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Estimates for Green's matrix of boundary value problems for the Stokes system in a polyhedral cone

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Abstract

The paper deals with a boundary value problem for the Stokes system in a polyhedral cone. The authors obtain regularity results for weak solutions in weighted L_2 Sobolev spaces and point estimates of Green's matrix.

1 Introduction

The present paper is concerned with a boundary value problem for the Stokes system

$$-\Delta u + \nabla p = f, \quad -\nabla \cdot u = g$$

in a three-dimensional polyhedral cone, where on each side Γ_j of the cone one of the following boundary conditions is given:

(C₁) $u = h$ (Dirichlet condition),

(C₂) $u_\tau = h, \quad -p + 2\varepsilon_{n,n}(u) = \phi,$

(C₃) $u_n = h, \quad \varepsilon_{n,\tau}(u) = \phi$ (free surface condition),

(C₄) $-pn + 2\varepsilon_n(u) = \phi$ (Neumann condition).

Here n denotes the exterior normal to Γ_j , $u_n = u \cdot n$ is the normal and $u_\tau = u - u_n n$ the tangential component of u . Furthermore, $\varepsilon(u)$ denotes the matrix with the components

$$\varepsilon_{i,j}(u) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

$\varepsilon_n(u)$ is the vector $\varepsilon(u)n$, $\varepsilon_{n,n} = \varepsilon_n(u) \cdot n$ its normal component and $\varepsilon_{n,\tau}(u)$ its tangential component.

Our goal is to obtain estimates for Green's matrix. For the case of the Dirichlet problem such estimates were obtained by Maz'ya and Plamenevskii [11], for the problem with boundary conditions (C₁) and (C₃) in a dihedron we refer to the paper of Maz'ya, Plamenevskii, Stupelis [12] and to the book of Stupelis [21]. As in [11, 12, 21] we obtain point estimates of Green's matrix by means of weighted L_2 estimates of the solutions and their derivatives. However, while the problem with Dirichlet boundary conditions can be

handled in weighted Sobolev spaces with so-called homogeneous norms, the more general boundary value problem considered in the present paper requires the use of weighted Sobolev spaces with inhomogeneous norms. Note that the estimates of Green's matrix of the Neumann problem to strongly elliptic second order systems in our previous paper [14] were also obtained by means of estimates in weighted Sobolev spaces with homogeneous norms.

We outline the main results of the paper. In Section 2 we deal with the boundary value problem for the Stokes system in a dihedron \mathcal{D} , where on both sides Γ^+ , Γ^- one of the boundary conditions (C_1) and (C_3) is given. We consider weak solutions $(u, p) \in \mathcal{H} \times L_2(\mathcal{D})$, where \mathcal{H} is the closure of $C_0^\infty(\bar{\mathcal{D}})^3$ (the set of infinitely differential vector functions on $\bar{\mathcal{D}}$ having compact support) with respect to the norm

$$\|u\|_{\mathcal{H}} = \left(\int_{\mathcal{D}} \sum_{j=1}^3 |\partial_{x_j} u|^2 dx \right)^{1/2}$$

and obtain regularity assertions for the solutions. The smoothness of the solutions near the edge depends on the smoothness of the data and on the eigenvalues of a certain operator pencil $A(\lambda)$ generated by the corresponding problem in a two-dimensional angle. These eigenvalues can be calculated as the zeros of certain transcendental functions which were established in [12, 21] for the boundary conditions (C_1) and (C_3) and by Orlt and Sändig [18] for the more general case. For example, in the case of the Dirichlet problem, the set of the eigenvalues of the pencil $A(\lambda)$ consists of the numbers $j\pi/\alpha$ (where α is the angle at the edge and j is an arbitrary nonzero integer) and of all nonzero solutions of the equation $\lambda \sin \alpha \pm \sin(\lambda\alpha) = 0$.

One of our results is the following. Let λ_1 be the eigenvalue of $A(\lambda)$ with smallest positiv real part, and let $W_\delta^l(\mathcal{D})$ be the closure of $C_0^\infty(\bar{\mathcal{D}})$ with respect to the norm

$$\|u\|_{W_\delta^l(\mathcal{D})} = \left(\int_{\mathcal{D}} \sum_{|\alpha| \leq l} r^{2\delta} |\partial_x^\alpha u(x)|^2 dx \right)^{1/2}, \quad (1.1)$$

where r denotes the distance to the edge. If $f \in W_\delta^{l-2}(\mathcal{D})^3$, $g \in W_\delta^{l-1}(\mathcal{D})$, and the boundary data belong to corresponding trace spaces, where $\max(l-1-\operatorname{Re} \lambda_1, 0) < \delta < l-1$, then we obtain $\zeta(u, p) \in W_\delta^l(\mathcal{D})^3 \times W_\delta^{l-1}(\mathcal{D})$ for an arbitrary smooth function ζ with compact support. In some cases, when $\lambda_1 = 1$ (e.g., in the case of the Dirichlet problem, $\alpha < \pi$), the eigenvalue λ_1 can be replaced by the second eigenvalue λ_2 .

By means of this result, we obtain point estimates for Green's matrix $(G_{i,j}(x, \xi))_{i,j=1}^4$. For example, if the edge of \mathcal{D} coincides with the x_3 -axis, then

$$\begin{aligned} |\partial_{x'}^\alpha \partial_{x_3}^\sigma \partial_{\xi'}^\beta \partial_{\xi_3}^\tau G_{i,j}(x, \xi)| &\leq c |x - \xi|^{-1-\delta_{i,4}-\delta_{j,4}-|\alpha|-|\beta|-\sigma-\tau} \\ &\times \left(\frac{|x'|}{|x - \xi|} \right)^{\min(0, \mu - |\alpha| - \delta_{i,4} - \varepsilon)} \left(\frac{|\xi'|}{|x - \xi|} \right)^{\min(0, \mu - |\beta| - \delta_{j,4} - \varepsilon)}, \end{aligned}$$

for $|x - \xi| \geq \min(|x'|, |\xi'|)$, where $x' = (x_1, x_2)$, $\xi' = (\xi_1, \xi_2)$, $\mu = \operatorname{Re} \lambda_1$, and ε is an arbitrarily small positive number. Again in some cases the eigenvalue λ_1 can be replaced by λ_2 what improves the estimates given in [11, 12, 21].

Section 3 is concerned with the boundary value problem in a polyhedral cone \mathcal{K} with vertex at the origin and edges M_1, \dots, M_n . The smoothness of solutions in a neighborhood of an edge point $x_0 \in M_k$ depends again on the eigenvalues of certain operator pencils $A_k(\lambda)$ which can be calculated as zeros of special transcendental functions. The smoothness of solutions in a neighborhood of the vertex depends additionally on the eigenvalues of a certain operator pencil $\mathfrak{A}(\lambda)$. Here $\mathfrak{A}(\lambda)$ is the operator of a parameter-depending boundary value problem on the intersection of the cone \mathcal{K} with the unit sphere. Spectral properties of this operator pencil are given in papers by Dauge [2], Kozlov, Maz'ya and Schwab [7] for the Dirichlet problem, by Kozlov and Maz'ya [4] for the Neumann problem and in the book by Kozlov, Maz'ya and Rossmann [6] for boundary conditions (C_1) - (C_3) . We prove the unique existence of a weak solution in $W_{\beta,0}^1(\mathcal{K})^3 \times W_{\beta,0}^0(\mathcal{K})$ if the line $\operatorname{Re} \lambda = -\beta - 1/2$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$, where $W_{\beta,0}^l(\mathcal{K})$ is the closure of $C_0^\infty(\bar{\mathcal{K}} \setminus \{0\})$ with respect to the norm

$$\|u\|_{W_{\beta,0}^l(\mathcal{K})} = \left(\sum_{|\alpha| \leq l} |x|^{2(\beta-l+|\alpha|)} |\partial_x^\alpha u(x)|^2 dx \right)^{1/2}.$$

The absence of eigenvalues of the pencil $\mathfrak{A}(\lambda)$ guarantees also the existence of a Green matrix $(G_{i,j}(x, \xi))_{i,j=1}^4$ of the boundary value problem in the cone \mathcal{K} such that the functions $x \rightarrow \zeta(|x - \xi|/r(\xi)) G_{i,j}(x, \xi)$ belong to $W_{\beta,0}^1(\mathcal{K})$ for every $\xi \in \mathcal{K}$, $i = 1, 2, 3$ and to $W_{\beta,0}^0(\mathcal{K})$ for $i = 4$. Here ζ is an arbitrary smooth function on $[0, \infty)$ equal to one in $(1, \infty)$ and to zero in $(0, \frac{1}{2})$. In the last subsection we derive point estimates of this Green matrix. In the case $|x|/2 < |\xi| < 2|x|$ we obtain analogous estimates to the case of a dihedron, while in the case $|\xi| < |x|/2$ the following estimate holds:

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\gamma G_{i,j}(x, \xi)| &\leq c |x|^{\Lambda_- - \delta_{i,4} - |\alpha| + \varepsilon} |\xi|^{-\Lambda_- - 1 - \delta_{j,4} - |\gamma| - \varepsilon} \\ &\times \prod_{k=1}^n \left(\frac{r_k(x)}{|x|} \right)^{\min(0, \mu_k - |\alpha| - \delta_{i,4} - \varepsilon)} \prod_{k=1}^n \left(\frac{r_k(\xi)}{|\xi|} \right)^{\min(0, \mu_k - |\gamma| - \delta_{j,4} - \varepsilon)}, \end{aligned}$$

Here $r_k(x)$ denotes the distance to the edge M_k , $\Lambda_- < \operatorname{Re} \lambda < \Lambda_+$ is the widest strip in the complex plane containing the line $\operatorname{Re} \lambda = -\beta - 1/2$ but no eigenvalues of the pencil $\mathfrak{A}(\lambda)$, and $\mu_k = \operatorname{Re} \lambda_1^{(k)}$, where $\lambda_k^{(1)}$ is the eigenvalue of the pencil $A_k(\lambda)$ with smallest positive real part. Note again that in some cases, when $\lambda_k^{(1)} = 1$, this eigenvalue can be replaced by the eigenvalue with smallest real part greater than 1.

In a forthcoming paper, the estimates of Green's matrix obtained in this paper will be used for the proof of weighted L_p and Schauder estimates of solutions to Stokes and Navier-Stokes equations in polyhedral domains.

2 The problem in a dihedron (L_2 -theory)

In the following let K be an infinite angle in the (x_1, x_2) -plane given in polar coordinates r, φ by the inequalities $0 < r < \infty$, $-\alpha/2 < \varphi < \alpha/2$. Furthermore, let \mathcal{D} be the dihedron $\{x = (x', x_3) : x' = (x_1, x_2) \in K, x_3 \in \mathbb{R}\}$. The sides $\{x : \varphi = \pm\alpha/2\}$ of \mathcal{D} are denoted by Γ^\pm and the edge $\bar{\Gamma}^+ \cap \bar{\Gamma}^-$ is denoted by M .

We consider a boundary value problem for the Stokes system, where on each of the sides Γ^\pm one of the boundary conditions (C_1) – (C_4) is given. Let $n^\pm = (n_1^\pm, n_2^\pm, 0)$ be the exterior normal to Γ^\pm , $\varepsilon_n^\pm(u) \varepsilon(u) n^\pm$ and $\varepsilon_{nn}^\pm(u) = \varepsilon_n^\pm(u) \cdot n^\pm$. Furthermore, let $d^\pm \in \{0, 1, 2, 3\}$ be integer numbers characterizing the boundary conditions on Γ^+ and Γ^- , respectively. We put

- $S^\pm u = u$ for $d^\pm = 0$,
- $S^\pm u = u - (u \cdot n^\pm) n^\pm$, $N^\pm(u, p) = -p + 2\varepsilon_{nn}^\pm(u)$ for $d^\pm = 1$,
- $S^\pm u = u \cdot n^\pm$, $N^\pm(u, p) = \varepsilon_n^\pm(u) - \varepsilon_{nn}^\pm(u) n^\pm$ for $d^\pm = 2$
- $N^\pm(u, p) = -pn^\pm + 2\varepsilon_n^\pm(u)$ for $d^\pm = 3$

and consider the boundary value problem

$$-\Delta u + \nabla p = f, \quad -\nabla \cdot u = g \quad \text{in } \mathcal{D}, \quad (2.1)$$

$$S^\pm u = h^\pm, \quad N^\pm(u, p) = \phi^\pm \quad \text{on } \Gamma^\pm. \quad (2.2)$$

Here the condition $N^\pm(u, p) = \phi^\pm$ is absent in the case $d^\pm = 0$, while the condition $S^\pm u = h^\pm$ is absent in the case $d^\pm = 3$.

2.1 Weighted Sobolev spaces

For arbitrary real δ we denote by $V_\delta^l(\mathcal{D})$ the closure of $C_0^\infty(\bar{\mathcal{D}} \setminus M)$ (the set of all infinitely differentiable functions with compact support in $\mathcal{D} \setminus M$) with respect to the norm

$$\|u\|_{V_\delta^l(\mathcal{D})} = \left(\int_{\mathcal{D}} \sum_{|\alpha| \leq l} r^{2(\delta - l + |\alpha|)} |\partial_x^\alpha u(x)|^2 dx \right)^{1/2},$$

where $r = |x'|$ denotes the distance to the edge. For real $\delta > -1$ let $W_\delta^l(\mathcal{D})$ be the closure of $C_0^\infty(\overline{\mathcal{D}})$ with respect to the norm (1.1). Furthermore, let $V_\delta^{l-1/2}(\Gamma^\pm)$ be the space of traces of functions from $V_\delta^l(\mathcal{D})$ on Γ^\pm . The norm in this space is

$$\|u\|_{V_\delta^{l-1/2}(\Gamma^\pm)} = \inf\{\|v\|_{V_\delta^l(\mathcal{D})} : v \in V_\delta^l(\mathcal{D}), v = u \text{ on } \Gamma^\pm\}.$$

Analogously, the norm in trace space $W_\delta^{l-1/2}(\Gamma^\pm)$ for $W_\delta^l(\mathcal{D})$ is defined. Note that the norm in $V_\delta^{l-1/2}(\Gamma^\pm)$ is equivalent (see [9, Le.1.4]) to

$$\begin{aligned} \|u\| &= \left(\int_{\gamma^\pm} \int_{\mathbb{R}} \int_{\mathbb{R}} r^{2\delta} |\partial_{x_3}^{l-1} u(r, x_3) - \partial_{y_3}^{l-1} u(r, y_3)|^2 \frac{dx_3 dy_3}{|x_3 - y_3|^2} dr \right. \\ &\quad + \int_{\mathbb{R}} \int_{\gamma^\pm} \int_{\gamma^\pm} |r_1^\delta (\partial_r^{l-1} u)(r_1, x_3) - (\partial_r^{l-1} u)(r_2, x_3)|^2 \frac{dr_1 dr_2}{|r_1 - r_2|^2} dx_3 \\ &\quad \left. + \int_{\Gamma^\pm} \sum_{j=0}^{l-1} r^{2(\delta-l+j)+1} |\partial_r^j u(r, x_3)|^2 dr dx_3 \right)^{1/2}. \end{aligned} \quad (2.3)$$

If δ is not integer, $-1 < \delta < l-1$, then the trace of an arbitrary function $u \in W_\delta^l(\mathcal{D})$ on the edge M belongs to the Sobolev space $W^{l-\delta-1}(M)$ (see, e.g., [13, Le.1.1]). Furthermore, the following lemma holds (see [13, Le.1.3]).

Lemma 2.1 1) *The space $W_\delta^l(\mathcal{D})$ is continuously imbedded into $W_{\delta-1}^{l-1}(\mathcal{D})$ if $\delta > 0$ and into $V_\delta^l(\mathcal{D})$ if $\delta > l-1$.*

2) *Let $u \in W_\delta^l(\mathcal{D})$, where δ is not integer, $\delta > -1$. Then for the inclusion $u \in V_\delta^l(\mathcal{D})$ it is necessary and sufficient that $\partial_x^\alpha u|_M = 0$ for $|\alpha| < l - \delta - 1$.*

Analogously to $V_\delta^l(\mathcal{D})$ and $W_\delta^l(\mathcal{D})$, we define the weighted Sobolev spaces $V_\delta^l(K)$ and $W_\delta^l(K)$ on the two-dimensional angle K . Here in the definition of the norms one has only to replace \mathcal{D} and x by K and x' , respectively. For the space $W_\delta^l(K)$ a result analogous to Lemma 2.1 holds.

2.2 Weak solutions of the boundary value problem

Let $L_2^1(\mathcal{D})$ be the closure of the set $C_0^\infty(\overline{\mathcal{D}})$ with respect to the norm

$$\|u\|_{L_2^1(\mathcal{D})} = \left(\int_{\mathcal{D}} \sum_{j=1}^3 |\partial_{x_j} u|^2 dx \right)^{1/2} \quad (2.4)$$

The closure of the set $C_0^\infty(\mathcal{D})$ with respect to this norm is denoted by $\mathring{L}_2^1(\mathcal{D})$.

Furthermore, let $\mathcal{H} = L_2^1(\mathcal{D})^3$ and $V = \{u \in \mathcal{H} : S^\pm u = 0 \text{ on } \Gamma^\pm\}$. In the case of the Dirichlet problem ($S^\pm u = u$), we have $V = \mathring{L}_2^1(\mathcal{D})^3$. Note that, by Hardy's inequality,

$$\int_{\mathcal{D}} |x|^{-2} |u|^2 dx \leq 12 \|u\|_{\mathcal{H}}^2 \quad \text{for } u \in C_0^\infty(\overline{\mathcal{D}}).$$

Therefore, the norm (2.4) is equivalent to the norm

$$\|u\| = \left(\int_{\mathcal{D}} (|x|^{-2} |u|^2 + \sum_{j=1}^3 |\partial_{x_j} u|^2) dx \right)^{1/2}, \quad (2.5)$$

and \mathcal{H} can be also defined as the closure of $C_0^\infty(\overline{\mathcal{D}})^3$ with respect to the norm (2.5). Obviously, every $u \in \mathcal{H}$ is quadratically summable on each compact subset of \mathcal{D} . From Hardy's inequality it follows that

$$\|u\|_{L_2(\mathcal{D})}^2 \leq \int_{\mathcal{D}} (r^{2\delta} + r^{2\delta-2}) |u|^2 dx \leq \int_{\mathcal{D}} r^{2\delta} (|u|^2 + c |\nabla u|^2) dx$$

for $u \in C_0^\infty(\bar{\mathcal{D}})$ and $0 < \delta < 1$, where c depends only on δ . Consequently, there are the continuous imbeddings $W_\delta^1(\mathcal{D}) \subset L_2(\mathcal{D})$ and $W_\delta^2(\mathcal{D})^3 \subset \mathcal{H}$ if $0 < \delta < 1$.

For the definition of weak solutions of problem (2.1), (2.2), we introduce the bilinear form

$$b(u, v) = 2 \int_{\mathcal{D}} \sum_{i,j=1}^3 \varepsilon_{i,j}(u) \varepsilon_{i,j}(v) dx. \quad (2.6)$$

Then the following Green formula is satisfied for all $u, v \in C_0^\infty(\bar{\mathcal{D}})^3$, $p \in C_0^\infty(\bar{\mathcal{D}})$:

$$b(u, v) - \int_{\mathcal{D}} p \nabla \cdot v dx = \int_{\mathcal{D}} (-\Delta u - \nabla \nabla \cdot u + \nabla p) \cdot v dx + \sum_{\pm} \int_{\Gamma^\pm} (-pn^\pm + 2\varepsilon(u)n^\pm) \cdot v dx. \quad (2.7)$$

Hence every solution $(u, p) \in W_\delta^2(\mathcal{D}) \times W_\delta^1(\mathcal{D})$ of problem (2.1), (2.2) satisfies the integral equality

$$b(u, v) - \int_{\mathcal{D}} p \nabla \cdot v dx = \int_{\mathcal{D}} (f + \nabla g) \cdot v dx + \sum_{\pm} \int_{\Gamma^\pm} \phi^\pm \cdot v dx.$$

for all $v \in C_0^\infty(\mathcal{D})^3$ (in the case $S^\pm v = v_\tau$, the function ϕ^\pm has to be replaced by $\phi^\pm n^\pm$).

By a weak solution of problem (2.1), (2.2) we mean a pair $(u, p) \in \mathcal{H} \times L_2(\mathcal{D})$ satisfying

$$b(u, v) - \int_{\mathcal{D}} p \nabla \cdot v dx = F(v) \quad \text{for all } v \in V, \quad (2.8)$$

$$-\nabla \cdot u = g \quad \text{in } \mathcal{D}, \quad S^\pm u = h^\pm \quad \text{on } \Gamma^\pm, \quad (2.9)$$

where

$$F(v) = \int_{\mathcal{D}} (f + \nabla g) \cdot v dx + \sum_{\pm} \int_{\Gamma^\pm} \phi^\pm \cdot v dx, \quad (2.10)$$

provided the functional (2.10) belongs to the dual space V^* of V . For example, $F \in V^*$ if $f \in W_\delta^0(\mathcal{D})^3$, $g \in W_\delta^1(\mathcal{D})$, $\phi^\pm \in W_\delta^{1/2}(\Gamma^\pm)$, $\delta < 1$, and the supports of f , g and ϕ^\pm are compact.

2.3 A property of the operator div

The goal of this subsection is to prove that the operator $\text{div} : \mathring{L}_2^1(\mathcal{D})^3 \rightarrow L_2(\mathcal{D})$ is surjective. For this end, we show that its dual operator is injective and has closed range. We start with an assertion in the two-dimensional angle K . Here $\mathring{W}^1(K)$ denotes the closure of $C_0^\infty(K)$ with respect to the norm

$$\|u\|_{W^1(K)} = \left(\int_K (|u|^2 + |\partial_{x_1} u|^2 + |\partial_{x_2} u|^2) dx \right)^{1/2}$$

and $W^{-1}(K)$ its dual space (with respect to the L_2 scalar product in K).

Lemma 2.2 *For arbitrary $f \in L_2(K)$ there is the estimate*

$$\|f\|_{L_2(K)} \leq c \left(\|\partial_{x_1} f\|_{W^{-1}(K)} + \|\partial_{x_2} f\|_{W^{-1}(K)} + \|f\|_{W^{-1}(K)} \right) \quad (2.11)$$

with a constant c independent of f .

Proof: For bounded Lipschitz domains the assertion of the lemma can be found e.g. in [3, Ch.2,§2]. Let \mathcal{U}_j , $j = 1, 2, \dots$, be pairwise disjoint congruent parallelograms such that $\bar{K} = \bar{\mathcal{U}}_1 \cup \bar{\mathcal{U}}_2 \cup \dots$. Then

$$\|f\|_{L_2(K)}^2 = \sum_{j=1}^{\infty} \|f\|_{L_2(\mathcal{U}_j)}^2 \leq c \sum_{j=1}^{\infty} \left(\|\partial_{x_1} f\|_{W^{-1}(\mathcal{U}_j)}^2 + \|\partial_{x_2} f\|_{W^{-1}(\mathcal{U}_j)}^2 + \|f\|_{W^{-1}(\mathcal{U}_j)}^2 \right). \quad (2.12)$$

By Riesz' representation theorem, there exist functions $g \in \mathring{W}^1(K)$, $g_j \in \mathring{W}^1(\mathcal{U}_j)$ such that

$$\begin{aligned}\|f\|_{W^{-1}(K)} &= \|g\|_{W^1(K)}, \quad \int_K f \bar{v} \, dx = (g, v)_{W^1(K)} \quad \text{for all } v \in \mathring{W}^1(K), \\ \|f\|_{W^{-1}(\mathcal{U}_j)} &= \|g_j\|_{W^1(\mathcal{U}_j)}, \quad \int_{\mathcal{U}_j} f \bar{v} \, dx = (g_j, v)_{W^1(\mathcal{U}_j)} \quad \text{for all } v \in \mathring{W}^1(\mathcal{U}_j), \quad j = 1, 2, \dots\end{aligned}$$

Let $g_{j,0}$ be the extension of g_j by zero. Then

$$\|g_j\|_{W^1(\mathcal{U}_j)}^2 = \int_{\mathcal{U}_j} f \bar{g}_j \, dx = \int_K f \bar{g}_{j,0} \, dx = (g, g_{j,0})_{W^1(K)} = (g, g_j)_{W^1(\mathcal{U}_j)}$$

and, therefore

$$\|g_j\|_{W^1(\mathcal{U}_j)} \leq \|g\|_{W^1(\mathcal{U}_j)}.$$

Consequently,

$$\sum_{j=1}^{\infty} \|f\|_{W^{-1}(\mathcal{U}_j)}^2 = \sum_{j=1}^{\infty} \|g_j\|_{W^1(\mathcal{U}_j)}^2 \leq \sum_{j=1}^{\infty} \|g\|_{W^1(\mathcal{U}_j)}^2 = \|g\|_{W^1(K)}^2 = \|f\|_{W^{-1}(K)}^2$$

The same inequality holds for $\partial_{x_j} f$. This together with (2.12) implies (2.11). ■

In the following lemma we give, in particular, an equivalent norm in the dual space $L_2^{-1}(\mathcal{D})$ of $L_2^1(\mathcal{D})$.

Lemma 2.3 *Let $F(y, \xi) = \tilde{f}(|\xi|^{-1}y, \xi)$, where $\tilde{f}(x', \xi)$ denotes the Fourier transform of $f(x', x_3)$ with respect to the last variable. Then*

$$\begin{aligned}\|f\|_{L_2(\mathcal{D})}^2 &= \int_{\mathbb{R}} \xi^{-2} \|F(\cdot, \xi)\|_{L_2(K)}^2 \, d\xi \quad \text{if } f \in L_2(\mathcal{D}), \\ \|f\|_{L_2^{-1}(\mathcal{D})}^2 &= \int_{\mathbb{R}} \xi^{-4} \|F(\cdot, \xi)\|_{W^{-1}(K)}^2 \, d\xi \quad \text{if } f \in L_2^{-1}(\mathcal{D}).\end{aligned}$$

Proof: The first equality follows immediately from Parseval's equality. We prove the second one. By Riesz' representation theorem, there exists a function $u \in \mathring{L}_2^1(\mathcal{D})$ such that

$$\|f\|_{L_2^{-1}(\mathcal{D})} = \|u\|_{L_2^1(\mathcal{D})}, \quad \int_{\mathcal{D}} f \bar{v} \, dx = \int_{\mathcal{D}} \nabla u \cdot \nabla \bar{v} \, dx \quad \text{for all } v \in \mathring{L}_2^1(\mathcal{D}). \quad (2.13)$$

Furthermore, for arbitrary $\xi \in \mathbb{R}$ there exists a function $W(\cdot, \xi) \in \mathring{W}^1(K)$ such that

$$\begin{aligned}\|F(\cdot, \xi)\|_{W^{-1}(K)} &= \|W(\cdot, \xi)\|_{W^1(K)}, \\ \int_K F(y, \xi) \overline{V(y, \xi)} \, dy &= \int_K (\nabla_y W(y, \xi) \cdot \nabla_y \overline{V(y, \xi)} + W(y, \xi) \overline{V(y, \xi)}) \, dy,\end{aligned}$$

where $V(y, \xi) = \tilde{v}(|\xi|^{-1}y, \xi)$ and \tilde{v} is the Fourier transform with respect to x_3 of an arbitrary function $v \in \mathring{L}_2^1(\mathcal{D})$. From the last equality it follows that

$$\int_K \tilde{f}(x', \xi) \overline{\tilde{v}(x', \xi)} \, dx' = \xi^{-2} \int_K \nabla_{x'} W(|\xi|x', \xi) \cdot \nabla_{x'} \overline{\tilde{v}(x', \xi)} + \xi^2 W(|\xi|x', \xi) \overline{\tilde{v}(x', \xi)} \, dx',$$

Comparing this with (2.13), we conclude that $W(|\xi|x', \xi) = \xi^2 \tilde{u}(x', \xi)$. Consequently,

$$\begin{aligned}\int_{\mathbb{R}} \|F(\cdot, \xi)\|_{W^{-1}(K)}^2 \, d\xi &= \int_{\mathbb{R}} \xi^{-4} \|W(\cdot, \xi)\|_{W^1(K)}^2 \, d\xi = \int_{\mathbb{R}} \int_K (|\nabla_{x'} \tilde{u}(x', \xi)|^2 + \xi^2 |\tilde{u}(x', \xi)|^2) \, dx' \, d\xi \\ &= \|u\|_{L_2^1(\mathcal{D})}^2 = \|f\|_{L_2^{-1}(\mathcal{D})}^2.\end{aligned}$$

The proof is complete. ■

Theorem 2.1 1) *There exists a constant c such that*

$$\|f\|_{L_2(\mathcal{D})} \leq c \sum_{j=1}^3 \|\partial_{x_j} f\|_{L_2^{-1}(\mathcal{D})} \quad \text{for all } f \in L_2(\mathcal{D}).$$

2) *For arbitrary $f \in L_2(\mathcal{D})$ there exists a vector function $u \in \mathring{L}_2^1(\mathcal{D})^3$ such that $\nabla \cdot u = f$ and*

$$\|u\|_{L_2^1(\mathcal{D})^3} \leq c \|f\|_{L_2(\mathcal{D})},$$

where the constant c is independent of f .

Proof: Let f be an arbitrary function in $L_2(\mathcal{D})$ and $F(y, \xi) = \tilde{f}(|\xi|^{-1}y, \xi)$, where \tilde{f} denotes the Fourier transform of f with respect to x_3 . Then, by Lemmas 2.2, 2.3, we have

$$\begin{aligned} \|f\|_{L_2(\mathcal{D})}^2 &= \int_{\mathbb{R}} \xi^{-2} \|F(\cdot, \xi)\|_{L_2(K)}^2 d\xi \leq c \int_{\mathbb{R}} \left(\sum_{j=1}^2 \|\partial_{y_j} F(\cdot, \xi)\|_{W^{-1}(K)}^2 + \|F(\cdot, \xi)\|_{W^{-1}(K)}^2 \right) d\xi \\ &= c \sum_{j=1}^3 \|\partial_{x_j} f\|_{L_2^{-1}(\mathcal{D})}^2. \end{aligned}$$

From this it follows, in particular, that the range of the mapping

$$L_2(\mathcal{D}) \ni f \rightarrow \nabla f \in L_2^{-1}(\mathcal{D})^3$$

is closed. Moreover, the kernel of this operator is obviously trivial. Consequently, by the closed range theorem, its dual operator $u \rightarrow -\nabla \cdot u$ maps $\mathring{L}_2^1(\mathcal{D})^3$ onto $L_2(\mathcal{D})$. This proves the theorem. ■

2.4 Existence and uniqueness of weak solutions

Lemma 2.4 *There exists a positive constant c such that $b(u, u) \geq c \|u\|_{\mathcal{H}}^2$ for all $u \in \mathcal{H}$.*

Proof: We have

$$b(u, u) + \|u\|_{L_2(\mathcal{D})^3}^2 \geq c \|u\|_{\mathcal{H}}^2$$

for all $u \in C_0^\infty(\overline{\mathcal{D}})^3$, $u(x) = 0$ for $|x| > 1$ (see, e.g., [3]). We consider the set of all $u \in C_0^\infty(\overline{\mathcal{D}})^3$ with support in the ball $|x| \leq \varepsilon$. For such u Hardy's inequality implies

$$\int_{\mathcal{D}} |u(x)|^2 dx \leq c_1 \int_{\mathcal{D}} |x|^2 \sum_{j=1}^3 |\partial_{x_j} u(x)|^2 dx \leq c_1 \varepsilon^2 \|u\|_{\mathcal{H}}^2$$

and, therefore,

$$b(u, u) \geq \frac{c}{2} \|u\|_{\mathcal{H}}^2$$

if ε is sufficiently small. Applying the similarity transformation $x = \alpha y$, we obtain the same inequality for arbitrary $u \in C_0^\infty(\overline{\mathcal{D}})^3$. The result follows. ■

Theorem 2.2 *Let $F \in V^*$, $g \in L_2(\mathcal{D})$, and let h^\pm be such that there exists a vector function $w \in \mathcal{H}$ satisfying the equalities $S^\pm w = h^\pm$ on Γ^\pm . Then there exists a unique solution $(u, p) \in \mathcal{H} \times L_2(\mathcal{D})$ of the problem (2.8), (2.9). Furthermore, (u, p) satisfies the estimate*

$$\|u\|_{\mathcal{H}} + \|p\|_{L_2(\mathcal{D})} \leq c \left(\|F\|_{V^*} + \|g\|_{L_2(\mathcal{D})} + \|w\|_{\mathcal{H}} \right)$$

with a constant c independent of F , g and w .

Proof: 1) We prove the existence of a solution. By our assumption on h^\pm and by Theorem 2.1, we may restrict ourselves to the case $g = 0$, $h^\pm = 0$. Let $V_0 = \{u \in V : \nabla \cdot u = 0\}$, and let V_0^\perp be the orthogonal complement of V_0 in V . Then, by Lax-Milgram's lemma, there exists a vector function $u \in V_0$ such that

$$b(u, v) = F(v) \quad \text{for all } v \in V_0, \quad \|u\|_{\mathcal{H}} \leq c \|F\|_{V_0^*} \leq c \|F\|_{V^*}.$$

By Theorem 2.1, the operator $B : u \rightarrow -\nabla \cdot u$ is an isomorphism from V_0^\perp onto $L_2(\mathcal{D})$. Hence, the mapping

$$L_2(\mathcal{D}) \ni q \rightarrow \ell(q) \stackrel{\text{def}}{=} F(B^{-1}q) - b(u, B^{-1}q)$$

defines a linear and continuous functional on $L_2(\mathcal{D})$. By Riesz representation theorem, there exists a function $p \in L_2(\mathcal{D})^3$ satisfying

$$\int_{\mathcal{D}} p q \, dx = \ell(q) \quad \text{for all } q \in L_2(\mathcal{D}), \quad \|p\|_{L_2(\mathcal{D})} \leq c \left(\|F\|_{V^*} + \|u\|_{\mathcal{H}} \right).$$

If we set $q = -\nabla \cdot v$, where v is an arbitrary element of V_0^\perp , we obtain

$$-\int_{\mathcal{D}} p \nabla \cdot v \, dx = F(v) - b(u, v) \quad \text{for all } v \in V_0^\perp.$$

Since both sides of the last equality vanish for $v \in V_0$, we get (2.8). Furthermore, $-\nabla \cdot u = 0$ and $S^\pm u|_{\Gamma^\pm} = 0$.

2) We prove the uniqueness. Suppose $(u, p) \in \mathcal{H} \times L_2(\mathcal{D})$ is a solution of problem (2.8), (2.9) with $F = 0$, $g = 0$, $h^\pm = 0$. Then, in particular, $u \in V_0$ and $b(u, u) = 0$ what, by Lemma 2.4, implies $u = 0$. Consequently,

$$\int_{\mathcal{D}} p \nabla \cdot v \, dx = 0 \quad \text{for all } v \in V.$$

Since v can be chosen such that $\nabla \cdot v = p$, we obtain $p = 0$. The proof is complete. ■

Remark 2.1 The assumption on h^\pm in Theorem 2.2 is satisfied e.g. for $h^\pm \in V_0^{1/2}(\Gamma^\pm)^{3-d^\pm}$. Then, by [9, Le.1.2], there exists a vector function $w \in V_0^1(\mathcal{D})^3 \subset \mathcal{H}$ satisfying $S^\pm w = h^\pm$ on Γ^\pm . (Note that h^\pm is a vector-function if $d^\pm \leq 1$ and a scalar function if $d^\pm = 2$. In the case $d^\pm = 1$ the vector h^\pm is tangential to Γ^\pm , therefore, the corresponding function space can be identified with $V_0^{1/2}(\Gamma^\pm)^2$.)

Furthermore, for $h^\pm \in W_\delta^{3/2}(\Gamma^\pm)^{3-d^\pm}$, $0 < \delta < 1$, satisfying a compatibility condition on M there exists a vector function $w \in W_\delta^2(\mathcal{D})^3 \subset \mathcal{H}$ such that $S^\pm w = h^\pm$ on Γ^\pm (see Lemma 2.6) below.

2.5 Reduction to homogeneous boundary conditions

In the sequel, we will consider weak solutions of problem (2.1), (2.2) with data $f \in W_\delta^1(\mathcal{D})^3$, $g \in W_\delta^1(\mathcal{D})$, $h^\pm \in W_\delta^{3/2}(\Gamma^\pm)^{3-d^\pm}$ and $\phi^\pm \in W_\delta^{1/2}(\Gamma^\pm)^{d^\pm}$, $0 < \delta < 1$. The following two lemmas are concerned with the question about the existence of a pair $(u, p) \in W_\delta^{2,p}(\mathcal{D})^3 \times W_\delta^1(\mathcal{D})$ satisfying the boundary condition (2.2).

Lemma 2.5 *For arbitrary $h^\pm \in V_\delta^{3/2}(\Gamma^\pm)^{3-d^\pm}$ and $\phi^\pm \in V_\delta^{1/2}(\Gamma^\pm)^{d^\pm}$ there exists a vector function $u \in V_\delta^2(\mathcal{D})^3$ satisfying the boundary conditions*

$$S^\pm u = h^\pm, \quad N^\pm(u, 0) = \phi^\pm \quad \text{on } \Gamma^\pm \tag{2.14}$$

and the estimate

$$\|u\|_{V_\delta^2(\mathcal{D})^3} \leq c \sum_{\pm} \left(\|h^\pm\|_{V_\delta^{3/2}(\Gamma^\pm)^{3-d^\pm}} + \|\phi^\pm\|_{V_\delta^{1/2}(\Gamma^\pm)^{d^\pm}} \right). \tag{2.15}$$

Moreover, if $\text{supp } h^\pm \in \Gamma^\pm \cap \mathcal{U}$ and $\text{supp } \phi^\pm \in \Gamma^\pm \cap \mathcal{U}$, where \mathcal{U} is an arbitrary domain in \mathbb{R}^3 , then u can be chosen such that $\text{supp } u \in \mathcal{U}$.

Proof: For simplicity, we assume that Γ^- coincides with the half-plane $\varphi = 0$ (i.e., $x_1 > 0, x_2 = 0$) and Γ^+ with the half-plane $\varphi = \alpha$. Then the boundary conditions

$$S^- u = h^-, \quad N^-(u, 0) = \phi^\pm \quad \text{on } \Gamma^- \quad (2.16)$$

have the form $u = h^-$ on Γ^- if $d^- = 0$,

$$\begin{aligned} u_1 &= h_1^-, \quad u_3 = h_3^-, \quad 2\partial_{x_2} u = \phi^- \quad \text{on } \Gamma^- \quad \text{if } d^- = 1, \\ u_2 &= h^-, \quad -\varepsilon_{1,2}(u) = \phi_1^-, \quad -\varepsilon_{3,2}(u) = \phi_3^- \quad \text{on } \Gamma^- \quad \text{if } d^- = 2, \\ -2\varepsilon_{j,2}(u) &= \phi_j^- \quad \text{on } \Gamma^- \quad \text{for } j = 1, 2, 3 \quad \text{if } d^- = 3. \end{aligned}$$

In all these cases, the existence of a vector function $u \in V_\delta^2(\mathcal{D})^3$ satisfying (2.16) can be easily deduced from [9, Le.3.1]. Analogously, there exists a function $v \in V_\delta^2(\mathcal{D})^3$ satisfying $S^+ v = h^+$ and $N^+(v, 0)v = \phi^+$ on Γ^+ . Let $\zeta = \zeta(\varphi)$ be a smooth function on $[0, \alpha]$ equal to 1 for $\varphi < \alpha/2$ and to zero for $\alpha > 3\pi/4$. Then the function $w(x) = \zeta(\varphi) u(x) + (1 - \zeta(\varphi)) v(x)$ satisfies (2.14). ■

The analogous result in the space W_δ^2 holds only under additional assumptions on h^+ and h^- . If $u \in W_\delta^2(\mathcal{D})^3$, then there exists the trace $u|_M \in W^{1-\delta}(M)^3$, and from the boundary conditions (2.2) it follows that $S^\pm u|_M = h^\pm|_M$. Here S^+ and S^- are considered as operators on $W^{1-\delta}(M)^3$. Consequently, the boundary data h^+ and h^- must satisfy the compatibility condition

$$(h^+|_M, h^-|_M) \in R(T), \quad (2.17)$$

where $R(T)$ is the range of the operator $T = (S^+, S^-)$. For example, in the case of the Dirichlet problem ($d^+ = d^- = 0$) condition (2.17) is satisfied if and only if $h^+|_M = h^-|_M$, while in the case $d^- = 0, d^+ = 2$ condition (2.17) is equivalent to $h^-|_M \cdot n^+ = h^+|_M$.

Lemma 2.6 *Let $h^\pm \in W_\delta^{3/2}(\Gamma^\pm)^{3-d^\pm}$ and $\phi^\pm \in W_\delta^{1/2}(\Gamma^\pm)^{d^\pm}$, $0 < \delta < 1$, be functions vanishing for $r(x) > C$. Suppose that h^+ and h^- satisfy the compatibility condition (2.17) on M . Then there exists a vector function $u \in W_\delta^2(\mathcal{D})^3$ satisfying (2.14) and an estimate analogous to (2.15). Moreover, if $\text{supp } h^\pm \in \Gamma^\pm \cap \mathcal{U}$ and $\text{supp } \phi^\pm \in \Gamma^\pm \cap \mathcal{U}$, where \mathcal{U} is an arbitrary domain in \mathbb{R}^3 , then u can be chosen such that $\text{supp } u \in \mathcal{U}$.*

Proof: By (2.17), there exists a vector function $\psi \in W^{1-\delta}(M)^3$ such that $S^\pm \psi = h^\pm|_M$. Let $v \in W_\delta^2(\mathcal{D})^3$ be an extension of ψ . Then the trace of $h^\pm - S^\pm v|_{\Gamma^\pm}$ on M is equal to zero and, consequently, $h^\pm - S^\pm v|_{\Gamma^\pm} \in V_\delta^{3/2}(\Gamma^\pm)^{3-d^\pm}$ (cf. Lemma 2.1). Furthermore, $\phi^\pm - N^\pm(v, 0)|_{\Gamma^\pm} \in W_\delta^{1/2}(\Gamma^\pm)^{d^\pm} \subset V_\delta^{1/2}(\Gamma^\pm)^{d^\pm}$. Thus, according to Lemma 2.5, there exists a function $w \in V_\delta^2(\mathcal{D})^3$ such that $S^\pm w = h^\pm - S^\pm v$ and $N^\pm(w, 0) = \phi^\pm - N^\pm(v, 0)$ on Γ^\pm . Then $u = v + w$ satisfies (2.14). ■

2.6 A priori estimates for the solutions

The following lemma is essentially proved in [9, Th.4.1].

Lemma 2.7 *Let (u, p) be a solution of problem (2.1), (2.2), $u \in W_{loc}^2(\overline{\mathcal{D}} \setminus M)^3 \cap V_{\delta-1}^{l-1}(\mathcal{D})^3$, $p \in W_{loc}^1(\overline{\mathcal{D}} \setminus M) \cap V_{\delta-1}^{l-2}(\mathcal{D})$ $l \geq 2$. Suppose that $f \in V_\delta^{l-2}(\mathcal{D})^3$, $g \in V_\delta^{l-1}(\mathcal{D})$, and the components of h^\pm and ϕ^\pm are from $V_\delta^{l-1/2}(\Gamma^\pm)$ and $V_\delta^{l-3/2}(\Gamma^\pm)$, respectively. Then $u \in V_\delta^l(\mathcal{D})^3$ and $p \in V_\delta^{l-1}(\mathcal{D})$.*

Here $W_{loc}^l(\overline{\mathcal{D}} \setminus M)$ denotes the set of all functions u on \mathcal{D} such that ζu belongs to the Sobolev space $W^l(\mathcal{D})$ for arbitrary $\zeta \in C_0^\infty(\overline{\mathcal{D}} \setminus M)$. We prove the same result for $l = 1$, i.e., for data $f \in V_\delta^{-1}(\mathcal{D})^3$, $g \in V_\delta^0(\mathcal{D})$, $h^\pm \in V_\delta^{1/2}(\Gamma^\pm)^{3-d^\pm}$, $\phi^\pm \in V_\delta^{-1/2}(\Gamma^\pm)^{d^\pm}$. Here $V_\delta^{-1}(\mathcal{D})$ and $V_\delta^{-1/2}(\Gamma^\pm)$ denote the dual spaces of $V_{-\delta}^1(\mathcal{D})$ and $V_{-\delta}^{1/2}(\Gamma^\pm)$, respectively. First we prove the following lemma.

Lemma 2.8 *Let ζ, η be infinitely differentiable functions with support in $\{x : a < r < 4a, -2a < x_3 < 2a\}$ such that $\zeta\eta = \zeta$ and $|\partial_x^\alpha \zeta(x)| + |\partial_x^\alpha \eta(x)| < c a^{-|\alpha|}$ for $|\alpha| \leq 2$, where the constant c is independent of a . Furthermore, let $u \in W_{loc}^1(\overline{\mathcal{D}} \setminus M)^3, p \in W_{loc}^0(\overline{\mathcal{D}} \setminus M)$,*

$$b(u, v) - \int_{\mathcal{D}} p \nabla \cdot v \, dx = F(v) \quad \text{for all } v \in C_0^\infty(\overline{\mathcal{D}} \setminus M)^3, \quad (2.18)$$

$-\nabla \cdot u = g$ in \mathcal{D} , and $S^\pm u = 0$ on Γ^\pm . Then for arbitrary $\delta, 0 < \delta < 1$, there is the estimate

$$\begin{aligned} & \|\zeta u\|_{V_\delta^1(\mathcal{D})^3} + \|\zeta p\|_{V_\delta^0(\mathcal{D})} \\ & \leq c \left(\|\zeta F\|_{V_\delta^{-1}(\mathcal{D})^3} + \|\zeta g\|_{V_\delta^0(\mathcal{D})} + \|\eta u\|_{V_{\delta-1}^0(\mathcal{D})^3} + \|\eta p\|_{V_{\delta-1}^{-1}(\mathcal{D})} + \sum_{\pm} \|\eta u\|_{V_{\delta-1}^{-1/2}(\Gamma^\pm)} \right) \end{aligned}$$

with a constant c independent of u, p and a .

Proof: Partial integration yields

$$b(\zeta u, v) - \int_{\mathcal{D}} \zeta p \nabla \cdot v \, dx = \Phi(v) \quad \text{for all } v \in V,$$

where

$$\begin{aligned} \Phi(v) &= F(\zeta v) + \sum_{i,j=1}^3 \int_{\mathcal{D}} u_i \varepsilon_{i,j}(v) \partial_{x_j} \zeta \, dx + \frac{1}{2} \sum_{i,j=1}^3 \int_{\mathcal{D}} u_i \partial_{x_j} (v_i \partial_{x_j} \zeta + v_j \partial_{x_i} \zeta) \, dx \\ &\quad - \frac{1}{2} \sum_{\pm} \int_{\Gamma^\pm} \left(\frac{\partial \zeta}{\partial n} u \cdot v + (\nabla \zeta \cdot u) v_n \right) \, dx - \int_{\mathcal{D}} p v \cdot \nabla \zeta \, dx. \end{aligned}$$

Furthermore, $-\nabla \cdot (\zeta u) = \zeta g + u \cdot \nabla \zeta$ in \mathcal{D} and $S^\pm(\zeta u) = 0$ on Γ^\pm . Let $L_2^{1/2}(\Gamma^\pm)$ be the space of the traces of functions from $L_2^1(\mathcal{D})$ on Γ^\pm . From Theorem 2.2 it follows that

$$\begin{aligned} \|\zeta u\|_{L_2^1(\mathcal{D})^3} + \|\zeta p\|_{L_2(\mathcal{D})} &\leq c \left(\|\zeta F\|_{V^*} + \|\zeta g\|_{L_2(\mathcal{D})} + \sum_{i,j=1}^3 \|u_i \partial_{x_j} \zeta\|_{L_2(\mathcal{D})} + \sum_{i,j=1}^3 \|u_i \partial_{x_i} \partial_{x_j} \zeta\|_{(L_2^1(\mathcal{D}))^*} \right. \\ &\quad \left. + \sum_{i,j=1}^3 \sum_{\pm} \|u_i \partial_{x_j} \zeta\|_{(L_2^{1/2}(\Gamma^\pm))^*} + \sum_{j=1}^3 \|p \partial_{x_j} \zeta\|_{(L_2^1(\mathcal{D}))^*} \right). \quad (2.19) \end{aligned}$$

By the assumptions on the support of ζ and by Hardy's inequality, we have

$$\|\zeta u\|_{L_2^1(\mathcal{D})^3} \geq c a^{-\delta} \|u\|_{V_\delta^1(\mathcal{D})^3} \quad \text{and} \quad \|\zeta p\|_{L_2(\mathcal{D})} \geq (4a)^{-\delta} \|\zeta p\|_{V_\delta^0(\mathcal{D})}.$$

Using the last estimates and the inequality

$$\|\eta v\|_{L_2^1(\mathcal{D})}^2 = \int_{\mathcal{D}} |\nabla(\eta v)|^2 \, dx = \int_{\mathcal{D}} (|\eta \nabla v|^2 + |\nabla \eta|^2 |v|^2) \, dx \leq c \int_{\mathcal{D}} (|\nabla v|^2 + |x|^{-2} |v|^2) \, dx \leq c \|v\|_{L_2^1(\mathcal{D})}^2,$$

we further obtain

$$\|\zeta F\|_{V^*} \leq \sup_v \frac{|(\zeta F)(v)|}{\|v\|_{L_2^1(\mathcal{D})^3}} \leq c \sup_v \frac{|(\zeta F)(\eta v)|}{\|\eta v\|_{L_2^1(\mathcal{D})^3}} \leq c a^{-\delta} \sup_v \frac{|(\zeta F)(v)|}{\|v\|_{V_\delta^1(\mathcal{D})^3}} \leq c a^{-\delta} \|\zeta F\|_{V_\delta^{-1}(\mathcal{D})^3}.$$

Here the supremum was taken over all $v \in C_0^\infty(\overline{\mathcal{D}})^3$. Analogously, the inequalities

$$\begin{aligned} \|\zeta g\|_{L_2(\mathcal{D})} &\leq c a^{-\delta} \|\zeta g\|_{V_\delta^0(\mathcal{D})}, \quad \|u_i \partial_{x_j} \zeta\|_{L_2(\mathcal{D})} + \|u_i \partial_{x_i} \partial_{x_j} \zeta\|_{(L_2^1(\mathcal{D}))^*} \leq c a^{-\delta} \|\eta u_i\|_{V_{\delta-1}^0(\mathcal{D})}, \\ \|u_i \partial_{x_i} \zeta\|_{(L_2^{1/2}(\Gamma^\pm))^*} &\leq c a^{-\delta} \|\eta u_i\|_{V_{\delta-1}^{-1/2}(\Gamma^\pm)}, \quad \|p \partial_{x_j} \zeta\|_{(L_2^1(\mathcal{D}))^*} \leq c a^{-\delta} \|\eta p\|_{V_{\delta-1}^{-1}(\mathcal{D})} \end{aligned}$$

can be proved. Thus, the desired estimate follows immediately from (2.19). ■

Lemma 2.9 *Let $u \in W_{loc}^1(\overline{\mathcal{D}} \setminus M)^3 \cap V_{\delta-1}^0(\mathcal{D})^3$ and $p \in W_{loc}^0(\overline{\mathcal{D}} \setminus M) \cap V_{\delta-1}^{-1}(\mathcal{D})$ satisfy (2.18), $-\nabla u = g$ in \mathcal{D} and $S^\pm u = 0$ on Γ^\pm , where $F \in V_{\delta-1}^{-1}(\mathcal{D})^3$, $g \in V_{\delta-1}^0(\mathcal{D})$, $0 < \delta < 1$. Furthermore, we assume that $u|_{\Gamma^\pm} \in V_{\delta-1}^{-1/2}(\Gamma^\pm)^3$. Then $u \in V_{\delta-1}^1(\mathcal{D})^3$, $p \in V_{\delta-1}^0(\mathcal{D})$, and*

$$\|u\|_{V_{\delta-1}^1(\mathcal{D})^3} + \|p\|_{V_{\delta-1}^0(\mathcal{D})} \leq c \left(\|F\|_{V_{\delta-1}^{-1}(\mathcal{D})^3} + \|g\|_{V_{\delta-1}^0(\mathcal{D})} + \|u\|_{V_{\delta-1}^0(\mathcal{D})^3} + \|p\|_{V_{\delta-1}^{-1}(\mathcal{D})} + \|u\|_{V_{\delta-1}^{-1/2}(\Gamma^\pm)^3} \right).$$

Proof: Let $\zeta_{j,k}$ be infinitely differentiable functions such that

$$\text{supp } \zeta_{j,k} \subset \{x : 2^{k-1} < r(x) < 2^{k+1}, j-1 < 2^{-k}x_3 < j+1\}, \quad \sum_{j,k=-\infty}^{+\infty} \zeta_{j,k} = 1, \quad |\partial_x^\alpha \zeta_{j,k}(x)| \leq c 2^{-k|\alpha|}.$$

Furthermore, let $\eta_{j,k} = \sum_{i=j-1}^{j+1} \sum_{l=k-1}^{k+1} \zeta_{i,l}$. Obviously, $\eta_{j,k} = 1$ on $\text{supp } \zeta_{j,k}$. By Lemma 2.8, there is the estimate

$$\begin{aligned} & \|\zeta_{j,k} u\|_{V_{\delta-1}^1(\mathcal{D})^3}^2 + \|\zeta_{j,k} p\|_{V_{\delta-1}^0(\mathcal{D})}^2 \\ & \leq c \left(\|\zeta_{j,k} F\|_{V_{\delta-1}^{-1}(\mathcal{D})^3}^2 + \|\zeta_{j,k} g\|_{V_{\delta-1}^0(\mathcal{D})}^2 + \|\eta_{j,k} u\|_{V_{\delta-1}^0(\mathcal{D})^3}^2 + \|\eta_{j,k} p\|_{V_{\delta-1}^{-1}(\mathcal{D})}^2 + \sum_{\pm} \|\eta_{j,k} u\|_{V_{\delta-1}^{-1/2}(\Gamma^\pm)^3}^2 \right), \end{aligned}$$

with a constant c independent of j and k . It can be proved analogously to [5, Le.6.1.1] that the norm in $V_{\delta-1}^1(\mathcal{D})$ is equivalent to

$$\|u\| = \left(\sum_{j,k=-\infty}^{+\infty} \|\zeta_{j,k} u\|_{V_{\delta-1}^1(\mathcal{D})}^2 \right)^{1/2}.$$

The same is true (cf. [5, Sec.6.1]) for the norms of the dual space $V_{\delta-1}^{-l}(\mathcal{D})$ and the trace space $V_{\delta-1}^{l-1/2}(\Gamma^\pm)$. Using this and the last inequality, we conclude that $u \in V_{\delta-1}^1(\mathcal{D})^3$, $p \in V_{\delta-1}^0(\mathcal{D})$. Moreover, the desired estimate for u and p holds. ■

Furthermore, there is the following analogon of Lemma 2.7 (cf. [14, Le.2.3]).

Lemma 2.10 *Let (u, p) be a solution of problem (2.1), (2.2), and let ζ, η be infinitely differentiable functions on $\overline{\mathcal{D}}$ with compact supports such that $\eta = 1$ in a neighborhood of $\text{supp } \zeta$. Suppose that $\eta u \in W_{loc}^2(\overline{\mathcal{D}} \setminus M)^3 \cap W_{\delta-1}^{l-1}(\mathcal{D})^3$, $\eta p \in W_{loc}^1(\overline{\mathcal{D}} \setminus M) \cap W_{\delta-1}^{l-2}(\mathcal{D})$, $l \geq 2$, $\eta f \in W_{\delta-1}^{l-2}(\mathcal{D})^3$, $\eta g \in W_{\delta-1}^{l-1}(\mathcal{D})$, and the components of ηh^\pm and $\eta \phi^\pm$ are from $W_{\delta-1}^{l-1/2}(\Gamma^\pm)$ and $W_{\delta-1}^{l-3/2}(\Gamma^\pm)$, respectively. Then $\zeta u \in W_{\delta-1}^l(\mathcal{D})^3$ and $\zeta p \in W_{\delta-1}^{l-1}(\mathcal{D})$.*

2.7 Smoothness of x_3 -derivatives

Our goal is to show that the solution (u, p) of problem (2.1), (2.2) belongs to $W_{\delta-1}^2(\mathcal{D})^3 \times W_{\delta-1}^1(\mathcal{D})$ if $f \in W_{\delta-1}^1(\mathcal{D})^3$, $g \in W_{\delta-1}^1(\mathcal{D})$, $h^\pm \in W_{\delta-1}^{3/2}(\Gamma^\pm)^{3-d^\pm}$ and $\phi^\pm \in W_{\delta-1}^{3/2}(\Gamma^\pm)^{d^\pm}$, $0 < \delta < 1$. For this end, we show in this subsection that $\partial_{x_3} u \in V_{\delta-1}^1(\mathcal{D})^3$ and $\partial_{x_3} p \in V_{\delta-1}^0(\mathcal{D})$ under the above assumptions on the data. Due to Lemma 2.6, we may restrict ourselves to the case of homogeneous boundary conditions, i.e., $h^\pm = 0$, $\phi^\pm = 0$.

Lemma 2.11 *Let $(u, p) \in \mathcal{H} \times L_2(\mathcal{D})$ be a solution of problem (2.8), (2.9). Suppose that $g \in V_{\delta-1}^1(\mathcal{D})$, $0 < \delta < 1$, $h^\pm = 0$, and the functional F has the form*

$$F(v) = \int_{\mathcal{D}} f \cdot v \, dx,$$

where $f \in V_{\delta-1}^0(\mathcal{D})^3$. We assume further that f and g have compact support. Then $\partial_{x_3} u \in V_{\delta-1}^0(\mathcal{D})^3$, $\partial_{x_3} u|_{\Gamma^\pm} \in V_{\delta-1}^{-1/2}(\Gamma^\pm)^3$, $\partial_{x_3} p \in V_{\delta-1}^{-1}(\mathcal{D})$ and

$$\|\partial_{x_3} u\|_{V_{\delta-1}^0(\mathcal{D})^3} + \|\partial_{x_3} u\|_{V_{\delta-1}^{-1/2}(\Gamma^\pm)^3} + \|\partial_{x_3} p\|_{V_{\delta-1}^{-1}(\mathcal{D})} \leq c \left(\|f\|_{V_{\delta-1}^0(\mathcal{D})^3} + \|g\|_{V_{\delta-1}^1(\mathcal{D})} \right).$$

Proof: First note that $F \in \mathcal{H}^* \subset V^*$ and $g \in L_2(\mathcal{D})$ under the assumptions of the lemma. For arbitrary real h let $u_h(x) = h^{-1}(u(x', x_3 + h) - u(x', x_3))$. Obviously,

$$b(u_h, v) - \int_{\mathcal{D}} p_h \nabla \cdot v \, dx = b(u, v_{-h}) - \int_{\mathcal{D}} p \nabla \cdot v_{-h} \, dx = F(v_{-h}) = \int_{\mathcal{D}} f_h \cdot v \, dx$$

for all $v \in V$, $-\nabla \cdot u_h = g_h$ in \mathcal{D} , and $S^\pm u_h = 0$ on Γ^\pm . Consequently, by Theorem 2.2, there exists a constant c independent of u, p and h such that

$$\|u_h\|_{\mathcal{H}}^2 + \|p_h\|_{L_2(\mathcal{D})}^2 \leq c \left(\|f_h\|_{V^*}^2 + \|g_h\|_{L_2(\mathcal{D})}^2 \right). \quad (2.20)$$

We prove that

$$\int_0^\infty h^{2\delta-1} \left(\|f_h\|_{V^*}^2 + \|g_h\|_{L_2(\mathcal{D})}^2 \right) dh \leq c \left(\|f\|_{V_\delta^0(\mathcal{D})^3}^2 + \|g\|_{V_\delta^1(\mathcal{D})}^2 \right). \quad (2.21)$$

Indeed, let $\tilde{g}(x', \xi)$ be the Fourier transform of $g(x', x_3)$ with respect to the variable x_3 . Then

$$\begin{aligned} \int_0^\infty h^{2\delta-1} \|g_h\|_{L_2(\mathcal{D})}^2 dh &= \int_0^\infty \int_{\mathbb{R}} \int_K h^{2\delta-3} |e^{i\xi h} - 1|^2 |\tilde{g}(x', \xi)|^2 dx' d\xi dh \\ &= c \int_{\mathbb{R}} \int_K |\xi|^{2-2\delta} |\tilde{g}(x', \xi)|^2 dx' d\xi \leq \int_{\mathbb{R}} \int_K r^{2\delta-2} (1 + r^2 \xi^2) |\tilde{g}(x', \xi)|^2 dx' d\xi \leq c \|g\|_{V_\delta^1(\mathcal{D})}^2 \end{aligned}$$

Furthermore, if χ is a smooth function, $0 \leq \chi \leq 1$, $\chi = 1$ for $r > h$, $\chi = 0$ for $r < h/2$, $|\nabla \chi| \leq c h^{-1}$, and ε is an arbitrary positive number, then

$$\begin{aligned} \left| \int_{\mathcal{D}} f_h \cdot v \, dx \right| &= \left| \int_{\mathcal{D}} f \cdot v_{-h} \, dx \right| \\ &\leq \|\chi f\|_{L_2(\mathcal{D})^3} \|v_{-h}\|_{L_2(\mathcal{D})^3} + \|r^{1-\varepsilon} h^{\varepsilon-1} f\|_{L_2(\mathcal{D}_h)^3} \|r^{\varepsilon-1} h^{1-\varepsilon} (1-\chi) v_{-h}\|_{L_2(\mathcal{D}_h)^3}, \end{aligned}$$

where $\mathcal{D}_h = \{x \in \mathcal{D} : r(x) < h\}$. Here

$$\|v_{-h}\|_{L_2(\mathcal{D})^3}^2 = \int_{\mathcal{D}} \left| \int_0^1 \partial_{x_3} v(x', x_3 - th) dt \right|^2 dx \leq \|\partial_{x_3} v\|_{L_2(\mathcal{D})^3}^2 \leq \|v\|_{\mathcal{H}}^2.$$

Using Hardy's inequality, we further obtain

$$\|r^{\varepsilon-1} h^{1-\varepsilon} v_{-h}\|_{L_2(\mathcal{D}_h)^3}^2 \leq c \int_{\mathcal{D}} r^{2\varepsilon} h^{2-2\varepsilon} |\partial_r (1-\chi) v_{-h}|^2 dx \leq c \int_{r < h} (|v_{-h}|^2 + |\partial_r v_{-h}|^2) dx \leq c \|v\|_{\mathcal{H}}^2.$$

Consequently, for $0 < \varepsilon < 1 - \delta$ we obtain

$$\begin{aligned} \int_0^\infty h^{2\delta-1} \|f_h\|_{V^*}^2 &\leq c \int_0^\infty \left(\|\chi f\|_{L_2(\mathcal{D})^3}^2 + \|r^{1-\varepsilon} h^{\varepsilon-1} f\|_{L_2(\mathcal{D}_h)^3}^2 \right) dh \\ &\leq \int_{\mathcal{D}} |f(x)|^2 \left(\int_0^{2r} h^{2\delta-1} dh + r^{2-2\varepsilon} \int_r^\infty h^{2\delta-3+2\varepsilon} dh \right) dx \leq c \|f\|_{V_\delta^0(\mathcal{D})^3}^2. \end{aligned}$$

This proves (2.21). Next we prove that

$$\|u_z\|_{V_{\delta-1}^0(\mathcal{D})^3}^2 + \|p_z\|_{V_{\delta-1}^{-1}(\mathcal{D})}^2 \leq c \int_0^\infty h^{2\delta-1} \left(\|u_h\|_{\mathcal{H}}^2 + \|p_h\|_{L_2(\mathcal{D})}^2 \right) dh. \quad (2.22)$$

It can be easily shown (see [19, Le.3]) that

$$\begin{aligned} \|\partial_{x_3} u\|_{V_{\delta-1}^0(\mathcal{D})^3}^2 &= \int_{\mathbb{R}} \int_K |x'|^{2\delta-2} |\xi|^2 |\tilde{u}(x', \xi)|^2 dx' d\xi \\ &\leq c \int_{\mathbb{R}} \int_K |\xi|^{2-2\delta} \left(|\xi|^2 |\tilde{u}(x', \xi)|^2 + \sum_{j=1}^2 |\partial_{x_j} \tilde{u}(x', \xi)|^2 \right) dx' d\xi \\ &= c \int_{\mathbb{R}} \int_K \left(\int_0^\infty h^{2\delta-3} |e^{i\xi h} - 1|^2 dh \right) \left(|\xi|^2 |\tilde{u}(x', \xi)|^2 + \sum_{j=1}^2 |\partial_{x_j} \tilde{u}(x', \xi)|^2 \right) dx' d\xi = \int_0^\infty h^{2\delta-1} \|u_h\|_{\mathcal{H}}^2. \end{aligned}$$

Furthermore, since

$$\begin{aligned}
\left| \int_{\mathcal{D}} \partial_{x_3} p v dx \right|^2 &= \left| \int_{\mathbb{R}} \int_K \xi \tilde{p} \tilde{v} dx' d\xi \right|^2 \leq \int_{\mathbb{R}} \int_K |\xi|^{2\delta} |\tilde{v}(x', \xi)|^2 dx' d\xi \int_{\mathbb{R}} \int_K |\xi|^{2-2\delta} |\tilde{p}(x', \xi)|^2 dx' d\xi \\
&\leq \int_{\mathbb{R}} \int_K r^{-2\delta} (1 + r^2 |\xi|^2) |\tilde{v}(x', \xi)|^2 dx' d\xi \int_{\mathbb{R}} \int_K |\xi|^{2-2\delta} |\tilde{p}(x', \xi)|^2 dx' d\xi \\
&\leq \|v\|_{V_{1-\delta}^1(\mathcal{D})}^2 \int_{\mathbb{R}} \int_K |\xi|^{2-2\delta} |\tilde{p}(x', \xi)|^2 dx' d\xi
\end{aligned}$$

and

$$\int_{\mathbb{R}} \int_K |\xi|^{2-2\delta} |\tilde{p}(x', \xi)|^2 dx' d\xi = c \int_0^\infty \int_{\mathbb{R}} \int_K h^{2\delta-3} |e^{i\xi h} - 1|^2 |\tilde{p}(x', \xi)|^2 dx' d\xi dh = c \int_0^\infty h^{2\delta-1} \|p_h\|_{L_2(\mathcal{D})}^2 dh,$$

we obtain

$$\|\partial_{x_3} p\|_{V_{\delta-1}^{-1}(\mathcal{D})}^2 \leq \int_{\mathbb{R}} \int_K |\xi|^{2-2\delta} |\tilde{p}(x', \xi)|^2 dx' d\xi \leq c \int_0^\infty h^{2\delta-1} \|p_h\|_{L_2(\mathcal{D})}^2 dh.$$

This proves (2.22). Finally, we show that

$$\|\partial_{x_3} u\|_{V_{\delta-1/2}^{-1/2}(\Gamma^\pm)^3}^2 \leq c \int_0^\infty h^{2\delta-1} \|u_h\|_{\mathcal{H}}^2 dh. \quad (2.23)$$

First let $1/2 \leq \delta < 1$. Since

$$\begin{aligned}
&\int_{\mathbb{R}} \int_0^\infty |\xi|^{2\delta-1} |\tilde{v}(r, \xi)|^2 dr d\xi \leq \int_{\mathbb{R}} \int_0^\infty r^{1-2\delta} (1 + r|\xi|) |\tilde{v}(r, \xi)|^2 dr d\xi \\
&= \int_{\mathbb{R}} \int_0^\infty r^{1-2\delta} |v(r, x_3)|^2 dr dx_3 + \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} r^{2-2\delta} h^{-2} |e^{i\xi h} - 1|^2 |\tilde{v}(r, \xi)|^2 dh d\xi dr \\
&= \int_{\mathbb{R}} \int_0^\infty r^{1-2\delta} |v|^2 dr dx_3 + \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} r^{2-2\delta} h^{-2} |v(r, x_3 + h) - v(r, x_3)|^2 dh dx_3 dr \leq \|v\|_{V_{1-\delta}^{1/2}(\Gamma^\pm)^3}^2
\end{aligned}$$

(here we used the equivalence of the norm in $V_\delta^{1/2}(\Gamma^\pm)$ to (2.3)), we obtain

$$\begin{aligned}
\left| \int_{\Gamma^\pm} \partial_{x_3} u \cdot v dr dx_3 \right|^2 &\leq \int_{\mathbb{R}} \int_0^\infty |\xi|^{2\delta-1} |\tilde{v}(r, \xi)|^2 dr d\xi \int_{\mathbb{R}} \int_0^\infty |\xi|^{3-2\delta} |\tilde{u}(r, \xi)|^2 dr d\xi \\
&\leq \|v\|_{V_{1-\delta}^{1/2}(\Gamma^\pm)^3}^2 \int_{\mathbb{R}} \int_0^\infty |\xi|^{3-2\delta} |\tilde{u}(r, \xi)|^2 dr d\xi.
\end{aligned}$$

Furthermore, setting $\tilde{w} = |\xi|^{1-\delta} \tilde{u}$, we get

$$\begin{aligned}
&\int_{\mathbb{R}} \int_0^\infty |\xi|^{3-2\delta} |\tilde{u}(r, \xi)|^2 dr d\xi = \int_{\mathbb{R}} \int_0^\infty |\xi| |\tilde{w}(r, \xi)|^2 dr d\xi \\
&= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}} h^{-2} |w(r, x_3) - w(r, x_3 + h)|^2 dh dx_3 dr \leq c \|w\|_{L_2^{1/2}(\Gamma^\pm)^3}^2 \leq c \|w\|_{\mathcal{H}}^2 \\
&= \int_{\mathbb{R}} \int_K |\xi|^{2-2\delta} \left(|\xi|^2 |\tilde{u}(x', \xi)|^2 + \sum_{j=1}^2 |\partial_{x_j} \tilde{u}(x', \xi)|^2 \right) dx' d\xi.
\end{aligned}$$

The last expression is equal to the right-hand side of (2.23) (see the proof of (2.22)). This implies (2.23) for $1/2 \leq \delta < 1$. Now let $0 < \delta < 1/2$ and let $V_0^{1-\delta}(\mathcal{D})$ the weighted Slobodetskiĭ space with the norm

$$\begin{aligned}
\|u\|_{V_0^{1-\delta}(\mathcal{D})} &= \left(\|u\|_{V_{\delta-1}^0(\mathcal{D})}^2 + \int_K \int_{\mathbb{R}} \int_{\mathbb{R}} h^{2\delta-3} |u(x', x_3 + h) - u(x', x_3)|^2 dh dx_3 dx' \right. \\
&\quad \left. + \int_{\mathbb{R}} \int_K \int_K |y'|^{2\delta-4} |u(x' + y', x_3) - u(x', x_3)|^2 dy' dx' dx_3 \right)^{1/2}. \quad (2.24)
\end{aligned}$$

The trace space $V_0^{1/2-\delta}(\Gamma^\pm)$ of $V_0^{1-\delta}(\Gamma^\pm)$ is continuously imbedded into the space $V_{\delta-1/2}^0(\Gamma^\pm)$ and, therefore, also into $V_{\delta-1}^{-1/2}(\Gamma^\pm)$ (see, e.g., [20]). Consequently, it suffices to show that

$$\|\partial_{x_3} u\|_{V_0^{1-\delta}(\mathcal{D})^3}^2 \leq c \int_0^\infty h^{2\delta-1} \|u_h\|_{\mathcal{H}}^2 dh. \quad (2.25)$$

The analogous estimate for the norm of $\partial_{x_3} u$ in $V_{\delta-1}^0(\mathcal{D})^3$ was proved above (see (2.22)). Furthermore,

$$\begin{aligned} & \int_K \int_{\mathbb{R}} \int_{\mathbb{R}} h^{2\delta-3} |\partial_{x_3} u(x', x_3 + h) - \partial_{x_3} u(x', x_3)|^2 dh dx_3 dx' \\ &= \int_K \int_{\mathbb{R}} \int_{\mathbb{R}} h^{2\delta-3} |e^{ix'\xi} - 1|^2 |\xi \tilde{u}(x', x_3)|^2 dh dx_3 dx' = c \int_{\mathbb{R}} \int_K |\xi|^{4-2\delta} |\tilde{u}(x', \xi)|^2 dx' d\xi \\ &\leq c \int_0^\infty h^{2\delta-1} \|u_h\|_{\mathcal{H}}^2 dh. \end{aligned}$$

The second integral on the right of (2.24) coincides with

$$\int_{\mathbb{R}} \int_K \int_K |y'|^{2\delta-4} |\xi|^2 |\tilde{u}(x' + y', \xi) - \tilde{u}(x', \xi)|^2 dy' dx' dx_3.$$

Here

$$\int_{\mathbb{R}} \int_K \int_{\substack{K \\ |y'| > 1/|\xi|}} |y'|^{2\delta-4} |\xi|^2 |\tilde{u}(x', \xi)|^2 dy' dx' dx_3 \leq c \int_{\mathbb{R}} \int_K \int_K |\xi|^{4-2\delta} |\tilde{u}(x', \xi)|^2 dx' dx_3$$

what is majorized by the right-hand side of (2.25). The same estimate holds for $\tilde{u}(x' + y', \xi)$. Moreover, from the inequality

$$|\tilde{u}(x' + y', \xi) - \tilde{u}(x', \xi)| \leq 2|y'|^2 \int_0^1 \sum_{j=1}^2 |\partial_{x_j} \tilde{u}(x' + ty', \xi)|^2 dt$$

it follows that

$$\begin{aligned} & \int_{\mathbb{R}} \int_K \int_{\substack{K \\ |y'| > 1/|\xi|}} |y'|^{2\delta-4} |\xi|^2 |\tilde{u}(x', \xi)|^2 dy' dx' dx_3 \\ &\leq 2 \int_0^1 \int_{\mathbb{R}} \int_K \int_{\substack{K \\ |y'| > 1/|\xi|}} |y'|^{2\delta-2} |\xi|^2 \sum_{j=1}^2 |\partial_{x_j} \tilde{u}(x' + ty', \xi)|^2 dy' dx' dx_3 dt \\ &\leq c \int_{\mathbb{R}} \int_K |\xi|^{2-2\delta} \sum_{j=1}^2 |\partial_{x_j} \tilde{u}(x', \xi)|^2 dx' d\xi \leq c \int_0^\infty h^{2\delta-1} \|u_h\|_{\mathcal{H}}^2 dh. \end{aligned}$$

This proves (2.25). Now the assertion of the lemma follows immediately from (2.20)–(2.23). ■

Corollary 2.1 *Let the assumptions of Lemma 2.11 be satisfied. Then $\partial_{x_3} u \in V_\delta^1(\mathcal{D})^3$, $\partial_{x_3} p \in V_\delta^0(\mathcal{D})$ and*

$$\|\partial_{x_3} u\|_{V_\delta^1(\mathcal{D})^3} + \|\partial_{x_3} p\|_{V_\delta^0(\mathcal{D})} \leq c \left(\|f\|_{V_\delta^0(\mathcal{D})^3} + \|g\|_{V_\delta^1(\mathcal{D})} \right).$$

with a constant c independent of u and p .

Proof: From Lemma 2.11 and well-known local regularity results for solutions of elliptic boundary value problems (see [1]) we conclude that $\partial_{x_3} u \in W_{loc}^1(\overline{\mathcal{D}} \setminus M)^3 \cap V_{\delta-1}^0(\mathcal{D})^3$ and $\partial_{x_3} p \in L_{2,loc}(\overline{\mathcal{D}} \setminus M) \cap V_{\delta-1}^{-1}(\mathcal{D})$. Obviously,

$$b(\partial_{x_3} u, v) - \int_{\mathcal{D}} \partial_{x_3} p \nabla \cdot v dx = \int_{\mathcal{D}} \partial_{x_3} f \cdot v dx \quad \text{for all } v \in C_0^\infty(\overline{\mathcal{D}} \setminus M)^3,$$

where $\partial_{x_3} f \in V_\delta^{-1}(\mathcal{D})^3$, $-\nabla \cdot \partial_{x_3} u = \partial_{x_3} g \in V_\delta^0(\mathcal{D})$, and $S^\pm \partial_{x_3} u = 0$ on Γ^\pm . Applying Lemma 2.9, we obtain $\partial_{x_3} u \in V_\delta^1(\mathcal{D})^3$, $\partial_{x_3} p \in V_\delta^0(\mathcal{D})$ and the desired inequality. ■

2.8 Auxiliary problems in the angle K , operator pencils

Suppose (u, p) is a solution of the Stokes system (2.1) with homogeneous boundary conditions (2.2) which is independent of x_3 . Then u_3 is a solution of the problem

$$-\Delta_{x'} u_3 = f_3 \text{ in } K, \quad u_3 = 0 \text{ on } \Gamma^\pm \text{ for } d^\pm \leq 1, \quad \frac{\partial u_3}{\partial n^\pm} = 0 \text{ on } \Gamma^\pm \text{ for } d^\pm \geq 2, \quad (2.26)$$

where $\Delta_{x'}$ denotes the Laplace operator in the coordinates $x' = (x_1, x_2)$, whereas the vector $(u', p) = (u_1, u_2, p)$ is a solution of the two-dimensional Stokes system

$$-\Delta_{x'} u' + \nabla_{x'} p = f', \quad -\nabla_{x'} \cdot u' = g \text{ in } K, \quad (2.27)$$

with the corresponding boundary conditions

$$\tilde{S}^\pm u' = 0, \quad \tilde{N}^\pm(u', p) = 0 \text{ on } \gamma^\pm. \quad (2.28)$$

Here $\tilde{S}^\pm u' = u'$ if $d^\pm = 0$, $\tilde{S}^\pm u' = u' \cdot \tau^\pm$ if $d^\pm = 1$, $\tilde{S}^\pm u' = u' \cdot n^\pm$ if $d^\pm = 2$, $\tilde{N}^\pm(u', p) = -p + 2(\varepsilon(u')n^\pm) \cdot n^\pm$ if $d^\pm = 1$, $\tilde{N}^\pm(u', p) = (\varepsilon(u')n^\pm) \cdot \tau^\pm$ if $d^\pm = 2$, and $\tilde{N}^\pm(u', p) = -pn^\pm + 2\varepsilon(u')n^\pm$ if $d^\pm = 3$, by $\varepsilon(u')$ we denote the matrix with the components $\varepsilon_{i,j}(u)$, $i, j = 1, 2$.

Setting $u = r^\lambda U(\varphi)$, $p = r^{\lambda-1} P(\varphi)$ we obtain a boundary value problem for the vector function (U, P) on the interval $(-\alpha/2, +\alpha/2)$ quadratically depending on the parameter $\lambda \in \mathbb{C}$. The operator $A(\lambda)$ of this problem is a continuous mapping

$$W^2(-\frac{\alpha}{2}, +\frac{\alpha}{2}) \times W^1(-\frac{\alpha}{2}, +\frac{\alpha}{2}) \rightarrow L_2(-\frac{\alpha}{2}, +\frac{\alpha}{2}) \times W^1(-\frac{\alpha}{2}, +\frac{\alpha}{2}) \times \mathbb{C}^3$$

for arbitrary λ . As is known, the spectrum of this pencil $A(\lambda)$ consists only of eigenvalues with finite geometric and algebraic multiplicities. We give here a description of the spectrum for different d^- and d^+ . Without loss of generality, we may assume that $d^- \leq d^+$.

1. In the case of the Dirichlet problem ($d^- = d^+ = 0$), the spectrum of the pencil $A(\lambda)$ consists of the numbers $\frac{j\pi}{\alpha}$, where j is an arbitrary nonzero integer, and of the nonzero solutions of the equation

$$\lambda \sin \alpha \pm \sin(\lambda \alpha) = 0. \quad (2.29)$$

2. $d^- = 0$, $d^+ = 1$: Then the spectrum consists of the numbers $\frac{j\pi}{\alpha}$, where $j = \pm 1, \pm 2, \dots$, and of the nonzero solutions of the equation

$$\lambda \sin(2\alpha) + \sin(2\lambda \alpha) = 0. \quad (2.30)$$

3. $d^- = 0$, $d^+ = 2$: Then the spectrum consists of the numbers $\frac{j\pi}{2\alpha}$, $j = \pm 1, \pm 2, \dots$ and of the nonzero solutions of the equation

$$\lambda \sin(2\alpha) - \sin(2\lambda \alpha) = 0. \quad (2.31)$$

4. $d^- = 0$, $d^+ = 3$: Then additionally to the numbers $\frac{j\pi}{2\alpha}$, $j = \pm 1, \pm 2, \dots$, the solutions of the equation

$$\lambda \sin \alpha \pm \cos(\lambda \alpha) = 0 \quad (2.32)$$

are eigenvalues of the pencil $A(\lambda)$.

5. $d^- = d^+ = 1$: Then the spectrum consists of the numbers $\frac{j\pi}{\alpha}$, and $\frac{k\pi}{\alpha} \pm 1$, where j, k are arbitrary integers, $j \neq 0$.

6. $d^- = 1$, $d^+ = 2$: Then the spectrum consists of the numbers $\frac{j\pi}{2\alpha}$, and $\frac{k\pi}{2\alpha} \pm 1$, where j is an arbitrary integer and k is an arbitrary odd integer.

7. $d^- = 1$, $d^+ = 3$: Then the numbers $\frac{j\pi}{2\alpha}$, $j = 0, \pm 1, \pm 2, \dots$, and all solutions of (2.31) belong to the spectrum.

8. $d^- = d^+ = 2$: Then the spectrum consists of the numbers $\frac{j\pi}{\alpha}$, and $\frac{k\pi}{\alpha} \pm 1$, where j, k are arbitrary integers.
9. $d^- = 2, d^+ = 3$: Then additionally to the numbers $\frac{j\pi}{\alpha}, j = 0, \pm 1, \dots$, the solutions of (2.30) belong to the spectrum.
10. In the case of the Neumann problem ($d^- = d^+ = 3$) the spectrum of the pencil $A(\lambda)$ consists of the numbers $\frac{j\pi}{\alpha}$, where j is an arbitrary integer, and of all solutions of (2.29).

We refer to [18] and for the cases 1 and 3 also to [6, 21].

Note that $\lambda = 0$ is an eigenvalue in the following cases:

1. $d^+ = 3$ and $d^- \geq 1$ (or $d^- = 3$ and $d^+ \geq 1$),
2. $1 \leq d^+ = d^- \leq 2$ and $\alpha \in \{\pi, 2\pi\}$,
3. $d^+ = 1, d^- = 2$ (or $d^- = 1, d^+ = 2$) and $\alpha \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$.

The eigenvectors corresponding to the eigenvalue $\lambda = 0$ are of the form $(U, P) = (C, 0)$, where C is constant, and have rank 2 (i.e., have a generalized eigenvector).

We denote by $\lambda_1 = \lambda_1(\alpha)$ the eigenvalue with smallest positive real part. For example, in the case of the Dirichlet problem ($d^- = d^+ = 0$) or Neumann problem ($d^- = d^+ = 3$) we have $\lambda_1 = 1$ for $\alpha < \pi$ and $\lambda_1 = \frac{\xi_+(\alpha)}{\alpha}$, for $\alpha > \pi$, where ξ_+ is the smallest positive solution of the equation $\frac{\sin \xi}{\xi} + \frac{\sin \alpha}{\alpha} = 0$. In the case $d^- = 0, d^+ = 2$ (Dirichlet and free boundary conditions) we have $\lambda_1 = 1$ for $\alpha \leq \pi/2$ and $\lambda_1 = \frac{\pi}{2\alpha}$ for $\alpha > \pi/2$. Furthermore, let $\mu = \text{Re } \lambda_1$.

Lemma 2.12 *Let $(u, p) \in W_0^1(K)^3 \times L_2(K)$ be a solution of problem (2.26)–(2.28) vanishing outside the unit ball. Suppose that $f \in W_\delta^0(K)^3, g \in W_\delta^1(K), 0 < \delta < 1$ and that the strip $0 < \text{Re } \lambda \leq 1 - \delta$ does not contain eigenvalues of the pencil $A(\lambda)$. Then $u \in W_\delta^2(K)^3, p \in W_\delta^1(K)$, and*

$$\|u\|_{W_\delta^2(K)^3} + \|p\|_{W_\delta^1(K)} \leq c \left(\|f\|_{W_\delta^0(K)^3} + \|g\|_{W_\delta^1(K)} \right)$$

Proof: Since the support of (u, p) is compact, we have $(u, p) \in V_\varepsilon^1(K)^3 \times V_\varepsilon^0(K)$ with arbitrary $\varepsilon > 0$. Consequently, u admits the representation $u = c + d \log r + v$ with constant vectors c, d and $v \in V_\delta^2(K)^3$. Furthermore, $p \in V_\delta^1(K)$ and

$$\|v\|_{V_\delta^2(K)^3} + \|p\|_{V_\delta^1(K)} \leq c \left(\|f\|_{V_\delta^0(K)^3} + \|g\|_{V_\delta^1(K)} \right)$$

(see, e.g., [5, Th.8.2.2]). Since $u \in \mathcal{H}$, we conclude that $d = 0$. The result follows. ■

Analogously the following assertion holds (cf. [14, Le.2.5])

Lemma 2.13 *Let $(u, p) \in W_\delta^{l-1}(K)^3 \times W_\delta^{l-2}(K)$ be a solution of (2.26)–(2.28) vanishing outside the unit ball. Suppose that $f \in W_\delta^{l-2}(K)^3, g \in W_\delta^{l-1}(K), l \geq 2, 0 < \delta < 1, \delta$ is not integer, and that the strip $l - 2 - \delta \leq \text{Re } \lambda \leq l - 1 - \delta$ does not contain eigenvalues of the pencil $A(\lambda)$. Then $u \in W_\delta^l(K)^3$ and $p \in W_\delta^{l-1}(K)$.*

2.9 Regularity assertions for weak solutions

Lemma 2.14 *Let $(u, p) \in \mathcal{H} \times L_2(\mathcal{D})$ be a solutions of (2.8), (2.9). We assume that the support of (u, p) is compact, the functional F has the form (2.10) with $f \in W_\delta^0(\mathcal{D})^3, g \in W_\delta^1(\mathcal{D}), \phi^\pm \in W_\delta^{1/2}(\Gamma^\pm)^{d^\pm}$, and that $h^\pm \in W_\delta^{3/2}(\Gamma^\pm)$ satisfy the compatibility condition (2.17). If $\max(1 - \mu, 0) < \delta < 1$, then $(u, p) \in W_\delta^2(\mathcal{D})^3 \times W_\delta^1(\mathcal{D})$ and*

$$\|u\|_{W_\delta^2(\mathcal{D})^3} + \|p\|_{W_\delta^1(\mathcal{D})} \leq c \left(\|f\|_{W_\delta^0(\mathcal{D})^3} + \|g\|_{W_\delta^1(\mathcal{D})} + \sum_{\pm} \|h^\pm\|_{W_\delta^{3/2}(\Gamma^\pm)^{3-d^\pm}} + \sum_{\pm} \|\phi^\pm\|_{W_\delta^{1/2}(\Gamma^\pm)^{3-d^\pm}} \right), \quad (2.33)$$

where the constant c is independent of f, g, h^\pm , and ϕ^\pm .

Proof: Due to Lemma 2.6, we may assume without loss of generality that $h^\pm = 0$ and $\phi^\pm = 0$. By our assumptions on u and p , we have $u(\cdot, x_3) \in W_0^1(K)^3$ and $p(\cdot, x_3) \in L_2(K)$ for almost all x_3 . Furthermore, by Corollary 2.1, $(\partial_{x_3} u)(\cdot, x_3) \in W_\delta^1(K)^3$ and $(\partial_{x_3} p)(\cdot, x_3) \in W_\delta^0(K)$. Consequently, for almost all x_3 , the function $u_3(\cdot, x_3)$ is a solution of problem (2.26) with $f_3 + \partial_{x_3}^2 u_3 - \partial_{x_3} p \in W_\delta^0(K)$ on the right-hand side of the differential equation, while $u'(\cdot, x_3) = (u_1(\cdot, x_3), u_2(\cdot, x_3))$ is a solution of problem (2.27), where f' and g have to be replaced by $f' + \partial_{x_3}^2 u'$ and $g + \partial_{x_3} u_3$, respectively. Applying Lemma 2.12, we obtain $u(\cdot, x_3) \in W_\delta^2(K)^3$, $p(\cdot, x_3) \in W_\delta^1(K)$, and

$$\begin{aligned} \|u(\cdot, x_3)\|_{W_\delta^2(K)^3}^2 + \|p(\cdot, x_3)\|_{W_\delta^1(K)}^2 &\leq c \left(\|f(\cdot, x_3)\|_{W_\delta^0(K)^3}^2 + \|g(\cdot, x_3)\|_{W_\delta^1(K)}^2 \right. \\ &\quad \left. + \|(\partial_{x_3} u)(\cdot, x_3)\|_{W_\delta^1(K)^3}^2 + \|(\partial_{x_3} p)(\cdot, x_3)\|_{W_\delta^0(K)}^2 \right) \end{aligned}$$

with a constant c independent of x_3 . Integrating this inequality with respect to x_3 and using Corollary 2.1, we obtain the assertion of the lemma. ■

Theorem 2.3 *Let ζ, η be smooth cut-off functions with support in the unit ball such that $\eta = 1$ in a neighborhood of $\text{supp } \zeta$, and let $(u, p) \in \mathcal{H} \times L_2(\mathcal{D})$ be a solutions of (2.8), (2.9). Suppose that $\eta \partial_{x_3}^j g \in W_\delta^1(\mathcal{D})$, $\eta \partial_{x_3}^j h^\pm \in W_\delta^{3/2}(\Gamma^\pm)^{3-d^\pm}$ for $j = 0, 1, \dots, k$, h^\pm satisfy the compatibility condition (2.17), and that F is a functional on V which has the form (2.10) with $\eta \partial_{x_3}^j f \in W_\delta^0(\mathcal{D})^3$ and $\eta \partial_{x_3}^j \phi^\pm \in W_\delta^{1/2}(\Gamma^\pm)^{d^\pm}$ for $j = 0, 1, \dots, k$. If $\max(1 - \mu, 0) < \delta < 1$, then $\zeta \partial_{x_3}^k (u, p) \in W_\delta^2(\mathcal{D})^3 \times W_\delta^1(\mathcal{D})$ and there is the estimate*

$$\begin{aligned} \|\zeta \partial_{x_3}^k u\|_{W_\delta^2(\mathcal{D})^3} + \|\zeta \partial_{x_3}^k p\|_{W_\delta^1(\mathcal{D})} &\leq c \left(\sum_{j=0}^k \left(\|\eta \partial_{x_3}^j f\|_{W_\delta^0(\mathcal{D})^3} + \|\eta \partial_{x_3}^j g\|_{W_\delta^1(\mathcal{D})} + \sum_{\pm} \|\eta \partial_{x_3}^j h^\pm\|_{W_\delta^{3/2}(\Gamma^\pm)^{3-d^\pm}} \right. \right. \\ &\quad \left. \left. + \sum_{\pm} \|\eta \partial_{x_3}^j \phi^\pm\|_{W_\delta^{1/2}(\Gamma^\pm)^{d^\pm}} \right) + \|\eta u\|_{W_\delta^1(\mathcal{D})^3} + \|\eta p\|_{W_\delta^0(\mathcal{D})} \right). \end{aligned} \quad (2.34)$$

Proof: 1) First we prove the theorem for $k = 0$. From (2.8), (2.9) it follows that $(\zeta u, \zeta p)$ satisfies the equations

$$\begin{aligned} b(\zeta u, v) - \int_{\mathcal{D}} \zeta p \nabla \cdot v \, dx &= \int_{\mathcal{D}} \tilde{f} v + \sum_{\pm} \int_{\Gamma^\pm} \tilde{g} v \, dx \quad \text{for all } v \in V, \\ -\nabla \cdot (\zeta u) &= \zeta g - (\nabla \zeta) \cdot u \quad \text{in } \mathcal{D}, \quad S^\pm(\zeta u) = \zeta h^\pm \quad \text{on } \Gamma^\pm, \end{aligned}$$

where

$$\begin{aligned} \tilde{f}_i &= \zeta f_i - 2 \sum_{j=1}^3 (\partial_{x_j} \zeta) \varepsilon_{i,j}(u) - \sum_{j=1}^3 \partial_{x_j} (u_i \partial_{x_j} \zeta + u_j \partial_{x_i} \zeta) - p \partial_{x_i} \zeta \in W_\delta^0(\mathcal{D}) \\ \tilde{g}_i &= \zeta g_i + \frac{\partial \zeta}{\partial n} u_i + u_n \partial_{x_i} \zeta \in W_\delta^{1/2}(\Gamma^\pm). \end{aligned}$$

Applying Lemma 2.14, we obtain the assertion of the theorem for $k = 0$.

2) Let the conditions of the theorem on F, g and h^\pm with $k = 1$ be satisfied. Moreover, we suppose that $\eta \partial_{x_3}^j h^\pm \in V_\delta^{3/2}(\Gamma^\pm)^{3-d^\pm}$ for $j = 0$ and $j = 1$. By χ and χ_1 we denote smooth function such that $\chi = 1$ in a neighborhood of $\text{supp } \zeta$, $\chi_1 = 1$ in a neighborhood of $\text{supp } \chi$, and $\eta = 1$ in a neighborhood of $\text{supp } \chi_1$. Furthermore, for an arbitrary function v on \mathcal{D} or Γ^\pm we set $v_h(x', x_3) = h^{-1}(v(x', x_3 + h) - v(x', x_3))$. Obviously, $(u_h, p_h) \in \mathcal{H} \times L_2(\mathcal{D})$ for arbitrary real h . Consequently, by Theorem 2.3, we have

$$\begin{aligned} \|\zeta u_h\|_{W_\delta^2(\mathcal{D})^3} + \|\zeta p_h\|_{W_\delta^1(\mathcal{D})} &\leq c \left(\|\chi f_h\|_{W_\delta^0(\mathcal{D})^3} + \|\chi g_h\|_{W_\delta^1(\mathcal{D})} + \sum_{\pm} \|\chi h_h^\pm\|_{V_\delta^{3/2}(\Gamma^\pm)^{3-d^\pm}} \right. \\ &\quad \left. + \sum_{\pm} \|\chi \phi_h^\pm\|_{V_\delta^{1/2}(\Gamma^\pm)^{d^\pm}} + \|\chi u_h\|_{W_\delta^1(\mathcal{D})^3} + \|\chi p_h\|_{W_\delta^0(\mathcal{D})} \right) \end{aligned} \quad (2.35)$$

with a constant c independent of h , where χ is a smooth function, $\chi = 1$ in a neighborhood of $\text{supp } \zeta$, $\eta = 1$ in a neighborhood of $\text{supp } \chi$. Here $\chi f_h = (\chi f)_h - \chi h f$ and, for sufficiently small $|h|$ we have

$$\begin{aligned} \|(\chi f)_h\|_{W_\delta^0(\mathcal{D})^3}^2 &= \int_{\mathcal{D}} r^{2\delta} \left| \int_0^1 \frac{\partial(\chi f)}{\partial x_3}(x', x_3 + th) dt \right|^2 dx \leq \|\partial_{x_3}(\chi f)\|_{W_\delta^0(\mathcal{D})}^2, \\ \|\chi h f\|_{W_\delta^0(\mathcal{D})^3}^2 &\leq c \|\eta f\|_{W_\delta^0(\mathcal{D})^3}. \end{aligned}$$

Analogous estimates hold for the norms of χg_h , χh_h^\pm and $\chi \phi_h^\pm$, χu_h , and χp_h on the right-hand side of (2.35). Here one can use the equivalence of the norm in $V_\delta^{l-1/2}(\Gamma^\pm)$ with (2.3). Hence the right-hand side and, therefore, also the limit (as $h \rightarrow 0$) of the left-hand of (2.35) are majorized by

$$\begin{aligned} c \sum_{j=0}^1 &\left(\|\eta \partial_{x_3}^j f\|_{W_\delta^0(\mathcal{D})^3} + \|\eta \partial_{x_3}^j g\|_{W_\delta^1(\mathcal{D})} + \sum_{\pm} \|\eta \partial_{x_3}^j h^\pm\|_{V_\delta^{3/2}(\Gamma^\pm)^{3-d^\pm}} + \sum_{\pm} \|\eta \partial_{x_3}^j \phi^\pm\|_{V_\delta^{1/2}(\Gamma^\pm)^{3-d^\pm}} \right. \\ &\left. + \|\chi_1 \partial_{x_3}^j u\|_{W_\delta^1(\mathcal{D})^3} + \|\chi_1 \partial_{x_3}^j p\|_{W_\delta^0(\mathcal{D})} \right). \end{aligned}$$

By the first part of the proof, the norm of $\chi_1 \partial_{x_3}(u, p)$ in $W_\delta^1(\mathcal{D})^3 \times W_\delta^0$ is majorized by the right-hand side of (2.34) with $k = 0$. This implies (2.34) for $k = 1$.

Suppose now that $\eta \partial_{x_3}^j h^\pm \in W_\delta^{3/2}(\Gamma^\pm)^{3-d^\pm}$ for $j = 0, 1$ and the compatibility condition (2.17) is satisfied. Then there exists a vector function $\psi \in W^{k+1-\delta}(M)^3$ such that $S^\pm \psi = (\eta h^\pm)|_M$. Let $v \in W_\delta^{k+2}(\mathcal{D})^3$ be an extension of ψ . Then $\chi \partial_{x_3}^j (h^\pm - S^\pm v)|_M = 0$ for $j = 0, \dots, k$ and, consequently, $\chi \partial_{x_3}^j (h^\pm - S^\pm v)|_{\Gamma^\pm} \in V_\delta^{3/2}(\Gamma^\pm)^{3-d^\pm}$. Now we can apply the result proved above to the vector function $(u - v, p)$ and obtain $\zeta \partial_{x_3}(u - v, p) \in W_\delta^2(\mathcal{D})^3 \times W_\delta^1(\mathcal{D})$.

3) For $k > 1$ the theorem can be easily proved by induction. ■

Now it is easy to prove the following generalization of Theorem 2.3.

Theorem 2.4 *Let ζ , η be the same cut-off functions as in Theorem 2.3, and let $(u, p) \in \mathcal{H} \times L_2(\mathcal{D})$ be a solutions of (2.8), (2.9). Suppose that $\eta \partial_{x_3}^j g \in W_\delta^{l-1}(\mathcal{D})$, $\eta \partial_{x_3}^j h^\pm \in W_\delta^{l-1/2}(\Gamma^\pm)^{3-d^\pm}$ for $j = 0, 1, \dots, k$, h^\pm satisfy the compatibility condition (2.17), and that F is a functional on V which has the form (2.10) with $\eta \partial_{x_3}^j f \in W_\delta^{l-2}(\mathcal{D})^3$ and $\eta \partial_{x_3}^j \phi^\pm \in W_\delta^{l-3/2}(\Gamma^\pm)^{d^\pm}$ for $j = 0, 1, \dots, k$. If δ is not integer and $\max(l-1-\mu, 0) < \delta < l-1$, then $\zeta \partial_{x_3}^j(u, p) \in W_\delta^l(\mathcal{D})^3 \times W_\delta^{l-1}(\mathcal{D})$ for $j = 0, 1, \dots, k$.*

Proof: 1) We consider first the case $k = 0$, $\max(l-1-\mu, 0) < \delta < 1$ and prove the theorem for this case by induction in l . For $l = 2$ we can refer to Theorem 2.3. Suppose the assertion is proved for a certain integer $l = s \geq 2$ and the conditions of the theorem are satisfied for $l = s + 1$. We denote by χ and χ_1 the same cut-off functions as in the proof of theorem 2.3. Then, by the induction hypothesis, we have $\chi_1(u, p) \in W_\delta^s(\mathcal{D})^3 \times W_\delta^{s-1}(\mathcal{D})$ and $\chi_1 \partial_{x_3}(u, p) \in W_\delta^{s-1}(\mathcal{D})^3 \times W_\delta^{s-2}(\mathcal{D})$. If $s \geq 3$, then this implies that $\chi_1 \partial_{x_3}(u, p) \in \mathcal{H} \times L_2(\mathcal{D})$ and, by the induction hypothesis, we obtain $\chi \partial_{x_3}(u, p) \in W_\delta^s(\mathcal{D})^3 \times W_\delta^{s-1}(\mathcal{D})$. For $s = 2$ the last inclusion follows from Theorem 2.3. Consequently, we have $\chi \partial_{x_3}^j(u, p) \in W_\delta^s(\mathcal{D})^3 \times W_\delta^{s-1}(\mathcal{D})$ for $j = 0$ and $j = 1$. Using this result and Lemma 2.13, we can show, analogously to the first part of the proof of Theorem 2.3, that $\zeta(u, p) \in W_\delta^{s+1}(\mathcal{D})^3 \times W_\delta^s(\mathcal{D})$.

2) Now let $k = 0$ and $\max(l-1-\mu, \sigma) < \delta < \sigma + 1$ for a certain integer σ , $1 \leq \sigma \leq l-2$. Since $\max(l-\sigma-1-\mu, 0) < \delta - \sigma < 1$, $\eta f \in W_{\delta-\sigma}^{l-\sigma-2}(\mathcal{D})^3$, $\eta g \in W_{\delta-\sigma}^{l-\sigma-1}(\mathcal{D})$, $\eta h^\pm \in W_{\delta-\sigma}^{l-\sigma-1/2}(\Gamma^\pm)^{3-d^\pm}$, and $\eta \phi^\pm \in W_{\delta-\sigma}^{l-\sigma-3/2}(\Gamma^\pm)^{d^\pm}$, it follows from the first part of the proof that $\chi(u, p) \in W_{\delta-\sigma}^{l-\sigma}(\mathcal{D})^3 \times W_{\delta-\sigma}^{l-\sigma-1}(\mathcal{D})$. Using Lemma 2.10, we obtain the assertion of the theorem for $k = 0$.

3) We prove the assertion for $k > 0$. Let first $\max(l-1-\mu, l-2) < \delta < l-1$. Then $\max(1-\mu, 0) < \delta - l + 2 < 1$, and according to Theorem 2.3, we have $\chi \partial_{x_3}^j(u, p) \in W_{\delta-l+2}^2(\mathcal{D})^3 \times W_{\delta-l+2}^1(\mathcal{D})$ for $j = 0, 1, \dots, k$. Using Lemma 2.10, we obtain $\chi \partial_{x_3}^j(u, p) \in W_\delta^l(\mathcal{D})^3 \times W_\delta^{l-1}(\mathcal{D})$ for $j = 0, 1, \dots, k$.

In the case $\max(l-1-\mu, 0) < \delta < l-2$ we prove the assertion by induction in k . Suppose the corollary is proved for $k-1$. From parts 1) and 2) of the proof we conclude that $\chi(u, p) \in W_\delta^l(\mathcal{D})^3 \times W_\delta^{l-1}(\mathcal{D})$ and $\chi \partial_{x_3}(u, p) \in W_\delta^{l-1}(\mathcal{D})^3 \times W_\delta^{l-2}(\mathcal{D})$. The last inclusion implies $\chi \partial_{x_3}(u, p) \in \mathcal{H} \times L_2(\mathcal{D})$. Consequently, by the induction hypothesis, we obtain $\zeta \partial_{x_3}^j \partial_{x_3}(u, p) \in W_\delta^l(\mathcal{D})^3 \times W_\delta^{l-1}(\mathcal{D})$ for $j = 0, \dots, k-1$. The

theorem is proved. ■

2.10 The case when $\lambda = 1$ is the smallest positive eigenvalue of $A(\lambda)$

The number $\lambda = 1$ belongs to the spectrum of the pencil $A(\lambda)$ for all angles α if $d^+ + d^-$ is even. Here $\lambda = 1$ is the eigenvalue with smallest real part if additionally $\alpha < \pi$ for $d^+ = d^-$ and $\alpha < \pi/2$ for $d^+ \neq d^-$. In these cases the eigenvalue $\lambda = 1$ has geometric and algebraic multiplicity 1. For even d^+ and d^- there is the eigenvector $(U, P) = (0, 1)$, for odd d^+ and d^- the eigenvector $(U, P) = (\sin \varphi, -\cos \varphi, 0, 0)$. Under the above restrictions on α , generalized eigenvectors do not exist.

In these cases the result of Theorem 2.4 can be improved. However, then additional compatibility conditions on the edges must be satisfied. We restrict ourselves in the proof to the Dirichlet problem

$$-\Delta u + \nabla p = f, \quad -\nabla \cdot u = g \text{ in } \mathcal{D}, \quad u = h^\pm \text{ on } \Gamma^\pm. \quad (2.36)$$

Suppose that $(u, p) \in W_\delta^3(\mathcal{D})^3 \times W_\delta^1(\mathcal{D})$ is a solution of problem (2.36). Then the traces of $\partial_{x_j} u$, $j = 1, 2, 3$ on M exist. From the equations $-\nabla \cdot u = g$ and $u|_{\Gamma^+} = h^+$ it follows that

$$-\partial_{x_1} u_1|_M - \partial_{x_2} u_2|_M = g|_M + \partial_{x_3} h_3^+|_M. \quad (2.37)$$

Furthermore, the equations $u = h^\pm$ on Γ^\pm imply that

$$\cos \frac{\alpha}{2} \partial_{x_1} u_j|_M \pm \sin \frac{\alpha}{2} \partial_{x_2} u_j|_M = \partial_r h_j^\pm|_M \quad \text{for } j = 1, 2, 3. \quad (2.38)$$

The algebraic system (2.37), (2.38) with the unknowns $\partial_{x_1} u_j|_M$ and $\partial_{x_2} u_j|_M$, $j = 1, 2, 3$ is solvable if and only if

$$n^- \cdot \partial_r h^+|_M + n^+ \cdot \partial_r h^-|_M = -(g|_M + \partial_{x_3} h_3^+|_M) \sin \alpha. \quad (2.39)$$

Theorem 2.5 *Let (u, p) be a solution of problem (2.36), and let ζ, η be the same cut-off functions as in Theorem 2.3. Suppose that $\eta(u, p) \in \mathcal{H} \times L_2(\mathcal{D})$, $\eta f \in W_\delta^{l-2}(\mathcal{D})^2$, $\eta g \in W_\delta^{l-1}(\mathcal{D})$, $\eta h^\pm = \eta(h_1^\pm, h_2^\pm, h_3^\pm) \in W_\delta^{l-1/2}(\Gamma^\pm)^3$, $l \geq 3$, $0 < \delta < 1$, $\alpha < \pi$, $\lambda = 1$ is the only eigenvalue of the pencil $A(\lambda)$ in the strip $0 < \text{Re } \lambda \leq l - 1 - \delta$ and that the compatibility conditions $h^+|_M = h^-|_M$ and (2.39) are satisfied. Then $\zeta u \in W_\delta^l(\mathcal{D})^3$ and $\zeta p \in W_\delta^{l-1}(\mathcal{D})$.*

Proof: We prove the theorem first for $l = 3$. Let χ, χ_1 be smooth functions on $\overline{\mathcal{D}}$ such that $\chi = 1$ in a neighborhood of $\text{supp } \zeta$, $\chi_1 = 1$ in a neighborhood of $\text{supp } \chi$, and $\eta = 1$ in a neighborhood of $\text{supp } \chi_1$. Then, by Theorem 2.3, we have $\chi_1 \partial_{x_3} u \in W_\delta^2(\mathcal{D})$ and $\chi \partial_{x_3} p \in W_\delta^1(\mathcal{D})$. Let $c(x_3)$, $d(x_3)$ be vectors satisfying

$$-c_1(x_3) - d_2(x_3) = g(0, x_3) + (\partial_{x_3} h_3^+)(0, x_3), \quad \cos \frac{\alpha}{2} c(x_3) \pm \sin \frac{\alpha}{2} d(x_3) = (\partial_r h^\pm)(0, x_3)$$

for $x_3 \in M \cap \text{supp } \zeta$ and the estimate

$$|c(x_3)| + |d(x_3)| \leq C \left(|g(0, x_3)| + |(\partial_{x_3} h_3^+)(0, x_3)| + |(\partial_r h^\pm)(0, x_3)| \right)$$

with a constant independent of x_3 . Furthermore, let

$$v(x', x_3) = u(x', x_3) - h^+(0, x_3) - c(x_3)x_1 - d(x_3)x_2$$

Then with the notation $v' = (v_1, v_2)$, $f' = (f_1, f_2)$ we have

$$-\Delta_{x'} v'(\cdot, x_3) + \nabla_{x'} p(\cdot, x_3) = F'(\cdot, x_3), \quad -\nabla_{x'} \cdot v'(\cdot, x_3) = G(\cdot, x_3), \quad -\Delta_{x'} v_3(\cdot, x_3) = F_3(\cdot, x_3) \quad \text{in } K$$

and $v(\cdot, x_3) = H^\pm(\cdot, x_3)$ on γ^\pm , where

$$\begin{aligned} F'(x) &= f'(x) + \partial_{x_3}^2 u'(x), & F_3(x) &= f_3(x) + \partial_{x_3}^2 u - \partial_{x_3} p, \\ G(x', x_3) &= g(x', x_3) - g(0, x_3) + \partial_{x_3} (u_3(x', x_3) - u_3(0, x_3)), \\ H^\pm(r, x_3) &= h^\pm(r, x_3) - h^\pm(0, x_3) - (\partial_r h^\pm)(0, x_3) r \end{aligned}$$

Obviously, $\chi(\cdot, x_3) F(\cdot, x_3) \in W_\delta^1(K)^3 \subset V_\delta^1(K)^3$ for almost all $x_3 \in M \cap \text{supp } \chi$. Furthermore, since $G(0, x_3) = 0$ and $H^\pm(0, x_3) = (\partial_r H^\pm)(0, x_3) = 0$, we have $\chi(\cdot, x_3) G(\cdot, x_3) \in V_\delta^2(K)$ and $\chi(\cdot, x_3) H^\pm(\cdot, x_3) \in V_\delta^{5/2}(\Gamma^\pm)^3$. Since in the strip $0 < \text{Re } \lambda \leq 2 - \delta$ there is only the eigenvalues $\lambda = 1$ of the pencil $A(\lambda)$ with the corresponding eigenvector $(U, P) = (0, 1)$ and without generalized eigenvectors, we conclude that $\zeta(\cdot, x_3)(u(\cdot, x_3) - u(0, x_3)) \in V_\delta^3(K)^3$ and $\zeta(\cdot, x_3)(p(\cdot, x_3) - p(0, x_3)) \in V_\delta^3(K)$ (see, e.g., [5, Th.8.2.2]). In particular, $\zeta(\cdot, x_3)u(\cdot, x_3) \in W_\delta^3(K)^3$ and $\zeta(\cdot, x_3)p(\cdot, x_3) \in W_\delta^2(K)$. Analogously to the proof of Lemma 2.14, it follows that $\zeta(u, p) \in W_\delta^3(\mathcal{D})^3 \times W_\delta^2(\mathcal{D})$. Thus, the theorem is proved for $l = 3$. For $l > 3$ the result holds analogously to Theorem 2.4 by means of Lemma 2.13. ■

Furthermore, the following assertion holds (see the third part of the proof of Theorem 2.4).

Corollary 2.2 *Let (u, p) be a solution of problem (2.36), and let ζ, η be the same cut-off functions as in Theorem 2.3. Suppose that $\eta(u, p) \in \mathcal{H} \times L_2(\mathcal{D})$, $\eta \partial_{x_3}^j f \in W_\delta^{l-2}(\mathcal{D})^2$, $\eta \partial_{x_3}^j g \in W_\delta^{l-1}(\mathcal{D})$, $\eta \partial_{x_3}^j h^\pm \in W_\delta^{l-1/2}(\Gamma^\pm)^3$ for $j = 0, 1, \dots, k$, where $l \geq 3$, $0 < \delta < 1$, and $\alpha < \pi$. Suppose furthermore that $\lambda = 1$ is the only eigenvalue of the pencil $A(\lambda)$ in the strip $0 < \text{Re } \lambda \leq l - 1 - \delta$ and that the compatibility conditions $h^+|_M = h^-|_M$ and (2.39) are satisfied. Then $\zeta \partial_{x_3}^j(u, p) \in W_\delta^l(\mathcal{D})^3 \times W_\delta^{l-1}(\mathcal{D})$ for $j = 0, 1, \dots, k$.*

Remark 2.2 The results of Theorem 2.5 and Corollary 2.2 are also valid for the boundary value problem (2.1), (2.2) if $d^+ + d^-$ is even, $\alpha < \pi$ for $d^+ = d^-$, $\alpha < \pi/2$ for $d^+ \neq d^-$. Then, of course, the boundary data and g must satisfy other compatibility conditions on the edge M . For example, in the case of the Neumann problem ($d^+ = d^- = 3$) the boundary data ϕ^\pm must satisfy the condition $n^+ \cdot \phi^- = n^- \cdot \phi^+$ on M .

2.11 Estimates of Green's matrix

A matrix $G(x, \xi) = (G_{j,k}(x, \xi))_{j,k=1}^4$ is called Green's matrix for problem (2.1), (2.2) if

$$-\Delta_x \vec{G}_k(x, \xi) + \nabla_x G_{4,k}(x, \xi) = \delta(x - \xi) \vec{e}_k, \quad -\nabla_x \cdot \vec{G}_k(x, \xi) = 0 \quad \text{for } x, \xi \in \mathcal{D}, \quad k = 1, 2, 3 \quad (2.40)$$

$$-\Delta_x \vec{G}_4(x, \xi) + \nabla_x G_{4,4}(x, \xi) = 0, \quad -\nabla_x \cdot \vec{G}_4(x, \xi) = \delta(x - \xi) \quad \text{for } x, \xi \in \mathcal{D}, \quad (2.41)$$

$$S^\pm \vec{G}_k(x, \xi) = 0, \quad N^\pm(\partial_x)(\vec{G}_k(x, \xi), G_{4,k}(x, \xi)) = 0 \quad \text{for } x \in \Gamma^\pm, \quad \xi \in \mathcal{D}, \quad k = 1, 2, 3, 4. \quad (2.42)$$

Here \vec{G}_k denotes the vector with the components $G_{1,k}, G_{2,k}, G_{3,k}$, while \vec{e}_k is the k -th unit vector in \mathbb{R}^3 .

Theorem 2.6 1) *There exists a unique Green matrix $G(x, \xi)$ such that the function $x \rightarrow G_{i,j}(x, \xi)$ belongs to the Sobolev space $W^1(\mathcal{D} \setminus \bar{U}_\xi)$ for $i = 1, 2, 3$ and to $L_2(\mathcal{D} \setminus \bar{U}_\xi)$ for $i = 4$, where U_ξ is an arbitrary neighborhood of ξ , $\bar{U}_\xi \subset \mathcal{D}$.*

2) *The functions $G_{i,j}(x, \xi)$ are infinitely differentiable with respect to $x, \xi \in \bar{\mathcal{D}} \setminus M$, $x \neq \xi$. For $|x - \xi| < \min(|x'|, |\xi'|)$ there are the estimates*

$$|\partial_x^\alpha \partial_\xi^\beta G_{i,j}(x, \xi)| \leq c |x - \xi|^{-T - |\alpha| - |\beta|}, \quad (2.43)$$

where $T = 1$ for $i, j = 1, 2, 3$, $T = 3$ for $i = j = 4$, $T = 2$ else.

According to (2.7), the Green formula

$$\begin{aligned} & \int_{\mathcal{D}} (-\Delta u - \nabla \nabla \cdot u + \nabla p) \cdot v \, dx - \int_{\mathcal{D}} (\nabla \cdot u) q \, dx + \sum_{\pm} \int_{\Gamma^\pm} (-pn^\pm + 2\varepsilon(u)n^\pm) \cdot v \, dx \\ &= \int_{\mathcal{D}} u \cdot (-\Delta v - \nabla \nabla \cdot v + \nabla q) \, dx - \int_{\mathcal{D}} u (\nabla \cdot v) \, dx + \sum_{\pm} \int_{\Gamma^\pm} u \cdot (-qn^\pm + 2\varepsilon(v)n^\pm) \, dx \end{aligned} \quad (2.44)$$

is valid for all $u, v \in C_0^\infty(\bar{\mathcal{D}})^3$, $p, q \in C_0^\infty(\bar{\mathcal{D}})$.

Consequently, problem (2.1), (2.2) is formally self-adjoint. From this it follows that

$$G_{i,j}(x, \xi) = G_{j,i}(\xi, x) \quad \text{for } i, j = 1, 2, 3, 4 \quad (2.45)$$

(cf. [21, Prop.3.5.3] for the case of Dirichlet and free boundary conditions). Furthermore, it can be easily shown that

$$G_{i,j}(tx, t\xi) = t^{-T} G_{i,j}(x, \xi) \quad \text{for arbitrary } t > 0, \quad x, \xi \in \mathcal{D}, \quad (2.46)$$

where T is the same integer number as in Theorem 2.6.

Lemma 2.15 *Let \mathcal{B} be a ball with radius 1 and center at x_0 , $\text{dist}(x_0, M) \leq 4$. Furthermore, let ζ, η be smooth functions with support in \mathcal{B} such that $\eta = 1$ in a neighborhood of $\text{supp } \zeta$. If $\eta(u, p) \in \mathcal{H} \times L_2(\mathcal{D})$, $-\Delta u + \nabla p = 0$, $\nabla \cdot u = 0$ in $\mathcal{D} \cap \mathcal{B}$ and $S^\pm u = 0, N^\pm(u, p) = 0$ on $\Gamma^\pm \cap \mathcal{B}$, then*

$$\sup_{x \in \mathcal{D}} \left(|x'|^{\max(|\alpha| - \mu + \varepsilon, 0)} |\zeta(x) \partial_{x'}^\alpha \partial_{x_3}^j u(x)| + |x'|^{\max(|\alpha| + 1 - \mu + \varepsilon, 0)} |\zeta(x) \partial_{x'}^\alpha \partial_{x_3}^j p(x)| \right) \leq c (\|\eta u\|_{\mathcal{H}} + \|\eta p\|_{L_2(\mathcal{D})})$$

Proof. Let ε be such that $\mu - \varepsilon \in (k, k + 1)$. Then $\delta = k + 1 - \mu + \varepsilon \in (0, 1)$. Furthermore, let χ be a function from $C_0^\infty(\mathcal{B})$ such that $\zeta \chi = \zeta$ and $\eta \chi = \chi$. From Theorem 2.4 it follows that $\partial_{x_3}^j(\chi u) \in W_\delta^{k+2}(\mathcal{D})^\ell$, $\partial_{x_3}^j(\chi p) \in W_\delta^{k+1}(\mathcal{D})^\ell$ for $j = 0, 1, \dots$. Using Lemma 2.10, we even get $\partial_{x_3}^j(\chi u) \in W_{\delta+\nu}^{k+\nu+2}(\mathcal{D})^\ell$, $\partial_{x_3}^j(\chi p) \in W_{\delta+\nu}^{k+\nu+1}(\mathcal{D})^\ell$, for $\nu = 0, 1, \dots$ and

$$\|\partial_{x_3}^j(\chi u)\|_{W_{\delta+\nu}^{k+\nu+2}(\mathcal{D})^\ell} + \|\partial_{x_3}^j(\chi p)\|_{W_{\delta+\nu}^{k+\nu+1}(\mathcal{D})^\ell} \leq c (\|\psi u\|_{\mathcal{H}} + \|\eta p\|_{L_2(\mathcal{D})}). \quad (2.47)$$

In particular, for $k \geq 1$ and $|\alpha| \leq k - 1$ we have $\partial_{x'}^\alpha \partial_{x_3}^j(\chi p) \in W_\delta^2(\mathcal{D})$. Since $W_\delta^2(K)$ is continuously imbedded into $C(\bar{K})$, we conclude that

$$\sup_{x' \in K, x_3 \in \mathbb{R}} |\partial_{x'}^\alpha \partial_{x_3}^j(\chi p)(x', x_3)| \leq c \sup_{x_3 \in \mathbb{R}} \|\partial_{x'}^\alpha \partial_{x_3}^j(\chi p)(\cdot, x_3)\|_{W_\delta^2(K)}$$

Furthermore, using the continuity of the imbedding $W_2^1(M) \subset C(M)$, we obtain

$$\sup_{x_3 \in \mathbb{R}} \|\partial_{x'}^\alpha \partial_{x_3}^j(\chi p)(\cdot, x_3)\|_{W_\delta^2(K)} \leq c \left(\|\partial_{x'}^\alpha \partial_{x_3}^j(\chi p)\|_{W_\delta^2(\mathcal{D})} + \|\partial_{x'}^\alpha \partial_{x_3}^{j+1}(\chi p)\|_{W_\delta^2(\mathcal{D})} \right)$$

This implies

$$\sup_{x \in \mathcal{D}} |\partial_{x'}^\alpha \partial_{x_3}^j(\chi p)(x)| \leq c (\|\psi u\|_{\mathcal{H}} + \|\eta p\|_{L_2(\mathcal{D})})$$

If $|\alpha| \geq k$, then we conclude from (2.47) that $\partial_{x'}^\alpha \partial_{x_3}^j(\chi p) \in W_{\delta-k+1+|\alpha|}^2(\mathcal{D}) \subset V_{\delta-k+1+|\alpha|}^2(\mathcal{D})$. Using the inequality

$$\sup_{x' \in K} |x'|^{\beta-1} |v(x')| \leq c \|v\|_{V_\beta^2(K)},$$

which can be easily deduced from Sobolev's lemma, the continuity of the imbedding $W_2^1(M) \subset C(M)$ and (2.47), we obtain

$$\begin{aligned} \sup_{x' \in K, x_3 \in \mathbb{R}} |x'|^{\delta-k+|\alpha|} |\partial_{x'}^\alpha \partial_{x_3}^j(\chi p)(x', x_3)| &\leq c \sup_{x_3 \in \mathbb{R}} \|\partial_{x'}^\alpha \partial_{x_3}^j(\chi p)(\cdot, x_3)\|_{V_{\delta-k+1+|\alpha|}^2(K)} \\ &\leq c \left(\|\partial_{x'}^\alpha \partial_{x_3}^j(\chi p)\|_{V_{\delta-k+1+|\alpha|}^2(\mathcal{D})} + \|\partial_{x'}^\alpha \partial_{x_3}^{j+1}(\chi p)\|_{V_{\delta-k+1+|\alpha|}^2(\mathcal{D})} \right) \leq c (\|\psi u\|_{\mathcal{H}} + \|\eta p\|_{L_2(\mathcal{D})}) \end{aligned}$$

for $|\alpha| \geq k$. Consequently,

$$\sup_{x \in \mathcal{D}} |x'|^{\max(0, |\alpha| + 1 - \mu + \varepsilon)} |\zeta(x) \partial_{x'}^\alpha \partial_{x_3}^j p(x)| \leq c (\|\psi u\|_{\mathcal{H}} + \|\eta p\|_{L_2(\mathcal{D})}).$$

Analogously, it can be shown (cf. [14, Le.2.9]) that

$$\sup_{x \in \mathcal{D}} |x'|^{\max(0, |\alpha| - \mu + \varepsilon)} |\zeta(x) \partial_{x'}^\alpha \partial_{x_3}^j p(x)| \leq c (\|\psi u\|_{\mathcal{H}} + \|\eta p\|_{L_2(\mathcal{D})}).$$

The proof is complete. ■

Remark 2.3 If $d^+ + d^-$ is even, $\alpha < \pi$ for $d^+ = d^-$, and $\alpha < \pi/2$ for $d^+ \neq d^-$ (in this case we have $\mu = 1$), then the number μ in Lemma 2.15 can be replaced by the real part μ' of the first eigenvalue of $A(\lambda)$ on the right of the line $\text{Re } \lambda = 1$. To prove this, one has to use Corollary 2.2 and Remark 2.2 instead of Theorem 2.4.

Theorem 2.7 For $|x - \xi| \geq \min(|x'|, |\xi'|)$ there is the estimate

$$|\partial_{x'}^\alpha \partial_{x_3}^\sigma \partial_{\xi'}^\beta \partial_{\xi_3}^\tau G_{i,j}(x, \xi)| \leq c |x - \xi|^{-T - |\alpha| - |\beta| - \sigma - \tau} \left(\frac{|x'|}{|x - \xi|} \right)^{\min(0, \mu - |\alpha| - \delta_{i,4} - \varepsilon)} \left(\frac{|\xi'|}{|x - \xi|} \right)^{\min(0, \mu - |\beta| - \delta_{j,4} - \varepsilon)},$$

where T is the same number as in Theorem 2.6 and ε is an arbitrarily small positive number.

Proof: Due to (2.46), it suffices to prove the estimate for $|x - \xi| = 2$. Then, under the assumption $\min(|x'|, |\xi'|) \leq |x - \xi|$, we have $\max(|x'|, |\xi'|) \leq 4$. Let $\mathcal{B}_x, \mathcal{B}_\xi$ be balls with centers x and ξ , respectively, and radius 1. Furthermore, let ζ and η be infinitely differentiable functions with supports in \mathcal{B}_x and \mathcal{B}_ξ equal to one in neighborhoods of x and ξ , respectively. By Lemma 2.15 and (2.45), we have

$$\begin{aligned} & \sum_{j=1}^4 |\xi'|^{\max(|\beta| + \delta_{j,4} - \mu + \varepsilon, 0)} |\partial_{x'}^\alpha \partial_{x_3}^\sigma \partial_{\xi'}^\beta \partial_{\xi_3}^\tau G_{i,j}(x, \xi)| \\ & \leq c \left(\sum_{j=1}^3 \|\eta(\cdot) \partial_{x'}^\alpha \partial_{x_3}^\sigma G_{i,j}(x, \cdot)\|_{\mathcal{H}} + \|\eta(\cdot) \partial_{x'}^\alpha \partial_{x_3}^\sigma G_{i,4}(x, \xi)\|_{L_2(\mathcal{D})} \right) \end{aligned} \quad (2.48)$$

for $i = 1, 2, 3, 4$. Let u be the vector with the components

$$u_i(x) = \int_{\mathcal{D}} \eta(\xi) F(\xi) \cdot \vec{G}_i(\xi, x) d\xi + \int_{\mathcal{D}} \eta(\xi) g(\xi) G_{4,i}(\xi, x) d\xi, \quad i = 1, 2, 3,$$

and let

$$p(x) = \int_{\mathcal{D}} \eta(\xi) F(\xi) \cdot \vec{G}_4(\xi, x) d\xi + \int_{\mathcal{D}} \eta(\xi) g(\xi) G_{4,4}(\xi, x) d\xi.$$

By (2.7) and (2.44), the vector (u, p) is a solution of the problem

$$b(u, v) - \int_{\mathcal{D}} p \nabla \cdot v dx = \int_{\mathcal{D}} \eta(x) F(x) \cdot v(x) dx \quad \text{for all } v \in V, \quad -\nabla \cdot u = \eta g \quad \text{in } \mathcal{D}, \quad S^\pm u = 0 \quad \text{on } \Gamma^\pm.$$

Since ηF vanishes in \mathcal{B}_x , we conclude from Lemma 2.15 that

$$|x'|^{\max(|\alpha| - \mu + \varepsilon, 0)} |\partial_{x'}^\alpha \partial_{x_3}^\sigma u(x)| + |x'|^{\max(|\alpha| + 1 - \mu + \varepsilon, 0)} |\partial_{x'}^\alpha \partial_{x_3}^\sigma p(x)| \leq c (\|\zeta u\|_{\mathcal{H}} + \|\zeta p\|_{L_2(\mathcal{D})})$$

Consequently, the mappings

$$\begin{aligned} (F, g) & \rightarrow |x'|^{\max(|\alpha| - \mu + \varepsilon, 0)} \partial_{x'}^\alpha \partial_{x_3}^\sigma u_i(x) \\ & = |x'|^{\max(|\alpha| - \mu + \varepsilon, 0)} \int_{\mathcal{D}} \eta(\xi) \left(\sum_{j=1}^3 F_j(\xi) \partial_{x'}^\alpha \partial_{x_3}^\sigma G_{i,j}(x, \xi) + g(\xi) \partial_{x'}^\alpha \partial_{x_3}^\sigma G_{i,4}(x, \xi) \right) d\xi, \end{aligned}$$

$i = 1, 2, 3$, and

$$\begin{aligned} (F, g) & \rightarrow |x'|^{\max(|\alpha| + 1 - \mu + \varepsilon, 0)} \partial_{x'}^\alpha \partial_{x_3}^\sigma p(x) \\ & = |x'|^{\max(|\alpha| + 1 - \mu + \varepsilon, 0)} \int_{\mathcal{D}} \eta(\xi) \left(\sum_{j=1}^3 F_j(\xi) \partial_{x'}^\alpha \partial_{x_3}^\sigma G_{4,j}(x, \xi) + g(\xi) \partial_{x'}^\alpha \partial_{x_3}^\sigma G_{4,4}(x, \xi) \right) d\xi \end{aligned}$$

represent linear and continuous functionals on $V^* \times L_2(\mathcal{D})$ for arbitrary $x \in \mathcal{D}$. The norms of these functionals are bounded by constants independent of x . Therefore,

$$\sum_{j=1}^3 \|\eta(\cdot) \partial_{x'}^\alpha \partial_{x_3}^\sigma G_{i,j}(x, \cdot)\|_{\mathcal{H}} + \|\eta(\cdot) \partial_{x'}^\alpha \partial_{x_3}^\sigma G_{i,4}(x, \cdot)\|_{L_2(\mathcal{D})} \leq c |x'|^{\min(\mu - |\alpha| - \delta_{i,4} - \varepsilon, 0)}$$

for $i = 1, 2, 3, 4$. From this and from (2.48) we obtain the assertion of the theorem. ■

Remark 2.4 If $d^+ + d^-$ is even, $\alpha < \pi$ for $d^+ = d^-$, and $\alpha < \pi/2$ for $d^+ \neq d^-$, then

$$|\partial_{x'}^\alpha \partial_{x_3}^\sigma \partial_{\xi'}^\beta \partial_{\xi_3}^\tau G_{i,j}(x, \xi)| \leq c |x - \xi|^{-T - |\alpha| - |\beta| - \sigma - \tau} \left(\frac{|x'|}{|x - \xi|} \right)^{\min(0, \mu' - |\alpha| - \delta_{i,4} - \varepsilon)} \left(\frac{|\xi'|}{|x - \xi|} \right)^{\min(0, \mu' - |\beta| - \delta_{j,4} - \varepsilon)}$$

for $|x - \xi| \geq \min(|x'|, |\xi'|)$, where μ' is the real part of the first eigenvalue on the right of the line $\operatorname{Re} \lambda = 1$ (cf. Remark 2.3).

3 The boundary value problem in a polyhedral cone

We consider the boundary value problem

$$-\Delta u + \nabla p = f, \quad -\nabla \cdot u = g \quad \text{in } \mathcal{K}, \quad (3.1)$$

$$S_j u = h_j, \quad N_j(u, p) = \phi_j \quad \text{on } \Gamma_j, \quad j = 1, \dots, n. \quad (3.2)$$

Here S_j is one of the following operator: $S_j u = u$ (then we set $d_j = 0$), $S_j u = u_n = u \cdot n$ (then we set $d_j = 2$), $S_j u = u_\tau = u - u_n n$ (then we set $d_j = 1$). The operators N_j are defined as $N_j(u, p) = -p + 2\varepsilon_{n,n}(u)$ if $d_j = 1$, $N_j(u, p) = \varepsilon_{n,\tau}(u)$ if $d_j = 2$, and $N_j(u, p) = -pn + 2\varepsilon_n(u)$ if $d_j = 3$. In the case $d_j = 0$ the condition $N_j(u, p) = \phi_j$ does not appear in (3.2), whereas the condition $S_j u = h_j$ does not appear if $d_j = 3$.

3.1 Weighted Sobolev spaces

For an arbitrary point $x \in \mathcal{K}$ let $\rho(x) = |x|$ be the distance to the vertex of the cone and $r_j(x)$ the distance to the edge M_j . Let l be a nonnegative integer, $\beta \in \mathbb{R}$, $\vec{\delta} = (\delta_1, \dots, \delta_n) \in \mathbb{R}^n$, $\delta_j > -1$ for $j = 1, \dots, n$. By $W_{\beta, \vec{\delta}}^l(\mathcal{K})$ we denote the weighted Sobolev space with the norm

$$\|u\|_{W_{\beta, \vec{\delta}}^l(\mathcal{K})} = \left(\int_{\mathcal{K}} \sum_{|\alpha| \leq l} \rho^{2(\beta - l + |\alpha|)} \prod_{j=1}^n \left(\frac{r_j}{\rho}\right)^{2\delta_j} |\partial_x^\alpha u|^2 dx \right)^{1/2}.$$

The corresponding trace space on the side Γ_j of \mathcal{K} is denoted by $W_{\beta, \vec{\delta}}^{l-1/2}(\Gamma_j)$.

Note that $W_{\beta+1, \vec{\delta}}^{l+1}(\mathcal{K})$ is continuously imbedded into $W_{\beta, \vec{\delta}}^l(\mathcal{K})$ if $\delta'_j \leq \delta_j + 1$ for $j = 1, \dots, n$ (see [14, Le.4.1]).

Using Lemma 2.6, one can prove the following assertion analogously to [14, Le.4.2].

Lemma 3.1 *Let $h_j \in W_{\beta, \vec{\delta}}^{3/2}(\Gamma_j)^{3-d_j}$ and $\phi_j \in W_{\beta, \vec{\delta}}^{1/2}(\Gamma_j)^{d_j}$, $j = 1, \dots, n$, be given, where $\delta_j > 0$ for all j . For every $j = 1, \dots, n$ let Γ_{j+} and Γ_{j-} be the sides of \mathcal{K} adjacent to the edge M_j . We suppose that the functions h_j satisfy the compatibility conditions*

$$(h_{j+}|_{M_j}, h_{j-}|_{M_j}) \in R(T_j), \quad (3.3)$$

where $R(T_j)$ is the range of the operator $T_j = (S_{j+}, S_{j-})$ (cf. condition (2.17)). Then there exists a vector function $u \in W_{\beta, \vec{\delta}}^2(\mathcal{K})^3$ such that $S_j u = h_j$, $N_j(u, 0) = \phi_j$ on Γ_j , $j = 1, \dots, n$ and

$$\|u\|_{W_{\beta, \vec{\delta}}^2(\mathcal{K})^3} \leq c \sum_{j=1}^n \left(\|h_j\|_{W_{\beta, \vec{\delta}}^{3/2}(\Gamma_j)^{3-d_j}} + \|\phi_j\|_{W_{\beta, \vec{\delta}}^{1/2}(\Gamma_j)^{d_j}} \right)$$

with a constant c independent of h_j and ϕ_j .

3.2 Operator pencils generated by the boundary value problem

We introduce the following operator pencils \mathfrak{A} and A_j .

1. Let $\Gamma_{j\pm}$ be the sides of \mathcal{K} adjacent to the edge M_j , and let α_j be the angle at the edge M_j and let $A_j(\lambda)$ be the operator pencil introduced in Section 2.5, where $S^\pm = S_{j\pm}$ and $N^\pm = N_{j\pm}$. We denote by $\lambda_1^{(j)}$ the eigenvalue with smallest positive real part and set $\mu_j = \text{Re } \lambda_1^{(j)}$. In the case when $d_{j-} + d_{j+}$ is even and $(|d_{j-} - d_{j+}| + 2)\alpha_j < 2\pi$, we have $\mu_j = \lambda_1^{(j)} = 1$. Then let $\mu'_j = \text{Re } \lambda_2^{(j)}$, where $\lambda_2^{(j)}$ is the eigenvalue of $A_j(\lambda)$ with smallest real part greater than 1.

2. Let $\rho = |x|$, $\omega = x/|x|$, $V_\Omega = \{u \in W^1(\Omega)^3 : S_j u = 0 \text{ on } \gamma_j \text{ for } j = 1, \dots, n\}$, and

$$a\left(\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix}; \lambda\right) = \frac{1}{\log 2} \int_{\substack{\mathcal{K} \\ 1 < |x| < 2}} \left(2 \sum_{i,j=1}^3 \varepsilon_{i,j}(U) \cdot \varepsilon_{i,j}(V) - P \nabla \cdot V - (\nabla \cdot U) Q \right) dx,$$

where $U = \rho^\lambda u(\omega)$, $V = \rho^{-1-\lambda} v(\omega)$, $P = \rho^{\lambda-1} p(\omega)$, $Q = \rho^{-2-\lambda} q(\omega)$, $u, v \in V_\Omega$, $p, q \in L_2(\Omega)$, and $\lambda \in \mathbb{C}$. The bilinear form $a(\cdot, \cdot; \lambda)$ generates the linear and continuous operator

$$\mathfrak{A}(\lambda) : V_\Omega \times L_2(\Omega) \rightarrow V_\Omega^* \times L_2(\Omega)$$

by

$$\int_\Omega \mathfrak{A}(\lambda) \begin{pmatrix} u \\ p \end{pmatrix} \cdot \begin{pmatrix} v \\ q \end{pmatrix} d\omega = a\left(\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix}; \lambda\right), \quad u, v \in V_\Omega, \quad p, q \in L_2(\Omega).$$

As is known, the spectrum of the pencil $\mathfrak{A}(\lambda)$ consists of isolated points, the eigenvalues of this pencil. Detailed information on the spectrum can be found in [6, Sec.5,6].

We consider the restriction of the operator $\mathfrak{A}(\lambda)$ to the space $W_{\vec{\delta}}^2(\Omega)^3 \times W_{\vec{\delta}}^1(\Omega)$. Here $W_{\vec{\delta}}^l(\Omega)$ is the weighted Sobolev space with the norm

$$\|u\|_{W_{\vec{\delta}}^l(\Omega)} = \left(\int_{\mathcal{K}} \sum_{|\alpha| \leq l} \prod_{j=1}^n r_j^{2\delta_j} |\partial_x^\alpha u(x)|^2 dx \right)^{1/2}$$

where the function u is extended by $u(x) = u(x/|x|)$ to \mathcal{K} . The corresponding trace space on the side γ_j is denoted by $W_{\vec{\delta}}^{l-1/2}(\gamma_j)$. By $\mathfrak{A}_{\vec{\delta}}(\lambda)$ we denote the operator

$$W_{\vec{\delta}}^2(\Omega)^3 \times W_{\vec{\delta}}^1(\Omega) \ni (u, p) \rightarrow (f, g, \{h_j\}, \{\phi_j\}) \in W_{\vec{\delta}}^0(\Omega)^3 \times W_{\vec{\delta}}^1(\Omega) \times \prod_{j=1}^n W_{\vec{\delta}}^{3/2}(\gamma_j)^{3-d_j} \times \prod_{j=1}^n W_{\vec{\delta}}^{1/2}(\gamma_j)^{d_j}$$

where $f(\omega) = \rho^{2-\lambda}(-\Delta U + \nabla P)$, $g(\omega) = -\rho^{1-\lambda} \nabla \cdot U$, $h_j = S_j u$, and $\phi_j = \rho^{1-\lambda} N_j(U, P)$ (U, P as above). It can be proved that the spectra of the pencils $\mathfrak{A}(\lambda)$ and $\mathfrak{A}_{\vec{\delta}}(\lambda)$ coincide if $\max(0, 1 - \mu_j) < \delta_j < 1$. Furthermore, there exist positive constants N and ε such that for all λ in the set $\{\lambda \in \mathbb{C} : |\lambda| > N, |\operatorname{Re} \lambda| < \varepsilon |\operatorname{Im} \lambda|\}$ the operator $\mathfrak{A}_{\vec{\delta}}(\lambda)$ is an isomorphism onto the subset of all $(f, g, \{h_j\}, \{\phi_j\})$ satisfying the compatibility conditions $(h_{j+}, h_{j-}) \in R(T_j)$ on the corners $M_j \cap S^2$ of Ω . For every λ in this set and every solution $(u, p) \in W_{\vec{\delta}}^2(\Omega)^3 \times W_{\vec{\delta}}^1(\Omega)$ of the equation $\mathfrak{A}_{\vec{\delta}}(\lambda)(u, p) = (f, g, \{h_j\}, \{\phi_j\})$ the estimate

$$\begin{aligned} \sum_{j=0}^2 |\lambda|^{2-j} \|u\|_{W_{\vec{\delta}}^j(\Omega)^3} + \sum_{j=0}^1 |\lambda|^{1-j} \|p\|_{W_{\vec{\delta}}^j(\Omega)} &\leq c \left(\|f\|_{W_{\vec{\delta}}^0(\Omega)^3} + \sum_{j=0}^1 |\lambda|^{1-j} \|g\|_{W_{\vec{\delta}}^j(\Omega)} \right. \\ &\left. + \sum_{j=1}^n \left(\|h_j\|_{W_{\vec{\delta}}^{3/2}(\gamma_j)^{3-d_j}} + |\lambda|^{3/2} \|h_j\|_{W_{\vec{\delta}}^0(\gamma_j)^{3-d_j}} + \|\phi_j\|_{W_{\vec{\delta}}^{1/2}(\gamma_j)^{d_j}} + |\lambda|^{1/2} \|\phi_j\|_{W_{\vec{\delta}}^0(\gamma_j)^{d_j}} \right) \right) \end{aligned} \quad (3.4)$$

holds with a constant c independent of (u, p) and λ . For the proof we refer to [14, Th.3.2].

3.3 Solvability of the boundary value problem

The following theorem can be proved in a standard way (see, e.g., [5, Ch.6]) using the estimate (3.4).

Theorem 3.1 1) *Suppose that there are no eigenvalues of the pencil \mathfrak{A} on the line $\operatorname{Re} \lambda = -\beta + 1/2$ and that the components of $\vec{\delta}$ satisfy the inequalities $\max(1 - \mu_j, 0) < \delta_j < 1$. Then the boundary value problem (3.1), (3.2) is uniquely solvable in $W_{\beta, \vec{\delta}}^2(\mathcal{K})^3 \times W_{\beta, \vec{\delta}}^1(\mathcal{K})$ for arbitrary $f \in W_{\beta, \vec{\delta}}^0(\mathcal{K})^3$, $g \in W_{\beta, \vec{\delta}}^0(\mathcal{K})$, $h_j \in W_{\beta, \vec{\delta}}^{3/2}(\Gamma_j)^{3-d_j}$ satisfying (3.3), and $\phi_j \in W_{\beta, \vec{\delta}}^{3/2}(\Gamma_j)^{d_j}$.*

2) *Let $(u, p) \in W_{\beta, \vec{\delta}}^2(\mathcal{K})^3 \times W_{\beta, \vec{\delta}}^1(\mathcal{K})$ be a solution of the boundary value problem (3.1), (3.2), where $f \in W_{\beta', \vec{\delta}'}^0(\mathcal{K})^\ell$, $g \in W_{\beta', \vec{\delta}'}^1(\mathcal{K})$, $h_j \in W_{\beta', \vec{\delta}'}^{3/2}(\Gamma_j)^{3-d_j}$, and $\phi_j \in W_{\beta', \vec{\delta}'}^{1/2}(\Gamma_k)^{d_j}$. Suppose that the components of $\vec{\delta}$ and $\vec{\delta}'$ satisfy the inequality $\max(1 - \mu_j, 0) < \delta'_j \leq \delta_j < 1$. If there are no eigenvalues of the pencil \mathfrak{A} on the lines $\operatorname{Re} \lambda = -\beta + 1/2$ and $\operatorname{Re} \lambda = -\beta' + 1/2$, then*

$$(u, p) = \sum_{\nu=1}^N \sum_{j=1}^{I_\nu} \sum_{s=0}^{\kappa_{\nu, j}-1} c_{\nu, j, s} \sum_{\sigma=0}^s \frac{1}{\sigma!} (\log \rho)^\sigma (\rho^{\lambda_\nu} u^{(\nu, j, s-\sigma)}(\omega), \rho^{\lambda_\nu-1} p^{(\nu, j, s-\sigma)}(\omega)) + (w, q) \quad (3.5)$$

where $(w, q) \in W_{\beta', \delta'}^2(\mathcal{K})^3 \times W_{\beta', \delta'}^1(\mathcal{K})$ is a solution of problem (3.1)–(3.2), λ_ν are the eigenvalues of the pencil \mathfrak{A} between the lines $\operatorname{Re} \lambda = -\beta + 1/2$ and $\operatorname{Re} \lambda = -\beta' + 1/2$ and $(u^{(\nu, j, s)}, p^{(\nu, j, s)})$ are eigenvectors and generalized eigenvectors corresponding to the eigenvalue λ_ν .

Furthermore, analogously to [14, Le.4.3], the following assertion holds.

Lemma 3.2 *Let $(u, p) \in W_{\beta, \delta}^2(\mathcal{K})^3 \times W_{\beta, \delta}^1(\mathcal{K})$ be a solution of problem (3.1), (3.2). We assume that $(\rho \partial_\rho)^\nu f \in W_{\beta, \delta}^0(\mathcal{K})$, $(\rho \partial_\rho)^\nu g \in W_{\beta, \delta}^1(\mathcal{K})$, $(\rho \partial_\rho)^\nu h_j \in W_{\beta, \delta}^{3/2}(\Gamma_j)^{3-d_j}$, and $(\rho \partial_\rho)^\nu \phi_j \in W_{\beta, \delta}^{1/2}(\Gamma_j)$ for $\nu = 0, \dots, k$, $j = 1, \dots, n$. If $\max(1 - \mu_j, 0) < \delta_j < 1$ for $j = 1, \dots, n$ and the line $\operatorname{Re} \lambda = -\beta + 1/2$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$, then $(\rho \partial_\rho)^\nu(u, p) \in W_{\beta, \delta}^2(\mathcal{K})^3 \times W_{\beta, \delta}^1(\mathcal{K})$ for $\nu = 0, \dots, k$.*

3.4 Existence of weak solutions

Let $V_\beta = \{u \in W_{\beta, 0}^1(\mathcal{K})^3 : S_j u = 0 \text{ on } \Gamma_j \text{ for } j = 1, \dots, n\}$, and let the operator

$$\mathcal{A}_\beta : V_\beta \times W_{\beta, 0}^0(\mathcal{K}) \rightarrow V_{-\beta}^* \times W_{\beta, 0}^0(\mathcal{K})$$

be defined as

$$\langle \mathcal{A}_\beta(u, p), (v, q) \rangle = b(u, v) - \int_{\mathcal{K}} p \nabla \cdot v \, dx - \int_{\mathcal{K}} (\nabla \cdot u) q \, dx \quad \text{for all } v \in V_{-\beta}^*, q \in W_{\beta, 0}^0(\mathcal{K}).$$

Lemma 3.3 *For arbitrary $u \in V_\beta$, $p \in V_\beta^0(\mathcal{K})$, $(f, g) = \mathcal{A}_\beta(u, p)$ there is the estimate*

$$\|u\|_{W_{\beta, 0}^1(\mathcal{K})^3} + \|p\|_{W_{\beta, 0}^0(\mathcal{K})} \leq c \left(\|f\|_{V_{-\beta}^*} + \|p\|_{W_{\beta, 0}^0(\mathcal{K})} + \|u\|_{W_{\beta-1, 0}^0(\mathcal{K})^3} + \|p\|_{W_{\beta-1, 0}^{-1}(\mathcal{K})} \right)$$

with a constant c independent of u and p . Here $W_{\beta-1, 0}^{-1}(\mathcal{K})$ denotes the dual space of $W_{1-\beta, 0}^1(\mathcal{K})$.

Theorem 3.2 *The operator \mathcal{A}_β is an isomorphism if there are no eigenvalues of the pencil $\mathfrak{A}(\lambda)$ on the line $\operatorname{Re} \lambda = -\beta - 1/2$.*

Proof: We show first that

$$\|u\|_{W_{\beta, 0}^1(\mathcal{K})^3} + \|p\|_{W_{\beta, 0}^0(\mathcal{K})} \leq c \left(\|f\|_{V_{-\beta}^*} + \|g\|_{W_{\beta, 0}^0(\mathcal{K})} \right) \quad (3.6)$$

for all $u \in V_\beta$, $p \in W_{\beta, 0}^0(\mathcal{K})$, $(f, g) = \mathcal{A}_\beta(u, p)$. Let $u \in V_\beta \subset W_{\beta-1, \varepsilon-1}^0(\mathcal{K})^3$, $p \in W_{\beta, 0}^0(\mathcal{K})$, $w \in W_{1-\beta, 1-\varepsilon}^0(\mathcal{K})^3$, and $\psi \in W_{1-\beta, 1-\varepsilon}^1(\mathcal{K})$ with sufficiently small positive ε . By Theorem 3.1, there exists a solution $(v, q) \in W_{1-\beta, 1-\varepsilon}^2(\mathcal{K})^3 \times W_{1-\beta, 1-\varepsilon}^1(\mathcal{K})$ of the problem

$$-\Delta v - \nabla \nabla \cdot v + \nabla q = w, \quad \nabla \cdot v = \psi \quad \text{in } \mathcal{K}, \quad S_j v = 0, \quad N_j(v, q) = 0 \quad \text{on } \Gamma_j, \quad j = 1, \dots, n$$

satisfying the estimate

$$\|v\|_{W_{1-\beta, 1-\varepsilon}^2(\mathcal{K})^3} + \|p\|_{W_{1-\beta, 1-\varepsilon}^0(\mathcal{K})} \leq c \left(\|w\|_{W_{1-\beta, 1-\varepsilon}^0(\mathcal{K})^3} + \|\psi\|_{W_{1-\beta, 1-\varepsilon}^1(\mathcal{K})} \right).$$

From Green's formula it follows that

$$\int_{\mathcal{K}} u \cdot w \, dx + \int_{\mathcal{K}} p \psi \, dx = b(u, v) - \int_{\mathcal{K}} p \nabla \cdot v \, dx - \int_{\mathcal{K}} (\nabla \cdot u) q \, dx = \langle \mathcal{A}_\beta(u, p), (v, q) \rangle.$$

Hence,

$$\begin{aligned} \left| \int_{\mathcal{K}} u \cdot w \, dx + \int_{\mathcal{K}} p \psi \, dx \right| &\leq \|\mathcal{A}_\beta(u, p)\|_{V_{-\beta}^* \times W_{\beta, 0}^0(\mathcal{K})} \left(\|v\|_{W_{1-\beta, 0}^1(\mathcal{K})^3} + \|q\|_{W_{\beta, 0}^0(\mathcal{K})} \right) \\ &\leq c \|\mathcal{A}_\beta(u, p)\|_{V_{-\beta}^* \times W_{\beta, 0}^0(\mathcal{K})} \left(\|v\|_{W_{1-\beta, 1-\varepsilon}^2(\mathcal{K})^3} + \|q\|_{W_{1-\beta, 1-\varepsilon}^0(\mathcal{K})} \right) \\ &\leq c \|\mathcal{A}_\beta(u, p)\|_{V_{-\beta}^* \times W_{\beta, 0}^0(\mathcal{K})} \left(\|w\|_{W_{1-\beta, 1-\varepsilon}^0(\mathcal{K})^3} + \|\psi\|_{W_{1-\beta, 1-\varepsilon}^1(\mathcal{K})} \right). \end{aligned}$$

Setting $w = \rho^{2(\beta-1)} \prod (r_j/\rho)^{2(\varepsilon-1)}$ and $\psi = 0$, we obtain

$$\|u\|_{W_{\beta-1,\varepsilon-1}^0(\mathcal{K})^3}^2 \leq c \|\mathcal{A}_\beta(u, p)\|_{V_{-\beta}^* \times W_{\beta,0}^0(\mathcal{K})} \|u\|_{W_{\beta-1,\varepsilon-1}^0(\mathcal{K})^3}$$

and, therefore,

$$\|u\|_{W_{\beta-1,0}^0(\mathcal{K})^3} \leq c \|u\|_{W_{\beta-1,\varepsilon-1}^0(\mathcal{K})^3} \leq c \|\mathcal{A}_\beta(u, p)\|_{V_{-\beta}^* \times W_{\beta,0}^0(\mathcal{K})}.$$

Analogously, for $w = 0$ and arbitrary $\psi \in W_{1-\beta,0}^1(\mathcal{K})$ we get

$$\left| \int_{\mathcal{K}} p \psi \, dx \right| \leq c \|\mathcal{A}_\beta(u, p)\|_{V_{-\beta}^* \times W_{\beta,0}^0(\mathcal{K})} \|\psi\|_{W_{1-\beta,1-\varepsilon}^1(\mathcal{K})} \leq c \|\mathcal{A}_\beta(u, p)\|_{V_{-\beta}^* \times W_{\beta,0}^0(\mathcal{K})} \|\psi\|_{W_{1-\beta,0}^1(\mathcal{K})}$$

what implies the inequality

$$\|p\|_{W_{\beta-1,0}^{-1}(\mathcal{K})} \leq c \|\mathcal{A}_\beta(u, p)\|_{V_{-\beta}^* \times W_{\beta,0}^0(\mathcal{K})}$$

Using Lemma 3.4, we arrive at (3.6).

From (3.6) it follows that \mathcal{A}_β is injective and its range is closed. By Theorem 3.1, the range of \mathcal{A}_β contains the set $W_{\beta-1,\varepsilon-1}^0(\mathcal{K})^3 \times W_{\beta-1,\varepsilon-1}^1(\mathcal{K})$ which is dense in $V_{-\beta}^* \times W_{\beta,0}^0(\mathcal{K})$. This proves the theorem. ■

Let $F \in V_{-\beta}^*$, $g \in W_{\beta,0}^0(\mathcal{K})$, and let $h_j \in W_{\beta,0}^{1/2}(\Gamma_j)$ be such that there exists a vector function $w \in W_{\beta,0}^1(\mathcal{K})^3$ satisfying the boundary condition $S_j w = h_j$ on Γ_j . By a weak solution $(u, p) \in W_{\beta,0}^1(\mathcal{K}) \times W_{\beta,0}^0(\mathcal{K})$ of problem (3.1), (3.2) we mean a pair (u, p) satisfying

$$b(u, v) - \int_{\mathcal{K}} p \nabla \cdot v \, dx = F(v) \quad \text{for all } v \in V_{-\beta} \quad (3.7)$$

$$-\nabla \cdot u = g \quad \text{in } \mathcal{K}, \quad S_j u = h_j \quad \text{on } \Gamma_j, \quad j = 1, \dots, n. \quad (3.8)$$

If $g \in W_{\beta+1,\delta}^1(\mathcal{K})$ with $\delta_j < 1$, $j = 1, \dots, n$, and the functional F has the form

$$F(v) = \int_{\mathcal{K}} (f + \nabla g) \cdot v \, dx + \sum_{j=1}^n \int_{\Gamma_j} \phi_j \cdot v \, dx, \quad (3.9)$$

with $f \in W_{\beta+1,\delta}^0(\mathcal{K})$, $\phi_j \in W_{\beta+1,\delta}^{1/2}(\Gamma_j)$, then (u, p) is a strong solution of problem (3.1), (3.2). Theorem 3.2 ensures the existence and uniqueness of weak solutions provided the line $\text{Re } \lambda = -\beta - 1/2$ does not contain eigenvalues of the pencil $\mathfrak{A}(\lambda)$.

3.5 Regularity of weak solutions

Analogously to [14, Th.4.4], the following result holds. The proof is essentially based on Theorem 2.3.

Theorem 3.3 *Let $(u, p) \in W_{\beta-l+1,0}^1(\mathcal{K})^3 \times W_{\beta,0}^0(\mathcal{K})$ be a solution of problem (3.7), (3.8). Suppose that $g \in W_{\beta,\delta}^{l-1}(\mathcal{K})$, $h_j \in W_{\beta,\delta}^{l-1/2}(\Gamma_j)$ satisfy the compatibility condition (3.3) on the edges M_j and that the functional $F \in V_{-\beta+l-1}^*$ has the form (3.9), where $f \in W_{\beta,\delta}^{l-2}(\mathcal{K})^3$ and $\phi_j \in W_{\beta,\delta}^{l-3/2}(\Gamma_j)$, $l \geq 2$, δ_j is not integer, $\max(l-1-\mu_j, 0) < \delta_j < l-1$. Then $u \in W_{\beta,\delta}^l(\mathcal{K})^3$ and $p \in W_{\beta,\delta}^{l-1}(\mathcal{K})$.*

Corollary 3.1 *Let the assumptions of Theorem 3.3 be valid. Additionally we assume that $(\rho \partial_\rho)^\nu f \in W_{\beta,\delta}^{l-2}(\mathcal{K})^3$, $(\rho \partial_\rho)^\nu g \in W_{\beta,\delta}^{l-1}(\mathcal{K})$, $(\rho \partial_\rho)^\nu h_j \in W_{\beta,\delta}^{l-1/2}(\Gamma_j)$, $(\rho \partial_\rho)^\nu \phi_j \in W_{\beta,\delta}^{l-3/2}(\Gamma_j)$ for $\nu = 1, \dots, k$ and $j = 1, \dots, n$. Then $(\rho \partial_\rho)^\nu u \in W_{\beta,\delta}^l(\mathcal{K})^3$ and $(\rho \partial_\rho)^\nu p \in W_{\beta,\delta}^{l-1}(\mathcal{K})$.*

Proof: It suffices to prove the assertion for $k = 1$. For $k > 1$ it can be proved by induction. Let first $\delta_j > l-2$ and, therefore, $\max(1-\mu_j, 0) < \delta_j - l + 2 < 1$ for $j = 1, \dots, n$. Then from Theorem 3.3

and Lemma 3.2 it follows that $(\rho\partial_\rho)^\nu u \in W_{\beta-l+2, \bar{\delta}-(l-2)\bar{1}}^2(\mathcal{K})^3$ and $(\rho\partial_\rho)^\nu p \in W_{\beta-l+2, \bar{\delta}-(l-2)\bar{1}}^1(\mathcal{K})$ for $\nu = 1, \dots, k$. Using Theorem 3.3, the equations

$$-\Delta(\rho\partial_\rho u) + \nabla(\rho\partial_\rho p) = (\rho\partial_\rho + 2)f - \nabla p \in W_{\beta, \bar{\delta}}^{l-2}(\mathcal{K})^3, \quad -\nabla \cdot (\rho\partial_\rho u) = (\rho\partial_\rho + 1) \cdot g \in W_{\beta, \bar{\delta}}^{l-1}(\mathcal{K}) \quad (3.10)$$

and analogous equations for $S_j(\rho\partial_\rho u)$ and $N_j(\rho\partial_\rho u, \rho\partial_\rho p)$, we obtain $\rho\partial_\rho u \in W_{\beta, \bar{\delta}}^l(\mathcal{K})^3$ and $\rho\partial_\rho p \in W_{\beta, \bar{\delta}}^{l-1}(\mathcal{K})$.

Now let $\delta_j < l - 2$ for $j=1, \dots, n$. By Theorem 3.3, we have $(u, p) \in W_{\beta, \bar{\delta}}^l(\mathcal{K})^3 \times W_{\beta, \bar{\delta}}^{l-1}(\mathcal{K})$ and, consequently, $\rho\partial_\rho(u, p) \in W_{\beta-1, \bar{\delta}}^{l-1}(\mathcal{K})^3 \times W_{\beta-1, \bar{\delta}}^{l-2}(\mathcal{K})$. Using again Theorem 3.3 and (3.10), we obtain $\rho\partial_\rho(u, p) \in W_{\beta, \bar{\delta}}^l(\mathcal{K})^3 \times W_{\beta, \bar{\delta}}^{l-1}(\mathcal{K})$.

Finally, let $l - 1 - \delta_j > 1$ for some, but not all, j . Then let ψ_1, \dots, ψ_n be smooth functions on $\bar{\Omega}$ such that $\psi_j \geq 0$, $\psi_j = 1$ near $M_j \cap S^2$, and $\sum \psi_j = 1$. We extend ψ_j to \mathcal{K} by the equality $\psi_j(x) = \psi_j(x/|x|)$. Then $\partial_x^\alpha \psi_j(x) \leq c|x|^{-|\alpha|}$. Consequently, the assumptions of the corollary are satisfied for $(\psi_j u, \psi_j p)$, and from what has been shown above it follows that $\rho\partial_\rho(\psi_j u, \psi_j p) \in W_{\beta, \bar{\delta}}^l(\mathcal{K})^3 \times W_{\beta, \bar{\delta}}^{l-1}(\mathcal{K})$ for $j = 1, \dots, n$. This completes the proof. ■

Remark 3.1 If $d_{j+} + d_{j-}$ is even and, moreover, $\alpha_j < \pi$ for $d_{j+} = d_{j-}$, $\alpha_j < \pi/2$ for $d_{j+} \neq d_{j-}$, then the number μ_j in the condition $\max(l - 1 - \mu_j, 0) < \delta_j < l - 1$ on δ_j of Theorem 3.3 and Corollary 3.1 is equal to 1 and can be replaced by μ'_j . However, if $\delta_j < l - 2$, then $h_{j\pm}$, $\phi_{j\pm}$ and g must satisfy certain additional compatibility condition on the edge M_j (cf. Theorem 2.5 and Remark 2.2).

The following theorem can be proved analogously to [14, Cor.4.3].

Theorem 3.4 Let $(u, p) \in W_{\beta, 0}^1(\mathcal{K})^3 \times W_{\beta, 0}^0(\mathcal{K})$ be solution of problem (3.7), (3.8), where $g \in W_{\beta, \bar{\delta}}^{l-1}(\mathcal{K})$, $h_j \in W_{\beta, \bar{\delta}}^{l-1/2}(\Gamma_j)$, and $F \in V_{-\beta}^* \cap V_{-\beta'}^*$. If there are no eigenvalues of the pencil $\mathfrak{A}(\lambda)$ on the lines $\text{Re } \lambda = -\beta - 1/2$ and $\text{Re } \lambda = -\beta' - 1/2$, then (u, p) admits the decomposition (3.5), where $w \in W_{\beta', 0}^1(\mathcal{K})^3$, $p \in W_{\beta', 0}^0(\mathcal{K})$, and λ_ν are the eigenvalues of $\mathfrak{A}(\lambda)$ between the lines $\text{Re } \lambda = -\beta - 1/2$ and $\text{Re } \lambda = -\beta' - 1/2$.

3.6 Estimates of Green's matrix

A matrix $G(x, \xi) = (G_{i,j}(x, \xi))_{i,j=1}^4$ is called Green's matrix for problem (3.1), (3.2) if

$$-\Delta_x \vec{G}_j(x, \xi) + \nabla_x G_{4,j}(x, \xi) = \delta(x - \xi) \vec{e}_j, \quad -\nabla_x \cdot \vec{G}_j(x, \xi) = 0 \quad \text{for } x, \xi \in \mathcal{K}, \quad j = 1, 2, 3 \quad (3.11)$$

$$-\Delta_x \vec{G}_4(x, \xi) + \nabla_x G_{4,4}(x, \xi) = 0, \quad -\nabla_x \cdot \vec{G}_4(x, \xi) = \delta(x - \xi) \quad \text{for } x, \xi \in \mathcal{K}, \quad (3.12)$$

$$S_k \vec{G}_j(x, \xi) = 0, \quad N_k(\partial_x) (\vec{G}_j(x, \xi), G_{4,j}(x, \xi)) = 0 \quad \text{for } x \in \Gamma_k, \quad \xi \in \mathcal{K}, \quad j = 1, 2, 3, 4, \quad k = 1, \dots, n. \quad (3.13)$$

Here \vec{G}_j denotes the vector with the components $G_{1,j}, G_{2,j}, G_{3,j}$, while \vec{e}_j is the j -th unit vector in \mathbb{R}^3 . Using Theorem 3.2, one can prove the following assertions analogously to [10, Th.2.1].

Theorem 3.5 Suppose that the line $\text{Re } \lambda = -\beta - 1/2$ is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$. Then there exists a unique Green matrix $G(x, \xi)$ such that the function $x \rightarrow \zeta(|x - \xi|/r(\xi)) G_{i,j}(x, \xi)$ belongs to $W_{\beta, 0}^1(\mathcal{K})$ for $i = 1, 2, 3$ and to $W_{\beta, 0}^0(\mathcal{K})$ for $i = 4$, where ζ is an arbitrary smooth function on $(0, \infty)$ equal to one in $(1, \infty)$ and to zero in $(0, \frac{1}{2})$. Furthermore, the functions $\xi \rightarrow \zeta(|x - \xi|/r(x)) G_{i,j}(x, \xi)$ belong to $W_{-\beta, 0}^1(\mathcal{K})$ for $j = 1, 2, 3$ and to $W_{-\beta, 0}^0(\mathcal{K})$ for $j = 4$. The vector functions $\vec{H}_i = (G_{i,1}, G_{i,2}, G_{i,3})$ and the functions $G_{i,4}$, $i = 1, 2, 3, 4$, are solutions of the problems

$$-\Delta_\xi \vec{H}_i(x, \xi) + \nabla_\xi G_{i,4}(x, \xi) = \delta(x - \xi) \vec{e}_i, \quad -\nabla_\xi \cdot \vec{H}_i(x, \xi) = 0 \quad \text{for } x, \xi \in \mathcal{K}, \quad i = 1, 2, 3$$

$$-\Delta_\xi \vec{H}_4(x, \xi) + \nabla_\xi G_{4,4}(x, \xi) = 0, \quad -\nabla_\xi \cdot \vec{H}_4(x, \xi) = \delta(x - \xi) \quad \text{for } x, \xi \in \mathcal{K},$$

$$S_k \vec{H}_i(x, \xi) = 0, \quad N_k(\partial_\xi) (\vec{H}_i(x, \xi), G_{i,4}(x, \xi)) = 0 \quad \text{for } x \in \mathcal{K}, \quad \xi \in \Gamma_k, \quad i = 1, 2, 3, 4, \quad k = 1, \dots, n.$$

Moreover, it can be easily shown that

$$G_{i,j}(tx, t\xi) = t^{-T} G_{i,j}(x, \xi) \quad \text{for } x, \xi \in \mathcal{K}, t > 0,$$

where $T = 1$ for $i, j = 1, 2, 3$, $T = 3$ for $i = j = 4$ and $T = 2$ else.

The following estimates can be deduced from the estimates of Green's matrix for the problem in a dihedron.

Theorem 3.6 *The components of Green's matrix satisfy the following estimates for $|x|/2 < |\xi| < 2|x|$.*

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta G_{i,j}(x, \xi)| &\leq c |x - \xi|^{-T - |\alpha| - |\beta|} \quad \text{if } |x - \xi| < \min(r(x), r(\xi)), \\ |\partial_x^\alpha \partial_\xi^\gamma G_{i,j}(x, \xi)| &\leq c |x - \xi|^{-T - |\alpha| - |\beta|} \prod_{k=1}^n \left(\frac{|x'|}{|x - \xi|} \right)^{\min(0, \mu_k - |\alpha| - \delta_{i,4} - \varepsilon)} \prod_{k=1}^n \left(\frac{|\xi'|}{|x - \xi|} \right)^{\min(0, \mu_k - |\gamma| - \delta_{j,4} - \varepsilon)} \\ &\quad \text{if } |x - \xi| > \min(r(x), r(\xi)). \end{aligned}$$

Here $T = 1 + \delta_{i,4} + \delta_{j,4}$ and $\delta_{i,j}$ denotes the Kronecker symbol.

Proof: Since $G_{i,j}(tx, t\xi) = t^{-T} G_{i,j}(x, \xi)$, we may assume, without loss of generality that $|x| = 1$. Then $1/2 < |\xi| < 2$, and we can apply Theorems 2.6 and 2.7. ■

For the following Lemma we refer to [14, Le.4.5].

Lemma 3.4 *If $u \in W_{\beta, \bar{\delta}}^l(\mathcal{K})$, $\rho \partial_\rho u \in W_{\beta, \bar{\delta}}^l(\mathcal{K})$, $l \geq 2$, $\delta_j \neq l - 1$ for $j = 1, \dots, n$, then there is the estimate*

$$\rho^{\beta - l + 3/2} \prod_{j=1}^n \left(\frac{r_j}{\rho} \right)^{\max(\delta_j - l + 1, 0)} |u(x)| \leq c \left(\|u\|_{W_{\beta, \delta}^l(\mathcal{K})} + \|\rho \partial_\rho u\|_{W_{\beta, \delta}^l(\mathcal{K})} \right)$$

with a constant c independent of u .

Finally we estimate Green's matrix and its derivatives in the cases $|x| > 2|\xi|$ and $|\xi| > 2|x|$.

Theorem 3.7 *Let $G(x, \xi)$ be Green's matrix introduced in Theorem 3.5. Furthermore, let $\Lambda_- < \text{Re } \lambda < \Lambda_+$ be the widest strip in the complex plane which is free of eigenvalues of the pencil $\mathfrak{A}(\lambda)$ and which contains the line $\text{Re } \lambda = -\beta - 1/2$. Then for $|x| > 2|\xi|$ there is the estimate*

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\gamma G_{i,j}(x, \xi)| &\leq c |x|^{\Lambda_- - \delta_{i,4} - |\alpha| + \varepsilon} |\xi|^{-\Lambda_- - 1 - \delta_{j,4} - |\gamma| - \varepsilon} \\ &\quad \times \prod_{k=1}^n \left(\frac{r_k(x)}{|x|} \right)^{\min(0, \mu_k - |\alpha| - \delta_{i,4} - \varepsilon)} \prod_{k=1}^n \left(\frac{r_k(\xi)}{|\xi|} \right)^{\min(0, \mu_k - |\gamma| - \delta_{j,4} - \varepsilon)}, \end{aligned}$$

where ε is an arbitrarily small positive number. Analogously,

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\gamma G_{i,j}(x, \xi)| &\leq c |x|^{\Lambda_+ - \delta_{i,4} - |\alpha| - \varepsilon} |\xi|^{-\Lambda_+ - 1 - \delta_{j,4} - |\gamma| + \varepsilon} \\ &\quad \times \prod_{k=1}^n \left(\frac{r_k(x)}{|x|} \right)^{\min(0, \mu_k - |\alpha| - \delta_{i,4} - \varepsilon)} \prod_{k=1}^n \left(\frac{r_k(\xi)}{|\xi|} \right)^{\min(0, \mu_k - |\gamma| - \delta_{j,4} - \varepsilon)} \end{aligned}$$

for $|\xi| > 2|x|$.

Proof: Suppose that $|x| = 1$. We denote by ζ and η smooth functions on $\bar{\mathcal{K}}$ such that $\zeta(x) = 1$ for $|x| < 1/2$, $\eta = 1$ in a neighborhood of $\text{supp } \zeta$, and $\eta(x) = 0$ for $|x| > 3/4$. Furthermore, let l be an integer, $l > \max \mu_k + 1$, $l \geq 3$. By Theorem 3.5, we have

$$\begin{aligned} \eta(\xi) (\Delta_\xi \partial_x^\alpha \bar{H}_i(x, \xi) + \nabla_\xi \partial_x^\alpha G_{i,4}(x, \xi)) &= 0 \quad \text{for } \xi \in \mathcal{K}, \\ \eta(\xi) S_k \partial_x^\alpha \bar{H}_i(x, \xi) = 0, \quad \eta(\xi) N_k (\partial_\xi) (\partial_x^\alpha \bar{H}_i(x, \xi), \partial_x^\alpha G_{i,4}(x, \xi)) &= 0 \quad \text{for } \xi \in \Gamma_k, \quad k = 1, \dots, n, \end{aligned}$$

for $i = 1, 2, 3, 4$. Since $\eta(\cdot)\partial_x^\alpha \vec{H}_i \in W_{-\beta,0}^1(\mathcal{K})^3$ and $\eta(\cdot)\partial_x^\alpha G_{i,4}(x, \cdot) \in W_{-\beta,0}^0(\mathcal{K})$, we conclude from Corollary 3.1 and Theorem 3.4 that $\zeta(\cdot) (|\xi|\partial_{|\xi|})^j \partial_\xi^\gamma \partial_x^\alpha \vec{H}_i(x, \cdot) \in W_{l-1-\beta'+|\gamma|, \vec{\delta}+|\gamma|\bar{1}}^l(\mathcal{K})^3$ and $\zeta(\cdot) (|\xi|\partial_{|\xi|})^j \partial_\xi^\gamma \partial_x^\alpha \vec{G}_{i,4}(x, \cdot) \in W_{l-1-\beta'+|\gamma|, \vec{\delta}+|\gamma|\bar{1}}^0(\mathcal{K})$ for $j = 0, 1, \dots$, where β' is an arbitrary number less than $-\Lambda_- - 1/2$ and the components of $\vec{\delta}$ satisfy the inequalities $\max(l-1-\mu_k, 0) < \delta_k < l-1$. Using Lemma 3.4, we obtain

$$\begin{aligned} & |\xi|^{-\beta'+|\gamma|+\delta_{j,4}+1/2} \prod_{k=1}^n \left(\frac{r_k(\xi)}{|\xi|} \right)^{\max(\delta_k+|\gamma|-l+1+\delta_{j,4}, 0)} \left| \partial_x^\alpha \partial_\xi^\gamma G_{i,j}(x, \xi) \right| \\ & \leq c \left(\|\eta(\cdot)\partial_x^\alpha \vec{H}_i(x, \cdot)\|_{W_{-\beta,0}^1(\mathcal{K})^3} + \|\eta(\cdot)\partial_x^\alpha G_{i,4}(x, \cdot)\|_{W_{-\beta,0}^0(\mathcal{K})} \right) \end{aligned} \quad (3.14)$$

for $i, j = 1, 2, 3, 4$, where c is independent of x and ξ . By Theorem 3.2, the problem

$$\begin{aligned} b(u, v) - \int_{\mathcal{K}} p \nabla \cdot v &= \int_{\mathcal{K}} \eta(x) F(x) \cdot v(x) dx \quad \text{for all } v \in V_{-\beta} \\ -\nabla \cdot e &= \eta g \quad \text{in } \mathcal{K}, \quad S_k u = 0 \quad \text{on } \Gamma_k \end{aligned}$$

has a unique solution $(u, p) \in V_\beta \times W_{\beta,0}^0(\mathcal{K})$ which can be written in the form

$$\begin{aligned} u_i(y) &= \int_{\mathcal{K}} \eta(\xi) (F(\xi) \cdot \vec{H}_i(y, \xi) + g(\xi) G_{i,4}(y, \xi)) d\xi, \quad i = 1, 2, 3, \\ p(y) &= \int_{\mathcal{K}} \eta(\xi) (F(\xi) \cdot \vec{H}_4(y, \xi) + g(\xi) G_{4,4}(y, \xi)) d\xi. \end{aligned}$$

Let χ_1 and χ_2 be smooth cut-off function, $\chi_2 = 1$ in a neighborhood of x , $\chi_1 = 1$ in a neighborhood of χ_2 , $\chi_1(y) = 0$ for $|x - y| > 1/4$. Since χ_1 and η have disjoint supports, we have

$$\chi_1(-\Delta u + \nabla p) = 0, \quad \chi_1 \nabla \cdot u = 0, \quad \chi_1 S_k u = 0, \quad \chi_1 N_k(u, p) = 0.$$

Consequently, from Corollary 3.1 and Theorem 3.4 it follows that $\chi_2(\rho \partial_\rho)^j \partial_x^\alpha u \in W_{\beta', \vec{\delta}+|\alpha|\bar{1}}^l(\mathcal{K})^3$ and $\chi_2(\rho \partial_\rho)^j \partial_x^\alpha p \in W_{\beta', \vec{\delta}+|\alpha|\bar{1}}^{l-1}(\mathcal{K})^3$ for $j = 0, 1, \dots$. Thus, by Lemma 3.4,

$$\begin{aligned} & \prod_{k=1}^n r_k(x)^{\max(\delta_k+|\alpha|-l+1, 0)} |\partial_x^\alpha u(x)| + \prod_{k=1}^n r_k(x)^{\max(\delta_k+|\alpha|-l+2, 0)} |\partial_x^\alpha p(x)| \\ & \leq c (\|F\|_{V_{-\beta}^*} + \|g\|_{W_{\beta,0}^0(\mathcal{K})}). \end{aligned}$$

This means that the functionals

$$\begin{aligned} V_{-\beta}^* \times W_{-\beta,0}^0(\mathcal{K}) \ni (F, g) &\rightarrow \prod_{k=1}^n r_k(x)^{\max(\delta_k+|\alpha|-l+1, 0)} \partial_x^\alpha u_i(x) \\ &= \prod_{k=1}^n r_k(x)^{\max(\delta_k+|\alpha|-l+1, 0)} \int_{\mathcal{K}} \eta(\xi) (F(\xi) \cdot \partial_x^\alpha \vec{H}_i(x, \xi) + g(\xi) \partial_x^\alpha G_{i,4}(x, \xi)) d\xi, \end{aligned}$$

$i = 1, 2, 3$, and

$$\begin{aligned} V_{-\beta}^* \times W_{-\beta,0}^0(\mathcal{K}) \ni (F, g) &\rightarrow \prod_{k=1}^n r_k(x)^{\max(\delta_k+|\alpha|-l+2, 0)} \partial_x^\alpha p(x) \\ &= \prod_{k=1}^n r_k(x)^{\max(\delta_k+|\alpha|-l+2, 0)} \int_{\mathcal{K}} \eta(\xi) (F(\xi) \cdot \partial_x^\alpha \vec{H}_4(x, \xi) + g(\xi) \partial_x^\alpha G_{4,4}(x, \xi)) d\xi, \end{aligned}$$

are continuous and their norms are bounded by constants independent of x . Consequently,

$$\|\eta(\cdot)\partial_x^\alpha \vec{H}_i(x, \cdot)\|_{V_{-\beta}^*} + \|\eta(\cdot)\partial_x^\alpha G_{i,4}(x, \cdot)\|_{W_{-\beta,0}^0(\mathcal{K})} \leq c \prod_{k=1}^n r_k(x)^{\min(l-1-|\alpha|-\delta_k, 0)}$$

for $i, 1, 2, 3$ and

$$\|\eta(\cdot)\partial_x^\alpha \vec{H}_4(x, \cdot)\|_{V_{-\beta}} + \|\eta(\cdot)\partial_x^\alpha G_{4,4}(x, \cdot)\|_{W_{-\beta,0}^0(\mathcal{K})} \leq c \prod_{k=1}^n r_k(x)^{\min(l-2-|\alpha|-\delta_k, 0)}.$$

Combining the last two inequalities with (3.14) and putting $\beta' = -\Lambda_- - \frac{1}{2} - \varepsilon$, $\delta_k = l - 1 - \mu_k + \varepsilon$, we obtain the assertion of the theorem for $|\xi| < |x|/2$. The proof for the case $|\xi| > 2|x|$ proceeds analogously.

■

Remark 3.2 The estimates in Theorems 3.6, 3.7 can be improved by means of Theorem 2.7 and Corollary 3.1 if the direction of the derivative is tangential to the edges. In particular, we have

$$|\partial_\rho G_{i,j}(x, \xi)| \leq c |x - \xi|^{-2-\delta_{i,4}-\delta_{j,4}} \prod_{k=1}^n \left(\frac{|x'|}{|x - \xi|} \right)^{\min(0, \mu_k - \delta_{i,4} - \varepsilon)} \prod_{k=1}^n \left(\frac{|\xi'|}{|x - \xi|} \right)^{\min(0, \mu_k - \delta_{j,4} - \varepsilon)}$$

if $|x|/2 < |\xi| < 2|x|$, $|x - \xi| > \min(r(x), r(\xi))$,

$$|\partial_\rho G_{i,j}(x, \xi)| \leq c |x|^{\Lambda_- - 1 - \delta_{i,4}} |\xi|^{-\Lambda_- - 1 - \delta_{j,4} - \varepsilon} \prod_{k=1}^n \left(\frac{r_k(x)}{|x|} \right)^{\min(0, \mu_k - \delta_{i,4} - \varepsilon)} \prod_{k=1}^n \left(\frac{r_k(\xi)}{|\xi|} \right)^{\min(0, \mu_k - \delta_{j,4} - \varepsilon)}$$

if $|x| > 2|\xi|$ and

$$|\partial_\rho G_{i,j}(x, \xi)| \leq c |x|^{\Lambda_+ - 1 - \delta_{i,4} - \varepsilon} |\xi|^{-\Lambda_+ - 1 - \delta_{j,4} + \varepsilon} \prod_{k=1}^n \left(\frac{r_k(x)}{|x|} \right)^{\min(0, \mu_k - \delta_{i,4} - \varepsilon)} \prod_{k=1}^n \left(\frac{r_k(\xi)}{|\xi|} \right)^{\min(0, \mu_k - \delta_{j,4} - \varepsilon)}$$

if $|\xi| > 2|x|$.

Remark 3.3 If $d_{k_+} + d_{k_-}$ is even and, moreover, $\alpha_k < \pi$ for $d_{k_+} = d_{k_-}$, $\alpha_k < \pi/2$ for $d_{k_+} \neq d_{k_-}$, then the numbers μ_k in the estimates of Theorems 3.6, 3.7 and Remark 3.2 can be replaced by μ'_k . To prove this, one has to employ the result of Remark 3.1.

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