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# ZERO INDEX OF PAIR OF PROJECTORS AND EXPANSIONS BY RESONANCE STATES FOR OPERATORS WITH BAND SPECTRUM

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The eigenfunctions expansions technique is well developed and largely used for selfadjoint and nonselfadjoint operators. But it could not be applied directly to operator-functions, since the main tool of it - the Hilbert identity - is absent there. The typical situation of this sort appears when considering resonances which arise as singularities of the analytical continuation of the compressed resolvent of given selfadjoint Hamiltonian $A$ from the 'physical sheet' $\Lambda_+$ of the spectral parameter ,

$$P_K(A-\lambda I)^{-1}/K \equiv (A(\lambda)-\lambda I)^{-1} \tag{1}$$

across absolutely continuous spectrum $\sigma_a$ of $A$ onto the so-called 'nonpysical sheet' $\{\Lambda_-\}$ of the spectral parameter $\lambda$. The choice of the subspace $K$ is defined by the physical content of the problem, and the operator-function $A(\lambda)$, formally defined by (1), is called the Livshiĉ-matrix. (see [1]). The bilinear decomposition of residues of the compressed resolvent at the poles of its analytical continuation onto $\{\Lambda_-\}$ defines the vectors in $K$, which are called resonance states, see [2], [3]. The problems of completeness of the family of all resonance states and of the corresponding spectral decomposition are typical nonstandard problems of spectral analysis. Other ones are connected with completeness and basis properties of exponential system $\{\exp(ik_n x)\}$ on a finite interval in $L_2(0,a)$, see [4,5,6]. An elegant approach to the spectral analysis of resonances was suggested by P. Lax and R. Phillips in [7]. Based on the harmonic analysis in the upper half plane $Im\lambda > 0$ (or in the unit disc), they developed spectral theory of compressed semigroup

$$P_K \exp(iAt)|K, \quad t>0,$$

supposing, that the unitary evolution group $exp(iAt)$ has orthogonal incoming and outgoing subspaces $\mathcal{D}_\pm, \mathcal{D}_+ - \mathcal{D}_-$

$$\begin{aligned} e^{iAt}\mathcal{D}_+ &\subset \mathcal{D}_+, \quad t>0 \\ e^{iAt}\mathcal{D}_- &\subset \mathcal{D}_-, \quad t<0 \\ e^{\pm iAt}\mathcal{D}_\pm &\to 0, \quad t\to\infty \end{aligned} \tag{2}$$

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In this case the Livshiĉ matrix proves to be constant, hence the Hilbert identity is valid for the compressed resolvent in $K$. Unfortunately the class of operators, for which this assumption is fulfilled looks rather narrow: it consists of the operators having a part, which is unitarily equivalent to the momentum $1/i\frac{d}{dx}$, hence the spectrum of $A$ contains a branch, filling the real axis with a constant multiplicity. Thus the spectral properties of resonances for band spectrum cannot be studied by this method.

In what follows, we develop an approach to the spectral analysis of resonances, which is applicable to the operators with the band spectrum. Our approach is based on a version of harmonic analysis on the Riemann surface of finite genus to substitute the standard Lax-Phillips theory, and one simple assertion concerning pairs of 'skew-connected' subspaces in Hilbert spaces (see[8]). The last is equivalent to the well known zero-index condition for pairs of orthogonal projectors (see[9,10,11,12,13,14]). Both ingredients are independent in some way, the second being more elementary. For this reason we demonstrate it first in a problem, relevant to the Friedrichs model, i.e. a compactly perturbed multiplication operator with one-band-spectrum .The second part contains the application of the developed techniques to the compactly perturbed semi infinite Jacobian matrix, which has a band spectrum of constant multiplicity. The appendix contains a review of results in harmonic analysis, which we systematically use in the text. Our approach is valid also for operators with a threshold spectrum,e.g. for a compactly perturbed orthogonal sum of semi infinite Jacobian matrices with star-like graphs for configuration space. The corresponding material will be published elsewhere.

## 1. Preliminaries. The T. Kato zero-index condition for the spectral analysis of resonances in frames of the standard Lax-Phillips theory

The index of the pair of projectors $P_1, P_2$ was initially defined as

$$\dim P_1 - \dim P_2 = \text{trace}(P_1 - P_2) = \text{ index } (P_1 - P_2)$$

It is obvious, that two finite dimensional projectors $P_1, P_2$ can be intertwined if and only if index $(P_1 - P_2) = 0$. In [9] T. Kato proved it for the infinite dimensional supposing

$$|P_1 - P_2| < 1 \tag{3}$$

New interesting properties of pairs of projectors, which have zero index, are described in [9,10,11]. In what follows we use the following simple fact.

**.** *The following statements are equivalent 1. $P_1, P_2$ is a Fredholm pair of orthogonal projectors and the norm-estimate is fulfilled:*

$$||P_1 - P_2|| < 1.$$

*2. Both operators $(P_1P_2)_{P_1H}$, $(_{P_2H}$ are invertible in $P_1H, P_2H$ correspondingly, and*

$$||(P_1P_2P_1)^{-1}_{P_1H}|| < \infty, ||(P_2P_1P_2)^{-1}_{P_2H}|| < \infty$$

*3. The skew projectors*

$$\mathcal{P}_{P_2H}^{||(1-P_1)H}$$

*onto* $P_2H$ *parallel to* $(1-P_1)H$, *and*

$$\mathcal{P}_{P_1H}^{||(1-P_2)H}$$

*onto* $P_1H$ *parallel to* $(1-P_2)H$ *are bounded and given by the next formulas*

$$\mathcal{P}_{P_2H}^{||(1-P_1)H} = (P_2P_1P_2)^{-1}P_2P_1,$$

$$\mathcal{P}_{P_1H}^{||(1-P_2)H} = (P_2P_1P_2)^{-1}P_1P_2.$$

*Proof.* It follows from 1, that $||P_1P_2P_1 - P_1|| < 1$, $||P_2P_1P_2 - P_2|| < 1$. Hence $1 \to 2$ by the Banach principle, $2 \to 3$ by the direct calculation, e.g.:

$$\begin{gathered}(P_2P_1P_2)^{-1}P_2P_1e_2 = e_2, \quad e_2 \in P_2H,\\ (P_1P_2P_1)^{-1}P_2P_1e_1^- = 0, \quad e_1 \in (1-P_1)H,\\ \mathcal{P}_{P_2H}^{||(1-P_1)H}|| \le ||(P_2P_1P_2)_{P_2H}^{-1}||\\ \le \frac{1}{1-||P_2P_1P_2-P_2||} \le \frac{1}{1-||P_1-P_2||}.\end{gathered}$$

Vice versa, if both skew projectors are bounded, then the following estimates for a minimal angle are true

$$\begin{gathered}\sin(P_2H,(1-P_1)H) \ge \delta_1 = ||\mathcal{P}_{P_2H}^{||(1-P_1)H}||^{-1},\\ \sin(P_1H,(1-P_2)H) \ge \delta_2 = ||\mathcal{P}_{P_1H}^{||(1-P_2)H}||^{-1}\end{gathered}$$

hence, $\cos(P_2H,(1-P_1)H) \le \sqrt{1-\delta_1^2}$, $\cos(P,H,(1-P_2)H) \le \sqrt{1-\delta_2^2}$. Thus, for every pair $e_1^- \in (1-P_1)H, e_2 \in P_2H$ we have the estimate $|<e_2, e_1^->| \le \sqrt{1-\delta_1^2}$ and for $e_1 \in P_1H, e_2^- \in (1-P_2)H$ similarly we have the estimate

$$|<e_1, e_2^->| \le \sqrt{1-\delta_2^2}$$

These estimates imply

$$||(1-P_1)P_2|| = ||P_2(1-P_1)|| \le \sqrt{1-\delta_1^2},$$

$$||(1-P_2)P_1|| = ||P_1(1-P_2)|| \le \sqrt{1-\delta_2^2}.$$

Let us write the operator $P_1 - P_2 = P_1(1-P_2) + (P_1-1)P_2$ in a form of a matrix, connecting two orthogonal decompositions of $H$, $H = P_2H \oplus (1-P_2)H = P_1H \oplus (1-P_1)H$:

$$P_1 - P_2 \Leftrightarrow \left\{ \begin{matrix} 0 & P_1(1-P_2) \\ -(1-P_1)P_2 & 0 \end{matrix} \right\}$$

This matrix representation implies obviously the estimate

$$||P_1 - P_2|| \leq \max\left(\sqrt{1-\delta_1^2}, \sqrt{1-\delta_2^2}\right).$$

Hence $3 \to 1$.

¿From the Assertion 1, and standard Lax-Phillips approach follows the representation for the Livshiĉ matrix of a selfadjoint operator with a one-band spectrum, having uniform multiplicity. We derive here the corresponding formula for the discrete Lax-Phillips picture only.

**Theorem 1.** *Let $U^l$ be a unitary group $l \in Z$, in a Hilbert space $H$, which fulfills the Lax-Phillips properties:*

*a. There exist two orthogonal invariant subspaces of $U^l$, $\mathcal{D}_\pm$, $\mathcal{D}_+ - \mathcal{D}_-$, such that*

$$U^*\mathcal{D}_+ \subset \mathcal{D}_+,$$
$$U^+\mathcal{D}_- \subset \mathcal{D}_-$$

*b. $U^l\mathcal{D}_+ \to 0$, $U^{-l}\mathcal{D}_- \to 0$, $l \to \infty$*

*then the operator $A = 1/2(U + U^+)$ is bounded and selfadjoint in $H$, and its resolvent, being compressed onto the translation invariant subspace $K = H \ominus \{\mathcal{D}_- \oplus \mathcal{D}_+\}$ of the group $\{U^l\}$ can be represented as a linear combination of resolvents of generators $T, T^+$ of the corresponding Lax-Phillips compressing semigroups*

$$T = P_k U|K, \quad T^+ = P_k U|K.$$

*and has singularities at eigenvalues of them on the nonphysical sheet.*

*Proof.* Denoting the spectral parameter of $U$ by $\zeta$, we can take the natural spectral parameter of $A$ in the form of $\lambda = 1/2(\zeta + 1/\zeta)$, assuming, that the outside of the unit disc $|\zeta| > 1$ corresponds to the spectral sheet of $\Lambda$

$$\zeta = \lambda + \sqrt{\lambda^2 - 1}.$$

Then the resolvent of $A$ can be written in the following form:

$$(A - \lambda I)^{-1} = 2[(U - \zeta I)^{-1}(1 - U^+\zeta^{-1})^{-1}] = \frac{2U\zeta}{\zeta^2 - 1}[(U - \zeta I)^{-1} + \zeta(1 - \zeta U)^{-1}].$$

Now we use the crucial fact of the Lax-Phillips theory that compressions of semi groups $\{U^l\}$, $l \geq 0$; $\{(U^+)^l\}$ $l \geq 0$; are contracting semigroups; e.g. for $l > 1$:

$$\begin{aligned}
P_K U^l P_K &= P_K U^{l-1}(P_{\mathcal{D}_+} + P_{\mathcal{D}_-} + P_K)UP_K \\
&= P_K U^{l-1}(P_{\mathcal{D}_-} + P_K)KP_K \\
&= P_K U^{l-1}P_K U P_K + (P_K U^+ P_{\mathcal{D}_-}(U^+)^{l-1}P_K) \\
&= P_K U^{l-1}P_K U P_K = (P_K U P_K)^l.
\end{aligned}$$

Hence for $|\zeta| > 1$

$$P_K U(U-\zeta I)^{-1}|_K = -P_K \sum_1^\infty \zeta^{-l} U^l|_K = T(T-\zeta I)^{-1},$$

$$P_K U\zeta(1-U\zeta)^{-1}|_K = -P_K \sum_0^\infty (U^+)^l \zeta^{-l}|_K = -(I-T^+\zeta^{-1})^{-1}$$

(4) $$P_K(A-\lambda I)^{-1}|_K = \frac{2\zeta}{(\zeta^2-1)}\left[\frac{T}{T-\zeta I} + \frac{\zeta}{T^+-\zeta I}\right]$$

The formula (4) gives the compressed resolvent representation on the spectral sheet $\Lambda_+(|\zeta| > |)$. The members, staying in the right side have analytical continuation onto $\Lambda_-$. The domain of analyticity of it is actually the intersection of the resolvent sets of $T, T^+$ hence the compressed resolvent staying on the left side can be continued analytically onto it. The residues of the compressed resolvent at the poles coincide with corresponding spectral projectors of $T$ of $T^+$, hence the resonance states are just the eigenfunctions of $T, T^+$.

Note, that the system of all resonance states, defined above, is obviously surplus complete. But when we confine the consideration exclusively by the discrete spectrum of $T$ and $T^+$, it could be complete and $\omega$-linearly independent (in the sense of M. Krein ), and even form a Riesz-Basis in $K$. Let us assume now, that the absolutely continuous spectrum of $T, T^+$ is empty. Then the characteristic function of $T$ is an inner function in the unit disc, see [15] . According to the main result of the functional model theory, see [15] the invariant subspaces of $T$, which correspond to a discrete and singular continuous spectrum, are defined by the Blaschke factor $\Pi$ and singular factor $\theta_s$ of characteristic function see correspondingly. In a scalar case when $\dim(1-T^+T) = 1$:

$$\Pi = \zeta^{l_0}\Pi_l \frac{\zeta_l - \zeta}{1-\bar{\zeta}_l\zeta}\frac{\bar{\zeta}_l}{\zeta}, \quad l_0 \in \{Z_+ \cup 0\},$$

$$\theta_s = \exp\left\{\int \frac{\zeta+t}{\zeta-t} d\nu(s)\right\}$$

where $\zeta_l$ are the eigenvalues of $T$ and $\nu$ is a singular measure on a circle, supported by a singular spectrum of $T$ (see appendix). If the absolutely continuous spectrum is absent,then the characteristic function of $T$ is just a product of Blaschke factor and singular factor

$$S(\zeta) = \Pi(\zeta)\theta_s(\zeta), |\zeta| < 1.$$

Writing down the operators $T$ $T^+$ in terms of 'incoming' spectral representation of $U$, we get in this case (see [1]5)

$$\mathcal{D}_- = H^2_-, \mathcal{D}_+ = SH^2_+, K = H^2_+ \ominus SH^2_+$$

and for the invariant subspaces of $T, T^+$, which correspond to the discrete spectrum

$$\mathcal{M}_d, \mathcal{M}_d^+$$

and singular continuous spectrum

$$\mathcal{M}_s, \mathcal{M}_s^+$$

we get (see [1]5)

$$\mathcal{M}_d^+ = H_+^2 \ominus \Pi H_+^2, \mathcal{M}_d = \theta_s \mathcal{M}_d^+,$$

$$\mathcal{M}_s^+ = H^2 \ominus \theta_s H_+^2, \mathcal{M}_s = \Pi \mathcal{M}_s^+$$

the conditions of the separability of spectral components is discussed in [8] , where sufficient conditions for positivity of minimal angles between $\mathcal{M}_d^+, \widehat{\ } \mathcal{M}_s+$; $\mathcal{M}_d, \widehat{\ } \mathcal{M}_s$ are proved. The problem of joint (combined) completeness

$$\mathcal{M}_d^+ + \mathcal{M}_d = K \tag{5}$$

with a positive angle between $\mathcal{M}_d, \mathcal{M}_d^+$ was discussed in [4] and in [5]. It is relevant to the classical problem of exponential system completeness and Riesz-basis properties on a finite interval.

**Theorem 2.** *The joint completeness condition* (5) *is fulfilled if and only if there exist an analytic function $f$ (with possible singularities on the unit circle $C$ and outside of unit disc) which admits the following factorizations*

$$f = \begin{cases} f_e^+ \Pi, & |\zeta| < 1, \\ f_e^- \theta_s, & |\zeta| > 1, \end{cases} \tag{6}$$

*where*

$$f_e^{\pm}(\zeta) = \exp\left\{ \pm \int_C \frac{s+\zeta}{s-\zeta} ln|f| dm(s) \right\}$$

*are corresponding outer factors inside* $(+)$ *and outside* $(-)$ *at the unit disc* $|\zeta| < 1$, $m = \frac{1}{2\pi} args$, *with modulo fulfilling the Muckenhaupt condition*

$$\sup \frac{1}{\Delta} \int_\Delta |f|^2 dm \quad \frac{1}{\Delta} \int |f|^{-2} dm < \infty \quad (A_2)$$

*Proof.* repeats basically the considerations from [5], being founded on the famous Muckenhaupt criterion of the boundedness of Gilbert transforms in weighted classes, see [1]6. Notice that the Cauchy type integral, which can be written in a form of a convolution with reproducing kernel $\|_s = (1 - \zeta \bar{s})^{-1}$ gives orthogonal projection from $L_2(C)$ onto the Hardy class $H_+^2$ of all analytic functions inside $\mathcal{D}$, square integrable over $C$.

$$<, \|_s >= P_{H_+^2}.$$

Being edged by multiplications by $f, f^{-1}$ it gives formally the skew projector onto $\Pi H_+^2$ parallel to $\theta_s H_-^2$

$$\mathcal{P}_{\Pi H_+^2}^{||\theta_s H_-^2} = f P_{H_+^2} f^{-1} \tag{7}$$

similarly

$$\mathcal{P}^{||\Pi H^2_-}_{\theta_s H^2_+} = \bar{f}^{-1} P_{H^2_+} \bar{f}. \tag{8}$$

The operators, given formally by formulae (7), (8) should be defined first on appropriate dense lineals $L_2 \cap f^e_+ L_2$, $L_2 \cap (\bar{f}^e_+)^{-1} L_2$ correspondingly. Then they should be continued by closure onto $L_2$, due to the Muckenhaupt condition, which guarantees the boundedness of the resulting operators (7), (8). On the other hand the boundedness of the skew projectors (7), (8), is equivalent to the sufficiant zero-index conditions for pairs of orthogonal projectors (see the assertion 1):

$$\{P_{\Pi H^2_+}, P_{\theta H^2_+}\}; \{P_{\Pi H^2_-}, P_{\theta H^2_-}\} :$$

$$|P_{\Pi H^2_+} - P_{\theta H^2_+}| < 1,$$
$$|P_{\Pi H^2_-} - P_{\theta H^2_-}| < 1.$$

Since $\Pi$ and $\theta$ are mutually prime (have no common factors), we have

$$\begin{aligned} &\Pi H^2_+ \cap \theta H^2_+ = \Pi\theta H^2_+, \\ &\Pi H^2_- \cap \theta H^2_- = (\Pi H^2_+)^- \cap (\theta H^2_+)^- \\ &= (\Pi H^2_- + \theta H^2_+)^- = H^2_- \end{aligned}$$

hence our zero-index conditions are fulfilled for the projectors onto complements of $\Pi\theta H^2_+$, $H^2_-$:

$$|P_{\Pi(H^2_+} \ominus \theta H^2_+) - P_{\theta(H^2_+} \ominus \Pi H^2_+)| < 1,$$
$$|P_{H^2_+ \ominus \theta H^2_+} - P_{H^2_+ \ominus \Pi H^2_+}| < 1.$$

since the normalized reproducing kernels

$$\sqrt{1 - |\zeta_l|^2} ||(z, \zeta_l) = \frac{\sqrt{1 - |\zeta_l|^2}}{1 - z\overline{\zeta_l}}, \quad \Pi(\zeta_l) = 0,$$

form a complete system in $H^2_+ \ominus \Pi H^2_+$, the orthogonal projections of them onto $H^2_+ \ominus \theta H^2_+$

$$\frac{1 - \theta\overline{\theta(\zeta_l)}}{1 - z\overline{\zeta_l}} \sqrt{1 - |\zeta_l|^2}$$

form a complete system there. On the other hand, also due to the Assertion 1, the joint system at eigenfunctions of $T$ and $T^+$ is complete in $K$

$$\overline{V\left\{ \frac{1}{1 - \overline{\zeta_e} z}, \frac{\theta\Pi}{z - \zeta_e} \right\}} = K$$

Conversely, if the joint completeness conditions (5) is fulfilled, then the angles between the components are positive and the corresponding parallel projectors are bounded. Hence Muckenhaupt condition is fulfilled.

**Note.**

The conditional statement of the preceding theorem can be reformed into explicite one, if we produce the generating function $f$ just from observation of inner factors $\Pi, \theta$.

**Theorem 3.** *If the inner factors $\theta, \Pi$ fulfil the following condition on $C$*

$$arg\Pi - arg\theta \subset L_1(C),$$

*then the generating function $f$ exists and is represented by each of the factorizations (6), with*

$$ln|f| = -\int ctg\left(\frac{\varphi_s - \varphi_3}{2}\right)[arg\Pi - arg\theta]dm(s)$$

*The generating function is defined uniquely up to the constant unitary factor.*

*Proof.* The factorizations (6) imply the equation for $a.e.$ $\zeta, |\zeta| = 1$

$$\Pi_{(\zeta)}\exp\left\{\int_C \frac{s+\zeta}{s-\zeta} ln|f|dm(s)\right\}$$
$$= \theta(\zeta)\exp\left\{\int \frac{s+\zeta}{s-\zeta} ln|f|dm(s)\right\}$$

where the integrations in the left side and in the right side are performed in opposite directions. Hence $a.e.$ on $C$

$$arg\Pi - arg\theta = \int ctg\left(\frac{\varphi_s - \varphi_\zeta}{2}\right) ln|f|dm(s)$$

Since the Hilbert transformation $G$ is an invertible operator in $L_1$ and $G^2 = -1$, we have

$$ln|f| = -\int ctg\left(\frac{\varphi_s - \varphi_3}{2}\right)[arg\Pi - arg\theta]dm(s).$$

then both $f_e^+, f_e^-$ are calculated by the usual formulae

$$f_e^{\pm}(3) = \exp\left\{\pm\int_c \frac{s+\zeta}{s-\zeta} ln|f|dm(s)\right\}.$$

$$\varphi_n(\zeta) = \begin{cases} \varphi_{n,e}^+(\zeta) & \Pi_n(\zeta), \quad |\zeta| < 1, \\ \varphi_{n,e}^-(\zeta) & \theta_n(\zeta), \quad |\zeta| > 1, \end{cases}$$

where $\varphi_n, e^+$ are outer functions inside and outside of the unit disc $\mathcal{D}$ ($\Pi_n$ is a Blaschke product, $\theta_n$ is a singular inner function).

**Theorem 4.** *If* $\sum |\alpha_n| < \infty$ *and infinite products*

$$\Pi = \Pi_{n=1}^{\infty}\Pi_n, \quad \theta = \Pi_{n=1}^{\infty}\theta_n$$

*are convergent on* $\mathcal{D}$ *and* $\int |ang\Pi - ang\theta| d\varphi < \infty$ *than the operator*

$$U = \zeta\Pi_n\left(1 + \frac{2i\alpha n}{1 - i\alpha n}P_n\right)$$

*exist and is unitary in* $L_2(C)$*. The corresponding unitary group* $\{U^l\}$ *possess incoming and outgoing subspaces*

$$\mathcal{D}_+ = \theta H_+^2$$

$$\mathcal{D}_- = \Pi H_+^2$$

*The spectrum of* $U$ *is absolutely continuous and covers the unit circle with a constant multiplicity 1. The corresponding eigenfunctions are represented in forms of scattered waves,which have an asymptotics at the singular points, parameterized by reflection coefficient* $R(\zeta)$

$$U\psi = \zeta\psi, \quad \psi \cong \frac{1}{z - \zeta(1-0)} + R(\zeta)\frac{1}{z - \zeta(1+0)} \quad z \to \zeta$$

$R(\zeta)$ *playing a role of Schrödinger scattering matrix. The reflection coefficient* $R$ *is unitary on* $C$ *and can be factorized in a form*

$$R(\zeta) = \frac{\overline{\mathcal{D}(\zeta)}}{\mathcal{D}(\zeta)}.$$

*where* $\mathcal{D}$ *is a perturbation determinant.*

*If considering Lax-Phillips scattering problems for the pair of orthogonal incoming and outgoing subspaces* $\mathcal{D}_\pm^\circ$

$$\mathcal{D}_+^\circ = \mathcal{D}_+ \cap \mathcal{D}_-^- = \theta\Pi H_+^2$$

$$\mathcal{D}_-^\circ = \mathcal{D}_+^- \cap \mathcal{D}_- = H_-^2,$$

*then the corresponding Lax-Phillips scattering matrix (in spectral incoming representation) is combined with* $\theta, \Pi, R$ *the following way*

$$\mathcal{S} = \bar{R}\Pi \cdot \theta = \frac{\mathcal{D}}{\overline{\mathcal{D}}}\Pi\theta$$

*It is analytic in the unit circle and can be factorized in the form*

$$\frac{\mathcal{D}\Phi}{\overline{\overline{\mathcal{D}\Phi}}},$$

*where* $\Phi$ *is a generating function for the pair of subspaces* $\Pi H_-^2, \theta H_+^2$*, constructed by the method described in Theorem 3.*

The proof of the similar theorem in a one dimensional case, $\alpha_1 - \alpha$, $\alpha_i = 0$, $i \neq 1$, is given in [1]7. The proof in a general case will be published elsewhere.

**Theorem 5.** *The operator formed in $L_2(C)$ from $U, U^+$ as*

$$A = \frac{1}{2}(U + U^+)$$

*is selfadjoint. The corresponding resolvent is compressed onto the translation-invariant subspace $K = L_2(C) \ominus [\mathcal{D}_-^\circ \oplus \mathcal{D}_+^\circ]$ is represented in the form of a linear combination of resolvents of generators $T, T^+$,*

$$T = P_K U|K,$$

$$P_K[A - 1/2(\zeta + \zeta^{-1})]^{-1}|K = \frac{2\zeta}{\zeta^2 - 1}\left[\frac{\zeta}{T - \zeta I} + \frac{T^+}{T^+ - \zeta}\right] |\zeta| < 1$$

*and possess an analytical continuation on the second sheet of the spectral parameter, which corresponds to the inside of the unit disc. $|\zeta| < 1$. The analytically continued compressed resolvent has poles at the eigenvalues $\zeta_l$ of $T$ and $\overline{\zeta_l}$ of $T^+$. The systems of resonance states is a combined system of eigenvectors $T, T^+$. It is complete in $K$ and direct decomposition is valid*

$$K = \mathcal{M}_{+d} \dot{+} \mathcal{M}_d^+, (\mathcal{M}_d \widehat{\ } \mathcal{M}_d^+ > 0),$$

*if and only if the Lax-Phillips scattering matrix factors $\mathcal{D}\varphi$ fulfil the Muckenhaupt condition:*

$$\Delta^{-1}\int_\Delta |\mathcal{D}\Phi|^2 d_\varphi \quad \Delta^{-1}\int_\Delta |\mathcal{D}\Phi|^{-2} d_\varphi \quad \leq C < \infty. \tag{9}$$

The proof is a combination of Theorems 2 and 4.

**Note.**

. All previous theorems concern the case of simple zeros of $\Pi$. Treating of the multiple zeros case is similar, with the only difference that then we should speak about completeness of root-vectors, forming a base for the Jordan form matrix.

Expansions by resonances require the completeness and the Riesz-basis property for the joint system of eigenvalues both $T$ and $T^+$. The conditions for that are given by the next statement, which follows from Theorems 2,4,5.

**Theorem 6.** *If the Blaschke factor of the analytical function $\mathcal{D}\Phi$ in the unit disc fulfills the Carleson condition*

$$\inf_m \Pi_{l \neq m} \left|\frac{\zeta_l - \zeta_m}{1 - \zeta_l \bar{\zeta}_m}\right| > 0$$

*and the outer factor of it fulfills the Muckenhaupt condition (9), then the joint system of eigenvectors of $T, T^+$ is complete in $K$ which is spectrally represented by $H_+^2 \ominus SH_+^2$, $S = \mathcal{D}\Phi(\overline{\mathcal{D}\Phi})^{-1}$, and forms the Riesz-basis there.*

Note at last, that the operator $A$ in consideration can be regarded as a generalized Friedrichs model - a compactely perturbed operator of multiplication by a real function in a weighted $L_2(\mu)$ space. In fact

$$A = z\Pi_l\left(1 + \frac{2i\alpha_l}{1 - i\alpha_l}P_l\right) + \overline{\Pi}\left(1 - \frac{2i\alpha_l}{1 + i\alpha_l}P_l\right) z^{-1} = \frac{1}{2}(z + \overline{z}) * + U *, \quad z = e^{i\varphi},$$

where $V$ is a compact selfadjoint operator in $L_2(C) = L_2(0, 2\pi)$. The change of variable $\cos\varphi = u$ produces a measure transformation

$$d\varphi = -\frac{du}{\sqrt{1-u^2}}$$

and corresponding isometry $J$ of the Hilbert spaces, resulting in similarity of $A$:

$$A \to A_u = u * + V_u, L_2(C) \to L_2(\frac{(1)}{\sqrt{1-u^2}})$$

where $V_u = JVJ^+$.

The spectrum of the Friedrich model $A$ or $A_u$ is purely continuous $\sigma = [-1, 1]$, but the structure of resonances could be highly nontrivial, especially if several accumulation points of them on the $[-1, 1]$ are present.

## 3. Functional model for periodical Jacobian matrix. Lax-Phillips approach

Let $A$ be three-diagonal hermitian $n$-periodic Jacobian matrix in $l_2(Z)$, with a block-period $A_0$

$$A_0 = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ 0 & a_{32} & \ddots & & 0 \\ 0 & 0 & & & \end{pmatrix}, A = \begin{pmatrix} A_0 & & & \\ & \bar{\alpha} & & \\ \alpha & & & \\ & & A_0 & \\ & & & \bar{\alpha} \\ & & \alpha & \\ & & & A_0 \end{pmatrix}$$

Then $A$ could be obviously Fourier-represented in $L_2[(0, 2\pi), E_n]$ by the matrix multiplication operator

$$A(\theta) = A_0 + \begin{pmatrix} & & \overline{\alpha\theta} \\ & 0 & \\ \alpha\theta & & \end{pmatrix}$$

$\theta = e^{i\varphi}$, $0 < \varphi < 2\pi$. The spectrum $\sigma(A)$ of $A$ is defined by solutions of the equation on the real axis $\lambda$:

$$\frac{1}{2}[\theta + \bar{\theta}] = \frac{1}{|\alpha|}P(\lambda) \quad \theta = e^{i(\varphi + arg\alpha)} \tag{13}$$

If the eigenvalues of $A_0$ are simple and the coupling constant $\alpha$ is small enough then the spectrum of $A$ is absolutely continuous and consists of $n$ bands, $\gamma_s$ lying near eigenvalues of $A_0$:

$$\sigma(A) = \bigcup_{s=1}^{n} \gamma_s$$

The first sheet of spectral variable $\Lambda_+$ is a complex plane with $n$ cuts $\gamma_s$ along the bands. For our considerations we need a double-Riemann surface $\Lambda = \Lambda_+ \cup \Lambda_-$ of two sheets, joined by cuts $\gamma_s$. Let us denote by $b_\kappa(\infty)$ a Blaschke factor on $\Lambda_+$, which has only one simple root at infinity (see Appendix), $\kappa$ is the corresponding quasimomentum vector $\kappa = (\kappa_1, \kappa_2, \dots \kappa_{n-1})$. The function $\theta$, defined on the first sheet $\Lambda_+$ as a compressing solution of the equation

$$\frac{1}{2}[\theta + \theta^{-1}] = \frac{1}{|\alpha|} P(\lambda) \tag{14}$$

$|\theta(\lambda)| < 1$, $|\theta||_\Gamma = 1$ ,should have a root of order $n$ at infinity $\infty_+$, hence

$$\theta = b_\kappa^n$$

up to the constant unitary factor, which we omit now. It means that $\kappa n = 0||_{\text{ mod } 2\pi}$. Let us form an $l_2$ vector of reproducing kernels $\langle_\mu(\lambda, \infty)$ for Hardy classes $H^2_{\mu,+}(\Lambda_+)$:

$$\chi = \{\dots, \langle_\mu(\lambda,\infty), \langle_{\mu-\kappa}(\lambda,\infty) b_\kappa, \dots \\ \langle_{\mu-(n-1)\kappa} b_\kappa^{n-1}, \langle_\mu(\lambda,\infty)\theta, \langle_{\mu-\kappa}(\lambda,\infty)\theta b_\kappa, \}$$

which can be regarded as a sequence of $n$-vectors:

$$\{\theta^l_\chi\} = \{\theta^l(\langle_\mu(\lambda,\infty), \langle_{\mu-\kappa}(\lambda,\infty) b_\kappa, \dots, \langle_{\mu-(n-1)\kappa} b_\kappa^{n-1})\}_{l=-\infty}^{\infty}$$

the following statement, proved in paper [1]8 describes the 'functional model' for the Jacobian matrix with a $n$-band spectrum.

**Theorem7.** *Let us consider the space $L_2$ of all functions on $\Gamma$, which are square integrables in respect to the harmonic measure $m$ on $\Gamma$,*

$$m = \frac{1}{2\pi i} \frac{db_\kappa}{b_\kappa} = \frac{1}{2\pi i n} \frac{d\theta}{\theta}.$$

*Then the transformation $\mathcal{T}$ for some $\mu$*

$$f \xrightarrow{\mathcal{T}} < f, \chi >_{l_\alpha} = \sum_{-\infty}^{\infty} \theta^l < \vec{f_l}, \vec{\chi} > = \tilde{f}$$

*maps $l_2$ into $L_2(m)$ isometrically in such a way that,*

$$f = \oint \chi < f, \chi >_{l_2} dm,$$
$$A_\mu f = \oint \lambda\chi < f, \chi >_{l_2} dm$$

Instead of $\chi = \chi_+$ one can use another system $\chi_- = \overline{\chi_+}$. Note that the eigenfunctions $\chi$ have a form of Bloch waves with the quasimomentum exponent $\theta$. They are obviously parameterized by the vector $\mu$, which is a parameter of the isospectral deformation. Changing $\mu$, we get a family of operators with the same spectrum $\sigma(A_\mu) = \Gamma$. It is obvious, but important, that a shift $T_n$ by $n$ steps (by period of $A$) can be represented in terms of spectral decomposition by the same Bloch waves:

$$T_n = \oint \theta \chi_+ < f, \chi > dm = \oint \theta^{-1} \chi_- < f, \chi_- > dm$$

¿From the spectral representations for both $T_n, A$ and the dispersion equation (14) follows the analog of the Caley equation

$$[T_n + T_n^+] = 2\mathcal{P}(A).$$

To develop the index approach for resonances in a case of band spectrum, we need a construction of an evolution group, generated by a discrete wave equation in corresponding energy-bound space of Cauchy-data.

Let us consider the discrete wave equation in $l_2(E_n)$. Denoting by $T$ the shift by one step in $l_2(E_n)$ (which is equivalent to the shift $T_n$ by the period in $l_2$) and by $\mathcal{T}$ the shift by one step in discrete time, we write the discrete wave equation in the form

$$(\mathcal{T} + \mathcal{T}^+)u = (T + T^+)u \tag{15}$$

The role of the Cauchy-data for it is played by the vectors in $l_2(E_n) \oplus l_2(E_n)$:

$$\vec{u} = \begin{pmatrix} u \\ (\mathcal{T} - \mathcal{T}^+)u \end{pmatrix} \equiv \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \tag{16}$$

The following statement belongs to P. Kurasov (private communication).

**Theorem 8.** *The quadratic form*

$$|\vec{u}|^2_\xi = \frac{1}{2}\left[ \left|(T - T^+)u^\circ\right|^2_{l^2(E_n)} + \left|u^1\right|_{l^2(E_n)} \right]$$

*plays a role of energy norm in a space $\mathcal{E}$ of all Cauchy-data. The solution of wave equation with fixed initial Cauchy-data having finite energy norm, exists and is uniquely defined.: The solution is generally represented in the form of a linear combination of D'Alambertian waves, which dependence of time is trivial*

$$\mathcal{T}u = Tu \text{ or } \mathcal{T}u = T^+u,$$

*for the waves moving to the right or to the left correspondingly, with the Cauchy-data*

$$\left\{ \begin{pmatrix} u \\ (T - T^+)u \end{pmatrix} \right\} \text{ or } \left\{ \begin{pmatrix} u \\ (T^+ - T)u \end{pmatrix} \right\}$$

*The energy norm of these solution is conserved: at any moment it is equal to the energy of initial data.*

*The Cauchy data of Dalambertian waves are orthogonal in energy norm.*

We'll need also the following result, containing in paper [2]0, which gives a base for Lax-Phillips approach to the discrete wave equation. Let us consider the one-step generator for Cauchy-data of the wave equation

$$\mathcal{L} : u(\tau) \to u(\tau + 1)$$
$$\mathcal{L} = \frac{1}{2}\begin{pmatrix} T + T^{+} & I \\ (T - T^{+})^2 & T + T^{+} \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 2\mathcal{P}(A) & I \\ 4[\mathcal{P}^2(A) - I] & 2\mathcal{P}(A) \end{pmatrix}$$

**Theorem 9.** . *The generator $\mathcal{L}$ is unitary in $\mathcal{E}$ and has a complete orthogonal in energy norm systems of eigenfunctions*

$$\Psi_{+} = \begin{pmatrix} \frac{1}{-\theta+\theta^{-1}} & \chi+ \\ \chi+ & \end{pmatrix}, \quad \mathcal{L}\Psi_{+} = \bar{\theta}\Psi_{+},$$
$$\Psi_{-} = \begin{pmatrix} \frac{1}{-\theta+\theta^{-1}} & \chi- \\ \chi- & \end{pmatrix}, \quad \mathcal{L}\Psi_{-} = \bar{\theta}\Psi_{-},$$

The spectral representation of $\mathcal{L}$ is given by the following formula

$$\vec{u} \to \begin{pmatrix} < \vec{u}, \Psi_{+} >_{\mathcal{E}} \\ < \vec{u}, \Psi_{-} >_{\mathcal{E}}> \end{pmatrix} = \begin{pmatrix} \tilde{u}_{+} \\ \tilde{u}_{-} \end{pmatrix},$$
$$\vec{u} = \oint (\Psi_{+}\tilde{u}_{+} + \Psi_{-}\tilde{u}_{-}) dm,$$
$$\mathcal{L}\vec{u} \to \theta \begin{pmatrix} \tilde{u}_{+} \\ \tilde{u}_{-} \end{pmatrix}.$$

## 4. Resonances for a compactly perturbed lattice

Let us consider the Jacobian matrix $A$, described in a previous section, restricted onto a non negative halt lattice $Z_{+}$

$$A_{+} = \begin{pmatrix} a_{00} & a_{01} & 0 & 0 & 0 & 0 & \dots \\ a_{10} & a_{11} & a_{12} & 0 & 0 & 0 & \dots \\ 0 & a_{21} & a_{22} & a_{23} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

To escape difficulties with denoting, we assume $A_{+}$ is real $a_{ik} = \overline{a_{ik}}$. Let $E$ be a separable Hilbert space, $B - a$ selfadjoint operator acting in it, and $e - a$ generating vector of $B$, $|e| = 1$. Denoting the first orth $(1, 0, 0, 0 \dots)$ in $l_{\alpha}(Z_{+})$ by $e_0$, we switch on the 'interaction' between $A_{+}$ and $B$, constructing the perturbed operator $A_B$ in $\xi = l_{\alpha}(Z_{+}) \oplus E$ the following way for given $\Phi = \Phi_{+} \oplus \Phi_E$:

$$A_B \begin{pmatrix} \Phi_E \\ \Phi_{+} \end{pmatrix} = \begin{pmatrix} B\Phi_E + \alpha e < \Phi_{+}, e_0 >_{l_2(Z_{+})} \\ \alpha e_0 < \Phi_E, e >_E + A_{+}\Phi_{+} \end{pmatrix}, \alpha = \bar{\alpha}$$

**Theorem 10.** *The operator $A_B$ is selfadjoint in $L_2 \oplus E$ and it's absolutely continuous spectrum $\sigma_a(A_B)$ coincides with $\sigma_a(A) \cup \sigma_a(B)$.*

*The rest part of the spectrum coincides with the support of the singular part of positive measure, defining the following R-type functions on $\Lambda_+$:*

$$\mathcal{D}(\lambda) = \{a_1 b_\kappa^{-1}(\lambda,\infty)\zeta_{\mu-(n-1)\kappa}(\lambda,\infty)\zeta_\mu^{-1}(\lambda,\infty) - |\alpha|^2 < (B-\lambda)^{-1}e, e >_E\}^{-1}.$$

*The eigenfunctions of the branch of absolutely continuous spectrum, coinciding with $\sigma_a(A)$, are represented in the form of scattered waves, which have a $Z_+$ component of the following form*

$$\Phi_{\lambda_+} = \chi_+ + R\chi_-$$

*and E-component of the form*

$$\Phi_{\lambda,E} = (B - \lambda I)^{-1} e < \Phi_{\lambda_+} e_0 > \tag{16}$$

*where the reflection coefficient R is equal to*

$$R = -\frac{\overline{\zeta_\mu(\lambda,\infty)}}{\zeta_\mu)\lambda,\infty} \cdot \frac{\mathcal{D}(\lambda)}{\overline{\mathcal{D}(\lambda)}} \tag{17}$$

*The absolutely continuous part of the spectrum has the multiplicity of 1, corresponding scattered waves $\Phi_\lambda$ are orthogonal and normalized in $\xi$, and the corresponding spectral projector is equal to*

$$P_+ = \int \Phi_\lambda <, \Phi_\lambda >_\xi dm, \tag{18}$$

*where m coincides with the spectral measure of Bloch waves for unperturbed operator $A$ , $m = (2i\pi n)^{-1} d\theta/\theta$.*

*Proof.* The proof is basically standard, but rather bohring, being based on the coordinates resolvent asymptotics and the Hilbert identity. In the two-band case, see [2]1 for details.

We discuss now the wave equation, connected with the operator $A_B$

$$(\mathcal{T}_+ + \mathcal{T})u = \alpha \mathcal{P}(A_B), \tag{19}$$

with the same polynomial $\mathcal{P}$ , which was used in a previous section for $A$ in the corresponding Caley identity. The Cauchy data and energy metric are defined by $\mathcal{P}$ in a usual way.

$$\vec{u} = \begin{pmatrix} u_0 \\ (\mathcal{T} - \mathcal{T}^+)u \end{pmatrix} \equiv \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \subset (l_2 \oplus E) \oplus (l_2 \oplus E), \tag{20}$$

$$|\vec{u}|^2_{\mathcal{E}} = \frac{1}{2}\left\{4 < (I - \mathcal{P}^2(A_B))u_0, u_0 > + < u_1, u_1 >\right\}$$

Generally the energy metric could be non positive, since the rest of the spectrum $\sigma(A_B)\backslash\sigma(A)$ could be nontrivial. But we confine our considerations by the part of the perturbed operator, which is generated by incoming and outgoing subspaces spanned by scattering waves in the spectral subspace $\mathcal{E}_+ = P_+\mathcal{E}$. The energy metric (20), associated with our problem, is automatically non-negative in $\mathcal{E}_+$. It can be easily checked, that the one-step generator $\mathcal{L}$ for the time-evolution, described by the wave-equation (19) is given by the formula

$$\mathcal{L} = \frac{1}{2}\begin{pmatrix} 2\mathcal{P}(A_B) & I \\ 2[\mathcal{P}^2(A_B) - 1] & 2\mathcal{P}(A_B) \end{pmatrix}$$

It is unitary in energy metric (20).

We consider only the part of $\mathcal{L}$, generated by corresponding scattered waves.

**Theorem 11.** *([2]0)The distribution Cauchy data*

$$\Phi_- = \begin{pmatrix} \frac{1}{\bar\theta - \theta} & \Phi \\ \Phi & \end{pmatrix},$$

*with $\Phi$ given by formula (16), are the eigenfunctions of absolutely continuous spectrum of $\mathcal{L}$*

$$\mathcal{L}\Phi_- = \bar\theta\Phi_-$$

*and the corresponding spectral representation of the part of $\mathcal{L}$ is given by the formula*

$$\mathcal{L}_- = \oint \theta\Phi_- <, \Phi_- >_\xi dm.$$

The similar spectral representation holds for the other part of $\mathcal{L}$, generated by another branch of distributions of Cauchy data

$$\Phi_+ = \begin{pmatrix} \frac{1}{\bar\theta - \theta} & \bar\Phi \\ \bar\Phi & \end{pmatrix},$$

which are eigenfunctions of $\mathcal{L}$ also,

$$\mathcal{L}\Phi_+ = \bar\theta\Phi_+.$$

The corresponding spectral representation of the part of $\mathcal{L}$ is given by

$$\mathcal{L}_+ = \oint \theta\Phi_+ <, \Phi_+ >_\xi d\mu$$

The minimal part of $\mathcal{L}$, containing both $\mathcal{L}_\pm$ can be written in symmetrical spectral representation

$$\mathcal{J}_s ym : u \to \begin{pmatrix} < u, \chi_+ >_\mathcal{E} \\ < u, \chi_- >_\mathcal{E} \end{pmatrix} \equiv \tilde u$$

in the form of multiplication by $\theta$ in corresponding weighted space of square integrable functions

$$L_2 \begin{pmatrix} I & \bar R \\ R & 1 \end{pmatrix}$$

with the norm

$$|\tilde{u}| = \left\{ \oint < \begin{pmatrix} 1 & \bar{R} \\ R & I \end{pmatrix} \tilde{u}, \tilde{u} > dm \right\}^{1/2}$$

Notice, that the suggested spectral representation is an analog of symmetrical spectral representation for the Lax-Phillips generators with a nontrivial absolutely continuous spectrum, see [2]0. It is used for constructing a symmetrical variant of Nagy- Foias functional model. In our case it opens a way for construction a functional model of a family of commuting contractions with an absolutely continuous spectrum, which could be interpreted as an absolutely continuous spectrum of resonances. We omit this attractive possibility now, and discuss the special case of a unitary scattering matrix, $|R| \quad |_\Gamma = 1$.

**Theorem 12.** *If the spectrum of the operator $B$ is discrete , then the reflection coefficient and corresponding scattering matrix are unitary. The scattering matrix is an analytic $(-2\mu)$-automorphic function on $\Lambda_+$, and the incoming and outgoing spaces of data in $\mathcal{E}$, supported by the positive halt lattice $Z_+$ are spectrally represented by $H^2_{-,(-\mu)}$, $S_{-2\mu}H^2_{+,(\mu)} \subset H^2_{+,-\mu}$.*

*Proof.* We calculate here only the spectral representation of $\mathcal{D}_-^{right}$

$$\mathcal{D}_-^{right} = \begin{pmatrix} u_0 \\ (T^+ - T)u_0 \end{pmatrix}, \sup u_0 \subset Z_+.$$

The calculation of the energy scalar product with distribution $\Phi_-$ gives

$$< u, \Phi_- >_{\mathcal{E}} = \frac{1}{2} \{ < (T - T^+)u_0, (T - T^+)(\bar{\theta} - \theta)^{-1}(\chi_+ + R\chi_-) >_{L_2(E_n)} + < (T^+ - T)u_0, \chi_+ + R\chi_- >_{L_2(E_n)} \}.$$

Using the equations $T\chi_+ = \bar{\theta}\chi_+, T\chi_- = \theta\chi_-$, we get

$$\begin{aligned} < u, \Phi_- >_\xi &= \frac{1}{2} \{ < u_0, (\bar{\theta} - \theta)^{-1}(\chi_+ + R\chi_-) >_{l_\alpha(E_n)} \\ &+ < u_0, (\bar{\theta} - \theta)\chi_+ + R(\theta - \bar{\theta})\chi_- >_{l_\alpha(E_n)} \\ &= (\bar{\theta} - \theta) < u_0, \chi_+ >_{L_\alpha(E_n)} \in (1 - \theta^2)H^2_{-,-\mu} \}. \end{aligned}$$

Using the choice of $u_0$, we see that lineal $< u, \Phi_- >_{\mathcal{E}}$ coveres $(1 - \theta^2)H^2_{-,-\mu}$ and the closure of it coincides with $H^2-,-\mu$, since $1 - \theta^2$is an outer function.Analysis concerning $D_+$ can be acomplished in a similar way. Following the Lax-Phillips idea we can compress the unitary evolution group $(\mathcal{L})^n$ onto the translation invariant subspace

$$K' = H^2_{+(-\mu)} \ominus \mathcal{S}_{-2\mu}H^2_{+,(+\mu)},$$

or

$$K = L_2 \ominus [S_{-2\mu}H^2_{+,(+\mu)} \oplus H^2_{-,(-\mu)}];$$

in the second case the translation invariant subspace is enlarged due to adding the defect $\mathcal{M}_0 = L_2 \ominus \{H^2_{-,(-\mu)} \oplus H^2_{+,(-\mu)}\}$ see Appendix. In both cases the restriction gives contracting oneparametrical semigroups $\{T'^n\}$ $\{T^n\}$

$$T = P_K\mathcal{L}|K, T' = P_{K'}\mathcal{L}|K'.$$

The spectra of generators $T, T'$ correspond to the singularities of the inverse scattering matrix $S = \bar{R}$ for $T'$ and, possibly contain some additional isolated eigenvalues, caused by the defect $\mathcal{M}_0$. Now we confine ourselves to the simplest case $T'$.

**Theorem 13.** *The compressed resolvent of $\mathcal{L}$*

$$P_{K'}(\mathcal{L} - \zeta I)^{-1}|K'$$

*coincides for $|\zeta| > |$, $|\zeta| < |$ with resolvents of $T', \{(T')^+\}^{-1}$ and hence possesses an analytical continuation from the outside of the unit disc onto the inside of it with poles at eigenvalues $\{t_e\}$ of $T$ and from the inside of the unit disc to the outside of it with the poles at $\{\bar{t}_e\}^{-1}$. The corresponding systems of eigenvectors form a Riesz-basis in linear hulls if and only if the Carleson condition for $\{t_e\}$ is fulfilled (see appendix).*

*The system of eigenvectors of $T', T^{'+}$ are jointly complete in $K'$*

$$\mathcal{M}'_d \dot{+} \mathcal{M}'_{d+} = K'$$

*with a positive angle between components, if and only if the factors of the reflection coefficient fulfil the Muckenhaupt condition,*

$$\frac{1}{\mu(\Delta)} \oint_\Delta \left|\langle_\mu(\lambda, \infty)^{-1}\mathcal{D}(\lambda)\right|^2 dm \cdot \frac{1}{\mu(\Delta)} \oint_\Delta \left|\langle_{\mu(\lambda,\infty)}^{-1}\mathcal{D}(\lambda)\right|^{-2} \cdot dm < \infty.$$

The proof is based on the $\mu$-automorphic version of the Muckenhaupt condition, which shall be published later on.

Let us consider the completeness and the expansion by resonances for the perturbed selfadjoint Jacobian matrix, which is represented spectrally by a multiplication operator in the space of Cauchy data:

$$A_B \sim \tilde{A}_B = \int z\Phi_- <, \Phi_- > dm$$

The resolvent of $\tilde{A}_B$ is obviously an analytic function on the first sheet $\omega_+$ of the Riemann surface and is represented by the Cauchy kernel

$$\frac{1}{z - \lambda}$$

Considering compressed resolvent

$$P_{K'}(z - \lambda)^{-1}|K'$$

we should prove, that it possesses an analytic continuation onto a nonphysical sheet $\omega_-$ with singularities defined by the scattering matrix.

According to the Stout-Fedorov result quoted in Appendix, the algebra of all bounded analytic functions on $\omega_+$ is generated by three multiplication operators $\theta*$, $\theta_1*, \theta - 2*$ by Blaschke products, which represent unitary one step evolution operators in the energy space

$$\theta* \sim \mathcal{L}_0$$

$$\theta * \sim \mathcal{L}_1$$

$$\theta * \sim \mathcal{L}_2$$

Let us denote by $T, T_1, T_2$ the compressions of them onto translation-invariant subspace $K'$

$$P_{K'}\mathcal{L}_0|K' = T, P_{K'}\mathcal{L}_1|K' = T_1 P_{K'}\mathcal{L}_2|K' = T_2$$

¿From the quoted Stout-Fedorov result follows the next Assertion.

. . *For every bounded analytic function* $\Phi$ *on* $\omega_+$, *represented through* $\theta, \theta_1\theta_2$

$$\Phi(z) = \Phi = \Phi(\theta, \theta_1\theta_2)$$

*the compressed multiplication operator is represented as an analytic function of* $T, T_1T_2$:

$$P_K\Phi(\theta, \theta_1, \theta_2)|K = \Phi(T, T_1T_2)$$

The similar result is valid for the class of bounded analytic functions on $\omega_-$.

**Theorem 14.** *The compressed resolvent of* $\tilde{A}_B$ *possesses an analytical continuation onto a nonphysical sheet as a linear combination of resolvents* $T, T^+$ *with coefficients, including* $T_1, T_1^+, T_2, T_2^+$ *correspondingly, with the poles at the roots* $\lambda_s$ *of the scattering matrix and conjugate points* $\{\bar{\lambda}_s\}$.

*Proof.* Using the Hamilton-Caley equation

$$\theta + \theta^{-1} = 2\mathcal{P}(z),$$

we can write the spectrally-represented resolvent of $\tilde{A}_B$ in the following form

$$\frac{1}{z-\lambda} = \frac{\mathcal{P}_{n-1}(z,\lambda)}{\theta + \theta^{-1} - [\theta(\lambda) + \theta^{-1}(\lambda)]}$$
$$= \mathcal{P}_{n-1}(z,\lambda)\left[\frac{1}{\theta - \theta(\lambda)} + \frac{\theta(\lambda)}{1 - \theta\cdot\theta(\lambda)}\right]$$
$$\frac{\theta\cdot\theta(\lambda)}{\theta^2(\lambda) - 1}.$$

Here $\mathcal{P}_{n-1} = 2(z-\lambda)^{-1}[\mathcal{P}(z) - \mathcal{P}(\lambda)]$ is a polynomial of the order $n-1$, which is real and symmetric in respect to the change of variables $z \leftrightarrow \lambda$

$$\mathcal{P}_{n-1} = \sum_{s=0}^{n-1} a_s(z^s + b_s^1 z^{s-1}\lambda + \cdots + b_s^1 z^{s-1}\lambda^{s-1} + \lambda^s)$$

The products $\mathcal{P}_{n-1}\theta$, $\mathcal{P}_{n-1}\bar{\theta}$ are bounded analytic functions of $z$ on $\omega_+$, $\omega_-$ correspondingly , hence the following decomposition is true:

$$\frac{1}{z-\lambda} = \frac{\theta(\lambda)}{\theta^2(\lambda)-1}\left[\frac{\mathcal{P}_{n-1}\theta}{1-\theta\cdot\theta(\lambda)}\theta(\lambda) + \mathcal{P}_{n-1} + \frac{\mathcal{P}_{n-1}\bar{\theta}}{1-\bar{\theta}\cdot\theta(\lambda)}\theta(\lambda)\right],$$

where the first term in the square bracket is an analytical function on $\omega_+$, the third term is an analytical function on $\omega_-$, and the middle one is a symmetric polynomial in variables $\lambda, z$. Using the previous assertion, we get the following representation for the compressed resolvent

$$P'_K \frac{1}{z-\lambda}|'_K = \frac{\theta(\lambda)}{\theta^2(\lambda)-1}\left[\frac{\theta(\lambda)}{1-T\cdot\theta(\lambda)} P'_K(\mathcal{P}_{n-1}\theta)|K' \right.$$
$$\left. + \frac{\theta(\lambda)}{1-T^+\theta(\lambda)} P_{K'}(\mathcal{P}_{n-1}\bar{\theta})|K' + P_{K'}\mathcal{P}_{n-1}P_{K'}\right]$$

The last term in a bracket is just an operator polynomial in $\lambda$. According to Stout-Fedorov result, quoted above the family of generators $\theta, \theta_1$ can be enlarged by adding a new generator $\theta_2$ to distinguish all the points on the first sheet $\omega_+$.Every analytic function $\Phi_+$ on the first sheet can be represented as a superposition of some function, analytic in polidisc $|\theta| < 1, |\theta_1| < 1 |\theta_2| < 1$ with functions $\theta\theta_1\theta_2$

$$\Phi_+ = \Phi_+(\theta\theta_1\theta_2)$$

Then the operator-functions

$$P_{K'}\mathcal{P}_{n-1}\theta|K' \equiv P_{K'}\Phi_+|K'$$
$$P_{K'}\mathcal{P}_{n-1}\bar{\theta}|K' \equiv P_{K'}\Phi_|K'$$

are represented as polynomials in $\lambda$ with operator coefficients

$$P'_K\Phi_+|K' = \Phi_+(T, T_1, T_2, \lambda)$$
$$P'_K\Phi_-|K' = \Phi_-(T^+, T_1^+, T_2, \lambda),$$

hence

$$P_{K'}\frac{1}{z-\lambda}|K' = \frac{\theta(\lambda)}{\theta^2(\lambda)-1}\left[\frac{\theta(\lambda)}{1-T\theta(\lambda)}\Phi_+(T, T_1, T_2, \lambda)\right.$$
$$\left. + \frac{\theta(\lambda)}{1-T^+\theta(\lambda)}\Phi_-(T^+, T_1^+, T_2^+\lambda) + P_K\mathcal{P}_{n-1}P_{K'}\right]$$

The right part of the last equation is obviously an analytical function in $\lambda$ on $\omega_+$ since the factor $\theta(\lambda)$ suppresses possible polynomial poles of order $n-1$ at infinity; the analytical continuation onto $\omega_-$ in $\lambda$ meets poles at the eigenvalues of $T, T^+$

$$T\varphi_s = \theta_+(\lambda_s)\varphi_s = \overline{\theta_-^{-1}}(\overline{\lambda_s})\varphi_s$$
$$T^+\varphi_s^+ = \overline{\theta_+}(\lambda_s)\varphi_s^+ = \frac{1}{\theta_-(\overline{\lambda_s})}\varphi_s^+$$

localized at the roots $\lambda_s$ of the scattering matrix and conjugate points. Thus we see, that the poles of the first addenda lie at the roots $\lambda_s$ of the scattering matrix, the poles of the second ones lie in complex conjugate points. The corresponding residues of the analytically continued resolvent are composed of eigenfunctions (or root functions) of $T$, $T^+$ respectively. The theorem is proved.

Let us discuss at last the completeness of the resonance states, basing on the Theorem 14, and the index approach.

According to the general index approach (see the first part of the paper), we should consider a multiplication operator by the analytic function $f$, which posesses two factorizations on the universal cover $\Omega_+ \cup \Omega_-$

$$f = \begin{cases} \Pi f^e_+ & Z \in \Omega_+, \\ \theta f^e_- & Z \in \Omega_-, \end{cases}$$

such, that in $L_2(\partial\Omega_+)$

$$fH^2_- = \theta H^2_-$$
$$fH^2_+ = \Pi H^2_+$$

We use this approach for the classes of $\mu$-automorphic functions (on the universal cover $\Omega_+$ of the first sheet). The essential difference of the theory of $\mu$- automorphic Hardy classes $H^2_{+,\mu}$ on $\Omega_+$ from the standard one is that there exists an $n$-dimensional defect $\mathcal{M}_0$ ,$n = 1 + genus\{\omega_- \cup \omega_+\}$

$$L_{2,\mu}(\partial l_+) = H^{\circ 2}_{+,\mu} \oplus H^{\circ 2}_{+,\mu} \oplus \mathcal{M}_{0(\mu)}$$

Here $H^{\circ 2}_{+,\mu}$ is the Hardy class of $\mu$-automorphic functions, vanishing at infinity $\infty_+ \subset \omega_+$, $H^{\circ}_{-,\mu} = \overline{H}^{\circ}_{+,-\mu}$ and $\mathcal{M}_0$ - a subspace, which consists of $\mu$-automorphic functions, which have poles on $\omega_+$ and $\omega_-$ as well. For $(-\mu)$- automorphic function f the following factorization are assumed

$$f_{-\mu} = \begin{cases} \Pi_{-\mu'} f^e_{+,-\mu+\mu'}, & z \in \Omega_+, \\ \theta_{-\mu''} f^e_{-,\mu+\mu''}, & z \in \Omega_-, \end{cases}$$

They imply

$$f_{-\mu} H^2_{+,+\mu} = \Pi_{-\mu'} H^2_{+,+\mu'},$$
$$f_{-\mu}(H^2_{-,+\mu} \oplus \mathcal{M}_{0,+\mu}) = \theta_{-\mu''}(H^2_{-,+\mu''} \oplus \mathcal{M}_{0,+\mu''})$$

Similarly, the factorizations for $(\bar{f}_{-\mu})^{-1}(\bar{z})$ are assumed

$$(\bar{f}_{-\mu})^{-1} = \begin{cases} \theta_{-\mu''}(\bar{f}^e_{-,-\mu+\mu''})^{-1} & = \theta_{-\mu''} \hat{f}^e_{+,-\mu+\mu''}, \\ \Pi_{-\mu'}(\bar{f}^e_{+,-\mu+\mu''})^{-1} & = \Pi_{-\mu'} \hat{f}^e_{-,-\mu+\mu'}, \end{cases}$$

They imply the equations

$$(\bar{f}_{-\mu})^{-1} H^2_{\mu} = \theta_{-\mu''} H^2_{+,+\mu''},$$
$$(\bar{f}_{-\mu})^{-1}(H^2_{\mu})^1 = \Pi_{-\mu'}(H^2_{-,+\mu'} \oplus \mathcal{M}_{0,+\mu'}).$$

We use for $f$ the renormalized perturbation determinant of our scattering problem.

$$f_{-\mu} = \frac{\mathcal{D}}{\langle_\mu}$$

The corresponding Muckenhoupt condition guarantees the completeness of resonance states of $T, T^+$ in $K'$ (see Theorem 14)

$$\mathcal{M}_a \dot{+} \mathcal{M}_a^+ = (H^2_{+,\mu} \ominus \Pi_{-\mu'} H_{+,\mu+\mu'})\dot{+}$$
$$(\theta_{-\mu''}(H^2_{+,\mu''+\mu} \ominus \Pi_{-\mu'} H_{+,\mu'+\mu''+\mu})) = K$$

Combining the results of Theorems 13 and 14, we get the following statement.

**Theorem 15.** *If the Muckenhoupt condition is fulfilled for the renormalized perturbation determinant $\mathcal{D}\langle_\mu^{-1}$, then the compressed resolvent of perturbed Jacobian matrix can be continued analytically onto the nonphysical sheet with poles at the roots $\lambda_s$ of the scattering matrix and at the conjugate points $\bar{\lambda}_s$. The system of resonance state, composed of eigenvectors of compressed Lax-Phillips generators $T, T^+$, forms a complete system in $K$.*

Note that additional Carleson conditions for $\theta(\lambda_s)$

$$\inf_s \Pi \left| \frac{\theta(\lambda_s) - \theta(\lambda_t)}{1 - \theta(\lambda_s)\overline{\theta(\lambda_t)}} \right| > \delta$$

on every sequence $\lambda_s$, accumulating on a spectral bands, implies Riesz-basis property for resonances states. This fact follows from the harmonic analysis, developed in [2]0,22, combined with spectral analysis of the generator $T$, similarly to the corresponding fact in [1]7.

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## Appendix. Some preliminary facts and assertions from harmonic analysis in the circle and on a Riemann surface

The orthogonal basis in a space $L_2(C)$ of all square integrable functions on the unit circle $C = \{z : |z| = 1\}$ is formed by powers of independent variable $z = \exp(i\varphi)$, $0 < \varphi \leq 2\pi$, $\{z^\nu\}_{\nu=-\infty}^{\infty}$. The Hardy classes of all functions from

$L_2(C)$, which can be continued analytically inside of the unite circle $H^2_+$, and outside of it, $H^2_-$ can be defined as linear hulls

$$H^2_+ = V^{\infty}_{\nu=0}\{z^\nu\},$$
$$H^2_- = V^{-1}_{\nu=-\infty}\{z^\nu\},$$

the reproducing kernel $\langle(z)$ for $H^2_+$ is given by the formula

$$\langle_{(}z,\zeta) = (1-\bar{\zeta}z)^{-1}, \quad |\zeta| < |,$$

and possesses the property of Cauchy integrals

$$< f, \langle_{(}\zeta) >_{L_2} = f(\zeta), \quad f \in H^2_+.$$

The analytic function $S$ is called an inner function in the unit disc $\mathcal{D} = \{z : |z| < 1\}$, if it is almost everywhere unitary on the unit circle and compressing inside $\mathcal{D}$ ,

$$|S(z)| \quad |_C = 1, \quad |S(Z)| \quad |_{\mathcal{D}} \leq 1.$$

Generally the inner functions are represented as a combination of 'Blaschke products',

$$B(z) = \Pi \frac{\zeta_k - z}{1-\bar{\zeta}_k z}\frac{\bar{\zeta}_k}{|\zeta_k|}, \quad \sum_k \{1 - B_n|^2\} < \infty,$$

and 'singular inner functions', generated by some positive measures $d\nu$, supported by the unit circle and singular in respect to the Lebesque measure $dm$

$$\theta(z) = \exp\left\{-\int_c \frac{\zeta+z}{\zeta-z}d\nu(\varphi)\right\}, Var\nu < \infty.$$

Thus every inner function is characterized by the distribution of its zeros inside $\mathcal{D}$ and the density of asymptotic zeros at the boundary, given by $\nu$:

$$S(z) = z^{l_0}B(z)\theta(z).$$

An analytical function $f$ is called outer function in $\mathcal{D}$, if the corresponding multiplication operator is quasi-invertible in $H^2_+$. The outer functions are represented in the form

$$f(\zeta) = \exp\left\{\int_c \frac{\zeta+z}{\zeta-z} ln|f(z)|\frac{d\varphi}{2\pi}\right\},$$

In what follows we consider the analytic functions in $\mathcal{D}$, which anyway fulfil the condition

$$\int_c \ln|f(z)||dm < \infty$$

These functions can be factorised in the form of the product of inner and outer factors

$$f(z) = S(z)\cdot f^e_+(z) \quad z \in \mathcal{D}.$$

The similar representation holds for functions analytic outside of $\mathcal{D}$. If $f \in H^2_\pm$, then $f^e_\pm \in H^2_\pm$. The shift operator in $L_2(C)$

$$U : f(z) \to z f(z)$$

is unitary and possesses a lattice of invariant subspaces $\mathcal{D}^s_+$, $\mathcal{D}^s_+ \subseteq H^2_+$, parameterized by

$$\sigma(T) = \sigma(T_K) = \{\zeta_e\} \cup supp\nu,$$

the eigenfunctions of the compression onto the translation-invariant subspace

$$K, K = H^2 \ominus SH^2$$

$$T = P_K U|_K$$

are given by

$$\Psi_l(z) = \frac{S(z)}{\zeta_l - z}, T\Psi_l = \zeta_l \Psi_l$$

The biorthogonal system is formed by reproducing kernels, which serve eigenfunctions of $T^+$:

$$\Psi^+_l(z) = \|_\zeta(z), T^+\Psi^+_l = \bar{\zeta}_l \Psi^+_l.$$

For multiple zeros of $S$ the Jordan cell basis is formed by derivatives of $\Psi, \Psi^+$ in respect of the spectral parameter 3.

The system of eigenvectors $\{\Psi\}$ (as far as $\Psi^+_e$) is complete in $K$ if and only if the singular factor in $S$ is absent. In this case each of the systems $\{\Psi_e\}$, $\{\Psi^+_e\}$ forms a Riesz-basis in $K$ if and only if the following Carleson condition is true

$$\inf_n \prod_{l \neq n} \left| \frac{\zeta_l - \zeta_n}{1 - \bar{\zeta}_l \zeta_n} \right| > 0,$$

the corresponding spectral decomposition for $T$ is given by the interpolation series

$$f = \sum_l \Psi_l < f, \Psi^+_l > \cdot \Psi^{-1}_l(\zeta_l).$$

For properly normalized eigenfunctions systems the additional factor $\Psi_l(\zeta_l)$ coincides with Carleson's constant

$$C_n = \prod_{l \neq n} \frac{\zeta_l - \zeta_n}{1 - \bar{\zeta}_e \zeta_n}.$$

Let us consider 'outgiong' subspace generated by the inner function $S$

$$\mathcal{D}^s_+ = SH^2_+$$

The orthogonal projections on it is given by Cauchy integrals edged by multiplications on $S, S^+$ from both sides:

$$P_{\mathcal{D}^s_+} = SP_{H^2_+} S^+ = S < *, \langle_\zeta > S^+, \quad P_{\mathcal{D}^s_+} f(\zeta) = \frac{S(\zeta)}{2i\pi} \int_C \frac{f(z)\bar{S}(z)}{z - \zeta} dz$$

Every simple contracting operator $T$ which is rank one non unitary

$$\dim(1 - T^{+}T) = 1, T^{n} \xrightarrow{s} 0,$$

is unitarily equivalent to the corresponding Nagy-Foias functional model, generated by some inner function $S$ on the 'translation-invariant' subspace $K = H^2_+ \ominus SH^2_+$:

$$T \sim P_K z|K \equiv T_K, T^n \sim P_K z^n|K, n \geq 0$$

The shifts group $\{z^n\}$ is a 'unitary dilation' of the model contracting semigroup $P_K z^n|K, n \geq 0$. The spectrum of the generator $T$ (or $T_K$) consists of all singularities of $S^{-1}$. In a case, when the condition

$$T^n \xrightarrow{s} 0$$

is not fulfilled, the absolutely continuous spectrum of $T$ is present. The corresponding spectral analysis see [8]. All mentioned facts are contained in [1]5 Not so much is known about harmonic analysis of operators on Riemann surfaces. Let $\Lambda_+$ be a double surface genus $r$, $r \geq 1$. We realize it as a joining of sheets $\Lambda_+ \cup \Lambda_-$ by $r+1$ two-sided cuts $\Gamma = \cup\gamma_s$ The Green function of the first sheet $\Lambda_\Gamma \backslash \Gamma$ with the zero boundary conditions on $\Gamma$ (Friedrichs extension of corresponding Laplacian) is a real part of an analytic function $G$

$$g(z, \zeta) = Re.G(z, \zeta),$$

defined on the universal cover $\Omega_+$ of $\Lambda_+$. The corresponding Blaschke factor defined on $\Omega_+$ as

$$b(z, \zeta) = \exp G(z, \zeta)$$

proves to be $\mu$-automorphic function on $\Omega_+$ in respect to the sheets overlapping group. It means, that being reduced to $\Lambda$ ,$b(z, \zeta)$ achieves a factor $\exp 2i\pi\mu s$ after $z$ going around the cut $\gamma_s$ $s = 1, 2, \ldots r$,

$$b(z, \zeta) = b_{\vec{\mu}}(z, \zeta)$$

with only $\mu_1, \ldots \mu_r$ independent, $\mu_{r+1} = -\sum_1^r \mu_s| \mod Z$. Here $\mu_s = \mu_s(\zeta)$ is a harmonic function of the parameter $\zeta$ on $\Lambda_+ \backslash \Gamma$, which fulfills the following boundary conditions

$$\mu_s(\zeta) = \begin{cases} 1, & \zeta \in \gamma_s \\ 0, & \zeta \in \gamma_i, \quad i \neq s \end{cases}$$

the corresponding harmonic measure is

$$\frac{1}{2i\pi}\frac{db}{b} = d\mu(\zeta) = -\frac{1}{2\pi}\frac{\partial g}{\partial n} d\gamma,$$

and

$$\mu(\gamma_s, \zeta) = \mu_s(\zeta)$$

Generally $\mu$-automorphic functions play a crucial role in harmonic analysis on the Riemann surface. For given vector $\mu$ the Hardy space of the all square integrable functions can be formed on $\Omega_+$ (and on $\Lambda_+$), $H^2_{+,\mu}(\Lambda_+)$. The corresponding reproducing kernel $\langle_\mu(z,\zeta)$ is given by averaging over the action of the overlapping group $G$ of the sheets on $\Gamma_+$:

$$\|_\mu(z,\zeta) = \sum_{\{G_s\}} e^{2\pi i<\mu,s>}\langle(z, G_s\zeta)$$

The $\mu$-automorphic inner and outer functions are defined on $\Omega_+$ (and $\Lambda_+$) as functions which are transformed under the overlapping group's action according to the rule

$$f(G_s Z) = e^{2i\pi<\mu,s>} f(z).$$

The invariant subspaces of the multiplication operator by the bounded analytic function $\cdot\theta$. (0-automorphic on $\Lambda_+$) are represented in the form

$$\mathcal{D}_+ = S_{-\kappa} H^2_{+,\mu}(\Lambda_+)$$

where $H_{+,\mu}+^2(\Lambda_+)$ is a space of $(\mu)$-automorphic functions on $\Gamma$, which are square-integrable in respect to the fixed harmonic measure $\mu = \mu(\zeta_0)$, which are 0-analytic on $\Omega_+$ and transformed properly under the overlapping group's actions, and $S_\mu$ is an inner $(-\mu)$-automorphic function on $\Omega_+(\Lambda_+)$.

The main difference of harmonic analysis on the Riemann surface from the harmonic analysis in a unit disc is the non triviality of the overlapping group. The operator of multiplication by bounded analytic (0-automorphic) function in $L_2(\Gamma)$ has a spectrum of multiplicity $\geq r+1$. That is why we need another function, to construct a 'complete system of quantum observables'- the family of multiplication operators which is a generating a double commutant, containing multiplication by $\theta_0$. In the simplest case of two symmetric cuts this family is formed by two functions, which are invariant in respect to some automorphisms of the first sheet,

$$\Gamma = [-1. -\alpha^2]U[\alpha^2, 1] \quad G_1 : z \to -z, G_2 : z \to -\alpha z,$$
$$\theta_0(G_1 z) = \theta_0(Z); \theta_1(G_2 z) = \theta_1(z)$$
$$\theta_1(G_1 z) = -\theta_1(Z) : \theta_0(G_2 z) = \theta_0 \frac{\theta_0 - \theta_0(0)}{1 - \theta_0\overline{\overline{\theta_0(0)}}}$$

the sheet's overlapping group is oneparametrical group represented by mobius transformations of universal cover $\Omega_+ = \mathcal{D}$. Then, defining $H^2_\pm$ as linear hulls of basis

$$\{\theta_0^l, \theta_1\theta_0^l\}_{l\in Z},$$
$$H^2_+(\Lambda) = V_{l\geq 0}\{\theta_0, \theta_1\theta_0^l\},$$
$$H^2_-(\Lambda) = V_{l>0}\{\theta_0^{-l}, \theta_1^{-1}, \theta_1^{-1}\theta^{-l}\},$$
$$L_2(\Gamma) = H^2_-(\Lambda_-) \oplus H^2_+(\Lambda_+) \oplus \{1/z\},$$

here the element $1/z = \frac{\theta_1}{\theta_0 - \theta_0(0)} \cdot const$ is a defect caused by nontrivial (oneparametrical) overlapping group. It is a derivative of the corresponding abelian differential of the second kind in respect to the harmonic measure . Generally for the double Riemann surface genus $r$, there exist $r$-parametrical overlapping groups acting on $\Omega_+$, and the defect of $H^2_+ \oplus H^2_-$ in $L_2(\Gamma)$ is $r$-dimensional and the base in it is calculated as derivatives of abelian differentials of the second kind in respect to the harmonic measure. In paper of E Stout [2]3 was shown, that generally on Riemann surface there exist three functions ,which are inner functions on the first sheet and separating all points of the first sheet. In the papers of S. Fedorov [2]2 an explicite construction for them was suggested. There was shown also, that any analytic function on the first sheet can be represented as a superposition of an analytic function in polydisc and the corresponding generators $\theta_0, \theta_1, \theta_2$ . We use these results as a basement for representation of Livshiĉ matrix in form of linear combination of resolvents of generators of Lax- Phillips semigroups.

For two cuts the analog of shift groups is given by $\{\theta_0^{m_0}\theta_1^{m_1}\} m \in \mathcal{Z}_2$. This discrete group possesses a nontrivial property

$$\theta_1^2 = \theta_0 \frac{\theta_0 - \theta_0(\theta)}{1 - \theta_0(0)\theta(a)} \equiv \theta_0 \cdot \theta_0(G_\alpha),$$

caused by specific behaviour of $\theta_{0,1}$ under transformations $G_{1,2}$. But being reduced to some translation-invariant subspace

$$K = H^2_+(\Lambda_+) \ominus S_{-\mu} H^{2,\mu}(\Lambda_+)$$

it produces contracting semigroups, similar to the Lax-Phillips ones

$$T_0^{l_0} T_1^{l_1} = P_K \theta_0^{l_0} \theta_1^{l_1} | K.$$

If the inner function $S_\mu$ is a Blaschke product, then the spectra of $T_0, T_1$ are discrete and are given by the formulae

$$t^s_{0,1} = \theta_{0,1}(\lambda_s),$$

where $\Lambda_s$ are the roots of $S_{-\mu}$. The corresponding adjoint operators $T^+_{0,1}$ have the complex conjugate eigenvalues and reproducing kernels $\mathcal{R}_0(Z, \Lambda_s)$ for eigenvectors. The basis property for them is guaranteed by 'splitted' Carleson conditions in terms of $\theta_0$

$$\prod_{l \neq m} \left| \frac{\theta_0(\lambda_l) - \theta_0(\lambda_m)}{1 - \overline{\theta_0(\lambda_l)}\theta_0(\lambda_m)} T_0^{l_0} \right| \geq 0,$$

for $\lambda_m \to \gamma_-$, and for $\lambda_m \to \gamma_+$ separately. The analysis of general hyperelliptic Riemann surface was done in [2]2 . For other results concerning Harmonic analysis on Riemann surfaces see papers [2]2,26,27 ,28, the book of S. Fisher [2]5 and the literature, quoted there . Another facts concerning Harmonic analysis of operators and applications to differential operators can be found in [2]9 30 .

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