$

where $A \in Gl(r)$, $a \in Gl(n-r)$, and $b \in M(k, n-r)$. If we were studying the geometry of distributions there would be no further restrictions. In order to preserve the metric restricted to $\mathcal{H}$, we must further insist that $A \in O(r)$. We would then have the starting point for the study of subriemannian geometry (see Montgomery [2002], Hughen [1995], or Moseley [2001]).

It is important to observe that that in nonholonomic geometry we need the full metric and not just its restriction to $\mathcal{H}$ (as in subriemannian geometry) to obtain the equations of motion. Cartan [1928]) showed that *in order to preserve the nonholonomic geodesic equations, we can only add covectors that are in the first derived ideal to the η^i.*

Since this fact is central to our analysis, we sketch the argument here (see Koiller, Rodrigues and Pitanga [2001]). Suppose $\bar\eta = g\eta$ with connection one-form defined by $d\bar\eta = -\bar\omega \wedge \bar\eta$. For simplicity assume that $A = id$, then $\eta^j \equiv \bar\eta^j \pmod{\mathcal{I}}$. The geodesic equations are preserved if and only if $\omega_{ij}(T) = \bar\omega_{ij}(T)$ for all $T \in \mathcal{H}$, in other words $\omega_{ij} \equiv \bar\omega_{ij} \pmod{\mathcal{I}}$. Note also that $\bar\eta^\nu \equiv 0 \pmod{\mathcal{I}}$. Subtracting the structure equations for $d\eta^i$ and $d\bar\eta^i$ we get

$$\begin{aligned} d\eta^i - d\bar\eta^i &= -\omega_{ij} \wedge \eta^j - \omega_{i\nu} \wedge \eta^\nu + \bar\omega_{ij} \wedge \bar\eta^j + \bar\omega_{i\nu} \wedge \bar\eta^\nu \\ &\equiv 0 \pmod{\mathcal{I}}. \end{aligned}$$

Now $\bar\eta^i = \eta^i + b_{i\nu}\eta^\nu$ so we also have

$$\begin{aligned} d\eta^i - d\bar\eta^i &= d\eta^i - (d\eta^i + db_{i\nu}\eta^\nu + b_{i\nu}d\eta^\nu) \\ &\equiv -b_{i\nu}d\eta^\nu \pmod{\mathcal{I}} \end{aligned}$$

Therefore $b_{i\nu}d\eta^\nu \equiv 0 \pmod{\mathcal{I}}$ or equivalently $b_{i\lambda}\eta^\lambda \in I^{(1)}$. This completes the argument. $\diamondsuit$

We further subdivide our indicial notation: let

$$\begin{aligned} r+1 \leq\ &\phi \leq s \\ s+1 \leq\ &\Phi \leq n. \end{aligned} \tag{2.12}$$

Adapted coframes. A covector $\eta = (\eta^i, \eta^\phi, \eta^\Phi)^{\mathrm{tr}})$ arranged so that

1. The η^ϕ and η^Φ generate I,
2. $ds^2|_{\mathcal{H}} = \sum \eta^i \otimes \eta^i$,
3. The η^Φ generate the first derived ideal $I^{(1)}$,

is said to be *adapted to the nonholonomic structure*. In matrix notation, the most general change of coframes that preserves the nonholonomic geodesic equations is of the form $\bar{\eta} = g\eta$ where

$$g = \begin{pmatrix} A & 0 & b \\ 0 & a_1 & a_2 \\ 0 & 0 & a_3 \end{pmatrix} \tag{2.13}$$

with $A \in O(k)$, $b \in M(n-s,k)$, $a_1 \in Gl(s-k)$, $a_2 \in Gl(n-s, s-k)$, and $a_3 \in Gl(n-s)$. The set of all such block matrices form a matrix subgroup of $Gl(n)$ which we shall denote G_0.

Definition 2.2. *The initial G-structure for nonholonomic geometry on $(Q, ds^2, \mathcal{H})$ is a subbundle $B_0(Q) \subset F^*(Q)$ (or simply B_0 if there is no risk of confusion) with structure group G_0 defined above. All local sections of $B_0(Q)$ lead to the same nonholonomic geodesic equations. In this way, the initial G-structure $B_0(Q)$ completely characterizes the nonholonomic geometry.*

Two G-structures, $B(M) \xrightarrow{\pi_M} M$ and $B(N) \xrightarrow{\pi_N} N$, are said to be *equivalent* if there is a diffeomorphism $f: M \to N$ for which $f_1(B(M)) = B(N)$ where f_1 is the induced bundle map. If we think of $b \in B(M)$ as a linear isomorphism $b: T_{\pi_M(b)}M \to \mathbf{R}^n$ then $f_1(b) = b \circ (f_*)^{-1}$ where f_* is the differential of f. Our original question of whether there is a local diffeomorphism that carries nonholonomic geodesics to nonholonomic geodesics can be answered by determining whether the two associated G-structures are locally equivalent.

2.3 A crash course on the method of equivalence

Necessary and sufficient conditions for the equivalence between G-structures are given in terms of *differential invariants* which are derived using the method of equivalence. In this section we briefly describe some of the main ideas behind the method of equivalence as it is applied in our example. Details and other facets of the method together with many examples can be found in the excellent text by Robert Gardner (Gardner [1989]).

One of the principal objects used in the method of equivalence is the *tautological one-form*. Let $B(M) \xrightarrow{\pi} M$ be a G-structure with structure group G whose Lie Algebra is $\mathcal{G}$. The tautological one-form Ω on $B(M)$ is an $\mathbf{R}^n$-valued one-form defined as follows. Let $\eta : U \subset M \to B(M)$ be a local section of $B(M)$ and consider the inverse trivialization $U \times G_0 \to B(M)$ defined by $(x, g) \to g^{-1}\eta(x)$. Relative to this section, the tautological one-form is defined by

$$\Omega(b) = g^{-1}(\pi^*\eta) \tag{2.14}$$

where $b = g^{-1}\eta$. From (2.14) one can verify that the tautological one-form is semi-basic (i.e. $\Omega(v) = 0$ for all $v \in ker(\pi_*)$), has the reproducing property $\bar{\eta}^*\Omega = \bar{\eta}$ where $\bar{\eta}$ is any local section of $B(M)$, and is equivariant: $R_g^*\Omega = g^{-1}\Omega$. The components of the tautological one-form provide a partial coframing for $B(M)$ and form a basis for the semi-basic forms on $B(M)$.

The following proposition reduces the problem of finding an equivalence between G-structures to finding a smooth map that preserves the tautological one-form. (See Gardner [1989] or Bryant [1994] for a proof.)

Proposition 2.3. *Let $B(M)$ and $B(N)$ be two G-structures with corresponding tautological one-forms Ω_M and Ω_N, and let $F : B(M) \to B(N)$ be a smooth map. If G is a connected and $F^*(\Omega_N) = \Omega_M$ then there exists a local diffeomorphism $f : M \to N$ for which $F = f_1$, i.e. the two G-structures are equivalent.*

To find the map F in this proposition we would like to apply *Cartan's technique of the graph* (cf. Warner [1971]): if we could find an integral manifold $\Sigma \subset B(M) \times B(N)$ of the one-form $\theta = \Omega_M - \Omega_N$ that projects diffeomorphically onto each factor, then Σ would be the graph of a function $h : M \to N$ for which $h_1^*\Omega^N = \Omega_N$. By the above proposition the G-structures would then be equivalent. We generally cannot apply this idea directly because Ω_M and Ω_N do not provide full coframes on $B(M)$ and $B(N)$ as is required in the technique of the graph. In the example of nonholonomic geometry on Engel manifolds, and indeed in many important examples (see Gardner [1989], Hughen [1995], Moseley [2001], Montgomery [2002], Ehlers [2002]), application of the method of equivalence leads to a new G-structure called an e-structure. An e-structure is a G-structure endowed with a canonical coframe.

Differentiating both sides of (2.14) one can verify that $d\Omega$ satisfies the structure equation

$$d\Omega = -\alpha \wedge \Omega + T \tag{2.15}$$

where T is a semi-basic two-form on $B(M)$ and α is a called a *pseudoconnection*: a $\mathcal{G}$-valued one-form on $B(M)$ that agrees with the Mauer-Cartan form on vertical vector fields. Here, $\mathcal{G}$ is the Lie Algebra of G. summarizing,

$$\text{Pseudoconnection}: \ \alpha = g^{-1}dg + \text{semibasic } \mathcal{G}\text{−valued one form.} \tag{2.16}$$

Intrinsic torsion, reduction and prolongation. The components of the pseudoconnection together with the tautological one-form do provide a full coframe on the G-structure, but unlike the tautological one-form, the pseudoconnection is not canonically defined. *Understanding how changes in the pseudoconnection affect the torsion is at the heart of the method of equivalence.* For any G-structure, that part of the torsion that is left unchanged under all possible changes of pseudoconnection is known as the *intrinsic torsion*. The intrinsic torsion is the only first order differential invariant of the G-structure (Gardner [1989]). As an exercise, we would encourage the interested reader to compute the intrinsic torsion for the G-structure for general distributions. See (2.11) above for the initial G-structure. The result, as found by Cartan [1910], is that the intrinsic torsion is the dual curvature of the distribution (see also Montgomery [2002]).

There are two major steps in the equivalence method: *reduction* and *prolongation* (see Gardner [1989] or Montgomery [2002]). In the case of nonholonomic geometry on an Engel manifold a sequence of reductions lead to an e-structure. A brief outline of the reduction procedure is as follows. The first step involves writing out the structure equations for the tautological one-form Ω. A semi-basic $\mathcal{G}$-valued one-form is added to the pseudoconnection to make the torsion as simple as possible. Gardner [1989] calls this step *absorption of torsion*. The action of G on the torsion is deduced by differentiating both sides of the identity $R_g^*(\Omega) = g^{-1}\Omega$. The action of G is used to simplify part of the torsion. The isotropy subgroup of that choice of simplified torsion is then the structure group of the reduced G-structure. In the case of nonholonomic geometry on an Engel manifold this procedure is repeated until an e-structure is obtained.

Suppose that Ω is the canonical coframing on the resulting a manifold B. The Ω^i form a basis for the one-forms on B so we can write

$$d\Omega^i = \sum_{j<k} c^i_{jk}\Omega^j \wedge \Omega^k \ . \tag{2.17}$$

Relationships between the c^i_{jk} are found by differentiating this equation. The resulting torsion functions provide the "complete invariants" for the geometric structure (see Gardner [1989] p.59, Bryant [1994] pp.9-10, or Cartan [2001]).

Many important examples have *integrable e-structures*. An e-structure is integrable if the c^i_{jk} are constant (Gardner [1989]). In this case we can apply the following result which we quote from Montgomery [2002]:

Lemma 2.4. *Let B be an n-dimensional manifold endowed with a coframing Ω. Then the (local) group G of diffeomorphisms of B that preserves this coframing is a finite-dimensional (local) Lie group of dimension at most n. The bound n is achieved if and only if the e-structure is integrable. In this case the c^i_{jk} are the structure constants of G, G acts freely and transitively on B, and the coframe can be identified with the left invariant one-forms on G.*

The Jacobi identities are found by differentiating $d\Omega^i = \sum_{j<k} c^i_{jk}\Omega^j \wedge \Omega^k$. Lie's third fundamental theorem then implies that we can, at least in principle, reconstruct the group G using the structure constants. In some circumstances one can also conclude that B itself is a Lie group (see Gardner [1989] p.72).

2.4 The nonholonomic geometry on an Engel manifold.

The initial G structure for nonholonomic geometry on $\{M, \mathcal{H}, ds^2\}$ where $\mathcal{H}$ is an Engel distribution on a four-dimensional manifold M is the subbundle $B_0 \subset F^*(M)$ with structure group G_0 consisting of matrices of the form

$$\begin{pmatrix} A_{11} & A_{12} & 0 & B_{14} \\ A_{21} & A_{22} & 0 & B_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \tag{2.18}$$

where $A = [A_{IJ}] \in O(2)$, $a_{33}a_{44} \neq 0$, and B_{14} and B_{24} are arbitrary.

Let $\Omega = (\Omega^1, \Omega^2, \Omega^3, \Omega^4)^{tr}$ be the tautological one-form on B_0. The structure equations are

$$d\begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{pmatrix} = -\begin{pmatrix} 0 & \gamma & 0 & \beta_{14} \\ -\gamma & 0 & 0 & \beta_{24} \\ 0 & 0 & \alpha_{33} & \alpha_{34} \\ 0 & 0 & 0 & \alpha_{44} \end{pmatrix} \wedge \begin{pmatrix} \Omega^1 \\ \Omega^2 \\ \Omega^3 \\ \Omega^4 \end{pmatrix} + \begin{pmatrix} T^1_{13}\Omega^1 \wedge \Omega^3 + T^1_{23}\Omega^2 \wedge \Omega^3 \\ T^2_{13}\Omega^1 \wedge \Omega^3 + T^2_{23}\Omega^2 \wedge \Omega^3 \\ T^3_{12}\Omega^1 \wedge \Omega^2 \\ T^4_{13}\Omega^1 \wedge \Omega^3 + T^4_{23}\Omega^2 \wedge \Omega^3 \end{pmatrix} \tag{2.19}$$

where we have chosen the pseudoconnection so that the remaining T^i_{jk} are zero. $\Omega^4 \in I^{(1)}$ so $d\Omega^4 = 0$ mod (Ω^3, Ω^4) and we must therefore have $T^4_{12} = 0$. Also, $\Omega^3 \notin I^{(1)}$ so $d\Omega^3 \neq 0$ mod (Ω^3, Ω^4) therefore the torsion function T^3_{12} cannot equal zero. The pseudo-connection for this choice of torsion is not unique. We can, for instance, add arbitrary multiples of Ω^4 to the β_{i4} and α_{i4}.

Following Cartan's prescription, we investigate the induced action of G_0 on the torsion. Let $g \in G_0$. To simplify notation, functions and forms pulled back by R_g will be indicated by a hat so, for instance, $R^*_g\Omega = \hat{\Omega} = (\hat{\Omega}^1, \hat{\Omega}^2, \hat{\Omega}^3, \hat{\Omega}^4)^{\mathrm{tr}}$ and $R^*_g(T^k_{ij}) = \hat{T}^k_{ij}$. We have

$$\begin{pmatrix} \hat{\Omega}^1 \\ \hat{\Omega}^2 \\ \hat{\Omega}^3 \\ \hat{\Omega}^4 \end{pmatrix} = \begin{pmatrix} \# \\ \# \\ \det(a^{-1})(a_{44}\Omega^3 - a_{34}\Omega^4) \\ \det(a^{-1})(a_{33}\Omega^4) \end{pmatrix} . \tag{2.20}$$

To determine the induced action of G_0 on the torsion we differentiate both sides of the identity $R^*_g\Omega^3 = \hat{\Omega}^3$. For Ω^3 we compute

$$\begin{aligned} R^*_g(d\Omega^3) &= \hat{\alpha}_{33} \wedge \hat{\Omega}^3 + \hat{\alpha}_{34} \wedge \hat{\Omega}^4 + \hat{T}^3_{12}\hat{\Omega}^1 \wedge \hat{\Omega}^2 \\ &= \det(A^{-1})\hat{T}^3_{12}\Omega^1 \wedge \Omega^2 \qquad (\mathrm{mod}\ \ \Omega^3, \Omega^4) \end{aligned}$$

and

$$\begin{aligned} d\hat{\Omega}^3 &= \det(a^{-1})(a_{44}d\Omega^3 - a_{34}d\Omega^4) \qquad (\text{mod} \ \ \Omega^3, \Omega^4) \\ &= \det(a^{-1})(a_{44}T^3_{12}\Omega^1 \wedge \Omega^2) \qquad (\text{mod} \ \ \Omega^3, \Omega^4) \ . \end{aligned}$$

The induced action of G_0 on T^3_{12} is therefore

$$R_g^*(T^3_{12}) = \frac{\det(A)}{a_{33}} T^3_{12} \tag{2.21}$$

Since $T^3_{12} \neq 0$ we can force it to equal 1 using the action of G_0. The stabilizer subgroup G_1 for this choice of torsion consists of matrices of the form (2.18) with $a_{33} = \epsilon$ where $\epsilon = \det(A)$.

The structure equations for the G_1-structure B_1 are

$$d\begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{pmatrix} = -\begin{pmatrix} 0 & \gamma & 0 & \beta_{14} \\ -\gamma & 0 & 0 & \beta_{24} \\ 0 & 0 & 0 & \alpha_{34} \\ 0 & 0 & 0 & \alpha_{44} \end{pmatrix} \wedge \begin{pmatrix} \Omega^1 \\ \Omega^2 \\ \Omega^3 \\ \Omega^4 \end{pmatrix} + \begin{pmatrix} T^1_{13}\Omega^1 \wedge \Omega^3 + T^1_{23}\Omega^2 \wedge \Omega^3 \\ T^2_{13}\Omega^1 \wedge \Omega^3 + T^2_{23}\Omega^2 \wedge \Omega^3 \\ T^3_{13}\Omega^1 \wedge \Omega^3 + T^3_{23}\Omega^2 \wedge \Omega^3 + \Omega^1 \wedge \Omega^2 \\ T^4_{13}\Omega^1 \wedge \Omega^3 + T^4_{23}\Omega^2 \wedge \Omega^3 \end{pmatrix} \tag{2.22}$$

Let $g \in G_1$. We write the inverse of g as

$$g^{-1} = \begin{pmatrix} A_{11} & A_{21} & 0 & \bar{B}_{14} \\ A_{12} & A_{22} & 0 & \bar{B}_{24} \\ 0 & 0 & \bar{a}_{33} & \bar{a}_{34} \\ 0 & 0 & 0 & \bar{a}_{44} \end{pmatrix} \tag{2.23}$$

so in particular $\bar{a}_{33} = \epsilon$, $\bar{a}_{34} = -\epsilon a_{34}(a_{44})^{-1}$, and $\bar{a}_{44} = (a_{44})^{-1}$. We have

$$R_g^*\Omega = \begin{pmatrix} \hat{\Omega}^1 \\ \hat{\Omega}^2 \\ \hat{\Omega}^3 \\ \hat{\Omega}^4 \end{pmatrix} = \begin{pmatrix} A_{11}\Omega^1 + A_{21}\Omega^2 + \bar{B}_{14}\Omega^4 \\ A_{12}\Omega^1 + A_{22}\Omega^2 + \bar{B}_{24}\Omega^4 \\ \bar{a}_{33}\Omega^3 + \bar{a}_{34}\Omega^4 \\ \bar{a}_{44}\Omega^4 \end{pmatrix} \tag{2.24}$$

For the next reduction we differentiate both sides of the identity $R_g^*\Omega^4 = \hat{\Omega}^4$. We have

$$\begin{aligned} R_g^* d\Omega^4 &= \hat{\alpha}_{44} \wedge \hat{\Omega}^4 + \hat{T}^4_{13}\hat{\Omega}^1 \wedge \hat{\Omega}^3 + \hat{T}^4_{23}\hat{\Omega}^2 \wedge \hat{\Omega}^3 \ \ (\text{mod} \ \Omega^4) \\ &= \bar{a}_{33}((A_{11}\hat{T}^4_{13} + A_{12}\hat{T}^4_{23})\Omega^1 \wedge \Omega^3 + (A_{21}\hat{T}^4_{13} + A_{22}\hat{T}^4_{23})\Omega^2 \wedge \Omega^3) \ \ (\text{mod} \ \Omega^4) \end{aligned}$$

On the other hand

$$\begin{aligned} d\hat{\Omega}^4 &= \bar{a}_{44} d\Omega^4 \\ &= \bar{a}_{44}(T^4_{13}\Omega^1 \wedge \Omega^3 + T^4_{23}\Omega^2 \wedge \Omega^3) \ \ (\text{mod} \ \Omega^4) \ . \end{aligned}$$

The induced action of G_1 on the torsion plane (T^4_{13}, T^4_{23}) is therefore

$$\begin{pmatrix} \hat{T}^4_{13} \\ \hat{T}^4_{23} \end{pmatrix} = \frac{\epsilon}{a_{44}} A^{-1} \begin{pmatrix} T^4_{13} \\ T^4_{23} \end{pmatrix} \ . \tag{2.25}$$

The torsion plane $(T^4_{13}, T^4_{23}) \neq (0,0)$ since $I^{(2)} = 0$ implies that $d\Omega^4 \wedge \Omega^4 \neq 0$ and we have already established that $T^4_{12} = 0$. We can therefore use the action to force $(T^4_{13}, T^4_{23}) = (0,1)$. To determine the subgroup that stabilizes this choice of torsion, we investigate

$$R^*_g \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\epsilon}{a_{44}} A^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.26}$$

As $A \in O(2)$ it must be of the form

$$\begin{pmatrix} \epsilon_1\epsilon_2 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \tag{2.27}$$

where $\epsilon_1, \epsilon_2 \in \{-1, 1\}$. We must also have $a_{44} = \epsilon_1\epsilon_2$ so that the stabilizer subgroup G_2 consists of matrices of the form

$$\begin{pmatrix} \epsilon_1\epsilon_2 & 0 & 0 & B_{14} \\ 0 & \epsilon_2 & 0 & B_{24} \\ 0 & 0 & \epsilon_1 & a_{34} \\ 0 & 0 & 0 & \epsilon_1\epsilon_2 \end{pmatrix} \tag{2.28}$$

where $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ and B_{14}, B_{24} and $a_{34} \in \mathbf{R}$. We compute

$$R^*_g \mathbf{\Omega} = \begin{pmatrix} \hat{\Omega}^1 \\ \hat{\Omega}^2 \\ \hat{\Omega}^3 \\ \hat{\Omega}^4 \end{pmatrix} = \begin{pmatrix} \epsilon_1\epsilon_2\Omega^1 - B_{14}\Omega^4 \\ \epsilon_2\Omega^2 - B_{24}\Omega^4 \\ \epsilon_1\Omega^3 - \epsilon_2 a_{34}\Omega^4 \\ \epsilon_1\epsilon_2\Omega^4 \end{pmatrix} \tag{2.29}$$

The structure equations are now

$$d\begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{pmatrix} = -\begin{pmatrix} \beta_{14} \wedge \Omega^4 \\ \beta_{24} \wedge \Omega^4 \\ \alpha_{34} \wedge \Omega^4 \\ 0 \end{pmatrix} + \begin{pmatrix} T^1_{12}\Omega^1 \wedge \Omega^2 + T^1_{13}\Omega^1 \wedge \Omega^3 + T^1_{23}\Omega^2 \wedge \Omega^3 \\ T^2_{12}\Omega^1 \wedge \Omega^2 + T^2_{13}\Omega^1 \wedge \Omega^3 + T^2_{23}\Omega^2 \wedge \Omega^3 \\ \Omega^1 \wedge \Omega^2 + T^3_{13}\Omega^1 \wedge \Omega^3 + T^3_{23}\Omega^2 \wedge \Omega^3 \\ T^4_{14}\Omega^1 \wedge \Omega^4 + \Omega^2 \wedge \Omega^3 + T^4_{24}\Omega^2 \wedge \Omega^4 + T^4_{34}\Omega^3 \wedge \Omega^4 \end{pmatrix} \tag{2.30}$$

B_2 is not an e-structure so again we differentiate both sides of the identity $R^*_g\Omega = \hat{\Omega}$ to determine the action of G_2 on the torsion. After some computation, we find that

$$\begin{aligned}
d\hat{\Omega}^1 &= \epsilon_1\epsilon_2(T^1_{13}\Omega^1 \wedge \Omega^3 + T^1_{23}\Omega^2 \wedge \Omega^3 + T^1_{12}\Omega^1 \wedge \Omega^2) - B_{14}\Omega^2 \wedge \Omega^3 \pmod{\Omega^4} \\
d\hat{\Omega}^2 &= \epsilon_2(T^2_{13}\Omega^1 \wedge \Omega^3 + T^2_{23}\Omega^2 \wedge \Omega^3 + T^2_{12}\Omega^1 \wedge \Omega^2) - \epsilon_1 B_{24}\Omega^2 \wedge \Omega^3 \pmod{\Omega^4} \\
d\hat{\Omega}^3 &= \epsilon_1(T^3_{13}\Omega^1 \wedge \Omega^3 + T^3_{23}\Omega^2 \wedge \Omega^3 + \Omega^1 \wedge \Omega^2) - \epsilon_2 a_{34}\Omega^2 \wedge \Omega^3 \pmod{\Omega^4}.
\end{aligned}$$

Also,

$$\begin{aligned}
R^*_g(d\Omega^1) &= \epsilon_2\hat{T}^1_{13}\Omega^1\Omega^3 + \epsilon_1\epsilon_2\hat{T}^1_{23} + \epsilon_1\hat{T}^1_{12}\Omega^1 \wedge \Omega^2 \pmod{\Omega^4} \\
R^*_g(d\Omega^2) &= \epsilon_2\hat{T}^2_{13}\Omega^1\Omega^3 + \epsilon_1\epsilon_2\hat{T}^2_{23} + \epsilon_1\hat{T}^2_{12}\Omega^1 \wedge \Omega^2 \pmod{\Omega^4} \\
R^*_g(d\Omega^3) &= \epsilon_1\hat{T}^3_{13}\Omega^1\Omega^3 + \epsilon_1\epsilon_2\hat{T}^3_{23} + \epsilon_1\Omega^1 \wedge \Omega^2 \pmod{\Omega^4}.
\end{aligned}$$

Matching the $\Omega^2 \wedge \Omega^3$ terms we find that

$$\begin{aligned}\hat{T}^1_{23} &= T^1_{23} - \epsilon_1\epsilon_2 B_{14},\\ \hat{T}^2_{23} &= \epsilon_1 T^2_{23} - \epsilon_2 B_{24},\\ \hat{T}^3_{23} &= \epsilon_2 T^3_{23} - \epsilon_1 a_{34}\ .\end{aligned} \tag{2.31}$$

We can therefore use the action of G_1 to force $T^1_{23} = T^2_{23} = T^3_{23} = 0$. The stabilizer subgroup G_{final} for this choice of torsion consists of matrices of the form

$$\begin{pmatrix} \epsilon_1\epsilon_2 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 0 \\ 0 & 0 & 0 & \epsilon_1\epsilon_2 \end{pmatrix} \tag{2.32}$$

The reduced structure group is discrete so we now have an e-structure B_{final}. The tautological one-form $(\Omega_1, \Omega_2, \Omega_3, \Omega_4)^{\mathrm{tr}}$ provides a full coframing for B_{final}. The B_{final} structure equations are

$$d\begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{pmatrix} = \begin{pmatrix} T^1_{12} & T^1_{13} & T^1_{14} & 0 & T^1_{24} & T^1_{34} \\ T^2_{12} & T^2_{13} & T^2_{14} & 0 & T^2_{24} & T^2_{34} \\ 1 & T^3_{13} & T^3_{14} & 0 & T^3_{24} & T^3_{34} \\ 0 & 0 & T^4_{14} & 1 & T^4_{24} & T^4_{34} \end{pmatrix} \begin{pmatrix} \Omega^1\wedge\Omega^2 \\ \Omega^1\wedge\Omega^3 \\ \Omega^1\wedge\Omega^4 \\ \Omega^2\wedge\Omega^3 \\ \Omega^2\wedge\Omega^4 \\ \Omega^3\wedge\Omega^4 \end{pmatrix} \tag{2.33}$$

where the T^k_{ij} are functions on B_{final}. What remains is to determine any second order relations between the torsion functions. To determine these we use the fact that $d^2 = 0$. After some computation, we find that $T^4_{14} = T^2_{12} + T^3_{13}$. $\diamondsuit$ We summarize these results in the following theorem:

Theorem 2.5. *Associated to any nonholonomic Engel structure $\{M, \mathcal{H}, ds^2 = \langle\ \cdot\ ,\ \cdot\ \rangle\}$ there is a canonical $G \cong \mathbf{Z}_2 \times \mathbf{Z}_2$-structure B_{final}. The tautological one-form $(\Omega_1, \Omega_2, \Omega_3, \Omega_4)^{tr}$ provides a canonical coframing for B_{final}. The B_{final} structure equations are*

$$d\begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{pmatrix} = \begin{pmatrix} T^1_{12} & T^1_{13} & T^1_{14} & 0 & T^1_{24} & T^1_{34} \\ T^2_{12} & T^2_{13} & T^2_{14} & 0 & T^2_{24} & T^2_{34} \\ 1 & T^3_{13} & T^3_{14} & 0 & T^3_{24} & T^3_{34} \\ 0 & 0 & T^2_{12}+T^3_{13} & 1 & T^4_{24} & T^4_{34} \end{pmatrix} \begin{pmatrix} \Omega^1\wedge\Omega^2 \\ \Omega^1\wedge\Omega^3 \\ \Omega^1\wedge\Omega^4 \\ \Omega^2\wedge\Omega^3 \\ \Omega^2\wedge\Omega^4 \\ \Omega^3\wedge\Omega^4 \end{pmatrix} \tag{2.34}$$

According to the framing lemma (lemma 2.4) the largest Lie group of symmetries of a nonholonomic structure on an Engel manifold is of dimension 4. An example of such a structure is given by the rolling penny.

The rolling penny (Continued). A B_{final}-adapted coframe for the penny-table system is

$$\begin{aligned} \eta^1 &= \sqrt{\frac{ma^2+I}{2}}d\phi \\ \eta^2 &= \sqrt{\frac{J}{2}}d\theta \\ \eta^3 &= \frac{\sqrt{J(ma^2+I)}}{2}(-\sin\theta dx + \cos\theta dy) \\ \eta^4 &= \sqrt{\frac{m}{2}}(\cos\theta dx + \sin\theta dy - d\phi) \end{aligned} \tag{2.35}$$

The structure equations are

$$\begin{aligned} d\eta^1 &= 0 \\ d\eta^2 &= 0 \\ d\eta^3 &= \eta^1\wedge\eta^2 - \sqrt{\frac{ma^2+I}{m}}\ \eta^2\wedge\eta^4 \\ d\eta^4 &= \frac{2}{J}\ \sqrt{\frac{m}{ma^2+I}}\ \eta^2\wedge\eta^3 \end{aligned} \tag{2.36}$$

The torsion functions are constant so by the framing lemma (lemma 2.4) we can identify these constants with the structure constants Lie group of symmetries of this system. We recognize them as the structure constants for the Lie algebra of the group $SE(2)\times SO(2)$ which is isomorphic to the configuration space of the penny-table system.

B_{final}-adapted frames and coframes. The e-structure B_{final} has a canonical coframing which descends to a coframing and hence a framing, up to signs, on M. It was pointed out by Richard Montgomery (personal communication) that there should be a relationship between this framing and a canonical line field possessed by any Engel manifold. In this section we briefly describe this relationship. If M and $\mathcal{H}$ are both oriented, then M is parallelizable and the following constructions can be made globally (Montgomery [2002]).

Let η be a B_{final} adapted coframe on $U\subset M$ with dual frame $X=\{X_I\}$ defined by $\eta^I(X_J)=\delta_{IJ}$. If $\bar\eta$ is any other B_{final}-adapted coframe with dual frame $\bar X$on U, then, by theorem 2.5, η is related to η by $\bar\eta^1=\epsilon_1\epsilon_2\eta^1$, $\bar\eta^2=\epsilon_2\eta^1$, $\bar\eta^3=\epsilon_1\eta^3$, $\bar\eta^4=\epsilon_1\epsilon_2\eta^4$. The dual frames are related in precisely the same way: $\bar X_1=\epsilon_1\epsilon_2X_1$, $\bar X_2=\epsilon_2X_2$, $\bar X_3=[\bar X_1,\bar X_2]=\epsilon_1X_3$, and $\bar X_4=[\bar X_2,\bar X_3]=\epsilon_1\epsilon_2X_4$.

An important feature of an Engel distribution is the presence of a canonical line field $L\subset\mathcal{H}$ (Montgomery [2002], Kazarian, Montgomery and Shapiro [1997]). L is defined by the condition that $[L,\mathcal{H}^{(1)}]\subset\mathcal{H}^{(1)}$. Here we are abusing notation, using L for the line field or a vector field spanning L. We have

Corollary 2.6. *Let $\eta=\eta^I$ be a B_{final}-adapted coframe. Let $X=\{X_I\}$ be the dual frame defined by $\eta^I(X_J)=\delta_{IJ}$, then $L=span(X_1)$.*

Proof. Suppose L is spanned by the vector field $Y=aX_1+bX_2$. Since η^4 annihilates $\mathcal{H}^{(1)}$ we have $\eta^4([X_3,Y])=0$. Then

$$0=\eta^4([X_3,Y])=X_3\eta^4(Y)-Y\eta^4(X_3)-d\eta^4(X_3,Y)=-d\eta^4(X_3,Y).$$

But $d\eta^4 \equiv \eta^2 \wedge \eta^3 \mod (\eta^4)$ so we must have

$$\begin{aligned} 0 = \eta^2 \wedge \eta^3(X_3, Y) &= \eta^2(X_3)\eta^3(Y) - \eta^3(X_3)\eta_2(Y) \\ &= -\eta^3(X_3)\eta_2(Y) \\ &= -b. \end{aligned}$$

L is therefore spanned by X_1. $\diamondsuit$

There is a natural metric, associated with B_{final}, on M given by $ds^2_{nat} = \tilde{\eta}^1 \otimes \tilde{\eta}^1 + \cdots \tilde{\eta}^4 \otimes \tilde{\eta}^4$ where $\tilde{\eta}$ is any B_{final}-adapted coframe. Clearly all B_{final}-adapted coframes induce this same metric; using the subriemannian metric $ds^2_{nat}|_{\mathcal{H}}$ we form $L^\perp$ within $\mathcal{H}$ so that $\mathcal{H} = L \oplus L^\perp$. By construction, X_2 spans $L^\perp$.

3 Symmetries, compression, reduction, Hamiltonization

We now change tack, looking at nH systems from the point of view of almost Hamiltonian structures.

3.1 Affine symplectic structures

Examples show Koiller and Rios [2001] that some interesting nH systems with symmetry, after compression, are conformally symplectic.

Almost Hamiltonian systems. Let Ω be a non-degenerate (but in general, non-closed) 2-form on M^{2n}, and H be a function on M. Denote by $X = X_H$ (as usual) the skew-gradient vectorfield defined by $i_X\Omega = dH$. We say X_H is almost Hamiltonian. If α is a closed 1-form, the vectorfield $X = X_\alpha$ defined by $i_X\Omega = \alpha$ is called *locally almost Hamiltonian.* We formalize a definition based on Stanchenko [1985], extending the notion of conformally symplectic structures.

Definition 3.1. *The 2-form Ω is called H (or α)- affinely symplectic if there is a function $f > 0$ on M and a two form Ω_o such that i) $i_X\Omega_o \equiv 0$; ii) $\Omega - \Omega_o$ is non-degenerate, and iii) $\tilde{\Omega} = f(\Omega - \Omega_o)$ is closed.*

The first condition implies that X does not "see" Ω_o and together with the third, we get $\tilde{\Omega}(X/f, \bullet) = dH$ so the vectorfield X/f is (truly) Hamiltonian with respect to the symplectic form $\tilde{\Omega}$. In other words,

X is Hamiltonian after a coordinate dependent time reparametrization.

Being affinely symplectic allows using the full power of symplectic geometry. To say the least, X/f admits a smooth invariant positive measure

$$f^n(\Omega - \Omega_o) \wedge ... \wedge (\Omega - \Omega_o)(\,n \text{ times}).$$

When $\Omega_o \equiv 0$, then this definition becomes of a conformally symplectic structure[9]. Let us take a quick look at the contraction condition. At any point where $X \neq 0$, one gets $d = 2n$ conditions on $d(d-1)/2$ unknowns (local coordinate coefficients of Ω_o). This allows an extra chance to Hamiltonize X rather than just requiring conformality of Ω. The closedness condition can be restated as

$$d(\Omega - \Omega_o) = (\Omega - \Omega_o) \wedge \theta \ , \ \text{ where } \ \theta = df/f \ . \tag{3.1}$$

When (3.1) holds with α a closed (but not necessarily exact) 1-form, we say that Ω is *locally* affine symplectic.

[9] A caveat: the hamiltonian plays a role in the construction of Ω_{NH}, so it is relevant also in the conformally symplectic case ($\Omega_o = 0$). One should not feel unconfortable with the presence of a specific function H in the definition of an affine symplectic structure. This is inherent in the nature of the game being played here.

Poisson and/or Dirac versions. Let $(P^{2n+r}, \{\,,\,\}, H)$ a manifold P equiped with a contravariant tensor of rank $2n$, but not satisfying the Jacobi identity. $\{\,,\,\}$ is called an *almost-Poisson structure.* The tensor $\{\,,\,\}$ is *H-affine Poisson* if there is a function f on P and a 2-contravariant tensor $\{\,,\,\}_o$ such that i) $\{dH, \bullet\}_o \equiv 0$, (H is a Casimir); and ii) $f(\{\,,\,\} - \{\,,\,\}_o)$ is Poisson (satisfies Jacobi). When $\{\,,\,\}_o \equiv 0$ we have a conformally Poisson structure. Mashke and van der Schaft [1994] introduced the almost-Poisson structure for nH systems and proved that the MS-bracket satisfies the Jacobi identity if and only if the constraints are holonomic. In the examples presented in Koiller and Rios [2001] this situation persists under a conformal change of the bracket; we conjecture that the MS no-go result is robust under affine transformations of $\{\,,\,\}$, but we have not attempted to check this fact.

The main result in this section is actually very easy to prove:

Theorem 3.2. *Given a locally almost hamiltonian system (Ω, α) and a candidate $f > 0$ for conformal factor, an affine term Ω_o exists so that $d(f\Omega - \Omega_o) = 0$ if and only if $i_X\, d(f\Omega) = 0$.*

Proof. The vectorfield X satisfies $i_X\Omega = \alpha$. The same equation holds by replacing X by X/f and Ω by $f\Omega$, so to expedite notation we may assume $f \equiv 1$. Let us prove that Ω_o exists under the hypothesis that $i_X d\Omega = 0$. Since $d(i_X\,\Omega) = d\alpha = 0$, we see that the Lie derivative $L_X\Omega = 0$. Consider a regular point of X. By the flow box theorem there are coordinates so that $X = \partial/\partial x_1$. Since $L_X\Omega = 0$, the coefficients of this 2-form do not depend on the coordinate x_1 (but there may exist terms with a dx_1 factor). However, our hypothesis $i_{\partial/\partial x_1} d\Omega = 0$ ensures that there are no terms containing a dx_1 factor in $d\Omega$. Thus $d\Omega$ can be thought as a 3-form in the space of the remaining coordinates. By Poincare's theorem $d\Omega = d\Omega_o$, where Ω_o is a 2-form in the space of the remaining coordinates. Hence $i_X\Omega_o = 0$ and $d(\Omega - \Omega_o) = 0$, as desired. $\diamondsuit$.

3.2 G-Chaplygin nonholonomic systems

Were recall the setting:

Definition 3.3. *A G-Chaplygin nH system is given by the data (G, Q, S, L, ϕ), where $G \hookrightarrow Q \to S$ is a principal bundle with connection given by the Lie algebra Lie(G)-valued 1-form ϕ, and $L = T - V$ a G-equivariant natural mechanical system on Q (we will assume $V = 0$ without great loss of generality). The constraint distribution is formed by the horizontal spaces of the connection.*

Hamiltonian viewpoint. Compressing to on T^*S, we obtain an almost Hamiltonian system

$$dH^\phi = \Omega_{NH}(X_H, \bullet),\ \Omega_{NH}\ \text{nondegenerate}\ ,\ d\Omega_{NH} \neq 0\ \text{(in general)}.$$

$H = H^\phi$ is called the *compressed Hamiltonian.* More precisely[10], $M = T^*S$ is endowed with a non-degenerate, non-closed 2-form

$$\Omega - NH := \Omega_{can} + \text{(J.K)} \tag{3.2}$$

where Ω_{can} is the canonical 2-form of T^*S and the (J.K) term is a semi-basic two form, which in general is non-closed, combining the following ingredients:

- The momentum mapping J of the G-action on T^*Q,
- The curvature K of the connection.

[10] For details, see Koiller, Rios and Ehlers [2002], Koiller and Rios [2001]. The Hamiltonian compression for Chaplygin systems was first explored by Stanchenko [1985], where the non-closed term was described as a semi-basic 2-form, depending linearly on the fiber coordinate in T^*S, but its geometric content was not indicated.

The clockwise diagram. The correspondence between tangent and cotangent spaces is given by the following "clockwise diagram", starting on a $p_s \in T^*S$ and ending in $P_q \in Leg(H) \subset T^*Q$, for every q on the fiber $\pi^{-1}(s)$ of Q over s.

$$\begin{array}{ccc} H \subset TQ & \underset{Leg}{\longrightarrow} & Leg(H) \subset T^*Q \\ \uparrow & & \\ h & & \\ | & & \\ TS & \underset{(Leg^{\phi})^{-1}}{\longleftarrow} & T^*S \end{array} \tag{3.3}$$

By taking differentials of all maps in (3.3) we obtain an induced principal connection $\hat{\phi}$ in the bundle

$$G \hookrightarrow Leg(H) \to T^*S\,.$$

Let $v, w, z \in T_{p_s}(T^*S)$, $\quad V, W, Z$ horizontal lifts at $P_q \in Leg(H)$, and denote by $\hat{K}$ the curvature of this induced connection. The following result is due to Bates and Śniatycki [1993].

Proposition 3.4.

$$d\,(\mathrm{J.K})(v, w, z) = \mathrm{cyclic}(dJ(V), K(W, Z))\ . \tag{3.4}$$

Lagrangian and connection viewpoints. A gyroscopic force corresponds to the (J.K) term ofthe Hamiltonian viewpoint. The dynamics in TS is written in terms of the Levi-Civita connection ∇^S of the induced metric in S, given by

$$\langle \dot{s}, \dot{s} \rangle := \langle hor(\dot{s}), hor(\dot{s}) \rangle_Q \tag{3.5}$$

(where *hor* means lifting to the horizontal spaces). The reduced equation of motion is given by (Koiller [1992])

$$\nabla^S_{\dot{s}}\, \dot{s} = -\mathrm{grad} V + F_{\mathrm{gyr}} \tag{3.6}$$

with an added gyroscopic like force F_{gyr}, whose expression was explicitly derived. This "nonholonomic" force represents, philosophically, a "conceiled force" in Hertz [1899] sense, having a definite geometric origin. This force vanishes in some special cases, not necessarily requiring the constraints being holonomic. In an equivalent viewpoint, the dynamics is TS is given by the geodesic spray of a *modified* connection. One adds to the Levi-Civita connection a certain tensor $B(X, Y)$, and we obtain the nH connection, which in general is non-metric.

The full system can be recovered from the compressed dynamics by horizontal lifting the trajectories via ϕ, since the admissible paths horizontal relative to the connection. This last step is not "just" a quadrature; actually, in the non-abelian case, it is a *path*-ordered integral. For the $G = SO(3)$, see Levi [1996].

3.3 Tutorial example: free particle on Heisenberg's distribution

Our "operational system" is somewhat roundabout, but has the advantage of being totally algorithmic; moreover, the dynamics come in a structured form. The methodology is based on moving frames for cotangent bundles, see Koiller, Rios and Ehlers [2002]. As a guide for deriving compressed system (T^*S, Ω_{NH}, H), we take the simplest possible example[11], a free particle $L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ under the contact (Heisenberg) constraint $\mathcal{H} : \dot{z} = x\dot{y}$. [First, we do a direct derivation. From d'Alembert-Lagrange principle one gets

$$\ddot{x} = 0\,,\ \ddot{y} = -\lambda x\,,\ \ddot{z} = \lambda\ .$$

The multiplier λ can be eliminated by differentiating the constraint equation, $\ddot{z} = x\ddot{y} + \dot{x}\dot{y}$, and

$$\lambda = \frac{\dot{x}\dot{y}}{1+x^2} \implies \ddot{x} = 0\,,\ \ddot{y} = -\frac{\dot{x}\dot{y}}{1+x^2}x\,,\ \dot{z} = x\dot{y}\ \ ]. \tag{3.7}$$

The configuration space $Q = \Re^3$ is the total space of the principal $G = \Re$ bundle (fibers are the z-lines) over $S = \Re^2$ (the xy-plane). The constraint distribution is given by the connection $\phi = dz - xdy$.

Step 1. The uncompressed Legendre transform Leg**.** We do the Legendre transform of from TQ to T^*Q via L, and take note of the image distribution $H^* = Leg(H)$. Here

$$P_x = \dot{x}\,,\ P_y = \dot{y}\,,\ P_z = \dot{z}$$

and compose with the horizontal lifts from TS to TQ. In the example:

$$P_x = \dot{x}\,,\ P_y = \dot{y}\,,\ P_z = x\dot{y}\ . \tag{3.8}$$

Step 2. Compressed Lagrangian and Hamiltonian. An induced metric on $\Re^2$ is defined by taking horizontal lifts of vectors. In practice, one replaces $\dot{z}$ by $x\dot{y}$, so

$$L^\phi = \frac{1}{2}(\dot{x}^2 + (1+x^2)\dot{y}^2)$$

Take the Legendre transform of L^ϕ from TS to T^*S. We get

$$\begin{aligned} H^\phi &= \tfrac{1}{2}(p_x^2 + \tfrac{1}{1+x^2}\,p_y^2) \\ p_x &= \dot{x}\ ,\ \ p_y = (1+x^2)\dot{y}\,. \end{aligned} \tag{3.9}$$

Step 3. Going around clockwise in (3.3). We lift a cotangent vector $p_s \in T^*S$ to an uniquely defined $P_q \in T^*Q$ for every q on the fiber $\pi^{-1}(s)$ of Q over s. In our example,

$$P_x = p_x\,,\ P_y = \frac{1}{1+x^2}\,p_y\ ,\ \ P_z = \frac{x}{1+x^2}\,p_y \tag{3.10}$$

Step 4. The semi-basic 2-form (J.K) **on** T^*S**.** Recall that $J : T^*Q \to \mathrm{Lie(G)}^*$ is the Ad^*-equivariant momentum map associated to the action of G on Q, and K is the curvature of the connection, an Ad-equivariant Lie(G)-valued 2-form on Q. We pull back this 2-form to T^*Q via the projection $T^*Q \to Q$. In this example, $J = P_z$, and $K = -dxdy$, so

$$(\mathrm{J.K}) = -\frac{x}{1+x^2}\,p_y\,dx\wedge dy$$

The compressed almost-Hamiltonian system is $(H^\phi\,,\,\Omega_{NH})$ with

$$\Omega_{NH} = \Omega_{can} + (\mathrm{J.K})$$

and where Ω_{can} is the canonical 2-form in T^*S.

[11] This example has been used *at nauseum* and we are no exception.

Step 5. The equations of motion. It is easy to check that equations (3.11) are indeed equivalent to (3.7) under the compressed Legendre transform (3.9).

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{p_x} \\ \dot{p_y} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -\frac{xp_y}{1+x^2} \\ 0 & -1 & \frac{xp_y}{1+x^2} & 0 \end{pmatrix} \begin{pmatrix} -\frac{xp_y^2}{(1+x^2)^2} \\ 0 \\ p_x \\ \frac{p_y}{1+x^2} \end{pmatrix} \tag{3.11}$$

One observes that the conservation law

$$\dot{p}_x = \frac{xp_y^2}{(1+x^2)^2} - \frac{xp_y^2}{(1+x^2)^2} = 0 \tag{3.12}$$

seems to come from a miraculous cancellation.

Hamiltonization. In this example Ω_{NH} is conformally symplectic, Koiller and Rios [2001]. Indeed

$$d\Omega_{NH} = d(\mathrm{J.K}) = -\frac{x}{1+x^2}\, dp_y\, dx\, dy\ . \tag{3.13}$$

Our ansatz is that the conformal factor f is a function of x only. Forcing $d(f\Omega_{NH}) = 0$ yields

$$\frac{f'}{f} = -x/(1+x^2) \implies f = (1+x^2)^{-1/2}\ . \tag{3.14}$$

With the change of position variables and momentum

$$x = \sinh(u)\ ,\ \ p_y = I\cosh(u) \tag{3.15}$$

we get

$$(1+x^2)^{-1/2}(dp_x dx + dp_y dy - \frac{x}{1+x^2}\, p_y\, dx dy) = dp_x du + dI dy\ ,\ \ H = \frac{1}{2}(p_x^2 + I^2) \tag{3.16}$$

the fee particle on the plane (u, y).

The conserved quantity p_x comes of course from the nH-Noether theorem, since $\xi_1 = \partial/\partial x$ is Killing for T and belongs to $\mathcal{H}$. But the other vectorfield $x\partial/\partial z + \partial/\partial y$ is not a symmetry of the metric. So, from where does the other conserved quantity I comes from? It turns out that $\xi_2 = f(x\partial/\partial z + \partial/\partial y)$ *is* an infinitesimal symmetry of T (the verification is straightforward but not automatic) and Noether's theorem gives

$$I = (\dot{x}, \dot{y}, \dot{x}) \cdot (1+x^2)^{-1/2}(x\partial/\partial z + \partial/\partial y) = p_y/\sqrt{1+x^2}\ . \tag{3.17}$$

Although ξ_2 looks ugly, it is no worse than ξ_1.

A lesson to be learned here. In the new time scale τ such that $dt/d\tau = \sqrt{1+x^2}$, there are two equally valid ways to reduce the compressed, time-rescaled system to a one degree of freedom system, either to (x, p_x) or to (y, I). Since we want actually to have the solution written in terms of the original time, the former reduction must be chosen, because the time rescaling depends only on x.

"Exercise". Consider the following example from Iliyev [1985]:

$$T = \frac{1}{2}\left(\dot q_1^2 + \dot q_2^2 + \dot q_3^2 + \dot q_4^2 + \dot q_5^2\right) \ , \ \dot q_4 = \dot q_2 \tan q_1 \ , \ \dot q_5 = \dot q_3 \tan q_1 \tag{3.18}$$

This is a Chaplygin system with $G = \{(q_4, q_5)\}$ and $S = \{(q_1, q_2, q_3)\}$. The momentum map is $J = (P_4, P_5)$ and the connection form $\phi = (dq_4 - \tan q_1\, dq_2\,,\ dq_5 - \tan q_1\, dq_3)$ has curvature $K = -\sec^2 q_1\,(dq_1 dq_2, dq_1 dq_3)$. Th compressed Hamiltonian is $H^\phi = \frac{1}{2}\left(p_1^2 + \cos^2 q_1 (p_2^2 + p_3^2)\right)$ and

$$\Omega_{NH} = p_1 dq_1 + p_2 dq_2 + p_3 dq_3 + (\mathrm{J.K})\ , \quad (\mathrm{J.K}) = -\tan q_1\,(p_2\, dq_1 \wedge dq_2 + p_3\, dq_1 \wedge dq_3) \tag{3.19}$$

This system is also conformally symplectic with $f = \cos q_1$.

3.4 Chaplygin's sphere

Geometrical setting, equations of motion, invariant measure. Chaplygin's sphere is a perfectly round sphere of radius r and mass μ rolling without slipping on a horizontal plane. The constraints are $\dot x = r\omega_1\,,\ \dot y = -r\omega_2\,,$ where $\vec\omega$ is the angular velocity (in the space frame). The center of mass is assumed to be at the geometric center; but the inertia matrix $A = \mathrm{diag}(I_1, I_2, I_3)$ may have unequal entries. The configuration space is the product of the plane $\Re^2$ with the orthogonal group $SO(3)$. The constraints define a $\Re^2$-equivariant distribution of three dimensional spaces. The sub-distribution with vertical rotations removed has Cartan's (2-)3-5 growth numbers. We regard the constraint distribution as an abelian *connection* on the bundle with base space $S = SO(3)$ and fiber $G = \Re^2$; it is given by

$$\phi := (dx - r\rho_2\,,\ dy + r\rho_1) \tag{3.20}$$

We will denote by ρ_1 the right invariant forms and by λ_i the left invariant forms on $SO(3)$. The standard basis $X_i \in sO(3) = T_I SO(3)$, $i = 1, 2, 3$ (infinitesimal rotations around the x, y, z-axis), can be either right or left transported as moving frames for $SO(3)$. The ρ's and λ's are the respective dual coframes.

Poisson action of S^1 on $SO(3)$. Consider the *left* S^1 action on $SO(3)$ given by

$$\phi \cdot R := S_\phi\, R \tag{3.21}$$

where S_ϕ is the rotation matrix about the z-axis:

$$S_\phi := \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \quad S(-\phi)S'(\phi) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = X_3\ . \tag{3.22}$$

Two matrices are in the same equivalence class if their third row, denoted γ, are the same:

$$R_1 \sim R_2 \Longleftrightarrow R_1^{-1}\hat k = R_1^{-1}\hat k = \gamma \in S^2\ .$$

So we have a principal bundle $\pi : SO(3) \to S^2$,

$$\gamma = \pi(R) = R^{-1}\hat k = R^\dagger\, \hat k \tag{3.23}$$

γ (called the "Poisson vector") is the third *row* of R, so we really have a "forgetting map". The derivative of π is

$$\dot\gamma = \pi_*(\dot R) = -(R^{-1}\dot R R^{-1})k = -(R^{-1}\dot R)(R^{-1})k = -[\Omega]\gamma = -\vec\Omega \times \gamma = \gamma \times \vec\Omega \tag{3.24}$$

where we used the costumary identification $[\Omega] \in SO(3) \leftrightarrow \vec\Omega \in \Re^3$, Arnold [1989]. We will drop the $[\bullet]$ and $\vec\bullet$ in the sequel, and mix the notations. Equation (3.24) is one half of Chaplygin's ODEs.

Connection on the $S^1 \hookrightarrow SO(3) \to S^2$ principal bundle. We take the bi-invariant metric $<<,>>$ on $SO(3)$ so that both $\{X_i^{left}\}$ and $\{X_i^{right}\}$ are orthonormal frames. The tangent vectors to the fibers are $(d/d\phi)S(\phi)\cdot R = X_3^{right}$. We consider the mechanical connection associated to $<<,>>$, so the horizontal spaces are generated by X_1^{right} and X_2^{right}. Therefore the connection form is $\phi = \rho_3$. The horizontal lift of $\dot\gamma$ to R is the tangent vector $\dot R$ such that

$$\Omega_{hor} = R^{-1}\dot R = [\dot\gamma \times \gamma] \tag{3.25}$$

Note that Ω_{hor} *is the -90 degrees rotation of* $\dot\gamma$ *inside* $T_\gamma S^2$. In the sequel, the +90 rotation operator (from the Kahler structure in S^2) will be denoted J_γ. The curvature of this connection is the area form of the sphere,

$$\kappa_\gamma(\dot\gamma_1\,,\,\dot\gamma_2) = (\gamma\,,\,\gamma_1\times\gamma_2\,) \tag{3.26}$$

Equations of motion and invariant measure. The angular momentum *at the contact point*, $\vec\ell$ (in the space frame), is constant. An engineer would say that this is because both gravity and friction produce no torque at that point; a mathematician would use the fact that the admissible vectorfields $V_i \in \mathcal{H}$ given by

$$V_1 := -r\,\partial/\partial y + X_1^{\rm right}\ ,\ V_2 := r\,\partial/\partial x + X_2^{\rm right}\ ,\ V_3 := X_3^{\rm right} \tag{3.27}$$

preserve the Lagrangian; he would then apply the nH version of Noether's theorem. Using Arnold's notation (vectors in body frame denoted by capital letters), we have $\ell = RL$. Differentiating, we get $0 = R\dot L + \dot R L = 0$, so $\dot L = -R^{-1}\dot R L = -\Omega\times L$, where Ω is the angular velocity vector in the body frame. Likewise, since $R\gamma = k$ (we should use the notation $\gamma = K$, but we won't), we get $\dot\gamma = -\Omega\times\gamma$. These two differential equations form a coupled system, since Ω *is a linear function of* L *depending only on* γ. This can be derived in several ways. One is to look at the total energy

$$2T = (\omega,\ell) = (\Omega, L) = (A\Omega,\Omega) + \mu(\dot x^2+\dot y^2) = (A\Omega,\Omega) + \mu r^2(\omega_1^2+\omega_2^2) \tag{3.28}$$

which can be also written as

$$2T = (A\Omega,\Omega) + \mu r^2(\omega\times k)^2 = (A\Omega,\Omega) + \mu r^2(\Omega\times\gamma)^2$$

or

$$2T = (\Omega, L) = (A\Omega,\Omega) + \mu r^2(\Omega\,,\,\gamma\times(\Omega\times\gamma)\,) = (\,\Omega\,,\,A\Omega + \mu r^2\gamma\times(\Omega\times\gamma)\,)\,) \tag{3.29}$$

The expression $\gamma\times(\bullet\times\gamma)$ represents the projection in the plane perpendicular to γ. Thus defining

$$\tilde A := A + \mu r^2 \mathrm{id} \tag{3.30}$$

we have

$$L = \quad L = A\Omega + \mu r^2\gamma\times(\Omega\times\gamma) = \tilde A\Omega - \mu r^2(\gamma,\Omega)\gamma\,. \tag{3.31}$$

This allows an *ansatz* for the inverse,

$$\Omega = \Omega(L,\gamma) = \tilde A^{-1}L + \alpha(L)\tilde A^{-1}(\gamma)\ . \tag{3.32}$$

Substituting, after a simple calculation one gets

$$\alpha(L) = \mu r^2\,\frac{(\gamma,\tilde A^{-1}L)}{1-\mu r^2(\gamma,\tilde A^{-1}\gamma)}\ . \tag{3.33}$$

The denominator

$$f(\gamma) := 1 - \mu r^2(\gamma, \tilde{A}^{-1}\gamma) \tag{3.34}$$

was found by Chaplygin to appear in the density of an invariant measure in $\Re^6$. We summarize in the following

Proposition 3.5. *Chaplygin's equations are*

$$\dot{\gamma} = \gamma \times \Omega \, , \; \dot{L} = L \times \Omega \; , \tag{3.35}$$

where

$$\Omega = \Omega(L, \gamma) = \tilde{A}^{-1}L + \mu r^2 \, \frac{(\gamma, \tilde{A}^{-1}L)}{1 - \mu r^2(\gamma, \tilde{A}^{-1}\gamma)} \tilde{A}^{-1}(\gamma) \; . \tag{3.36}$$

This system has the invariant measure

$$\nu_{\Re^6} = [1 - \mu r^2(\gamma, \tilde{A}^{-1}\gamma)]^{-1/2} \, d\gamma_1 d\gamma_2 d\gamma_3 dL_1 dL_2 dL_3 \tag{3.37}$$

For the proof, see Duistermaat [2000] or Fedorov and Kozlov [1995].

Almost Hamiltonian description. We know that after compression we get an almost Hamiltonian system, with H given by the Legendre transform of (3.29), and a non-closed, non-degenerate 2-form Ω_{NH} in $T^*SO(3)$. Were $(T^*SO(3), \Omega_{NH}, H)$ hamiltonizable, obviously it would be integrable because of the three independent first integrals $H, \ell_3, \ell_1^2 + \ell_2^2, \ell_3^2$. *We will show in the next section that (unfortunately?)* Ω_{NH} is *not affine symplectic, even in the homogeneus case.* At any rate, Stanchenko [1985], showed that Chaplygin's density function of the system in $\Re^6$ also gives an invariant measure on $T^*SO(3)$,

$$\nu_{T^*SO(3)} = [f(\gamma)]^{-1/2} d\lambda_1 d\lambda_2 d\lambda_3 dL_1 dL_2 dL_3 \; .$$

See also Duistermaat [2000], section 7. Integrability follows from "Jacobi's last multiplier" procedure: the invariant measure, taken together with the three integrals of motion $(H^*, \ell_3, |\ell|^2)$.

The clockwise map is given here by

$$\begin{array}{ccccccc} T(SO(3)\times\Re^2) & \xrightarrow[Leg]{} & T^*(SO(3)\times\Re^2) & \quad & (\Omega,\dot{x},\dot{y}) & \mapsto & (M = A\Omega, P_x = \mu\dot{x}, P_y = \mu\dot{y}) \\ \uparrow h & & & & \uparrow h & & \\ TSO(3) & \xleftarrow[(Leg^{\phi})^{-1}]{} & T^*SO(3) & & \Omega & \leftarrow & L \end{array} \tag{3.38}$$

where $(\dot{x}, \dot{y}) = a\omega \times k$, and $(P_x, P_y) = \mu a\omega \times k$. We now evaluate the "gyroscopic" 2-form:

$$(\mathrm{J}, \mathrm{K}) = a(-p_x d\rho_2 + p_y d\rho_1) = \mu a(-\dot{x} d\rho_2 + \dot{y} d\rho_1) = -\mu a^2(\omega_2 d\rho_2 + \omega_1 d\rho_1) \tag{3.39}$$

We must write ω_1 and ω_2 as functions on $T^*SO(3)$, using the Legendre transformation. We have

$$\omega = R\Omega = RA^{-1}M \tag{3.40}$$

so that, as expected, (J, K) will be combinations of the basic forms $\rho_3 \wedge \rho_1$, $\rho_2 \wedge \rho_3$, with coefficients linear in M and functions of R. A crucial observation is that (3.39), written in terms of the left-invariant forms, will depend only on the Poisson vector γ:

Proposition 3.6. *An equivalent expression for* (J,K) *is:*

$$(\mathrm{J},\mathrm{K}) = \mu a^2 \left(\gamma \times (\Omega(L,\gamma)\times\gamma)\,,\, d\lambda\right) \tag{3.41}$$

where $\lambda = (\lambda_1,\lambda_2,\lambda_3)^\dagger$ *are the standard left-invariant 1-forms in* $SO(3)$*.*

Proof. As we present the derivation, we clarify our notations. We write

$$[\rho] = dRR^{-1} = \begin{pmatrix} 0 & -\rho_3 & \rho_2 \\ * & 0 & -\rho_1 \\ * & * & 0 \end{pmatrix} \Longleftrightarrow \rho = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} \tag{3.42}$$

$$[\lambda] = R^{-1}dR = \begin{pmatrix} 0 & -\lambda_3 & \lambda_2 \\ * & 0 & -\lambda_1 \\ * & * & 0 \end{pmatrix} \Longleftrightarrow \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \tag{3.43}$$

In passing: if one takes the exterior derivative of the identity $R\lambda = \rho$ well known results will be reproduced. Indeed, we get

$$d\rho = (d\rho_1,\rho_2,\rho_3)^\dagger = dR\wedge\lambda + Rd\lambda$$

so

$$R^{-1}d\rho = R^{-1}dR\wedge\lambda + d\lambda\ .$$

Now from the Cartan structure equations of $SO(3)$,

$$d\lambda = -(\lambda_2\wedge\lambda_3,\ \lambda_3\wedge\lambda_1,\ \lambda_1\wedge\lambda_2\,)^\dagger \tag{3.44}$$

hence

$$R^{-1}d\rho = [\lambda]\wedge\lambda - (\lambda_2\wedge\lambda_3,\ \lambda_3\wedge\lambda_1,\ \lambda_1\wedge\lambda_2\,)^\dagger$$

which gives, after a short examination,

$$R^{-1}d\rho = -d\lambda \tag{3.45}$$

We know that $R[\lambda]R^{-1} = [\rho]$ is equivalent to $R\lambda = \rho$ (the latter written as a column vector). Now, denote

$$p = (p_x,p_y,0)^\dagger\,,\ d\rho = (d\rho_1,d\rho_2,d\rho_3)^\dagger\ .$$

We rewrite (3.39) as

$$(\mathrm{J},\mathrm{K}) = -ak\cdot(p\times d\rho)\ .$$

Since the mixed product is invariant under the action of $SO(3)$ we have

$$(\mathrm{J},\mathrm{K}) = -a\gamma\cdot(R^{-1}p\times d\lambda)\ .$$

But $R^{-1}p = \mu a\Omega\times\gamma$ and (3.41) follows from inserting (3.45), and correctly keeping track of signs.

S^1 **invariance.** We claim that Ω_{NH} is invariant under the S^1 action lifted to $T^*SO(3)$. For the canonical part this is a standard symplectic fact. Let us show that the (J,K) term is invariant as well. Recall that the S^1 action is generated by the right invariant vectorfield X_3^{right}. This action maintains the projection γ fixed, and by general nonsense, we know that the right invariant vectorfields preserve the left invariant forms: $R_\phi^*\lambda_i = \lambda_i$. (A direct proof: $(R_\phi^*\lambda_i)(\dot R) = \lambda_i(R_\phi R[\Omega]) = \Omega_i$). Since under the left S^1 action (actually under the left action of $SO(3)$ on $SO(3)$) the *value* of Ω remains unchanged, the (J,K) term is preserved.

Therefore it is possible to reduce by the S^1 symmetry. However, the usual MW reduction method must be modified. But *not* because $d\Omega_{NH}\neq 0$. The trouble is that the X_3^{right} vectorfield is the Hamiltonian vectorfield of $H = J_{X_3} = \ell_3$, relative to the canonical symplectic form, not to Ω_{NH}. This issue is discussed in the next section.

4 Comments on Chaplygin's sphere

Chaplygin showed that the 3d problem is integrable in elliptic coordinates; for $n > 3$ the problem is open. For basic informations, see Fedorov and Kozlov [1995], p. 147-149, on the 3-d case and p. 153-156 for the general n-dimensional case. Schneider [2002] analyzed control theoretical aspects. For a detailed account on the algebraic integrability of "Chaplygin's Chaplygin sphere", see Duistermaat [2000].

Disclaimer. Hopefully we give here additional geometrical contextualization, and highlight some aspects of Duistermaat's paper. Our elementary observations are "elementary" also in P. Erdös sense: we do not make use of its explicit integration in terms of hyperelliptic integrals and Jacobi Theta functions.

4.1 Reduction to T^*S^2.

Equations (3.35) reduces to TS^2 using the connection $\phi = \rho_3$ described above. The induced kinetic energy is

$$T^{\phi}(\dot{\gamma}) = T(\Omega_{hor}) = \frac{1}{2}(\tilde{A}(J_\gamma(\dot{\gamma})\,,\, J_\gamma(\dot{\gamma})) = -\frac{1}{2}(J_\gamma \tilde{A} J_\gamma\,\dot{\gamma}\,,\,\dot{\gamma})\ . \tag{4.1}$$

We will now do a seemingly *adhoc* construction. Consider the tangent vector $a \in TS^2$ defined by

$$a := \gamma \times L \in T_\gamma S^2\ . \tag{4.2}$$

Then L can be recovered via

$$L = a \times \gamma + b\gamma\,,\ \text{ where } b = (L,\gamma) = \ell_3\ . \tag{4.3}$$

We compute

$$\dot{a} = \dot{\gamma} \times L + \gamma \times \dot{L} = (\gamma \times \Omega) \times L + \gamma \times (L \times \Omega)$$

Proposition 4.1. *Chaplygin's equations reduce to T^*S^2:*

$$\dot{\gamma} = \gamma \times \Omega\ ,\ \ \dot{a} = -2H\,\gamma + (\gamma,\Omega)\cdot(a \times \gamma + \ell_3\,\gamma) \tag{4.4}$$

whith

$$\Omega = \Omega(a,\gamma;\ell_3) = \tilde{A}^{-1}L + \mu r^2\,\frac{(\gamma,\tilde{A}^{-1}L)}{1-\mu r^2(\gamma,\tilde{A}^{-1}\gamma)}\,\tilde{A}^{-1}(\gamma)\ .$$

and where insert (4.3).

Geometrical meaning of a. Consider the "mixed" coframe in $SO(3)$, given by the forms ρ_3 and $d\gamma$, where the latter is interpreted as the pullback via π^*. We get (4.2) and (4.3) from the identity

$$L_1\lambda_1 + L_2\lambda_2 + L_3\lambda_3 = a\cdot d\gamma + b\ell_3\ . \tag{4.5}$$

One question in which we are interested (but have not yet solved) is the following: *find the S^1 reduction of Ω_{NH}, giving an almost Hamiltonian formulation for equations (4.4). Note: This is easy to do if we allow "right hand sides". We would like to avoid them.*

4.2 Hamiltonization of the homogeneous case

In this case it is simpler to write everything in terms of the right coframe:

$$\Omega_{NH} = d\ell_1\rho_1 + d\ell_2\rho_2 + d\ell_3\rho_3 + \ell_1\rho_2\rho_3 + \ell_2\rho_3\rho_1 + \ell_3\rho_1\rho_2 - \mu r^2(\omega_2\rho_3\rho_1 + \omega_1\rho_2\rho_3) \tag{4.6}$$

This formula holds in general (i.e, also in the nonhomogeneous case), but one must write ω_1 and ω_2 in terms of L (or ℓ) and $R \in SO(3)$:

$$\omega = R\Omega = R\Omega_\gamma(R^{-1}\ell)$$

which is a quite involved expression. But in the homogeneous case, life is much easier. ¿From (3.40), we get

$$\omega = R\frac{1}{\kappa}IR^{-1}m = \frac{1}{I}m \tag{4.7}$$

so the dependence of ω on R disappears. The Hamiltonian is given by

$$H = \frac{1}{2}\left(\frac{\ell_1^2 + \ell_2^2}{I + \mu r^2}\right) + \frac{\ell_3^2}{I} \tag{4.8}$$

where

$$\ell_1 = (1 + \frac{\mu r^2}{I})m_1\,,\ \ell_2 = (1 + \frac{\mu r^2}{I})m_2\,,\ \ell_3 = m_3 \quad , \quad \omega_1 = \frac{m_1}{I}\,,\ \omega_2 = \frac{m_2}{I}\,,\ \omega_3 = \frac{m_3}{I}\,.$$

The 3×3 matrix E which gives the non-Darboux block is skew-symmetric with

$$\begin{aligned} E_{12} &= \ell_3 \\ E_{13} &= -\ell_2 + \mu r^2\omega_2 = -m_2 - m_2\mu r^2/I + \mu r^2/Im_2 = -I\omega_2 \\ E_{23} &= \ell_1 - \mu r^2\omega_1 = m_1 + m_1\mu r^2/I - m_1\mu r^2/I = I\omega_1 \end{aligned} \tag{4.9}$$

and the equations of motion become

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \dot{\ell}_1 \\ \dot{\ell}_2 \\ \dot{\ell}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & \ell_3 & -I\omega_2 \\ 0 & -1 & 0 & -\ell_3 & 0 & -I\omega_1 \\ 0 & 0 & -1 & I\omega_2 & -I\omega_1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{4.10}$$

where we have used

$$H_{\ell_1} = \ell_1/(I + \mu r^2) = m_1/I = \omega_1 \ ,$$

and similarly, $H_{\ell_2} = \omega_2\,, H_{\ell_2} = \omega_2$. This gives, as one should expect:

$$\dot{\ell}_1 = (I\omega_3)H_{\ell_2} - I\omega_2 H_{\ell_3} = 0\,,\ \dot{\ell}_2 = ... = 0\,,\ \dot{\ell}_3 = ... = 0 \ .$$

Thus

$$\omega_i = m_i/I = \text{const} \ , \ i = 1, 2, 3$$

and the vectorfield is simply (no components in the fiber directions $\partial/\partial m_i$):

$$X = \omega_1 X_1^{\text{right}} + \omega_2 X_2^{\text{right}} + \omega_3 X_3^{\text{right}} \ . \tag{4.11}$$

Summarizing, the dynamics in the homogeneous case is embarrassingly simple. The angular velocity in space being constant, the attitude matrix R evolves as a 1-parameter group $R = \exp([\omega]t)$, so $\vec{\Omega} = \vec{\omega} =$ const. The vector $\gamma(t)$ describes a circle in the sphere perpendicular to ω, and $L(t)$ the curve given by $L(t) = (I + \mu r^2)\,\omega - \omega_3\,\gamma(t)$. Provided ℓ is not vertical, L and γ are never parallel. The invariant tori are always foliated by closed curves and the two frequencies coincide. ¿From the constraint equations we see that the motion of the contact point in the plane is a straight line. Shooting pool with a perfect Chaplygin ball is very dull[12].

Hamiltonization is not possible in $T^*SO(3)$. We use criterion 3.2. The nonholonomic two form is given by

$$\Omega_{NH} = d\ell_1\rho_1 + d\ell_2\rho_2 + d\ell_3\rho_3 + I(\omega_1\rho_2\rho_3 + \omega_2\rho_3\rho_1 + \omega_3\rho_1\rho_2) \tag{4.12}$$

or, using m as coordinates,

$$\Omega_{NH} = (1 + \frac{\mu r^2}{I})\,(dm_1\rho_1 + dm_2\rho_2) + dm_3\rho_3 + (m_1\rho_2\rho_3 + m_2\rho_3\rho_1 + m_3\rho_1\rho_2)$$

so that

$$d\Omega_{NH} = -\frac{\mu r^2}{I}(dm_1\rho_2\rho_3 + dm_2\rho_3\rho_1)\ . \tag{4.13}$$

It is easy to see that the equation $d\Omega_{nH} = \Omega_{NH} \wedge \alpha$ has no solution. Indeed, suppose

$$\alpha = A_1 dm_1 + A_2 dm_2 + A_3 dm_3 + B_1\rho_1 + B_2\rho_2 + B_3\rho_3\ .$$

Taking the exterior product, and looking at terms like $dm_1 dm_2 \rho_2$ we see that all the A's must be zero. Examining the coefficient of $\rho_1\rho_2\rho_3$ we see that $B_1 m_1 + B_2 m_2 + B_3 m_3 \equiv 0$ so all the B's are also zero.

This shows that the homogeneous Chaplygin sphere has no conformal symplectic structure. In fact, it does not have an affine symplectic structure either. A simple calculation shows that

$$i_X d\Omega_{NH} = \frac{\mu r^2}{I^2}(-dm_1 m_2\rho_3 + dm_1\rho_2 m_3 - dm_2 m_3\rho_1 + dm_2\rho_3 m_1) \neq 0.$$

We have also done the calculation for the non homogeneous case and things only get worse. But, it still remains a possibility: is the *reduced* system to T^*S^2 Hamiltonizable? The answer is yes, at least for the homogeneous case.

Hamiltonization of the S^1 reduction to T^*S^2. Alan Weinstein commented in more than one occasion that "unreduction" is sometimes even nicer than reduction: unreducing a non-trivial system may lead to an trivial one. Alan credits this moto to Guillemin and Sternberg; one reference could be Guillemin and Sternberg [1980]. We just arrived at the following conclusion:

[12] We found the following relevant information in www.ot.com/skew/five/myths.html (Top Ten Myths in Pool or the Laws of Physics Do Apply): "4. If the cue is kept level, contacting the cueball purely left or right of its center will make it curve as it rolls. (**No!** The rolling cue ball can have two completely independent components to its angular momentum. Basically, this means that it can rotate in the manner of a top while rolling slowly forward along a straight line. In general, spin on a cue ball is of two types; follow/draw is the spin like tires on a car, while English is the spin like a child's toy 'top'. Seperately, neither one will make a ball curve! If they are combined - e.g., strike low-left giving left English and draw - then the spin is called masse (mass-ay), and the ball will curve as it travels.)"

Proposition 4.2. *The homogeneous Chaplygin sphere is hamiltonizable when reduced to T^*S^2. Equations (4.4) become*

$$\dot{\gamma} = \frac{1}{I+\mu r^2}\, a \quad , \quad \dot{a} = \omega_3\, a \times \gamma - \frac{1}{I+\mu r^2}\, |a|^2 \gamma \tag{4.14}$$

We get an isotropic 3d-oscillator with an arbitrary (including zero) Lorenz force.

Proof. One observes immediately that $(a, \gamma) = 0$ and that $|a|^2$ is conserved.

Question. Since Kepler's problem in space is equivalent (up to time reparametrization) to the isotropic oscillator in 4-dimensions, via the Kustaanheimo-Stiefel transformation. Is there a relation of the homogeneous 4d Chaplygins sphere with the 4d oscillator?

4.3 Bobylev's case, and the general case of three unequal inertia.

Bobylev's case $I_1 = I_2 \neq I_3$ is discussed in Cendra, Lacomba and Reartes [2001], but the authors did not write a detailed description of the reduced equations. Bobylev's dynamics is described in Duistermaat [2000], section 6.

The following formulae, expressing Ω_{NH} in terms of ρ_3 and $d\gamma$, could be useful for reduction. In the mixed coframe for $SO(3)$ (4.5) we impose the relations $(a, \gamma) \equiv 0$ and $(\gamma, d\gamma) \equiv 0$. ¿From (4.5) and (4.3) we get, taking $L = i, j, k$ respectively:

$$\lambda_1 = \gamma_1\, \rho_3 + (\gamma \times i) \cdot d\gamma \ , \ \lambda_2 = \gamma_2\, \rho_3 + (\gamma \times j) \cdot d\gamma \ , \ \lambda_3 = \gamma_3\, \rho_3 + (\gamma \times k) \cdot d\gamma \tag{4.15}$$

The canonical form writes as

$$\Omega_{can}^{T^*SO(3)} = d(L \cdot \lambda) = da \wedge d\gamma + d\ell_3 \wedge \rho_3 + \ell_3 d\rho_3 \ . \tag{4.16}$$

A short calculation gives, as expected,

$$d\rho_3 = \gamma \cdot \frac{1}{2}(d\gamma \times d\gamma) \ \ (\mathrm{S}^2 \text{ area form of } \mathrm{S}^2\ ).$$

Moreover,

$$(\mathrm{J.K}) = \mu r^2\, (\gamma \times (\Omega \times \gamma), d\lambda) = \mu r^2\, (\gamma \times \Omega) \cdot (\gamma \times d\lambda) \tag{4.17}$$

where a direct calculation gives

$$\gamma \times d\lambda = (\gamma \times d\gamma)\, \rho_3 \ . \tag{4.18}$$

Question. Is the reduced system (4.4) affine symplectic also in the case of different moments of inertia?

We give one more argument for that possibility that Chaplygins sphere could be Hamiltonizable when reduced to T^*S^2. We make a dimension count. Suppose the base S of a Chaplygin system has dimension m. The reduced symplectic form Ω_{NH} in T^*S satisfies

$$(\Omega_{NH})^m = \nu_{\mathrm{Liouville}} = dp_1 ... dp_m ds_1 ... ds_m \tag{4.19}$$

Suppose $F \cdot \nu_{\mathrm{Liouville}}$ is an invariant measure for nH vectorfield X in T^*S, so that

$$d(F i_X \nu_{\mathrm{Liouville}}) = 0$$

(since $i_X d$ will give zero in the maximum dimension). If g is a candidate for a conformal factor (i.e, we hope that $g\Omega_{NH}$ is closed) , then

$$(g\Omega_{NH})^m = g^m \nu_{\text{Liouville}}$$

is preserved by X/g . Hence

$$d(i_{X/g} g^m \, \nu_{\text{Liouville}}) = d(g^{m-1} i_X \, \nu_{\text{Liouville}}) = 0 \; .$$

Comparing, we get : $F = g^{m-1}$.

In the case of Chaplygin's sphere, the candidate for conformal factor must be

$$g = F = [1 - \mu r^2 (\gamma, \tilde{A}^{-1}\gamma)]^{-1/2} \, ,$$

because from Duistermaat [2000], proposition 8.3 and corollary 8.4, the vectorfield X/F describe the tori linearly. Hence: $n = 2$. Hamiltonization, if it ever occurs, must happen at this level.

4.4 Reconstruction

In this section we highlight some results in Duistermaat [2000].

Average motion in the plane. Chaplygin's system is integrable, so after a time reparametrization $dt/d\tau = \sqrt{f(\gamma)}$ (see (3.34), most trajectories $(\gamma(\tau), \Omega(\tau))$ are quasi-periodic, filling 2-tori densely and linearly. One could think[13] that by lifting the nonperiodic trajectories to the plane (x, y) we would get trajectories that, in average, return always to the starting point. Here is the argument, although (perhaps unfortunately) is it flawed. We write the constraint condition as:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ 0 \end{pmatrix} = a\omega \times k = \exp(i\psi)(R(\gamma)\Omega \times k) \qquad (4.20)$$

where $R(\gamma)$ is any section of the bundle $SO(3) \to S^2$, for instance, the matrix with rows $e_1, e_2, e_3 = \gamma$, with

$$e_1 = \frac{1}{\sqrt{\gamma_1^2 + \gamma_2^2}}(\gamma_2, -\gamma_1, 0) \;\; , \;\; e_2 = e_3 \times e_1 = \frac{1}{\sqrt{(\gamma_1^2 + \gamma_2^2)}}(\gamma_1\gamma_3, \gamma_2\gamma_3, -(\gamma_1^2 + \gamma_2^2)) \; . \qquad (4.21)$$

We replace time averages by space averages Arnold [1989]. Integration is done on 3-dimensional tori, but the integral over coordinate ϕ separates out. For every fixed (γ, Ω), $R(\gamma)\Omega \times k$ is a vector on the plane. Then, averaging over ϕ would give zero.

Where is the flaw? It is not necessarily true that ψ covers the whole circle, and in fact it does not if ℓ is not vertical[14]. So ψ has a mean value $\overline{\psi}$, and

$$\overline{(\dot{x}, \dot{y})} = R(\overline{\psi}) \cdot \overline{R(\gamma)\Omega \times k} \neq 0 \;\; .$$

This can be seen even more clearly in the case of periodic motion, using a trick we learned in Montgomery [1991]. Suppose $\gamma(T) = \gamma(0)$ and $L(T) = L(0)$. We may assume that $R(0) =$ identity so $R(T)$ preserves both k and ℓ. If we assume ℓ is not vertical, then $R(T)$ must be also the identity (the only orthogonal matrix with two different eigenvectors with eigenvalue 1). Since the initial conditions are reproduced after time T, then there is a "geometric phase" (here meaning a translation) in the plane, $\Delta z = (\Delta x, \Delta y)$. From Duistermaat [2000], section 11, one knows this direction:

[13] Actually, we did; thanks to Hans Duistermaat (personal communication) much embarrassment was avoided.

[14] The case where ℓ is vertical is analyzed in detail in Duistermaat [2000], section 5.

Proposition 4.3. *In average, Δz moves in the direction of $\ell \times k$.*

In the normal direction $k\times(\ell\times k)$ there is a "swaying motion", with zero average, see Duistermaat [2000], (11.71), and remark 11.11. This result depends on the explicit solution in terms of elliptic coordinates, but the zero average can be proved in a more elementary way, see Duistermaat, section 8.2. In the direction $\ell \times k$ one has

$$\frac{d}{dt}\,(z(t), \ell \times k) = r\,(\omega \times k\,,\, \ell \times k) = r\,(\omega\,,\, \ell - \ell_3\, k) = r\ (2T - \ell_3\omega_3\,) > 0\ .$$

Duistermaat [2000] shows (section 9.2) that by a suitable change of coordinates, one may assume that $\ell_3 = 0$, so in this equivalent problem, the velocity in this direction is simply $2rT$. *We will indicate below a possible alternative route to estimate directly the average value of ω_3.*

Solution in terms of elliptic coordinates. The invariant tori have two commuting vectorfields, given by

$$\sqrt{f}\,(\gamma \times \Omega\,,\, L \times \Omega)\quad ,\quad \sqrt{f}\,(\gamma \times \tilde{A}\Omega\,,\, L \times \tilde{A}\Omega\,) \tag{4.22}$$

see Duistermaat, section 8.1. Linearizing cordinates are the elliptic coordinates in the sphere, given by (Duistermaat, section 11):

$$\sum_{i=1}^{3} \frac{x_i^2}{a_i - \lambda} = 1\ ,\ \ \gamma_i = x_i/\sqrt{a_i}\ ,\ \ a_i = 1/I_i\ . \tag{4.23}$$

The Poisson sphere corresponds to $\lambda_1 = 0$, and the confocal quadric coordinates λ_2, λ_2 give the separation of variables coordinates. Chaplygin's sphere relates to a geodesic flow on SE(3) (section 9.1) and a Lax pair (section 9.4).

Reconstruction of the rotational motion from $\gamma(t)$, $a(t)$. Denote $R_h(t)$ the lift of $\gamma(t)$ to $SO(3)$ using the connection $\phi = \rho_3$. Write the rotational motion as $R(t) = R(\phi(t))\,R_h(t)$. Then

$$[\Omega(t)] = \dot{\phi}\,R_h^{-1}\,X_3\,R_h + R_h^{-1}\dot{R}_h = \dot{\phi}\,\gamma + \Omega_h \tag{4.24}$$

so that

$$\dot{\phi} = (\Omega\,,\,\gamma) = \omega_3(t)\ ,\ \Omega = \omega_3\,\gamma + \dot{\gamma} \times \gamma \tag{4.25}$$

In $\omega_3 = (\Omega\,\gamma)$, Ω can be writen as a function of $a(t)$ and $\gamma(t)$, depending on ℓ_3 as a parameter.

All this looks rather cumbersome. But Levi [1996] has shown that the non-commutativity of $SO(3)$ can be circumvented, by taking parallel transports along certain spherical curves composed together with total rotations

$$\int^t \omega_3(t)\,dt\ \ ,\ \ \int^t |\dot{\gamma} \times \gamma| = |\dot{\gamma}|\,dt = \text{arc length}\ .$$

For closed curves the first integral can be related to the (algebraic) *area* enclosed bu $\gamma(t)$, see Montgomery [1991]. Finding a geometrical description of recosntruction could be an interesting project.

5 Questions

5.1 Almost Poisson, almost Dirac approaches

Mashke and van der Schaft [1994] were the first to describe a nH system via an almost-Poisson structure[15],

$$\dot{x_i} = \{x_i, H\}_{MS}$$

with a "non Jacobi" bracket on the manifold $P = Leg(\mathcal{H}) \subset T^*Q$, where $Leg : TQ \to T^*Q$ is the Legendre transformation. They proved the following "no-go" result: the MS-bracket satisfies the Jacobi identity if and only if the constraints are integrable (so the system was already Hamiltonian). In Koiller, Rios and Ehlers [2002] we gave a moving frames based approach to derive the brackets. For some recent work in the study of the MS-bracket and Dirac estructures (the latter introduced in Courant [1990]), see Cantrijn, de Léon, de Diego [1999], Koon and Marsden [1998], Ibort, de Léon, Marrero and de Diego [1999], Clemente-Gallardo, Maschke and van der Schaft [2001]. In spite of these advances, the general theory for the general NH bracket geometry (i.e, regardless of special features) is still in order[16].

5.2 G-Chaplygin systems via affine connections

It was shown in Koiller [1992], see also Vershik and Fadeev [1981] that the trajectories of the compressed almost Hamiltonian system in T^*S can be described in an equivalent way as the geodesics of an affine connection ∇^{NH} in S. For background in this approach, see Lewis [1998] and references therein. Consider the parallel transport operator along closed curves; if the holonomy group is always conjugate to a subgroup of $S0(m)$, then the connection is metrizable. This means that there is a metric such that ∇^{NH} is precisely the Levi-Civita connection of this metric. More generally, one may want to know when the geodesics of ∇^{NH} are, up to time reparametrization, the geodesics of a Riemannian metric. This is a traditional area in differential geometry, whose roots go back to the 19th century, and goes under the name of *projectively equivalent connections* (Cartan [1937], Eisenhart [1925], Kobayashi and Nomizu [1963], Sharpe [1997]). For a recent paper, studying integrability via the equivalence method, see Grossman [2000]. In our setting, one would be interested to find conditions for the nH connection to be projectively equivalent to a riemannian connection, in terms of the original data $(G, Q, \langle\, ,\rangle, \mathcal{H})$. It would be also interesting to tie the hamiltonization question with the canonical system and invariants of the Cartan equivalence method. When an internal symmetry group is present, it would be desirable to construct a projected connection in S for each set of conserved momenta, and address these issues in the reduced level.

5.3 Integrability of nH systems

Although a number of interesting problems have been solved using, say, Abelian functions, a precise definition for integrability of a nH system is still lacking (Bates and Cushman [1999]).

Examples suggest that the existence of an invariant measure must be imposed as a necessary (although not sufficient) condition, see Kozlov [2002]. We list a few papers: Veselov and Veselova [1988], Veselov and Veselova [1986], Fedorov [1989], Cushman, Hermans and Kemppainen [1995], Zenkov [1995], Zenkov and

[15] Physicists are never shy to use the word "super" in their endeavours; on the other hand we, mathematicians, prefer to use low key terminology, like "almost-quasi-twisted-(freakaz-)-'oid's"; this certainly does not help our image problem with applied people, see Papastavridis [2002] and Koiller [2003].

[16] Remarks at their talks at Alanfest by J. Marsden (joint work with H. Yoshimura), and by C.Marle, in the Courant-Weinstein-Dirac context, may represent important steps in this direction.

Bloch [2000], Dragovic, Gajic and Jovanovic [1998], Jovanovic [2003], Fedorov and Jovanovic [2003]. Our views are very congenial to the latter. Most examples have enough integrals of motion (due to symmetries) that the dynamics occurs on invariant 2-dimensional tori. Moreover, due to the invariant measure, *the flow becomes linear in these tori after a time rescaling.*, in view of Kolmogorov's theorem Arnold [1989] and Jacobi's "last multiplier trick". Time reparametrization sinalizes the possibility of an affine symplectic structure, but this seems to be an excessively strong requirement. Nonetheless, we believe that characterizing nH systems possessing an affine symplectic structure (after some reduction) could be an interesting project. As a first step, one coud re-examine the literature to see which examples fit.

5.4 Reduction of nH Chaplygin systems

As we mentioned in the introduction, it seems natural to do compression (of external symmetries) first and reduction (of internal) symmetries on a second stage. *For Hamiltonizable systems, Marsden-Weinstein reduction (Marsden and Weinstein [1974]) is directly appliable procedure, specialized to the cotangent bundle Kummer [1981].*

In general, what is the difficulty for reduction of Ω_{NH}? Suppose G is a Lie group of internal symmetries for (1.4). Attempting to mimic the MW reduction procedure, weencounter the following problem: $X_{J\xi}$, $\xi \in \mathcal{G}$, the vectorfields used for cotangent bundle symplectic reduction, are defined via the canonical symplectic form $\Omega_{\rm can}$, but Ω_{NH} has the extra term (J.K).

The difficulties for reduction of general nH systems are explained in Śniatycki [2002]. A few references in this rapidly developing theme, besides those already mentioned in the text, are: Bates and Śniatycki [1993], Bates [2002], Cushman, Kemppainen, Śniatycki and Bates [1995], Sńiatycki [2001], Śniatycki [1998], Cushman and Śniatycki [2002], Koon and Marsden [1997], Koon and Marsden [1998], Cantrijn, de Léon, Marrero and de Diego [1998], Cortés and de Léon [1999], Marle [1998], Marle [2003]. So we feel it is appropriate to finish this paper with

A quote from Marsden and Weinstein [2001].

> "Nonholonomic mechanical systems (such as systems with rolling contraints) provide a very interesting class of systems where the reduction procedure has to be modified. In fact this provides a class of systems that give rise to an almost Poisson structure, i.e, a bracket which does not necessarily satisfy the Jacobi identity. Reduction theory for nonholonomic systems has made a lot of progress, but many interesting questions still remain".

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