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ON THE HILBERT TRANSFORM AND $C^{1+\epsilon}$ FAMILIES OF LINES

MICHAEL LACEY AND XIAOCHUN LI

Abstract. We study the operator

$$H_v f(x) := \text{p.v.} \int_{-1}^{1} f(x - yv(x)) \, \frac{dy}{y}$$

defined for smooth functions on the plane and measurable vector fields v from the plane into the unit circle. We prove that if v has $1+\epsilon$ derivatives, then H_v extends to a bounded map from $L^2(\mathbb{R}^2)$ into itself. What is noteworthy is that this result holds in the absence of some additional geometric condition imposed upon v, and that the smoothness condition is nearly optimal. Whereas H_v is a Radon transform, for which there is an extensive theory, see e.g. [5], our methods of proof are necessarily those associated to Carleson's theorem on Fourier series [3], and the proof given by Lacey and Thiele [10]. A previous paper of the authors [9], has shown how to adapt these ideas to H_v; herein these ideas are combined with a crucial maximal function estimate that is particular to the smooth vector field in question.

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1. Introduction

We are interested in singular integral operators on functions of two variables, which act by performing a one dimensional transform along a particular line in the plane. The choice of lines is to be variable. Thus, for a measurable map, v from $\mathbb{R}^2$ to the unit circle in the plane, that is a vector field, and a Schwartz function f on $\mathbb{R}^2$, define

$$H_v f(x) := \text{p.v.} \int_{-1}^{1} f(x - yv(x))\, \frac{dy}{y}.$$

This is a truncated Hilbert transform performed on the line segment $\{x+tv(x) : |t| < 1\}$. We prove norm inequalities for H_v, requiring only that v has $1+\epsilon$ derivatives.

1.1. **Theorem.** *Let v be C^α map for $\alpha > 1$. Then H_v maps $L^2(\mathbb{R}^2)$ into itself. The norm of the transform is at most*

$$\|H_v\|_{2\to 2} \lesssim (1 + \log(1 + \|v\|_{C^\alpha}))^{3/2}.$$

The essential step towards proving this Theorem is the next Proposition, in which we restrict the frequency support of the functions acted upon. Let $\boldsymbol{\lambda}_t f = \lambda_t * f$ where λ is a Schwartz function with $\widehat{\lambda}$ supported in $1/2 \le |\xi| \le 3/2$, and $\lambda_t(y) = t^{-2}\lambda(y/t)$.

1.2. **Proposition.** *If v is Lipschitz, then we have the estimate*

$$\|H_v \boldsymbol{\lambda}_t\|_2 \lesssim 1 + \log(1 + t\|v\|_{\mathrm{Lip}}). \tag{1.3}$$

Constructions of the Besicovitch set show that the Theorem is false under the assumption that v is Hölder continuous for any index strictly less than 1. These constructions, known since the 1920's, were the inspiration for A. Zygmund to ask if integrals of, say, $L^2(\mathbb{R}^2)$ functions could be differentiated in a Lipschitz choice of directions. That is, for Lipschitz v, and $f \in L^2$, is it the case that

$$\lim_{\epsilon\to 0} (2\epsilon)^{-1} \int_{-\epsilon}^{\epsilon} f(x - yv(x))\, dy = f(x) \qquad \text{a.e.}(x)$$

Our Theorem gives a partial answer to the singular integral version of this question, as posed by E. M. Stein [15]. The methods of this paper are not by themselves strong enough to answer the differentiation question.

Prior results have a subtle relationship with these results. The form of our Theorem pertain to, in the standard parlance, singular Radon transforms. Such results have been under investigation for roughly forty years, with a subtle exposition of that theory being the work of Christ, Nagel, Stein and Wainger [5]. The focus of that theory concerns results with singular integrals over hypersurfaces of arbitrary co–dimension, which vary in a smooth manner, and satisfy some minimal geometric conditions. In contrast, a primary interest of the current result is that the theorem is phrased in complete absence of geometric conditions. Our theorem is of co–dimension one, and might rely in some critical way upon such a formulation. And finally, we work in the arena of only $1+\epsilon$ derivatives.

The results of Christ, Nagel, Stein and Wainger [5] apply to certain vector fields v. Earlier, a positive result for analytic vector fields followed from Nagel, Stein and Wainger [12]. E.M. Stein [15] specifically raised the question of the boundedness of H_v for smooth vector fields v. And the results of D. Phong and Stein [13,14] also give results about H_v. J. Bourgain [1] considered real–analytic vector field. N. H. Katz [8] has made an interesting contribution to maximal function question. Also see the partial results of Carbery, Seeger, Wainger and Wright [2].

An example pointed out to us by M. Christ [4] shows that under the assumption that the vector field is measurable, the sharp conclusion is that $H_v\boldsymbol{\lambda}_1$ maps L^2 into $L^{2,\infty}$. And a variant of the approach to Carleson's theorem by Lacey and Thiele [10] will prove this norm inequality. This method will also show that under only the measurability assumption, that $H_v\boldsymbol{\lambda}_1$ maps L^p into itself for $p>2$, as is shown by the current authors [9]. The results and techniques of that paper are critical to this one. Note that the Proposition is an essential strengthening of what is known in the measurable vector field case. We do not know if the norm estimates above continues to hold for $1<p<2$.

It is known that the Theorem and Proposition above have as a corollary Carleson's theorem [3] on the pointwise convergence of Fourier series. Set $\sigma(\xi)=\int_{-1}^{1} e^{i\xi y}\,dy/y$. For a C^2 function $N:\mathbb{R}\to\mathbb{R}$, we should deduce that the operator with symbol $\sigma(\xi-N(x))$ maps $L^2(\mathbb{R})$ into itself, with norm that is independent of N, the C^2 norm of N. Take the vector field to be $v(x_1,x_2)=(1,-N(x_1)/n)$. Then, H_v is bounded on $L^2(\mathbb{R}^2)$, with norm bounded by an absolute constant. The symbol of H_v is $\sigma((\xi_1,\xi_2)\cdot v(x_1))=\sigma(\xi_1-\xi_2 N(x_1)/n)$. The trace of this symbol along the line $\xi_2=N$ defines a symbol of a bounded operator on $L^2(\mathbb{R})$, which is the fact we needed to prove.

A novel point of this paper is a particular maximal function result, detailed in Section 5. This is the key point at which the Lipschitz character of the vector field is exploited. We use this inequality to carry out an interpolation argument to prove Proposition 1.2, and find that this argument must be carried out with some care. The reason for this is that the logarithmic bound in the Lipschitz norm we prove in this Proposition fails utterly below L^2.

Our Theorem requires additional smoothness of the vector field beyond Lipschitz. This additional smoothness can be used to show that the spatial scales of the operator H_v decouple in a strong way. Namely, that $H_v\boldsymbol{\lambda}_{2^j}$ are essentially orthogonal operators for $j\in\mathbb{N}$, namely that for $j\neq j'$ we have $\boldsymbol{\lambda}_{2^{j'}}H_v\boldsymbol{\lambda}_{2^j}\simeq 0$. This is quantified by technical lemmas of Section 6, and is largely an L^2 estimate.

2. Definitions and Principle Lemma

We begin with some conventions. We do not keep track of the value of generic absolute constants, instead using the notation $A \lesssim B$ iff $A \le KB$ for some constant K. And $A \simeq B$ iff $A \lesssim B$ and $B \lesssim A$. We use the notation $\mathbf{1}_A$ to denote the indicator function of the set A. And the Fourier transform on $\mathbb{R}^2$ is denoted by $\widehat{f}(\xi) = \int_{\mathbb{R}^2} e^{-2\pi i x\cdot\xi} f(x)\, dx$, with a similar definition on the real line. We use the notation

$$\fint_A f\, dx := |A|^{-1} \int_A f\, dx.$$

For an operator T, $\|T\|_p$ denotes the norm of T as an operator from $L^p(\mathbb{R}^2)$ to itself.

Throughout this paper, κ will denote a fixed small positive constant, whose exact value need not concern us. κ of the order of 10^{-3} would suffice. The following definitions are as in the authors' previous paper [9].

2.1. **Definition.** *A* grid *is a collection of intervals $\mathcal{G}$ so that for all $I, J \in \mathcal{G}$, we have $I \cap J \in \{\emptyset, I, J\}$. The dyadic intervals are a grid.*

Let ρ be rotation on $\mathbb{T}$ by an angle of $\pi/2$. Coordinate axes for $\mathbb{R}^2$ are a pair of unit orthogonal vectors $(e, e_\perp)$ with $\rho e = e_\perp$.

2.2. **Definition.** *We say that $\omega \subset \mathbb{R}^2$ is a* rectangle *if it is a product of intervals with respect to a choice of axes $(e, e_\perp)$ of $\mathbb{R}^2$. We will say that ω is an* annular rectangle *if $\omega = (-2^{l-1}, 2^{l-1}) \times (a, 2a)$ for an integer l with $2^l < \kappa a$, with respect to the axes $(e, e_\perp)$. The dimensions of ω are said to be $2^l \times a$. Notice that the face $(-2^{l-1}, 2^{l-1}) \times a$ is tangent to the circle $|\xi| = a$ at the midpoint to the face, $(0, a)$. We say that the* scale of ω *is $\operatorname{scl}(\omega) := 2^l$ and that the* annular parameter of ω *is $\operatorname{ann}(\omega) := a$. In referring to the coordinate axes of an annular rectangle, we shall always mean $(e, e_\perp)$ as above.*

Annular rectangles will decompose our functions in the frequency variables. But our methods must be sensitive to spatial considerations; it is this and the uncertainty principle that motivate the next definition.

2.3. **Definition.** *Two rectangles R and R are said to be* dual *if they are rectangles with respect to the same basis $(e, e_\perp)$, thus $R = r_1 \times r_2$ and $\mathsf{R} = \mathsf{r}_1 \times \mathsf{r}_2$ for intervals r_i, r_i, $i = 1, 2$. Moreover, $1 \le |r_i| \cdot |\mathsf{r}_i| \le 4$ for $i = 1, 2$. The product of two dual rectangles we shall refer to as a* phase rectangle. *The first coordinate of a phase rectangle we think of as a frequency component and the second as a spatial component.*

We consider collections of phase rectangles $\mathcal{AT}$ which satisfy these conditions. For $s, s' \in \mathcal{AT}$ we write $s = \omega_s \times R_s$, and require that

$$\omega_s \text{ is an annular rectangle,} \tag{2.4}$$

$$R_s \text{ and } \omega_s \text{ are dual,} \tag{2.5}$$

$$\{R \,:\, R\times\omega_s \in \mathcal{AT}\} \quad \text{partitions } \mathbb{R}^2\text{, for all } \omega_s. \tag{2.6}$$

$$\operatorname{ann}(\omega_s) = 2^j \quad \text{for some integer } j, \tag{2.7}$$

$$\sharp\{\omega_s \,:\, \operatorname{scl}(s)=2^l,\ \operatorname{ann}(s)=2^j\} \ge c2^{j-l}, \qquad j,\, l\in\mathbb{Z}, \tag{2.8}$$

$$\operatorname{scl}(s) \le \kappa \operatorname{ann}(s). \tag{2.9}$$

We assume that there are auxiliary sets $\boldsymbol{\omega}_s, \boldsymbol{\omega}_{s1}, \boldsymbol{\omega}_{s2} \subset \mathbb{T}$ associated to s—or more specifically ω_s—which satisfy these properties.

$$\boldsymbol{\Omega} := \{\boldsymbol{\omega}_s, \boldsymbol{\omega}_{s1}, \boldsymbol{\omega}_{s2} \,:\, s\in\mathcal{AT}\} \text{ is a grid in } \mathbb{T}, \tag{2.10}$$

$$\boldsymbol{\omega}_{s1}\cap\boldsymbol{\omega}_{s2} = \emptyset, \qquad |\boldsymbol{\omega}_s| \ge 32(|\boldsymbol{\omega}_{s1}| + |\boldsymbol{\omega}_{s2}| + \operatorname{dist}(\boldsymbol{\omega}_{s1}, \boldsymbol{\omega}_{s2})) \tag{2.11}$$

$$\boldsymbol{\omega}_{s1} \text{ lies clockwise from } \boldsymbol{\omega}_{s2} \text{ on } \mathbb{T}, \tag{2.12}$$

$$|\boldsymbol{\omega}_s| \le K\frac{\operatorname{scl}(\omega_s)}{\operatorname{ann}(\omega_s)}, \tag{2.13}$$

$$\{\tfrac{\xi}{|\xi|} \,:\, \xi\in\omega_s\} \subset \rho\boldsymbol{\omega}_{s1}. \tag{2.14}$$

In the top line, the intervals $\boldsymbol{\omega}_{s1}$ and $\boldsymbol{\omega}_{s2}$ are small subintervals of the unit circle, and we can define their dilate by a factor of 2 in an obvious way. Recall that ρ is the rotation that takes e into $e_\perp$. Thus, $e_{\omega_s} \in \boldsymbol{\omega}_{s1}$.

Note that $|\boldsymbol{\omega}_s| \ge |\boldsymbol{\omega}_{s1}| \ge \operatorname{scl}(\omega_s)/\operatorname{ann}(\omega_s)$. Thus, e_{ω_s} is in $\boldsymbol{\omega}_{s1}$, and $\boldsymbol{\omega}_s$ serves as "the angle of uncertainty associated to R_s." Let us be more precise about the geometric information encoded into the angle of uncertainty. Let $R_s = r_s \times r_{s\perp}$ be as above. Choose another set of coordinate axes $(e', e'_\perp)$ with $e' \in \boldsymbol{\omega}_s$ and let R' be the product of the intervals r_s and $r_{s\perp}$ in the new coordinate axes. Then $K_0^{-1}R' \subset R_s \subset K_0 R'$ for an absolute constant $K_0 > 1$.

We say that *annular tiles* are collections $\mathcal{AT}$ satisfying the conditions (2.4)—(2.14) above. We extend the definition of e_ω, $e_{\omega\perp}$, $\operatorname{ann}(\omega)$ and $\operatorname{scl}(\omega)$ to annular tiles in the obvious way, using the notation e_s, $e_{s\perp}$, $\operatorname{ann}(s)$ and $\operatorname{scl}(s)$.

A phase rectangle will have two distinct functions associated to it. In order to define these functions, set

$$\begin{aligned}
\mathrm{T}_y\, f(x) &:= f(x-y), \quad y\in\mathbb{R}^2 \quad \text{(Translation operator)}\\
\mathrm{Mod}_\xi\, f(x) &:= e^{i\xi\cdot x} f(x), \quad \xi\in\mathbb{R}^2 \quad \text{(Modulation operator)}\\
\mathrm{Dil}^p_{R_1\times R_2}\, f(x_1,x_2) &:= \frac{1}{(|R_1||R_2|)^{1/p}} f\Big(\frac{x_1}{|R_1|}, \frac{x_2}{|R_2|}\Big),\\
& \qquad 0<p<\infty, \quad \text{(Dilation operator).}
\end{aligned}$$

In the last display, $R_1\times R_2$ is a rectangle, and the coordinates (x_1, x_2) are those of the rectangle. Note that the definition depends only on the side lengths of the rectangle, and not the location. And that it preserves L^p norm.

For a function φ and tile $s \in \mathcal{AT}$ set

$$\varphi_s := \operatorname{Mod}_{c(\omega_s)} \operatorname{Dil}^2_{R_s} \operatorname{T}_{c(R_s)} \varphi \tag{2.15}$$

We shall consider φ to be a Schwartz function for which $\widehat{\varphi} \geq 0$ is supported in a small ball, of radius κ, about the origin in $\mathbb{R}^2$, and is identically 1 on another smaller ball around the origin. (Recall that κ is a fixed small constant.)

We introduce the tool to decompose the singular integral kernels. Fix a Schwartz function ψ on $\mathbb{R}$ with frequency support in a small neighborhood of 1. More specifically, we take $\widehat{\psi} \geq 0$, and supported on $[1, 1+\kappa]$. Then, define

$$\phi_s(x) := \int_{\mathbb{R}} \varphi_s(x - yv(x)) \operatorname{scl}(s)\psi(\operatorname{scl}(s)y)\, dy. \tag{2.16}$$

An essential feature of this definition is that the support of ϕ_s is contained in the set $\{v(x) \in \boldsymbol{\omega}_{s2}\}$, a fact which is verify by restricting appropriately the Fourier support of φ and ψ. That is we have $\phi_s(x) = \mathbf{1}_{\boldsymbol{\omega}_{s2}}(v(x))\phi_s(x)$. The set $\boldsymbol{\omega}_{s2}$ serves to localize the vector field, while $\boldsymbol{\omega}_{s1}$ serves to identify the location of φ_s in the frequency coordinate.

The model operators we consider act on a Schwartz functions f, and sends it into a sequences of functions. It is defined by

$$\mathcal{C}_{\mathsf{ann}} f := \sum_{\substack{s \in \mathcal{AT}(\mathsf{ann}) \\ \operatorname{scl}(s) \geq 1}} \langle f, \varphi_s \rangle \phi_s. \tag{2.17}$$

In this display, $\mathcal{AT}(\mathsf{ann}) := \{s \in \mathcal{AT} \,:\, \operatorname{ann}(s) = \mathsf{ann}\}$. As much of our analysis concentrated on a single annulus, this is a very commonly used notation.

2.18. **Lemma.** *Assume that the vector field is Lipschitz. The operator $\mathcal{C}_{\mathsf{ann}}$ extends to a bounded map from L^2 into itself. The norm $\mathcal{C}_{\mathsf{ann}}$ depend upon the vector field v in the following way.*

$$\|\mathcal{C}_{\mathsf{ann}}\|_2 \lesssim 1 + \log(1 + \mathsf{ann}^{-1}\|v\|_{\mathrm{Lip}}).$$

We remind the readers that for $2 < p < \infty$ the only condition needed for the boundedness of $\mathcal{C}_{\mathsf{ann}}$ is the measurability of the vector field, a principal result of Lacey and Li [9]. It is of course of great importance to add up the $\mathcal{C}_{\mathsf{ann}}$ over ann. The methods for doing this are purely L^2 in nature, and lead to the estimate for $\mathcal{C} := \sum_{j=1}^{\infty} \mathcal{C}_{2^j}$.

2.19. **Lemma.** *Assume that the vector field is C^α for some $\alpha > 1$. Then $\mathcal{C}$ maps L^2 into itself. And the norm depends upon v in the following way.*

$$\|\mathcal{C}\|_2 \lesssim (1 + \log(1 + \|v\|_{C^\alpha}))^2.$$

Moreover, the sum is unconditionally convergent in $s \in \mathcal{AT}$.

These are the principal steps towards the proof of Proposition 1.2 and Theorem 1.1. In the course of the proof, we shall not invoke the additional notation needed to account for the unconditional convergence, as it entirely notational. They can be added in by the reader.

The proof of Theorem 1.1 and Proposition 1.2 from these two lemmas is an argument in which one averages over translations, dilations and rotations of grids. The specifics of the approach are very close to the corresponding argument in [9]. The details are omitted.

3. Truncation and an Alternate Model Sum

There are significant obstacles to proving the boundedness the model sum of Proposition 1.2 on an L^p space, for $1 < p < 2$. In this section, we rely upon some naive L^2 estimates to define a new model sum which is bounded on L^p, for some $1 < p < 2$.

Our next Lemma is indicative of the estimates we need. For choices of $\mathsf{scl} < \kappa\mathsf{ann}$, set

$$\mathcal{AT}(\mathsf{ann},\mathsf{scl}) := \{s \in \mathcal{AT}(\mathsf{ann}) \,:\, \mathrm{scl}(s) = \mathsf{ann}\}.$$

3.1. **Lemma.** *For measurable vector fields v and all choices of* ann *and* scl*.*

$$\Big\| \sum_{s\in\mathcal{AT}(\mathsf{ann},\mathsf{scl})} \langle f,\varphi_s\rangle\phi_s \Big\|_2 \lesssim \|f\|_2$$

Proof. The scale and annulus are fixed in this sum, making the Bessel inequality

$$\sum_{s\in\mathcal{AT}(\mathsf{ann},\mathsf{scl})} |\langle f,\varphi_s\rangle|^2 \lesssim \|f\|_2^2$$

evident. For any two tiles s and s' that contribute to this sum, if $\boldsymbol{\omega}_s \neq \boldsymbol{\omega}_{s'}$, then ϕ_s and $\phi_{s'}$ are disjointly supported. And if $\boldsymbol{\omega}_s = \boldsymbol{\omega}_{s'}$, then R_s and R'_s are disjoint, but share the same dimensions and orientation in the plane. The rapid decay of the functions ϕ_s then gives us the estimate

$$\begin{aligned}\Big\| \sum_{s\in\mathcal{AT}(\mathsf{ann},\mathsf{scl})} \langle f,\varphi_s\rangle\phi_s \Big\|_2 &\lesssim \Big[\sum_{s\in\mathcal{AT}(\mathsf{ann},\mathsf{scl})} |\langle f,\varphi_s\rangle|^2\Big]^{1/2} \\ &\lesssim \|f\|_2\end{aligned}$$

□

Consider the variant of the operator (2.17) given by

$$\Phi f = \sum_{\substack{s\in\mathcal{AT}(\mathsf{ann}) \\ \mathrm{scl}(s)\geq\kappa^{-1}\|v\|_{\mathrm{Lip}}}} \langle f,\varphi_s\rangle\phi_s. \tag{3.2}$$

As ann is fixed, we shall begin to suppress it in our notations for operators. The difference between Φ and $\mathcal{C}_{\mathsf{ann}}$ is the absence of the initial $\lesssim \log(1+\|v\|_{\mathrm{Lip}})$ scales in the former. The L^2 bound for these missing scales is clearly provided by Lemma 3.1, and so it remains for us to establish

$$\|\Phi\|_2 \lesssim 1, \tag{3.3}$$

the implied constant being independent of ann, and the Lipschitz norm of v.

It is an important fact, the main result of Lacey and Li [9], that

$$\|\Phi\|_p \lesssim 1, \qquad 2 < p < \infty. \tag{3.4}$$

This holds without the Lipschitz assumption.

We are now at a point where we can be more directly engaged with the construction of our alternate model sum. We only consider tiles with $\kappa^{-1}\|v\|_{\mathrm{Lip}} \leq \mathrm{scl}(s) \leq \kappa\mathsf{ann}$. Set

$$\gamma_s^2 := \frac{\mathrm{scl}(s)}{\|v\|_{\mathrm{Lip}}} \tag{3.5}$$

Write $\varphi_s = \alpha_s + \beta_s$ where $\alpha_s = (D^0_{\gamma_s R_s} T_{c(R_s)}\zeta)\varphi_s$, and ζ is a smooth Schwartz function supported on $|x| < 1/2$, and equal to 1 on $|x| < 1/4$.

Recall that the kernel function ψ is a Schwartz function on $\mathbb{R}$ with compact frequency support. Write for choices of tiles s,

$$\mathrm{scl}(s)\psi(\mathrm{scl}(s)y) = \psi_{s-}(y) + \psi_{s+}(y) \tag{3.6}$$

where $\psi_{s-}(y)$ is a Schwartz function on $\mathbb{R}$, supported on $|y| < \frac{1}{2}\gamma_s$ and equal to $\mathrm{scl}(s)\psi(\mathrm{scl}(s)y)$ for $|y| < \frac{1}{4}\gamma_s$. Then define

$$a_{s\pm}(x) = \mathbf{1}_{\boldsymbol{\omega}_{s2}}(v(x)) \int \alpha_s(x - yv(x))\psi_{s\pm}(y)\, dy.$$

Thus, $\phi_s = a_{s-} + a_{s+}$. Recalling the notation $\boldsymbol{\lambda}_{\mathsf{ann}}$ in Proposition 1.2, define

$$A_\pm f := \sum_{\substack{s \in \mathcal{AT}(\mathsf{ann}) \\ \mathrm{scl}(s) \geq \kappa^{-1}\|v\|_{\mathrm{Lip}}}} \langle \boldsymbol{\lambda}_{\mathsf{ann}} f, \alpha_s \rangle a_{s\pm} \tag{3.7}$$

We will write $\Phi = \Phi\boldsymbol{\lambda}_{\mathsf{ann}} = A_+ + A_- + B$, where B is an operator defined in (3.10) below. The main fact we need concerns A_-.

3.8. **Lemma.** *There is a choice of $1 < p_0 < 2$ so that*

$$\|A_-\|_p \lesssim 1, \qquad p_0 < p < \infty.$$

The implied constant is independent of the value of ann, and the Lipschitz norm of v.

The proof of this Lemma is given in the next section, modulo three additional Lemmata stated there in. The following Lemma is important for our approach to the previous Lemma. It is proved in Section 4.

3.9. **Lemma.** *For each choice of $\kappa^{-1}\|v\|_{\mathrm{Lip}} < \mathsf{scl} < \kappa\mathsf{ann}$, we have the estimate*

$$\sum_{s\in\mathcal{AT}(\mathsf{ann},\mathsf{scl})} |\langle \boldsymbol{\lambda}_{\mathsf{ann}} f, \alpha_s\rangle|^2 \lesssim \|f\|_2^2.$$

Define

$$Bf := \sum_{\substack{s\in\mathcal{AT}(\mathsf{ann})\\ \mathrm{scl}(s)\geq\kappa^{-1}\|v\|_{\mathrm{Lip}}}} \langle \boldsymbol{\lambda}_{\mathsf{ann}} f, \beta_s\rangle \phi_s \tag{3.10}$$

3.11. **Lemma.** *For a Lipschitz vector field v, we have*

$$\|B\|_p \lesssim 1, \qquad 2\leq p<\infty.$$

Proof. For choices of integers $\kappa^{-1}\|v\|_{\mathrm{Lip}} \leq \mathsf{scl} < \kappa\mathsf{ann}$, consider the vector valued operator

$$T_{j,k} f := \Big\{ \frac{\langle \boldsymbol{\lambda}_{\mathsf{ann}} f, \beta_s\rangle}{\sqrt{|R_s|}} \,:\, s\in\mathcal{AT}(\mathsf{ann},\mathsf{scl}) \Big\}.$$

Recall that β_s is supported off of $\frac12\gamma_s R_s$. This is bounded linear operator from $L^\infty(\mathbb{R}^2)$ to $\ell^\infty(\mathcal{AT}(\mathsf{ann},\mathsf{scl}))$. It has norm $\lesssim (\mathsf{scl}/\|v\|_{\mathrm{Lip}})^{-10}$. Routine considerations will verify that $T_{j,k}$ maps $L^2(\mathbb{R}^2)$ into $\ell^2(\mathcal{AT}(\mathsf{ann},\mathsf{scl}))$ with a similarly favorable estimate on it's norm. By interpolation, we achieve the same estimate for $T_{j,k}$ from $L^p(\mathbb{R}^2)$ into $\ell^p(\mathcal{AT}(\mathsf{ann},\mathsf{scl}))$, $2\leq p<\infty$.

It is now very easy to conclude the Lemma by summing over scales in a brute force way, and using the methods of Lemma 3.1.

□

We turn to A_+, as defined in (3.7).

3.12. **Lemma.** *We have the estimate*

$$\|A_+\|_p \lesssim 1 \qquad 2\leq p<\infty.$$

Proof. We redefine the vector valued operator $T_{j,k}$ to be

$$T_{j,k} f := \Big\{ \frac{\langle \boldsymbol{\lambda}_{\mathsf{ann}} f, \alpha_s\rangle}{\sqrt{|R_s|}} \,:\, s\in\mathcal{AT}(\mathsf{ann},\mathsf{scl}) \Big\}.$$

This is bounded $L^p(\mathbb{R}^2)$ into $\ell^p(\mathcal{AT}(\mathsf{ann},\mathsf{scl}))$, $2\leq p<\infty$ with norm $\lesssim 1$.

But, for $s\in\mathcal{AT}(\mathsf{ann},\mathsf{scl})$, we have

$$|a_{s+}| \lesssim (\mathsf{scl}/\|v\|_{\mathrm{Lip}})^{-10}|R_s|^{-1/2} M\mathbf{1}_{R_s}.$$

Here M denotes the strong maximal function in the plane in the coordinates determined by R_s. This permits one to again adapt the estimate of Lemma 3.1 to conclude the Lemma. □

Now we wish to conclude that $\|\Phi\|_2 \lesssim 1$. We have $\Phi = A_- + A_+ + B$, so from (3.4), Lemma 3.11 and Lemma 3.12, we deduce that $\|A_-\|_p \lesssim 1$ for all $2 < p < \infty$. And A_+ and B are also bounded on L^2. It remains for us to verify that A_- is of restricted weak type p_0 for some choice of $1 < p_0 < 2$. For then Lemma 3.8 will be true. That is, we should verify that for all sets $F, G \subset \mathbb{R}^2$ of finite measure

$$|\langle A_- \mathbf{1}_F, \mathbf{1}_G \rangle| \lesssim |F|^{1/p} |G|^{1-1/p}, \qquad p_0 < p < 2. \tag{3.13}$$

Since A_- maps L^p into itself for $2 < p < \infty$, it suffices to consider the case of $|F| < |G|$. Since we assume only that the vector field is Lipschitz, we can use a dilation to assume that $1 < |G| < 2$. We prove this inequality in the next section.

4. Proofs of Lemmata

4.1. **Proof of Lemma 3.8.** We fix the data $0 < \lambda < 1$, $F \subset \mathbb{R}^2$ of finite measure, ann, and vector field v with $\|v\|_{\mathrm{Lip}(1)} \le \kappa \mathsf{ann}$. and take $p_0 = 2 - \kappa^2$.

We need a set of definitions that are inspired by the approach of Lacey and Thiele [10], and are also used in Lacey and Li [9]. For subsets $\mathcal{S} \subset \mathcal{A}_v := \{s \in \mathcal{AT}(\mathsf{ann}) : \kappa^{-1}\|v\|_{\mathrm{Lip}} \le \mathrm{scl}(s) < \kappa \mathsf{ann}\}$, set

$$A^{\mathcal{S}} = \sum_{s \in \mathcal{S}} \langle \boldsymbol{\lambda}_{\mathsf{ann}} \mathbf{1}_F, \alpha_s \rangle a_{s-}$$

Set $\chi(x) = (1 + |x|)^{-1/\kappa}$. Define

$$\chi_{R_s}^{(p)} := \chi_s^{(p)} = T_{c(R_s)} D_{R_s}^p \chi, \qquad 0 \le p \le \infty. \tag{4.1}$$

And set $\widetilde{\chi}_s^{(p)} = \mathbf{1}_{\gamma_s R_s} \chi_s$.

As we are fixing attention to a single annulus, there is a natural partial order on tiles given by $s < s'$ iff $\boldsymbol{\omega}_s \supset \boldsymbol{\omega}_{s'}$, $R_{s1} \subset R_{s'1}$, and $R_{s2} = R_{s'2}$. We are free to restrict attention to a set of tiles for which we have the conclusion

$$\text{If } \boldsymbol{\omega}_s \times R_s \cap \boldsymbol{\omega}_{s'} \times R_{s'} \ne \emptyset, \text{ then } s \text{ and } s' \text{ are comparable under `}<\text{'}. \tag{4.2}$$

A *tree* is a collection of tiles $\mathbf{T} \subset \mathcal{A}_v$, for which there is a (non–unique) tile $\omega_{\mathbf{T}} \times R_{\mathbf{T}} \in \mathcal{AT}(\mathsf{ann})$ with $s < \omega_{\mathbf{T}} \times R_{\mathbf{T}}$ for all $s \in \mathbf{T}$. For $j = 1, 2$, call $\mathbf{T}$ a i–tree if the tiles $\{\boldsymbol{\omega}_{si} \times R_s : s \in \mathbf{T}\}$ are pairwise disjoint.

Our proof is organized around these parameters and functions associated to tiles and sets of tiles. We note in particular that the first definition is more restrictive than the corresponding definitions of Lacey and Thiele [10], which were adapted to the current setting by Lacey and Li [9].

$$\mathrm{dense}(s) := \int_{G \cap v^{-1}(\boldsymbol{\omega}_s)} \widetilde{\chi}_s^{(1)} \, dx, \tag{4.3}$$

$$\operatorname{dense}(\mathbf{S}) := \sup_{s\in\mathbf{S}} \operatorname{dense}(s), \tag{4.4}$$

$$\operatorname{sh}(\mathbf{S}) := \bigcup_{s\in\mathbf{S}} R_s \qquad (\text{the } \textit{shadow} \text{ of } \mathbf{S}), \tag{4.5}$$

$$\Delta(\mathbf{T})^2 := \sum_{s\in\mathbf{T}} \frac{|\langle \boldsymbol{\lambda}_{\mathsf{ann}} 1_F, \alpha_s\rangle|^2}{|R_s|} \mathbf{1}_{R_s}, \qquad \mathbf{T} \text{ is a 1–tree}, \tag{4.6}$$

$$\operatorname{size}(\mathbf{S}) := \sup_{\substack{\mathbf{T}\subset\mathbf{S} \\ \mathbf{T} \text{ is a 1–tree}}} \fint_{R_{\mathbf{T}}} \Delta(\mathbf{T})\, dx. \tag{4.7}$$

It is essential to note that if $\mathbf{T}$ is a 1–tree with $\fint_{R_{\mathbf{T}}} \Delta(\mathbf{T}) \ge \sigma$, then

$$|F \cap \min(\gamma_{\mathbf{T}}, \sigma^{-\kappa}) R_{\mathbf{T}}| \gtrsim \sigma^{1+\kappa} |R_{\mathbf{T}}|. \tag{4.8}$$

Here, K is an absolute constant, depending on any fixed choice of $\kappa > 0$, and by $\gamma_{\mathbf{T}}$ we mean the $\gamma_{\boldsymbol{\omega}_{\mathbf{T}}\times R_{\mathbf{T}}}$, with γ_s defined in (3.5). This follows, in part, from standard aspects of Calderón–Zygmund theory, and in part, to the fact that for all $s\in\mathbf{T}$, we have a_{s-} supported on $K\gamma_{\mathbf{T}} R_{\mathbf{T}}$. We shall comment on it again in the proof of Lemma 4.14, which is presented in Section 4. One should also note that $\operatorname{size}(\mathcal{A}_v(\mathsf{ann})) \lesssim 1$.

Concerning the definition of density, we need to make this comment. Call a set of tiles $\mathbf{S}$ *convex* if for all $s, s'' \in \mathbf{S}$, one also has $s' \in \mathbf{S}$ for any $s < s' < s''$. We will at each stage of the proof consider only convex sets of tiles.

Given a convex set of tiles, say that $\operatorname{count}(\mathbf{S}) < A$ iff $\mathbf{S}$ is a union of convex trees $\mathbf{T} \in \mathcal{T}$ for which

$$\sum_{T\in\mathcal{T}} |\operatorname{sh}(\mathbf{T})| < A.$$

We will also use the notation $\operatorname{count}(\mathbf{S}) \lesssim A$, implying the existence of an absolute constant K for which $\operatorname{count}(\mathbf{S}) \le KA$.

The principal organizational Lemma is

4.9. **Lemma.** *Any finite convex collection of tiles $\mathbf{S} \subset \mathcal{A}_v$ is a union of four convex subsets*

$$\mathbf{S}_{\text{light}}, \quad \mathbf{S}_{\text{small}}, \quad \mathbf{S}^{\ell}_{\text{large}}, \ \ell = 1, 2.$$

They satisfy these properties.

$$\operatorname{size}(\mathbf{S}_{\text{small}}) < \tfrac12 \operatorname{size}(\mathbf{S}), \tag{4.10}$$

$$\operatorname{dense}(\mathbf{S}_{\text{light}}) < \tfrac12 \operatorname{dense}(\mathbf{S}), \tag{4.11}$$

and both $\mathbf{S}^{\ell}_{\text{large}}$ are unions of convex trees $\mathbf{T} \in \mathcal{T}^{\ell}$,for which we have the estimates

$$\operatorname{count}(\mathbf{S}^1_{\text{large}}) \lesssim \begin{cases} \operatorname{size}(\mathbf{S})^{-2-\kappa} |F| \\ \operatorname{size}(\mathbf{S})^{-1-\kappa} \operatorname{dense}(\mathbf{S})^{-4-\kappa} |F| \\ \operatorname{dense}(S)^{-1} \end{cases} \tag{4.12}$$

$$\operatorname{count}(\mathbf{S}^2_{\text{large}}) \lesssim \begin{cases} \operatorname{size}(\mathbf{S})^{-2}(\log 1/\operatorname{size}(\mathbf{S}))^3|F| \\ \operatorname{size}(\mathbf{S})^{\kappa/50}\operatorname{dense}(\mathbf{S})^{-1} \end{cases} \tag{4.13}$$

The estimates that involve $\operatorname{size}(\mathbf{S})^{-2}|F|$ are those that follow from orthogonality considerations. The estimates in $\operatorname{dense}(\mathbf{S})^{-1}$ are those that follow from density considerations. Of particular note is the middle estimate of (4.12). For it we shall need the critical maximal function estimate for Lipschitz vector fields in Section 5. There is a new estimate that needs to be invoked to obtain the second estimate in (4.13).

For individual trees, we need the estimate below, which is essentially the tree lemma of [9].

4.14. **Lemma.** *For convex trees $\mathbf{T}$ we have the estimate*

$$\sum_{s\in\mathbf{T}} |\langle \boldsymbol{\lambda}_{\mathsf{ann}}\mathbf{1}_F, \alpha_s\rangle\langle a_{s-}, \mathbf{1}_G\rangle| \lesssim \operatorname{dense}(\mathbf{T})\operatorname{size}(\mathbf{T})|\operatorname{sh}(\mathbf{T})|. \tag{4.15}$$

Set

$$\operatorname{Sum}(\mathbf{S}) := \sum_{s\in\mathbf{S}} |\langle \boldsymbol{\lambda}_{\mathsf{ann}}\mathbf{1}_F, \alpha_s\rangle\langle a_{s-}, \mathbf{1}_G\rangle|$$

We want to bound $\operatorname{Sum}(\mathcal{A}_v)$ by $\lesssim |F|^{1/p}$ for $p_0 < p < 2$. And we have the trivial bound

$$\operatorname{Sum}(\mathbf{S}) \lesssim \operatorname{dense}(\mathbf{S})\operatorname{size}(\mathbf{S})\operatorname{count}(\mathbf{S}). \tag{4.16}$$

By inductive application of this Lemma 4.9, $\mathcal{A}_v$ is the union of $\mathbf{S}^\ell_{\delta,\sigma}$, $\ell = 1, 2$ for $\delta, \sigma \in \mathbf{2} := \{2^n : n \in \mathbb{Z}\}$, satisfying

$$\operatorname{dense}(\mathbf{S}^\ell_{\delta,\sigma}) \lesssim \delta, \tag{4.17}$$

$$\operatorname{size}(\mathbf{S}^\ell_{\delta,\sigma}) \lesssim \sigma, \tag{4.18}$$

$$\operatorname{count}(\mathbf{S}^\ell_{\delta,\sigma}) \lesssim \begin{cases} \min(\sigma^{-2-\kappa}|F|, \delta^{-4}\sigma^{-1-\kappa}|F|, \delta^{-1}) & \ell = 1, \\ \min(\sigma^{-2}(\log 1/\sigma)^3|F|, \delta^{-1}\sigma^{\kappa/50}) & \ell = 2 \end{cases} \tag{4.19}$$

Using (4.16), we see that

$$\operatorname{Sum}(\mathbf{S}^\ell_{\delta,\sigma}) \lesssim \begin{cases} \min(\delta\sigma^{-1-\kappa}|F|, \delta^{-3}\sigma^{-\kappa}|F|, \sigma) & \ell = 1, \\ \min(\delta\sigma^{-1}(\log 1/\sigma)^3|F|, \sigma^{1+2\kappa}) & \ell = 2 \end{cases}$$

It is a routine exercise, left to the reader, to check that for $\ell = 1, 2$,

$$\sum_{\delta,\sigma\in\mathbf{2}} \operatorname{Sum}(\mathbf{S}^\ell_{\delta,\sigma}) \lesssim |F|^{1/p}, \qquad p_0 < p < 2.$$

This completes the proof of Lemma 3.8, aside from the proof of Lemma 4.9.

It is interesting to note these points. The estimate for $\mathbf{S}^2_{\delta,\sigma}$ will prove an L^p estimate for $(2+\kappa)/(1+2\kappa) < p$. If one varies κ, the estimates for $\mathbf{S}^1_{\delta,\sigma}$ will prove an L^p estimate for $7/4 < p$. One may be able to avoid the use of the collections $\mathbf{S}^2_{\delta,\sigma}$, but at the cost of a more sophisticated proof than the one given for Lemma 4.9.

4.2. **Proof of Lemma 3.9.** We only consider tiles $s \in \mathcal{AT}(\mathsf{ann}, \mathsf{scl})$, and sets $\boldsymbol{\omega} \in \boldsymbol{\Omega}$ which are associated to one of these tiles. For an element $a = \{a_s\} \in \ell^2(\mathcal{AT}(\mathsf{ann}, \mathsf{scl}))$,

$$T_{\boldsymbol{\omega}} a = \sum_{s\,:\,\boldsymbol{\omega}_s = \boldsymbol{\omega}} a_s \boldsymbol{\lambda}_{\mathsf{ann}} \alpha_s$$

For $\boldsymbol{\omega}_s = \boldsymbol{\omega}_{s'}$, note that $\operatorname{dist}(\boldsymbol{\omega}_s, \boldsymbol{\omega}_{s'})$ is measured in units of $\mathsf{scl}/\mathsf{ann}$.

By a lemma of Cotlar and Stein, it suffices to provide the estimate

$$\|T_{\boldsymbol{\omega}} T^*_{\boldsymbol{\omega}'}\|_2 \lesssim \rho^{-3}, \qquad \rho = 1 + \frac{\mathsf{ann}}{\mathsf{scl}} \operatorname{dist}(\boldsymbol{\omega}, \boldsymbol{\omega}').$$

Now, the estimate $\|T_{\boldsymbol{\omega}}\|_2 \lesssim 1$ is obvious. For the case $\boldsymbol{\omega} \neq \boldsymbol{\omega}'$, by Schur's test, it suffices to see that

$$\sup_{s'\,:\,\boldsymbol{\omega}_{s'} = \boldsymbol{\omega}'} \sum_{s\,:\,\boldsymbol{\omega}_s = \boldsymbol{\omega}} |\langle \boldsymbol{\lambda}_{\mathsf{ann}} \alpha_s, \boldsymbol{\lambda}_{\mathsf{ann}} \alpha_{s'} \rangle| \lesssim \rho^{-3}. \tag{4.20}$$

For tiles s' and s as above, recall that $\langle \varphi_s, \varphi_{s'} \rangle = 0$, note that

$$\frac{|R_{s'} \cap R_s|}{|R_s|} \lesssim \frac{\mathsf{scl}}{\mathsf{ann}\,\operatorname{dist}(\boldsymbol{\omega}, \boldsymbol{\omega}')} = \rho^{-1},$$

and in particular, for a fixed s', let $\mathbf{S}_{s'}$ be those s for which $\rho\gamma_s R_s \cap \rho\gamma_{s'} R_{s'} \neq \emptyset$. Clearly,

$$\operatorname{card}(\mathbf{S}_{s'}) \lesssim \frac{|R_s|}{|R_{s'} \cap R_s|} \gamma_{s'}^2 \simeq \rho^3 \gamma_{s'}^2$$

If for $r > 1$, $r\gamma_s R_s \cap r\gamma_{s'} R_{s'} = \emptyset$ but $2r\gamma_s R_s \cap 2r\gamma_{s'} R_{s'} \neq \emptyset$, then it is routine to show that

$$|\langle \boldsymbol{\lambda}_{\mathsf{ann}} \alpha_s, \boldsymbol{\lambda}_{\mathsf{ann}} \alpha_{s'} \rangle| \lesssim (r\gamma_s)^{-10}$$

And so we may directly sum over those $s \notin \mathbf{S}_{s'}$,

$$\sum_{s \notin \mathbf{S}_{s'}} |\langle \boldsymbol{\lambda}_{\mathsf{ann}} \alpha_s, \boldsymbol{\lambda}_{\mathsf{ann}} \alpha_{s'} \rangle| \lesssim \rho^{-3}.$$

For those $s \in \mathbf{S}_{s'}$, we estimate the inner product in frequency variables. Recalling the definition of $\alpha_s = (D^0_{\gamma_s R_s} T_{c(R_s)} \zeta) \varphi_s$, we have

$$\widehat{\alpha}_s = (\mathrm{Mod}_{-c(R_s)}\, D^1_{\gamma_s^{-1} \omega_s} \widehat{\zeta}) * \widehat{\varphi}_s.$$

Recall that ζ is a smooth compactly supported Schwartz function. We estimate the inner product

$$|\langle \widehat{\boldsymbol{\lambda}_{\mathsf{ann}} \alpha_s}, \widehat{\boldsymbol{\lambda}_{\mathsf{ann}} \alpha_{s'}} \rangle|$$

without appealing to cancellation. Since we choose the function λ to be supported in an annulus $\frac{1}{2}\mathsf{ann} < |\xi| < 3\mathsf{ann}$, We can restrict our attention to this same range of ξ. In the region $|\xi| > \mathsf{ann}/4$, suppose, without loss of generality, that ξ is closer to ω_s than $\omega_{s'}$. Then since ω_s and $\omega_{s'}$ are separated by an amount $\gtrsim \mathsf{ann}\,\mathrm{dist}(\boldsymbol{\omega},\boldsymbol{\omega}')$,

$$
\begin{aligned}
|\widehat{\alpha}_s(\xi)\widehat{\alpha}_{s'}(\xi)| &\lesssim \chi^{(2)}_{\omega_s}(\xi)\chi^{(2)}_{\omega_{s'}}(\xi)\Big(\gamma_s\frac{\mathsf{ann}}{\mathsf{scl}}\mathrm{dist}(\boldsymbol{\omega},\boldsymbol{\omega})\Big)^{-20} \\
&\lesssim \chi^{(2)}_{\omega_s}(\xi)\chi^{(2)}_{\omega_{s'}}(\xi)(\gamma_s\rho)^{-10}.
\end{aligned}
$$

Here, χ is the non–negative bump function in (4.1). Hence, we have the estimate

$$
\int_{|\xi|>\mathsf{ann}/4} |\widehat{\alpha}_s(\xi)\widehat{\alpha}_{s'}(\xi)|\,d\xi \lesssim \frac{|R_{s'}\cap R_s|}{|R_s|}(\gamma_s\rho)^{-10}.
$$

This is summed over the $\lesssim \gamma_{s'}^2\rho^3$ possible choices of $s\in \mathbf{S}_{s'}$, giving the estimate

$$
(\gamma_s\rho)^{-10} \lesssim \rho^{-3}.
$$

This is the proof of (4.20). And this concludes the proof of Lemma 3.9.

4.3. Proof of the Key Organizational Lemma 4.9.

Recall that $\mathbf{S}$ is convex, and we are to decompose it into distinct convex subsets. For the remainder of the proof set $\mathrm{dense}(\mathbf{S}) := \delta$ and $\mathrm{size}(\mathbf{S}) := \sigma$. Take $\mathbf{S}_{\mathrm{light}}$ to be all those $s\in\mathbf{S}$ for which there is no tile s' of density at least $\delta/2$ for which $s<s'$. It is clear that this set so constructed has density at most $\delta/2$, that this is a convex set of tiles , and that $\mathbf{S}_1 := \mathbf{S}-\mathbf{S}_{\mathrm{light}}$ is also convex.

Issues related to orthogonality are important of the proof, and single out for a different treatment of orthogonality those tiles

$$
\mathbf{S}_0 := \{s\in\mathbf{S} \,:\, \gamma_s < C_0[|\langle \mathbf{1}_F,\alpha_s\rangle|/\sqrt{|R_s|}]^{-\kappa/5}\} \tag{4.21}
$$

where C_0 is a constant to be chosen. Since the terms γ_s increases as does $\mathrm{scl}(s)^{1/2}$, tiles $s\in\mathbf{S}_0$ can have only $\lesssim 1+k\log 1/\sigma$ possible values of scale, which is the decisive feature of this case. Observe that by lemma 3.9,

$$
\begin{aligned}
\Big(\frac{\sigma^2}{\log 1/\sigma}\Big)^2\sum_{s\in\mathbf{S}_0}|R_s| &\lesssim \sum_{s\in\mathbf{S}_0'}|\langle \boldsymbol{\lambda}_{\mathsf{ann}}\mathbf{1}_F,\alpha_s\rangle|^2 \\
&\lesssim k(\log 1/\sigma)|F|.
\end{aligned} \tag{4.22}
$$

We shall appeal to this below. We do not do so immediately, since S_0 is certainly not a convex set in general.

The following definition will be of use to us. Suppose that $\mathbf{T}$ is a tree. We say that $\mathrm{charge}(\mathbf{T})\ge\tau$ iff there is a 1–tree $\mathbf{T}'\subset\mathbf{T}$ for which

$$
⨍_{\mathrm{sh}(\mathbf{T})} \Delta_{\mathbf{T}'} \ge \tau. \tag{4.23}
$$

The tree $\mathbf{T}'$ is said to *achieve the charge of* $\mathbf{T}$.

We comment on the method we use to obtain the middle estimate in (4.12). This depends upon the novel maximal function estimate of the next section. Suppose we have a collection of trees $\mathbf{T} \in \mathcal{T}$, each contained in $\mathbf{S}_1$, with charge($\mathbf{T}$) comparable to $\sigma^{1+\kappa/100}$. Moreover, each tree has top element $s(\mathbf{T}) := R_{\mathbf{T}} \times \boldsymbol{\omega}_{\mathbf{T}}$ of density at least δ. Set

$$\mathsf{s}(\mathbf{T}) = \boldsymbol{\omega}_{\mathbf{T}} \times \sigma^{-\kappa/5} R_{\mathbf{T}}.$$

Observe that we can regard $\operatorname{ann}(\mathsf{s}(\mathbf{T})) \simeq \sigma^{-\kappa/5}\mathsf{ann}$ as a constant independent of $\mathbf{T}$. And, we have

$$\kappa^{-1}\|v\|_{\mathrm{Lip}} \leq \operatorname{scl}(\mathsf{s}(\mathbf{T})) \leq \kappa \operatorname{ann}(\mathsf{s}(\mathbf{T})), \qquad \mathbf{T} \in \mathcal{T},$$

since $C_0\sigma^{-\kappa/5} < \gamma_s$. We have in addition,

$$\begin{aligned} &\operatorname{dense}(\mathsf{s}(\mathbf{T})) \geq \delta\sigma^{2\kappa/5}, \\ &|F \cap R_{\mathsf{s}(\mathbf{T})}| \geq \sigma^{1+\kappa/4}|R_{\mathsf{s}(\mathbf{T})}|. \end{aligned}$$

The point of these observations is that our Lemma 5.1 applies to the maximal function formed over the set of tiles $\{\mathsf{s}(\mathbf{T}) : \mathbf{T} \in \mathcal{T}\}$. In particular, applying that Lemma for a choice of L^p for p very close to one, we have the estimate

$$\Big|\bigcup_{\mathbf{T}\in\mathcal{T}} \sigma^{-\kappa/5} R_{\mathbf{T}}\Big| \lesssim \delta^{-3}\sigma^{-1+\kappa}|F|.$$

In the argument below, we shall in addition have at our disposal the assumption that the tops of the trees $s(\mathbf{T})$ for $\mathbf{T} \in \mathcal{T}$ are pairwise disjoint. The sets $R_{\mathbf{T}} \cap v^{-1}(\boldsymbol{\omega}_{\mathbf{T}}) \cap G$ are then of measure $\gtrsim \delta|R_{\mathbf{T}}|$. Hence,

$$\sum_{\mathbf{T}\in\mathcal{T}} |\operatorname{sh}(\mathbf{T})| \lesssim \delta^{-4}\sigma^{-1+\kappa/2}|F|. \tag{4.24}$$

Observe that by maximality, and the fact that the measure of G is at most one, we can also conclude

$$\sum_{\mathbf{T}\in\mathcal{T}} |\operatorname{sh}(\mathbf{T})| \lesssim \delta^{-1}. \tag{4.25}$$

We can now begin the principal line of reasoning.

The Construction of $\mathbf{S}^1_{\mathrm{large}}$. We use an orthogonality, or TT^* argument that has been used many times before, especially in [10] and [9]. (There is a feature of the current application of the argument that is present due to the fact that we are working on the plane, and it is detailed by Lacey and Li [9].)

We may assume that all intervals $\boldsymbol{\omega}_s$ are contained in the upper half of the unit circle in the plane. Fix $\mathbf{S} \subset \mathcal{A}_v$, and $\sigma = \operatorname{size}(\mathbf{S})$.

We construct a collection of trees $\mathcal{T}^1_{\mathrm{large}}$ for the collection $\mathbf{S}_1$, and a corresponding collection of 1–trees $\mathcal{T}^{1,1}_{\mathrm{large}}$, with particular properties. The we begin the recursion by initializing

$$\mathcal{T}^1_{\text{large}} := \emptyset, \qquad \mathcal{T}^{1,1}_{\text{large}} := \emptyset,$$
$$\mathbf{S}^1_{\text{large}} := \emptyset, \qquad \mathbf{S}^{\text{stock}} := \mathbf{S}_1.$$

In the recursive step, if $\operatorname{size}(\mathbf{S}^{\text{stock}}) < \frac{1}{2}\sigma^{1+\kappa/100}$, then this recursion stops. Otherwise, we select a tree $\mathbf{T} \subset \mathbf{S}^{\text{stock}}$ such that three conditions are met: (a) the top of the tree $s(\mathbf{T})$ (which need not be in the tree) satisfies $\operatorname{dense}(s(\mathbf{T})) \geq \delta/4$; (b) $\mathbf{T}$ has charge greater than $\frac{1}{2}\sigma^{1+\kappa/100}$; (c) and that $\boldsymbol{\omega}_{\mathbf{T}}$ is both minimal and most counterclockwise among all possible choices of $\mathbf{T}$. (Since all $\boldsymbol{\omega}_s$ are in the upper half of the unit circle, this condition can be fulfilled.) We take $\mathbf{T}$ to be the maximal convex tree in $\mathbf{S}^{\text{stock}}$ which satisfies these conditions. We take $\mathbf{T}^1 \subset \mathbf{T}$ to be a subtree that achieves the charge of $\mathbf{T}$.

We then update

$$\mathcal{T}^1_{\text{large}} := \{\mathbf{T}\} \cup \mathcal{T}_{\text{large}}, \qquad \mathcal{T}^{1,1}_{\text{large}} := \{\mathbf{T}^1\} \cup \mathcal{T}^{1,1}_{\text{large}}, \qquad \mathbf{S}^{\text{stock}} := \mathbf{S}^{\text{stock}} - \mathbf{T}.$$

It is important to note that $\mathbf{T}$ is convex, and maximal, hence $\mathbf{S}^{\text{stock}}$ and the collection $\mathbf{S}^1_{\text{large}}$ so constructed will also be convex. The recursion then repeats. Once the recursion stops, we update

$$\mathbf{S}_1 := \mathbf{S}_{\text{stock}}$$

It is this collection that we analyze in the next subsection.

The bottom estimate of (4.12) is then immediate from the construction and (4.25).

We turn to the deduction of the first and middle estimates. The argument must be split into two cases, depending upon the behavior with respect to the set S_0 defined in (4.21). Let $\mathcal{T}'$ be those It fo $\mathbf{T} \in \mathcal{T}^1_{\text{large}}$ so that $T \cap \mathbf{S}_0$ has charge at least $\frac{1}{4}\sigma^{1+\kappa/100}$. It follows from (4.22) that

$$\sum_{\mathbf{T}\in\mathcal{T}'} |\operatorname{sh}(\mathbf{T})| \lesssim \sigma^{-2-\kappa/50}(\log 1/\sigma)^2|F|.$$

This is the top estimate of (4.12). In addition, we must have

$$\Big|\bigcup_{s\in\mathbf{T}\cap S_0} R_s\Big| \geq (\log 1/\sigma)^{-2}|\operatorname{sh}(\mathbf{T})|.$$

And since for each $s \in S_0$ we necessarily have $|F \cap \sigma^{-\kappa/5}R_s| \geq \sigma^{1+\kappa/5}|R_s|$, we conclude that

$$|\operatorname{sh}(\mathbf{T}) \cap R_s| \geq \sigma^{1+3\kappa/5}|\operatorname{sh}(\mathbf{T})|.$$

Therefore, we can follow the reasoning that leads to (4.24) to see the middle estimate of (4.12) in this case.

We hence forth assume that for each $\mathbf{T} \in \mathcal{T}^1_{\text{large}}$ that the tree $\mathbf{T} - \mathbf{S}_0$ achieves charge at least $\frac{1}{4}\sigma^{1+\kappa/100}$. It is important to observe that by choosing C_0 in (4.21) sufficiently large, we can then conclude that

$$|R_{\mathbf{T}^1}|^{-1} \sum_{s \in \mathbf{T}^1} |\langle \mathbf{1}_F, \varphi_s \rangle|^2 > C_2 \sigma^2,$$

for an absolute C_2. The replacement of α_s by φ_s in the inequality above is an important point for us. That we can then drop the $\boldsymbol{\lambda}_{\mathsf{ann}}$ is immediate.

With this construction and observation, the argument for "size Lemma" in [9] then shows that we have

$$\sum_{T \in \mathcal{T}^1_{\text{large}}} |\text{sh}(\mathbf{T})| \lesssim \sigma^{-2-\kappa/50} |F|. \tag{4.26}$$

This is the first estimate of (4.12). By (4.24), we deduce the middle estimate of (4.12). And the last estimate of (4.12) follows from (4.25).

The Construction of $\mathbf{S}^2_{\text{large}}$. We repeat the TT^* construction of the previous step in the proof, with two significant changes.

We construct a collection of trees $\mathcal{T}^1_{\text{large}}$ from the collection $\mathbf{S}_1$, and a corresponding collection of 1–trees $\mathcal{T}^{2,1}_{\text{large}}$, with particular properties. The we begin the recursion by initializing

$$\mathcal{T}^2_{\text{large}} := \emptyset, \qquad \mathcal{T}^{2,1}_{\text{large}} := \emptyset,$$
$$\mathbf{S}^2_{\text{large}} := \emptyset, \qquad \mathbf{S}^{\text{stock}} := \mathbf{S}_1.$$

In the recursive step, if size($\mathbf{S}^{\text{stock}}$) $< \sigma/2$, then this recursion stops. Otherwise, we select a tree $\mathbf{T} \subset \mathbf{S}^{\text{stock}}$ such that three conditions are met: (a) $\mathbf{T}$ has charge greater than $\delta/2$; (b) and that $\boldsymbol{\omega}_{\mathbf{T}}$ is both minimal and most counterclockwise among all possible choices of $\mathbf{T}$. We take $\mathbf{T}$ to be the maximal convex tree in $\mathbf{S}^{\text{stock}}$ which satisfies these conditions. We take $\mathbf{T}^1 \subset \mathbf{T}$ to be a subtree that achieves the charge of $\mathbf{T}$.

We then update

$$\mathcal{T}^2_{\text{large}} := \{\mathbf{T}\} \cup \mathcal{T}_{\text{large}}, \qquad \mathcal{T}^{2,1}_{\text{large}} := \{\mathbf{T}^1\} \cup \mathcal{T}^{2,1}_{\text{large}}, \qquad \mathbf{S}^{\text{stock}} := \mathbf{S}^{\text{stock}} - \mathbf{T}.$$

The recursion then repeats.

Once the recursion stops, it is clear that the size of $\mathbf{S}^{\text{stock}}$ is at most $\sigma/2$, and so we take $\mathbf{S}_{\mathsf{small}} := \mathbf{S}_{\mathsf{stock}}$.

The estimate

$$\sum_{\mathbf{T} \in \mathcal{T}^2_{\text{large}}} |\text{sh}(\mathbf{T})| \lesssim \sigma^{-2} (\log 1/\sigma)^2 |F|$$

then is a consequence of the TT^* method, as indicated in the previous step of the proof. That is the first estimate claimed in (4.13).

What is significant is the second estimate of (4.13). The point to observe is this. Consider any tile s of density at least $\delta/2$. Let $\mathcal{T}_s$ be those trees $\mathbf{T} \in \mathcal{T}^2_{\text{large}}$ with top $s(\mathbf{T}) < s$. By the construction of $\mathbf{S}_{\text{large}}$, we must have that the charge of $\bigcup_{\mathbf{T}\in\mathcal{T}_s} \mathbf{T}$ is at most $\sigma^{1+\kappa/100}$. But, in addition, the tops of the trees in $\mathcal{T}^2_{\text{large}}$ are pairwise incomparable with respect to $<$, hence we conclude that

$$\frac{\sigma^2}{4} \sum_{\mathbf{T}\in\mathcal{T}_s} |\text{sh}(\mathbf{T})| \lesssim \sigma^{2+\kappa/50} |R_s|.$$

Moreover, by the construction of $\mathbf{S}_{\text{light}}$, for each $\mathbf{T} \in \mathcal{T}^2_{\text{large}}$ we must be able to select some tile s with density at least $\delta/2$ and $s(\mathbf{T}) < s$.

Thus, we let $\mathbf{S}^*$ be the maximal tiles of density at least $\delta/2$. Then, the inequality (4.25) applies to this collection. And, therefore,

$$\sum_{\mathbf{T}\in\mathcal{T}^2_{\text{large}}} |\text{sh}(\mathbf{T})| \le \sigma^{\kappa/50} \sum_{s\in\mathbf{S}^*} |R_s| \lesssim \delta^{\kappa/50}\delta^{-1}.$$

This completes the proof of second estimate of (4.13).

5. The Maximal Function Estimate

Let $\mathbf{S} \subset \mathcal{AT}(\mathsf{ann})$ be a set of tiles satisfying $|v^{-1}(\boldsymbol{\omega}_{s2}) \cap R_s| \ge \delta |R_s|$ for all $s \in \mathbf{S}$. Define a maximal function by

$$M^{\mathbf{S}} g = \sup_{s\in\mathbf{S}} \mathbf{1}_{R_s} \fint_{R_s} |g| \, dx.$$

Notice that we do not concern ourselves with the expansion factor γ_s.

5.1. **Lemma.** *For any $1 < p < \infty$, the maximal function $M^{\mathbf{S}}$ maps L^p into $L^{p,\infty}$ with norm bounded by at most $\lesssim \delta^{-3}$. As a consequence, for all $1 < p < \infty$, and all $\epsilon > 0$, the maximal operator extends to a bounded operator on L^p into itself, with operator norm is $\|M^{\mathbf{S}}\|_p \lesssim \delta^{-3/p+\epsilon}$. What is most important is that the norm bound is independent of* ann.

For the proof of our main theorem, it is important that this Lemma hold for some $1 < p < 2$, with any finite power of δ^{-1}. A variant of the proof will apply to maximal functions constructed from a richer class of rectangles, with the caveat that one only gets the weak L^2 estimate. We note it here because of its potential use in subsequent investigations.

5.2. **Lemma.** *Assume that $\mathbf{S}$ is a set of tiles satisfying $|v^{-1}(\boldsymbol{\omega}_{s2}) \cap R_s| \ge \delta|R_s|$ for all $s \in \mathbf{S}$, and having varying values of* ann*, but always subject to the conditions $\kappa\mathsf{ann} \ge \kappa\, \text{scl}(\ge)\|v\|_{Lip}$. Then $M^{\mathbf{S}}$ maps L^2 into weak L^2 with norm δ^{-3}.*

It suffices to show that for any integer $n>1$, and any finite set of tiles $\mathbf{S}$, with $\operatorname{dense}(s)\gtrsim\delta$ for all $s\in\mathbf{S}$, there is a subset $\mathbf{S}'\subset\mathbf{S}$ for which

$$|\operatorname{sh}(\mathbf{S})|\lesssim\delta^{-1}|\operatorname{sh}(\mathbf{S}')|, \tag{5.3}$$

$$\Big\|\sum_{\substack{s\in\mathbf{S}'\\ \operatorname{scl}(s)\ge\operatorname{scl}(s_0)}}\mathbf{1}_{R_s\cap R_{s_0}}\Big\|_n\lesssim\delta^{-2}|R_{s_0}|^{1/n}\qquad s_0\in\mathbf{S}'. \tag{5.4}$$

Indeed, this implies that

$$\begin{aligned}\Big\|\sum_{s\in\mathbf{S}'}\mathbf{1}_{R_s}\Big\|_{n+1}^{n+1}&\lesssim\delta^{-2n}\Big\|\sum_{s\in\mathbf{S}'}\mathbf{1}_{R_s}\Big\|_1\\&\lesssim\delta^{-2(n+1)}|\operatorname{sh}(\mathbf{S})'|.\end{aligned}$$

The proof of the weak type bound for the maximal function is then straightforward. If $f\in L^{n/(n-1)}$ and $\lambda>0$, we can assume that for all $s\in\mathbf{S}$ we have $\fint_{R_s}f>\lambda$. Then

$$\begin{aligned}|\operatorname{sh}(\mathbf{S})|&\lesssim\delta^{-1}|\operatorname{sh}(\mathbf{S})'|\\&\lesssim\delta^{-1}\Big\|\sum_{s\in\mathbf{S}'}\mathbf{1}_{R_s}\Big\|_1\\&\lesssim\delta^{-1}\lambda^{-1}\sum_{s\in\mathbf{S}'}\int f\mathbf{1}_{R_s}\,dy\\&\lesssim\delta^{-1}\lambda^{-1}\|f\|_{n/(n-1)}\Big\|\sum_{s\in\mathbf{S}'}\mathbf{1}_{R_s}\Big\|_n\\&\lesssim\delta^{-3}\lambda^{-1}\|f\|_{n/(n-1)}|\operatorname{sh}(\mathbf{S})|^{1/n}.\end{aligned}$$

And this proves the maximal function estimate from $L^{n/(n-1)}$ to weak $L^{n/(n-1)}$ with norm bounded by $\lesssim\delta^{-3}$. Interpolation gives the remaining conclusions of the Lemma.

Let us specify the two ways in which the Lipschitz nature of the vector field enters into our argument.

5.5. **Proposition.** *Suppose that there is a* scl, *tiles* $s_j\in\mathbf{S}$, $j=1,\dots,n$, *for which* $\operatorname{scl}(s_j)=\mathsf{scl}$ *for all* j, *and*

$$\{\boldsymbol{\omega}_{s_j}\,:\,1\le j\le n\}\ \textit{are pairwise disjoint},\qquad\bigcap_{j=1}^n R_{s_j}\neq\emptyset.$$

Then $n\le\delta^{-1}$.

Proof. Suppose that the origin is common to all R_{s_j}, and that $n>\delta^{-1}$. Then the sets

$$\{r>0\,:\,\exists x\in v^{-1}(\boldsymbol{\omega}_s)\cap R_s,\ |x|=r\},\qquad 1\le j\le n$$

are contained in $[0,\mathsf{scl}^{-1}]$, pairwise disjoint, and have measure at least $\delta\,\mathsf{scl}^{-1}$. As $n>\delta^{-1}$, there are points $x,x'\in\mathbb{R}^2$, and tiles $s_j\neq s_{j'}$ with $|x|=|x'|$, and $v(x)\in\boldsymbol{\omega}_{s2}$, and $v(x') \in \boldsymbol{\omega}_{s'2}$. Recall the fact that $e_s \in \boldsymbol{\omega}_{s1}$ for all tiles s. And the assumption (2.11). Hence for the point x, we have

$$\big|v(x) - x/|x|\big| \le |v(x) - e_s| + |e_s - x/|x|| \le \tfrac{1}{8}|\boldsymbol{\omega}_s|.$$

There is a similar inequality for x'. It follows that

$$\begin{aligned} |v(x) - v(x')| &\ge \frac{|x - x'|}{|x|} - \tfrac{1}{8}(|\boldsymbol{\omega}_s| + |\boldsymbol{\omega}_{s'}|) \\ &\ge \frac{|x - x'|}{2|x|} \end{aligned}$$

We conclude that $\|v\|_{\mathrm{Lip}} \ge \frac{1}{2}\mathsf{scl}$, a contradiction. □

5.6. **Proposition.** *Suppose that for $s_0, s, s' \in \mathbf{S}$, we have $R_{s_0} \cap R_s \neq \emptyset$, $R_{s_0} \cap R_{s'} \neq \emptyset$ and $\mathrm{scl}(s') \ge \mathrm{scl}(s) > \mathrm{scl}(s_0)$. Suppose that the coordinates for R_{s_0} are the canonical ones, and the length of R_{s_0} is in the first coordinate. Suppose that there are points*

$$(x_0, y_0) \in R_{s_0}, \qquad (x_0, y) \in R_s \cap v^{-1}(\boldsymbol{\omega}_s), \qquad (x_0, y') \in R_{s'} \cap v^{-1}(\boldsymbol{\omega}_{s'}).$$

Then

$$\mathrm{dist}(\boldsymbol{\omega}_s, \boldsymbol{\omega}_{s'}) \le 4\mathrm{dist}(\boldsymbol{\omega}_s, \boldsymbol{\omega}_{s_0})\frac{\|v\|_{\mathrm{Lip}}}{\mathrm{scl}(s)}$$

Proof. Observe that

$$\begin{aligned} \mathrm{dist}(\boldsymbol{\omega}_s, \boldsymbol{\omega}_{s'}) &\le 2|v(x_0, y) - v(x_0, y')| \\ &\le \|v\|_{\mathrm{Lip}}|y - y'| \\ &\le 4\mathrm{dist}(\boldsymbol{\omega}_s, \boldsymbol{\omega}_{s_0})\frac{\|v\|_{\mathrm{Lip}}}{\mathrm{scl}(s)}. \end{aligned}$$

□

The principle line of argument begins with the selection of the subcollection $\mathbf{S}'$. Let M_{100} be a maximal function computed in 100 uniformly distributed directions of the plane. Initialize

$$\mathbf{S}^{\mathrm{stock}} := \mathbf{S}, \qquad \mathbf{S}' = \emptyset.$$

While $\mathbf{S}^{\mathrm{stock}} \neq \emptyset$, select $s' \in \mathbf{S}^{\mathrm{stock}}$ with $\mathrm{scl}(s)$ minimal (so the length of R_s is maximal) and that $\mathrm{ann}(s)$ is minimal among those tiles with that value of $\mathrm{scl}(s)$. Update, $\mathbf{S}' := \mathbf{S}' \cup \{s'\}$. Remove from $\mathbf{S}^{\mathrm{stock}}$ any tile s such that

$$R_s \subset \Big\{ M_{100} \sum_{s' \in \mathbf{S}'} \mathbf{1}_{R_s} \ge \delta^{-1} \Big\}.$$

Observe that

$$|\mathrm{sh}(\mathbf{S} - \mathbf{S}')| \lesssim \Big| \Big\{ M_{100} \sum_{s' \in \mathbf{S}'} \mathbf{1}_{R_s} \ge \delta^{-1} \Big\} \Big|$$

$$\lesssim \delta \Big\| \sum_{s \in \mathbf{S}'} \mathbf{1}_{R_s} \Big\|_1 .$$

We shall verify that $\mathbf{S}'$ satisfies (5.4), so that

$$\begin{aligned} \Big\| \sum_{s \in \mathbf{S}'} \mathbf{1}_{R_s} \Big\|_1 &\lesssim |\mathrm{sh}(\mathbf{S}')|^{1/2} \Big\| \sum_{s \in \mathbf{S}'} \mathbf{1}_{R_s} \Big\|_2 \\ &\lesssim \delta^{-2} |\mathrm{sh}(\mathbf{S}')|. \end{aligned}$$

Thus, (5.3) holds.

Our principal contention is (5.4). Fix an $s_0 \in \mathbf{S}'$, write $R_{s_0} = I_{s_0} \times J$ and

$$\mathbf{S}_0 = \{ s \in \mathbf{S} \,:\, \mathrm{scl}(s) \ge \mathrm{scl}(s_0),\ R_s \cap R_{s_0} \ne \emptyset \}.$$

We may normalize s_0 so that R_{s_0} is a rectangle in the canonical coordinates of the plane. Then the intervals $\boldsymbol{\omega}_s$, for $s \in \mathbf{S}_0$, can be identified with intervals in say $(1/4, 1/4) \subset \mathbb{R}$. In particular, $c(\boldsymbol{\omega}_s)$ is identified with a real number. Write $R_{s_0} = I_0 \times J$, where $|I_0| \simeq \mathrm{scl}(s_0)^{-1}$, and J has an endpoint j. In what follows, the product of intervals is to be understood in the canonical coordinates.

For $s \in \mathbf{S}_0$, recall that $|R_s \cap v^{-1}(\boldsymbol{\omega}_s)| \ge \delta |R_s|$. Denote by I_s the minimal dyadic subinterval of I_{s_0} that contains the projection of one of the long sides of R_s onto $I_{s_0} \times \{j\}$. And denote the projection of $R_s \cap v^{-1}(\boldsymbol{\omega}_s)$ onto the interval $I_{s_0} \times \{j\}$ by F_s. Then $|F_s| \gtrsim \delta \mathrm{scl}(s)^{-1} \simeq \delta |I_s|$.

Select a distinguished subset $\mathbf{S}_1$ of $\mathbf{S}_0$ by the following mechanism. Initialize

$$\mathbf{S}^{\mathrm{stock}} := \mathbf{S}_0, \qquad \mathbf{S}_1 := \emptyset.$$

While $\mathbf{S}^{\mathrm{stock}} \ne \emptyset$, select $s \in \mathbf{S}^{\mathrm{stock}}$ for which $\mathrm{scl}(s)$ is minimal. Update

$$\mathbf{S}_1 = \{s\} \cup \mathbf{S}_1, \qquad \mathbf{S}^{\mathrm{stock}} = \{ s' \in \mathbf{S}^{\mathrm{stock}} \,:\, F_{s'} \cap F_s = \emptyset \}.$$

Then, for $s_1 \in \mathbf{S}_1$, set $\mathbf{S}_1(s_1)$ to be the collection of tiles $s \in \mathbf{S}_0$ such that $F_s \cap F_{s_1} \ne \emptyset$ and $I_s \subset I_{s_1}$. We have that $\mathbf{S}_0$ is a union of the tiles in $\mathbf{S}_1(s_1)$, for $s_1 \in \mathbf{S}_1$. The next proposition is a central contention in this proof.

5.7. **Proposition.** *For any subinterval $I \subset I_{s_0}$, we have the two estimates*

$$\sum_{\substack{s_1 \in \mathbf{S}_1 \\ I_{s_1} \subset I}} |I_{s_1} \times J| \lesssim \delta^{-1} |I \times J|, \tag{5.8}$$

$$\sum_{\substack{s \in \mathbf{S}_1(s_1) \\ R_s \cap R_{s_0} \subset I \times J}} |R_s \cap R_{s_0}| \lesssim \delta^{-1} |I \times J|, \qquad s_1 \in \mathbf{S}_1. \tag{5.9}$$

Observe that both bounds are of the type associated with Carleson measures. In particular, a straightforward inductive argument, of the type associated to the John–Nirenberg inequality, then shows that

$$\Big\| \sum_{\substack{s_1 \in \mathbf{S}_1 \\ I_{s_1} \subset I_{s_0}}} \mathbf{1}_{I_{s_1} \times J} \Big\|_p \lesssim \delta^{-1} |R_{s_0}|^{1/p}, \qquad 2 < p < \infty,$$

Further observe that in the case that the annular parameter of all tiles is fixed, we have

$$|R_s \cap R_{s_0}| \simeq |I_s \times J|, \quad \text{and} \quad \mathbf{1}_{R_s \cap R_{s_0}} \lesssim (M \mathbf{1}_{I_s \times J})^2,$$

where M is the strong maximal function in the canonical coordinates. Thus, (5.4) follows from the Fefferman–Stein maximal inequalities. It remains to prove Proposition 5.7.

Concerning the lemma 5.2, our argument will prove (5.9) in the case when the value of $\operatorname{ann}(s)$ varies. But in this case it does not seem that the John–Nirenberg arguments apply. That is why this Lemma only asserts the weak–type inequality for $p = 2$.

Proof. The proof of (5.8) is nearly immediate. The projected sets $\{F_{s_1} : s_1 \in \mathbf{S}_1\}$ are disjoint, contained in I, and have measure at least $\gtrsim \delta |I_{s_1}|$. This gives (5.8), and we turn to the more subtle inequality (5.9).

Observe that by Proposition 5.6, we have

$$(5.10) \qquad |c(\boldsymbol{\omega}_s) - c(\boldsymbol{\omega}_{s_1})| \lesssim |c(\boldsymbol{\omega}_{s_1})| \frac{\|v\|_{\mathrm{Lip}}}{\operatorname{scl}(s_1)}, \qquad s \in \mathbf{S}(s_1).$$

That is, the angle of s is very close to the angle for s_1.

There is an essential geometric observation to make. Suppose that there is an interval $I \subset I_0$ and a choice of $s_1 \in \mathbf{S}_1$ such that

$$(5.11) \qquad \sum_{\substack{s \in \mathbf{S}(s_1) \\ \operatorname{scl}(s)^{-1} \geq 4|I|}} |R_s \cap I \times J| \geq 10^3 \delta^{-1} |I \times J|.$$

Then, for either $\varepsilon = +1$ or $\varepsilon = -1$, there can be no $s' \in \mathbf{S}(s_1)$ with $2 \operatorname{scl}(s')^{-1} < |I|$ and $R_{s'}$ intersects $\frac{1}{2}(I + \varepsilon |I|) \times J$.

Indeed, let π be the projection onto the first canonical coordinate. Choose $\varepsilon \in \{\pm 1\}$ so that

$$\sum_{\substack{s \in \mathbf{S}(s_1) \\ \operatorname{scl}(s)^{-1} \geq 4|I|}} \mathbf{1}_{\{I + \varepsilon |I| \subset \pi(R_s)\}} |R_s \cap I \times J| \geq \tfrac{1}{2} 10^3 \delta^{-1} |I \times J|$$

Let $(e, e_\perp)$ be the coordinate axes of R_{s_0}. Recall that M_{100} is a maximal function over 100 uniformly distributed directions of the plane. Choose the directions $(e', e'_\perp)$ from these 100 that are closest to $(e_\perp, -e)$, in that order. Consider a rectangle R' in the

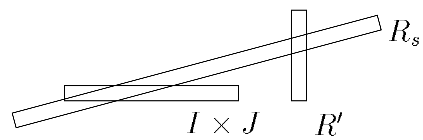

Figure 1: Proof of the essential geometric observation

$(e', e'_\perp)$ coordinates, of dimensions $10c(\boldsymbol{\omega}_{s_1})|I|$ by $|J|$, and whose center is contained in $I + \varepsilon|I| \times J$. For any $s \in \mathbf{S}(s_1)$ with $\operatorname{scl}(s)^{-1} \geq |I|$, and $R_s \cap I \times J = \emptyset$, we have

$$|R_s \cap (2I) \times J| \simeq \frac{|J|\operatorname{ann}(s)^{-1}}{|c(\boldsymbol{\omega}_{s_1})|}, \qquad |R_s \cap R'| \simeq |J|\operatorname{ann}(s)^{-1}.$$

The ratio between these two quantities is the ratio of $|I|$ to the length of R'. This is the main use of Proposition 5.6, and in particular (5.10). Hence

$$R' \subset \Big\{ M_{100} \sum_{s \in \mathbf{S}'} \mathbf{1}_{R_s} > \delta^{-1} \Big\}.$$

Letting R' vary, we see that if there were an $s' \in \mathbf{S}(s_1)$ with $2\operatorname{scl}(s') < |I|$ and $R_{s'} \cap \frac{1}{2}(I + \varepsilon|I|) \times J = \emptyset$, we would have contradicted the construction of $\mathbf{S}'$.

Set

$$\mathbf{S}(s_1, I) := \{ s \in \mathbf{S}_1(s_1) \,:\, R_s \cap R_{s_0} \subset I \times J \}.$$

We shall inductively decompose this collection as follows. Initialize

$$\mathbf{S}^{\text{stock}} := \mathbf{S}(s_1; I), \qquad \mathcal{I} := \emptyset.$$

While there is a dyadic interval $I \subset I_0$ for which

$$\sum_{\substack{s \in \mathbf{S}^{\text{stock}} \\ \operatorname{scl}(s)^{-1} \geq 4|I|}} |R_s \cap I \times J| \geq 10^4 \delta^{-1} |I \times J| \tag{5.12}$$

let I be a maximal dyadic interval satisfying this condition. Note three points, (a) that the bound we are requiring is somewhat larger than what occurs in (5.11), (b) with the power of δ^{-1} that occurs here, Proposition 5.5 implies that $I \subset_{\neq} I' \subset_{\neq} I_0$ for some I', and (c) therefore, we see that the sum above is at most $\leq 2 \cdot 10^4 \delta^{-1} |I \times J|$.

Define $\mathbf{S}(I) := \{ s \in \mathbf{S}^{\text{stock}} \,:\, \operatorname{scl}(s)^{-1} \geq |I|,\ R_s \cap I \times J \neq \emptyset \}$, and update

$$\mathbf{S}^{\text{stock}} := \mathbf{S}^{\text{stock}} - \mathbf{S}(I), \qquad \mathcal{I} := \mathcal{I} \cup \{I\}.$$

If there is no such interval I, set $\mathbf{S}_2 := \mathbf{S}^{\text{stock}}$, and the procedure stops.

Now, it is clear that

$$\sum_{s\in \mathbf{S}_2} |R_s \cap R_{s_0}| \le 10^4 \delta^{-1} |R_{s_1}|.$$

By our geometric observation, observe that $\mathcal{I}$ cannot contain three dyadic interval I, I', I'', with $I'' \subset_{\neq} I' \subset I + \varepsilon |I|$, where $\varepsilon \in \{\pm 1\}$ comes from our geometric observation, and is permitted to depend upon I. This particular condition implies that

$$\sum_{I\in\mathcal{I}} |I| \le 4|I_0|.$$

But then, it is follows that

$$\sum_{s\in \mathbf{S}(s_1,I)} |R_s \cap R_{s_0}| \lesssim \delta^{-1} |I\times J|,$$

with the implied constant being absolute. Our proof is complete. □

6. Orthogonality Between Annuli

We are to prove Lemma 2.19, and to do so rely upon a technical lemma on Fourier localization in the next subsection. We change scales, to assume that the vector field has C^α norm at most one. The inequality we are to establish is that

$$\Big\| \sum_{\mathsf{ann}>\mathsf{ann}_0} \mathcal{C}_{\mathsf{ann}} f \Big\|_2 \lesssim (1+(\log 1 + \mathsf{ann}_0^{-1}))^{3/2} \|f\|_2$$

And, here the principal estimate is

$$\Big\| \sum_{\mathsf{ann}>1} \mathcal{C}_{\mathsf{ann}} f \Big\|_2 \lesssim \|f\|_2, \qquad \|v\|_{C^\alpha} \le 1. \tag{6.1}$$

In the case that $\mathsf{ann}_0 < 1$, we use Cauchy–Schwarz and Lemma 2.18 to see that

$$\begin{aligned}\Big\| \sum_{\mathsf{ann}>\mathsf{ann}_0} \mathcal{C}_{\mathsf{ann}} f \Big\|_2 &\lesssim \sqrt{|\log \mathsf{ann}_0|} \Big[\sum_{\mathsf{ann}_0 \le \mathsf{ann} < 1} \|\mathcal{C}_{\mathsf{ann}}\|_2^2 \Big]^{1/2} \\ &\lesssim |\log \mathsf{ann}_0|^2 \|f\|_2.\end{aligned}$$

To establish (6.1), let λ be a Schwartz function on the plane with

$$\mathbf{1}_{\{\frac12<|\xi|<4\}} \le \widehat{\lambda}(\xi) < \mathbf{1}_{\{\frac14<|\xi|<8\}},$$

and set $\lambda_{\mathsf{ann}} = \mathsf{ann}^2 \lambda(\mathsf{ann}\, y)$. Since $\mathcal{C}_{\mathsf{ann}} f = \mathcal{C}_{\mathsf{ann}} \lambda_{\mathsf{ann}} * f$, we need only show that

$$\|\mathcal{C}_{\mathsf{ann},\mathsf{scl}} - \lambda_{\mathsf{ann}} * \mathcal{C}_{\mathsf{ann},\mathsf{scl}}\|_2 \lesssim \mathsf{ann}^{(1-\alpha)/2}, \qquad 1 \le \mathsf{scl} < \frac18 \mathsf{ann}.$$

$$\mathcal{C}_{\mathsf{ann},\mathsf{scl}} f := \sum_{\substack{s\in\mathcal{AT}(\mathsf{ann})\\ \mathsf{scl}(s)=\mathsf{scl}}} \langle f, \varphi_s\rangle \phi_s.$$

Recall that $\alpha > 1$, so the exponent on $\mathsf{ann} \geq 1$ is negative and so this is just summed over the $\log(\mathsf{ann})$ values of scl to prove (6.1).

The difference above is estimated by brute force

$$\sum_{\mathsf{scl}(s)=\mathsf{scl}} |\langle f, \varphi_s\rangle| |\lambda_{\mathsf{ann}} * \varphi_s - \varphi_s|$$

By Lemma 6.2, the last difference is dominated by a sum of three terms, as specified in (6.3)—(6.5). Of these, the most delicate is

$$\mathsf{ann}^{(1-\alpha)/2} \Big\| \sum_{\mathsf{scl}(s)=\mathsf{scl}} \frac{|\langle f, \varphi_s\rangle|}{|R_s|^{1/2}} [\operatorname{Max} \chi_{R_s}^{(\infty)} \mathbf{1}_{\boldsymbol{\omega}_s}(v(x))]^2 \Big\|_2$$

where Max is the usual maximal function. The leading term is less than one, since $\alpha > 1$. One uses the Fefferman–Stein maximal inequalities to estimate the norm as

$$\Big\| \sum_{\mathsf{scl}(s)=\mathsf{scl}} \frac{|\langle f, \varphi_s\rangle|}{|R_s|^{1/2}} \chi_{R_s}^{(\infty)} \mathbf{1}_{\boldsymbol{\omega}_s}(v(x)) \Big\|_2 \lesssim \Big[\sum_{\mathsf{scl}(s)=\mathsf{scl}} |\langle f, \varphi_s\rangle|^2 \Big]^{1/2} \lesssim \|f\|_2$$

This follows from the fact that with the scale fixed, the intervals $\boldsymbol{\omega}_s$ are either equal or disjoint, that the functions $\{\varphi_s : \mathrm{scl}(s) = \mathsf{scl}\}$ satisfy a Bessel inequality, and the decay of $\chi_{R_s}^{(\infty)}$.

There are still two more terms that arise from Lemma 6.2, but they are easier to control, and so the details are omitted.

6.1. **A technical estimate.** The precise form of the inequalities quantifying the Fourier localization effect follow.

6.2. **Lemma.** *Let $1 < \alpha < 2$, and v be a vector field with*

$$\|\nabla v\|_{C^\alpha} \leq 1.$$

Let s a tile with

$$V \leq \mathrm{scl}(s) = \mathsf{scl} \leq \mathrm{ann}(s) = \mathsf{ann} < \tfrac{1}{16} 2^k.$$

Let

$$f_s = \operatorname{Mod}_{-c(\omega_s)} \phi_s$$

Let ζ be a smooth function with $\mathbf{1}_{[0,2)}(|\xi|) \leq \widehat{\zeta} \leq \mathbf{1}_{[0,3)}(|\xi|)$ and set $\zeta_{2^k}(y) = 2^{2k}\zeta(y2^k)$. We have this inequality.

$$|f_s - \zeta_{2^k} * f_s| \lesssim |R_s|^{-1/2} 2^{(1-\alpha)k/2} \big(\operatorname{Max} \chi_s^{(\infty)} \mathbf{1}_{\boldsymbol{\omega}_s}(v)\big)^2 \tag{6.3}$$

$$+ 2^{-10k} \chi_s^{(2)} \tag{6.4}$$

$$+ |R_s|^{-1/2} \mathbf{1}_{F_s}. \tag{6.5}$$

In this estimate, Max *is the usual maximal function on the plane, and the sets $F_s \subset \mathbb{R}^2$ satisfy*

$$|F_s| \lesssim 2^{-k/3} |R_s|, \tag{6.6}$$

$$\Big\| \sum_{s\,:\,\mathsf{scl}(s)=\mathsf{scl}} \mathbf{1}_{F_s} \Big\|_\infty \lesssim 2^{k/200} \tag{6.7}$$

We rely upon the obvious estimate

$$\int_{|y|>t2^{-k}} |y2^k||\zeta_{2^k}(y)|\,dy \lesssim t^{-N}, \qquad t>1. \tag{6.8}$$

This estimate holds for all $N>1$. Likewise,

$$\int_{|u|>t\mathsf{scl}} |u\mathsf{scl}||\mathsf{scl}\,\psi(\mathsf{scl}\,u)|\,du \lesssim t^{-N}, \qquad t>1. \tag{6.9}$$

More significantly, we have

$$\int_{\mathbb{R}^2} e^{i\xi_0\cdot y}\varphi^{(2)}_{R_s}(x-y)\zeta_{2^k}(y)\,dy = \varphi^{(2)}_{R_s}(x) \qquad x\in\mathbb{R}^2,\ |\xi_0|<2^{k+1}. \tag{6.10}$$

This is clear from the frequency side. Likewise, for vectors v_0 of unit length,

$$\int_{\mathbb{R}} e^{iu\lambda_0}\varphi^{(2)}_{R_s}(x-uv_0)\mathsf{scl}\,\psi(\mathsf{scl}\,u)\,du \neq 0$$

implies that

$$\mathsf{scl} \le \lambda_0+\xi v_0 \le \tfrac{9}{8}\mathsf{scl}, \quad \text{for some } \xi\in\operatorname{supp}(\widehat{\varphi^{(2)}_{R_s}}). \tag{6.11}$$

At this point, it is useful to recall that we have specified the frequency support of φ to be in a small ball of radius κ in (2.15). This has the implication that

$$|\xi\cdot e_s| \le \kappa\mathsf{scl}, \quad |\xi\cdot e_{s\perp}| \le \kappa\mathsf{ann} \qquad \xi\in\operatorname{supp}(\widehat{\varphi^{(2)}_{R_s}}) \tag{6.12}$$

We begin the main line of the argument. Let $\varepsilon_1,\varepsilon_2$ be strictly positive quantities to be chosen. We have the estimate

$$|f_s(x)| + |\zeta_{2^k} * f_s(x)| \lesssim 2^{-10k}\chi^{(2)}_{R_s}(x), \qquad x\notin 2^{\varepsilon_1 k}R_s.$$

This follows from (6.8). And is as claimed in (6.4). We need only consider $x\in 2^{\varepsilon_1 k}R_s$.

Let us define the sets F_s, as in (6.5). Let

$$\lambda_s := \begin{cases} 8 & \text{if } \frac{\mathsf{ann}}{\mathsf{scl}} < 2^{(1-\varepsilon_1)k} \\ 2^{\varepsilon_1 k} & \text{otherwise} \end{cases} \tag{6.13}$$

Let $\lambda\omega_s$ denote the interval on the unit circle with length $\lambda|\boldsymbol{\omega}_s|$, and the same center as $\boldsymbol{\omega}_s$. We take

$$F_s = 2^{\varepsilon_1 k}R_s \cap v^{-1}(\lambda_s\boldsymbol{\omega}_s) \cap \Big\{\Big|\frac{\partial v}{\partial e_s}\cdot e_{s\perp}\Big| > 2^{(1-\varepsilon_2)k}\frac{\mathsf{scl}}{\mathsf{ann}}\Big\}. \tag{6.14}$$

It is clear that

$$\Big\| \sum_{s\,:\,\mathsf{scl}(s)=\mathsf{scl}} \mathbf{1}_{F_s} \Big\|_\infty \lesssim \lambda_s \lesssim 2^{\varepsilon_1 k}.$$

And so to satisfy (6.7), we should take $2\varepsilon_1 < 1/600$.

Let us argue that the measure of F_s satisfies (6.6). Fix a line ℓ in the direction of e_s. We should see that

$$|\ell \cap F_s| \lesssim 2^{-k(1/3+\varepsilon_1)}\mathsf{scl}^{-1}. \tag{6.15}$$

This set consists of open intervals $A_n = (a_n, b_n)$, $1 \le n \le N$. List them so that $b_n < a_{n+1}$ for all n. Partition the integers $\{1, 2, \dots, N\}$ into sets of consecutive integers $I_\sigma = [m_\sigma, n_\sigma] \cap \mathbb{N}$ so that for all points x between A_{m_σ} and A_{n_σ} the derivative $\partial v(x)/\partial e_s \cdot e_{s\perp}$ has the same sign. Take the intervals I_σ to be maximal with respect to this property.

For $x \in F_s$, the partial derivative of v, in the direction that is transverse to $\lambda_s \boldsymbol{\omega}_s$, is large with respect to the length of $\lambda_s \boldsymbol{\omega}_s$. Hence

$$\sum_{m \in I_\sigma} |A_m| \lesssim 2^{-(1-\varepsilon_1-\varepsilon_2)k} \qquad \text{for all } \sigma.$$

Now consider intervals A_{n_σ} and $A_{1+n_\sigma} = A_{m_{\sigma+1}}$. By definition, there must be a change of sign of $\partial v(x)/\partial e_s \cdot e_{s\perp}$ between these two intervals. And so there is a change in this derivative that is at least as big as $2^{(1-\varepsilon_2)k}\frac{\mathsf{scl}}{\mathsf{ann}}$. The partial derivative is also Hölder continuous of index $\alpha - 1$, which implies that

$$\operatorname{dist}(A_{n_\sigma}, A_{m_{\sigma+1}}) \ge \big(2^{(1-\varepsilon_2)k}\frac{\mathsf{scl}}{\mathsf{ann}}\big)^{1/(\alpha-1)}$$

As all of the intervals A_n lie in an interval of length $2^{\varepsilon_1 k}\mathsf{scl}^{-1}$, it follows that there can be at most

$$\lesssim 2^{\varepsilon_1 k}\mathsf{scl}^{-1}\big(2^{(1-\varepsilon_2)k}\frac{\mathsf{scl}}{\mathsf{ann}}\big)^{-1/(\alpha-1)}$$

intervals I_σ. Consequently,

$$\begin{aligned} |\ell \cap F_s| &\lesssim 2^{-(1-2\varepsilon_1-\varepsilon_2+(1-\varepsilon_2)/(\alpha-1))k}\mathsf{scl}^{-1}\big(\frac{\mathsf{ann}}{\mathsf{scl}}\big)^{1/(\alpha-1)} \\ &\lesssim 2^{-(1-2\varepsilon_1-\varepsilon_2/(\alpha-1))k}\mathsf{scl}^{-1} \end{aligned}$$

It is now clear that we can choose ε_1 and ε_2 to achieve the estimate (6.15). This completes the proof of (6.6).

For $x \in 2^{\varepsilon_1 k}R_s$, we always have the bound

$$|f_s(x) - \zeta_{2^k} * f_s(x)| \lesssim 2^{10\varepsilon_1 N k}|R_s|^{-1/2}\big(\operatorname{Max}\chi_{R_s}^{(\infty)}\mathbf{1}_{\boldsymbol{\omega}_s}\big)^{10}.$$

Here, N is only a function of κ as appears in (4.1). Note that we are still free to take ε_1 quite small. We establish the bound

$$|f_s(x) - \zeta_{2^k} * f_s(x)| \lesssim 2^{(1-\alpha)k}|R_s|^{-1/2}, \qquad x \in 2^{\varepsilon_1 k}R_s - F_s. \tag{6.16}$$

These two bounds will prove (6.3).

To ease the burden of notation, we set

$$e(x) := e^{2\pi i u c(\omega_s)\cdot v(x)}, \qquad \Phi(x, x') = \varphi_{R_s}^{(2)}(x - uv(x')),$$

with the dependency on u being suppressed, and define

$$w(du,dy) := \mathsf{scl}\,\psi(\mathsf{scl}\,u)\zeta_{2^k}(y)\,du\,dy.$$

In this notation, the difference we are to estimate is a linear combination of

$$\begin{aligned}\mathrm{Diff}_1(x) &:= \int_{\mathbb{R}^2}\int_{\mathbb{R}} e(x)\Phi(x,x) - e(x-y)\Phi(x-y,x)w(du,dy)\\ \mathrm{Diff}_2(x) &:= \int_{\mathbb{R}^2}\int_{\mathbb{R}} e(x)\{\Phi(x-y,x-y) - \Phi(x-y,x)\}w(du,dy)\end{aligned}$$

The analysis of both terms is quite similar. We begin with the first term.

Note that by (6.10), we have

$$\mathrm{Diff}_1(x) = \int_{\mathbb{R}^2}\int_{\mathbb{R}}\{e(x)-e(x-y)\}\Phi(x-y,x)w(du,dy).$$

Observe that

$$\begin{aligned} e(x)-e(x-y) &= e(x)\{1-e(x-y)\overline{e(x)}\}\\ &= e(x)\{1-e^{iuc(\omega_s)\nabla v(x)y}\} + O(|u|\mathsf{ann}|y|^{\alpha}).\end{aligned} \tag{6.17}$$

In the Big–Oh term, $|u|$ is typically of the order scl^{-1}, and $|y|$ is of the order 2^{-k}. Hence, direct integration leads to the estimate of this term by

$$\lesssim (\mathsf{scl}\,2^{\alpha k})^{-1}|R_s|^{-1/2}.$$

This is better than in (6.16).

The term left to estimate is

$$\mathrm{Diff}_1'(x) := \int_{\mathbb{R}^2}\int_{\mathbb{R}} e(x)(1-e^{iuc(\omega_s)\nabla v(x)y})\Phi(x-y,x)w(du,dy).$$

Observe that by (6.10), the integral in y is zero if

$$|uc(\omega_s)\nabla v(x)| = \tfrac{3}{2}|u|\mathsf{ann}\Big|\frac{\partial v}{\partial e_{s\perp}}(x)\Big| \le 2^k.$$

Here we recall that $c(\omega_s) = \frac{3}{2}\mathsf{ann}\,e_{s\perp}$. If $v(x)\in\lambda_s\boldsymbol{\omega}_s$, we conclude by the definition of F_s that

$$\Big|\frac{\partial v}{\partial e_{s\perp}}(x)\Big| \le 2^{(1-\varepsilon_1)k}\frac{\mathsf{scl}}{\mathsf{ann}}.$$

Hence, the integral in y in $\mathrm{Diff}_1'(x)$ can be non–zero only for

$$\mathsf{scl}|u| \gtrsim 2^{\varepsilon_1 k}.$$

By (6.9), it follows that in this case we have the estimate

$$|\mathrm{Diff}_1'(x)| \lesssim 2^{-2k}|R_s|^{-1/2}$$

This estimate holds for $x\in 2^{\varepsilon_1 k}R_s\cap v^{-1}(\lambda_s\boldsymbol{\omega}_s) - F_s$.

We must also consider the case of $x \in 2^{\varepsilon_1 k} R_s - v^{-1}(\lambda_s \boldsymbol{\omega}_s)$. Observe that by (6.11), the integral in u in $\mathrm{Diff}_1'(x)$ will be zero unless there is a $\xi \in \mathrm{supp}(\widehat{\varphi_{R_s}^{(2)}})$ for which

$$\mathsf{scl} \lesssim c(\omega_s)\{v(x) - \nabla v(x)y\} + \xi v(x) \le \tfrac{9}{8}\mathsf{scl}.$$

Recalling (6.12), we see that for any such ξ, we have $|(c(\omega_s)+\xi)v(x)| \gtrsim \lambda_s \mathsf{scl}$. Hence, this condition can only be satisfied for

$$|y| \gtrsim \lambda_s \frac{\mathsf{scl}}{\mathsf{ann}} \gtrsim 2^{-(1-\varepsilon_1)k}$$

by the definition of λ_s in (6.13). But then, we can appeal to (6.8) to see that Diff_1' satisfies (6.16).

We consider the term Diff_2. The term $v(x-y)$ occurs twice in this term, in $e(x-y)$, and in $\Phi(x-y, x-y)$. We will use the approximation (6.17), and similarly,

$$\begin{aligned}\Phi(x-y, x-y) - \Phi(x-y, x) &= \varphi_{R_s}^{(2)}(x-y-uv(x-y)) - \varphi_{R_s}^{(2)}(x-y-uv(x)) \\ &= \varphi_{R_s}^{(2)}(x-y-uv(x) - u\nabla v(x)y) \\ &\qquad - \varphi_{R_s}^{(2)}(x-y-uv(x)) \\ &\qquad + O(\mathsf{ann}\,|u||y|^{\alpha}) \\ &:= \Delta\Phi(x,y) + O(\mathsf{ann}\,|u||y|^{\alpha})\end{aligned}$$

The Big–Oh term gives us, upon integration in u and y,

$$\lesssim |R_s|^{-1/2} \frac{\mathsf{ann}}{\mathsf{scl}} 2^{-\alpha k} \lesssim |R_s|^{-1/2} 2^{-(\alpha-1)k}.$$

This is as required by (6.16). We are left with estimating

$$\mathrm{Diff}_2'(x) := \int_{\mathbb{R}^2} \int_{\mathbb{R}} e^{iuc(\omega_s)(v(x)-u\nabla v(x)y)} \Delta\Phi(x,y)\, w(du, dy).$$

By (6.10), the integral in y is zero if both of these conditions hold.

$$|uc(\omega_s)\nabla v(x)| < 2^{k+1},$$

$$|uc(\omega_s)\nabla v(x) - \xi - u\xi\nabla v(x)| < 2^{k+1}, \qquad \xi \in \mathrm{supp}(\widehat{\varphi_{R_s}^{(2)}})$$

Just as in the analysis of Diff_1, assuming that $x \in 2^{-\varepsilon_1 k} R_s \cap v^{-1}(\lambda_s \boldsymbol{\omega}_s)$, the first condition is satisfied for $|u| \lesssim 2^{\varepsilon_1 k}$. Recalling the conditions (6.12), the second condition is also satisfied for the same set of values for u. The application of (6.9) then yields a very small bound after integrating $|u| \gtrsim 2^{\varepsilon_1 k}$.

We can consider $x \in 2^{-\varepsilon_1 k} R_s - v^{-1}(\lambda_s \boldsymbol{\omega}_s) - F_s$. By (6.11), the integral in u is zero if both of these conditions hold.

$$\mathsf{scl} < c(\boldsymbol{\omega}_s)(v(x) - \nabla v(x)y) + \xi(v(x) - \nabla v(x)y) \le \frac{9}{8}\mathsf{scl},$$

$$\mathsf{scl} < c(\boldsymbol{\omega}_s)(v(x) - \nabla v(x)y) + \xi v(x) \le \frac{9}{8}\mathsf{scl}.$$

The second condition is as arose in the analysis of Diff_1', and it leads to the condition

$$2^k|y| \gtrsim 2^{-\varepsilon_1 k}.$$

The first condition leads to the same conclusion, so that by appeal to (6.8), we can conclude the proof that Diff_2 obeys the estimate (6.16). This completes the proof of our Lemma.

References

[1] J. Bourgain, *A remark on the maximal function associated to an analytic vector field*, Analysis at Urbana, Vol. I (Urbana, IL, 1986–1987), London Math. Soc. Lecture Note Ser., vol. 137, Cambridge Univ. Press, Cambridge, 1989, pp. 111–132. MR 90h:42028

[2] Anthony Carbery, Andreas Seeger, Stephen Wainger, and James Wright, *Classes of singular integral operators along variable lines*, J. Geom. Anal. **9** (1999), 583–605. MR 2001g:42026

[3] Lennart Carleson, *On convergence and growth of partial sumas of Fourier series*, Acta Math. **116** (1966), 135–157. MR 33 #7774

[4] Michael Christ, *Personal Communication.*

[5] Michael Christ, Alexander Nagel, Elias M. Stein, and Stephen Wainger, *Singular and maximal Radon transforms: analysis and geometry*, Ann. of Math. (2) **150** (1999), 489–577. MR 2000j:42023

[6] Charles Fefferman, *Pointwise convergence of Fourier series*, Ann. of Math. (2) **98** (1973), 551–571. MR 49 #5676

[7] Nets Hawk Katz, *Maximal operators over arbitrary sets of directions*, Duke Math. J. **97** (1999), 67–79. MR 2000a:42036

[8] ______, *A partial result on Lipschitz differentiation.*

[9] Michael Lacey and Xiaochun Li, *Maximal Theorems for Directional Hilbert Transform on the Plane*, Preprint.

[10] Michael Lacey and Christoph Thiele, *A proof of boundedness of the Carleson operator*, Math. Res. Lett. **7** (2000), 361–370. MR 2001m:42009

[11] Camil Muscalu, Terence Tao, and Christoph Thiele, *Multi-linear operators given by singular multipliers*, J. Amer. Math. Soc. **15** (2002), 469–496 (electronic). MR 2003b:42017

[12] Alexander Nagel, Elias M. Stein, and Stephen Wainger, *Hilbert transforms and maximal functions related to variable curves*, Harmonic Analysis in Euclidean Spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1, Proc. Sympos. Pure Math., XXXV, Part, Amer. Math. Soc., Providence, R.I., 1979, pp. 95–98. MR 81a:42027

[13] D. H. Phong and Elias M. Stein, *Hilbert integrals, singular integrals, and Radon transforms. II*, Invent. Math. **86** (1986), 75–113. MR 88i:42028b

[14] ______, *Hilbert integrals, singular integrals, and Radon transforms. I*, Acta Math. **157** (1986), 99–157. MR 88i:42028a

[15] Elias M. Stein, *Problems in harmonic analysis related to curvature and oscillatory integrals*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, pp. 196–221. MR 89d:42028

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