Algebras of Lukasiewicz's Logic and their Semiring Reducts

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ALGEBRAS OF LUKASIEWICZ'S LOGIC AND THEIR SEMIRING REDUCTS

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Abstract. In this paper we shall establish some links between the algebras of Lukasiewicz logic (MV-algebras) and the semirings. The relationship of these algebraic structures gives a hint on how to construct linear algebra starting from MV-algebras. Indeed here the role of sum and product is played respectively by a lattice operation and by an arithmetical operation. In this way, following the tradition of semirings, it makes full sense to consider "many-valued automata" and "many-valued formal languages" interpreted in Lukasiewicz logic.

1. Introduction

Semirings are algebraic structures with two associative binary operations, where one distribute over the other, and were introduced by Vandiver [13] in 1934. Later on, semirings have been studied by various researchers, especially in relation with applications. For example semirings have been used to model formal languages and automata theory (see [5]), and semirings over real numbers ($(\max, +)$-semirings) are the basis for the idempotent analysis [10].

In this paper we give a preliminary study addressed to establish a relationship between semirings and many-valued logics.

Formal many-valued logic begins with the work of Jan Lukasiewicz in 1920 and the independent work of Post in 1921, in which three-valued logic is studied. Few years later, Heyting introduced a three valued propositional calculus related with intuitionistic logic and Gödel proposed an infinite hierarchy of finitely-valued systems: his aim was to show that there is no finitely-valued propositional calculus which is sound and complete for intuitionistic logic. (For an overview, see [8].)

In the last few decades many-valued logics have been the object of a renewed interest: in 1965 Zadeh published the paper [14] where fuzzy sets are defined and the discipline of *Fuzzy Logic* began. Nowadays the various approaches to many-valued logics found in literature are competing as natural candidates to offer to the engineering discipline of Fuzzy Logic the theoretical foundations that have been lacking for several years.

The paper is organized as follows: In next section all the preliminaries on Łukasiewicz logic and MV-algebras are given. In Section 3, semiring and MV-semirings will be introduced and some results will be proven. In Section 4 we will show some application to automata theory and to fuzzy control.

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2. Preliminaries

Formulas of Łukasiewicz propositional calculus are built from the connectives of conjunction ($\odot$), disjunction ($\oplus$), negation ($\neg$) and implication $\to$ in the usual way. Interpretation of connectives of infinite-valued Łukasiewicz logic is given by the following definition.

Definition 2.1. An *assignment* is a function $v : Form \to [0,1]$ such that

- $v(\neg\varphi) = 1 - v(\varphi)$
- $v(\varphi \odot \psi) = \max(0, v(\varphi) + v(\psi) - 1)$
- $v(\varphi \oplus \psi) = \min(1, v(\varphi) + v(\psi))$.
- $v(\varphi \to \psi) = \min(1 - v(\varphi) + v(\psi), 1)$.

Every function ι from the set of variables to $[0,1]$ is uniquely extendible to an assignment v^ι. For each point $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$ let $\iota_{\mathbf{x}}$ be the function mapping each variable X_j into x_j. Fixed n, with each formula φ with at most n variables it is possible to associate the function

$$f_\varphi : \mathbf{x} \in [0,1]^n \mapsto v^{\iota_{\mathbf{x}}}(\varphi) \in [0,1]$$

satisfying the following conditions:

- $f_{X_i}(x_1, \ldots, x_n) = x_i =$ the ith projection.
- $f_{\neg\varphi} = 1 - f_\varphi$.
- $f_{(\varphi\oplus\psi)} = \min(1, f_\varphi + f_\psi)$
- $f_{(\varphi\to\psi)} = \min(1 - f_\varphi + f_\psi, 1)$.

The function f_φ is the *truth table* of the formula φ.

Definition 2.2. A formula φ is *satisfied* by an assignment v if $v(\varphi) = 1$. A formula φ is *valid* (or a *tautology*, if φ is satisfied by all assignments, that is, if for every v, $v(\varphi) = 1$, or equivalently its truth table f_φ is identically equal to 1.

The syntactic approach to many-valued logics is the same as for propositional classical logic. A set of formulas called *axioms* is fixed and the inference rule is *modus ponens*: from φ and $\varphi \to \psi$ we can infer ψ.

Axioms of infinite-valued Łukasiewicz logic are:

Ł1) $\varphi \to (\psi \to \varphi)$
Ł2) $(\varphi \to \psi) \to ((\psi \to \theta) \to (\varphi \to \theta))$
Ł3) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$
Ł4) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$

In order to prove the completeness of this schemata of axioms with respect to semantics of the interval $[0,1]$, Chang introduced *MV-algebras* in [1]. In the following we shall summarize some of the main results for the theory of MV-algebras. A standard reference is [2].

An MV-algebra is a structure $A = (A, \oplus, \odot, \neg, 0, 1)$ satisfying the following equations:

MV1 $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
MV2 $x \oplus y = y \oplus x$
MV3 $x \oplus 0 = x$
MV4 $x \oplus 1 = 1$
MV5 $\neg 0 = 1$ and $\neg 1 = 0$

MV6 $\neg(\neg x \oplus \neg y) = x \odot y$
MV7 $x \oplus (\neg x \odot y) = y \oplus (\neg y \odot x)$.

As proved by Chang, boolean algebras coincide with MV-algebras satisfying the additional equation $x \oplus x = x$ (idempotency). Each MV-algebra contains as a subalgebra the two-element boolean algebra $\{0, 1\}$. The set $B(A)$ of all idempotent elements of an MV-algebra A is the largest boolean algebra contained in A and is called the *boolean skeleton* of A.

Any MV-algebra A is equipped with the order relation

$$x \leq y \quad \text{if and only if} \quad \neg x \oplus y = 1, \tag{1}$$

and $(A, \leq, 0, 1)$ is a lattice where

$$x \vee y = x \oplus (\neg x \odot y) \tag{2}$$

$$x \wedge y = x \odot (\neg x \oplus y). \tag{3}$$

Example 2.3. (i) The set $[0, 1]$ equipped with operations

$$x \oplus y = \min\{1, x + y\}, \quad x \odot y = \max\{0, x + y - 1\}, \quad \neg x = 1 - x. \tag{4}$$

is an MV-algebra.

(ii) For each $k = 1, 2, \ldots$, the set

$$\text{Ł}_{k+1} = \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\}, \tag{5}$$

equipped with operations as in (4), is a linearly ordered MV-algebra (also called *MV-chain*).

(iii) If X is any set and A is an MV-algebra, the set of functions $f : X \to A$ obtained by pointwise application of operations in A is an MV-algebra.

(iv) The set of all functions from $[0, 1]^n$ into $[0, 1]$ that are continuous and piecewise linear, and such that each linear piece has integer coefficients, and operations are obtained as pointwise application of operation in (4), is an MV-algebra (actually, the *free* MV-algebra over n free generators).

Chang's Completeness Theorem states:

Theorem 2.4. *An equation holds in every MV-algebra if and only if it holds in the MV-algebra* $[0, 1]$ *equipped with operations* $x \oplus y = \min\{1, x + y\}$*,* $x \odot y = \max\{0, x + y - 1\}$ *and* $\neg x = 1 - x$*.*

This theorem was proved by Chang using quantifier elimination for totally ordered divisible abelian groups. There are several alternative proofs in literature: the syntactic proof by Rose and Rosser, the algebraic proof by Cignoli and Mundici and the geometric proof by Panti (see [2] for an overview).

Mundici in [11] constructed an equivalence functor $\mathbf{\Gamma}$ from the category of ℓ-groups with strong unit to the category of MV-algebras:

A lattice-ordered group (ℓ-group for short) $G = (G, 0, -, +, \wedge, \vee)$ is an abelian group $(G, 0, -, +)$ equipped with a lattice structure $(G, \wedge, \vee)$ such that, for every $a, b, c \in G$, $c + (a \wedge b) = (c + a) \wedge (c + b)$. An ℓ-group is said to be totally ordered if the lattice-order is total. An element $u \in G$ is a *strong unit* of G if for every $x \in G$ there exists $n \in \mathbb{N}$ such that $nu \geq x$. If G is an ℓ-group and u is a strong unit for G, the MV-algebra $\mathbf{\Gamma}(G, u)$ has the form $\{x \in G \mid 0 \leq x \leq u\}$ and operations are defined by $x \oplus y = u \wedge x + y$ and $\neg x = u - x$. If A is an MV-algebra we shall denote by G_A the ℓ-group corresponding to A via $\mathbf{\Gamma}$.

In [4] the author shows that every MV-algebra is an algebra of functions taking values in an ultrapower of the interval $[0,1]$.

The developing of many-valued logic is strictly connected with the notion of t-norm.

Definition 2.5. A *t-norm* is a binary operation $*$ on $[0,1]$ such that

- $*$ is commutative and associative, i.e., for all $x, y, z \in [0,1]$,

$$\begin{aligned} x * y &= y * x \\ (x * y) * z &= x * (y * z), \end{aligned}$$

- $*$ is non-decreasing in both arguments

$$\begin{aligned} x_1 \leq x_2 \quad &\text{implies} \quad x_1 * y \leq x_2 * y, \\ y_1 \leq y_2 \quad &\text{implies} \quad x * y_1 \leq x * y_2 \end{aligned}$$

- $1 * x = x$ and $0 * x = 0$ for all $x \in [0,1]$.

Example 2.6. The following are example of continuous (in the usual sense) t-norms

(i) Lukasiewicz t-norm: $x \odot y = \max(0, x + y - 1)$.
(ii) Product t-norm: $x \cdot y$ usual product between real numbers.
(iii) Gödel t-norm: $x \wedge y = \min(x, y)$.

An element $x \in [0,1]$ is *idempotent* with respect to a t-norm $*$, if $x * x = x$.

When choosing a continuous t-norm as the truth table for conjunction, the following proposition enable us to obtain a truth table for implication:

Proposition 2.7 (Residuum). *Let $*$ be a continuous t-norm. Then, for every $x, y, z \in [0,1]$, the operation*

$$x \rightarrow_* y = \max\{z \mid x * z \leq y\}$$

is the the unique operation satisfying the condition

$$(x * z) \leq y \quad \textit{if and ony if} \quad x \leq (x \rightarrow_* y)$$

The operation $\rightarrow_*$ is called *residuum* of the t-norm $*$.

Example 2.8. The following are residua of the three main continuous t-norms:

	T-norm	**Residuum**
Ł	$x \odot y = \max(x + y - 1, 0)$	$x \rightarrow_\odot y = \min(1, 1 - x + y)$
P	$x \cdot y$ usual product of reals	$x \rightarrow_\cdot y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$
G	$x \wedge y = \min(x, y)$	$x \rightarrow_\wedge y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases}$

The problem of finding an appropriate axiomatization of many-valued logics based on continuous t-norm has been approached by introducing suitable classes of algebraic structures. In [9], monoidal logic is introduced and Hájek in [8] defines Basic (fuzzy) Logic. In the following we shall briefly describe some important features of Basic Logic.

Basic Fuzzy infinite-valued Logic (BL) is defined from connectives & and $\rightarrow$ interpreted over the interval $[0,1]$ respectively by a continuous t-norm $*$ and its associates residuum $\rightarrow_*$.

Truth tables of other derived connectives are defined as follows:

$$x \wedge y = x * (x \rightarrow_* y) \tag{6}$$

$$x \vee y = ((x \rightarrow_* y) \rightarrow_* y) \wedge ((y \rightarrow_* x) \rightarrow_* x) \tag{7}$$

$$\neg x = x \rightarrow_* 0 \tag{8}$$

$$x \equiv y = (x \rightarrow_* y) * (y \rightarrow_* x) \tag{9}$$

An axiom is a formula that can be written in any one of the following ways, where φ, ψ and χ denote arbitrary formulas:

(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(A2) $(\varphi * \psi) \rightarrow \varphi$
(A3) $(\varphi * \psi) \rightarrow (\psi * \varphi)$
(A4) $(\varphi * (\varphi \rightarrow \psi) \rightarrow (\psi * (\psi \rightarrow \varphi))$
(A5a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi * \psi) \rightarrow \chi)$
(A5b) $((\varphi * \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
(A6) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
(A7) $0 \rightarrow \varphi$.

In order to prove completeness, in [8] the author also introduced BL-algebras:

Definition 2.9. A BL-algebra is an algebra

$$\mathbf{L} = (L, \cup, \cap, *, \rightarrow, 0, 1)$$

with four binary operations and two constants such that

(i) $(L, \cup, \cap, 0, 1)$ is a lattice with largest element 1 and least element 0 (with respect to the lattice ordering $\leq$),

(ii) $(L, *, 1)$ is a commutative semigroup with the unit element 1, i.e., $*$ is commutative, associative, $1 * x = x$ for all x (thus $\mathbf{L}$ is a residuated lattice),

(iii) the following conditions holds:

(1) $z \leq (x \rightarrow y)$ if and only if $x * z \leq y$, for all $x, y, z \in L$
(2) $x \cap y = x * (x \rightarrow y)$
(3) $x \cap y = ((x \rightarrow y) \rightarrow y) \cap ((y \rightarrow x) \rightarrow x)$
(4) $(x \rightarrow y) \cap (y \rightarrow x) = 1$

Example 2.10. The unit interval $[0, 1]$ equipped with a continuous t-norm and the corresponding residuum, is a BL-algebra.

At first step, Hájek showed that a propositional formula is provable in BL if and only if it is a tautology in any linearly ordered BL-algebra. However, the *completeness theorem* of BL with respect to BL-algebras, i.e., that a formula is provable in BL if and only if it is a tautology in $[0, 1]$, was left as an open problem in [8]. In [7] Hájek proved that such completeness theorem can be obtained provided two new axioms were added to the original axiomatic system of BL. In [3] the authors showed that these axioms are redundant, hence the original axiomatic of BL is sound and complete with respect to the algebraic structure induced by continuous t-norms on $[0, 1]$.

MV-algebras turn out to coincide with those BL-algebras satisfying the equation $\neg\neg x = x$.

Let m be a positive integer number. For every $i = 1, \ldots, m-1$ let

$$I_i^m = \left[\frac{i-1}{m}, \frac{i}{m}\right) \qquad \text{and} \qquad I_m^m = \left[\frac{m-1}{m}, 1\right]$$

and let $\sigma_i^m : [0,1] \to \mathrm{cl}(I_i^m)$ be the *resizing function* defined by

$$\sigma_i^m(x) = \frac{i-1}{m} + \frac{x}{m},$$

where $\mathrm{cl}(S)$ denotes the topological closure of the set S.

The ordinal sum of a family $\mathcal{B} = (B_i)_{i=1}^m = ([0,1], *_i, \to_i, 0, 1)_{i=1}^m$, of BL-algebras having $[0,1]$ as domain, is the BL-algebra $([0,1], *, \to, 0, 1)$ defined in the following way:

$$x *^m y = \begin{cases} \sigma_i^m((\sigma_i^m)^{-1}(x) *_i (\sigma_i^m)^{-1}(y)) & \text{if } x, y \in I_i \\ \min(x,y) & \text{otherwise} \end{cases}$$

$$x \to^m y = \begin{cases} 1 & \text{if } x \leq y \\ \sigma_i^m((\sigma_i^m)^{-1}(x) \to_i (\sigma_i^m)^{-1}(y)) & \text{if } x, y \in I_i \text{ and } x > y \\ y & \text{otherwise.} \end{cases}$$

3. Semirings and MV-algebras

Definition 3.1. A semiring $\mathcal{R} = (R, +, 0, \cdot, 1)$ is an algebraic structure such that 0 and 1 are distinct elements of R and $+$ and $\cdot$ are binary operations on R such that:

(1) $(R, +)$ is a commutative monoid with identity 0,
(2) $(R, \cdot)$ is a monoid with identity 1,
(3) Multiplication distributes over addition,
(4) $0r = r0 = 0$ for every $r \in R$.

A semiring is **commutative** if $(R, \cdot)$ is a commutative monoid.
A semiring is **(additively) idempotent** if for every $r \in R$, $r + r = r$.

Definition 3.2. A semiring R is *lattice-ordered* if and only if it also has the structure of a lattice such that for all $a, b \in R$:

- $a + b = a \vee b$
- $a \cdot b \leq a \wedge b$

A semiring R is *dual lattice-ordered* if and only if it also has the structure of a lattice such that for all $a, b \in R$:

- $a + b = a \wedge b$
- $a \cdot b \geq a \vee b$

Lattice-ordered semirings and dual-lattice ordered semirings are additively idempotent. In the following we shall use the name **lc-semiring** for lattice-ordered commutative semirings and **dual lc-semiring** for dual lattice-ordered commutative semirings.

Let R and S be semirings. A morphism between R and S is a map $f : R \to S$ such that

- $f(0) = 0$ and $f(1) = 1$
- $f(r + r') = f(r) + f(r')$ and $f(r \cdot r') = f(r) \cdot f(r')$ for all $r, r' \in R$.

The set $f^{-1}(0)$ is the *kernel* of the morphism f, and it is denoted by $ker(f)$. As usual, a bijective morphism is called isomorphism.

In the following propositions and definitions we shall show a relationship between lc-semirings and MV-algebras.

Definition 3.3. A *coupled semiring* $\mathcal{A}$ is a triple (R_1, R_2, α) such that

CS1) $R_1 = (A, \vee, 0, \cdot, 1)$ and $R_2 = (A, \wedge, 0', \cdot', 1')$ are respectively an lc-semiring and a dual lc-semiring.
CS2) $0' = 1$ and $1' = 0$
CS3) $\alpha : A \to A$ is a semiring isomorphism from R_1 into R_2.
CS4) $\alpha(\alpha(x)) = x$, for every $x \in A$.
CS5) $x \vee y = x \cdot' (\alpha(x) \cdot y)$

Lemma 3.4. *For every $x, y \in A$ we have: $x \wedge y = x \cdot (\alpha(x) \cdot' y)$*

Proof. By (7) we have

$$\alpha(x) + \alpha(y) = \alpha(x) \cdot' (\alpha(\alpha(x) \cdot \alpha(y))) =$$

by (6)

$$= \alpha(x) \cdot' (x \cdot \alpha(y))$$

Applying the map α we have:

$$\alpha(\alpha(x) + \alpha(y)) = \alpha(\alpha(x) \cdot' (x \cdot \alpha(y))$$

and since, by (5) and (6), α is an isomorphism of R_1 into R_2 and R_2 into R_1 then

$$\alpha(\alpha(x)) +' \alpha(\alpha(y)) = \alpha(\alpha(x)) \cdot' (x \cdot \alpha(y)) = x \cdot \alpha(x \cdot \alpha(y)) = x \cdot (\alpha(x) \cdot' y)$$

□

Proposition 3.5. *Let $\mathcal{A} = (R_1, R_2, \alpha)$ be a coupled semiring, where $R_1 = (A, \vee, 0, \cdot, 1)$ and $R_2 = (A, \wedge, 1, \cdot', 0)$. Then $(A, \cdot', \cdot, \alpha, 0, 1)$ is an MV-algebra.*

Proof. Axioms MV1), MV2) and MV3) are satisfied observing that $(A, \cdot, 1)$ and $(A, \cdot', 1')$ are commutative monoids. Let us verify the axiom MV4). Indeed we have $\alpha(x) \cdot 0 = 0$ and applying α:

$$\alpha(\alpha(x) \cdot 0) = \alpha(0)$$
$$\alpha(\alpha(x)) \cdot' \alpha(0) = \alpha(0)$$
$$x \cdot' \alpha(0) = \alpha(0)$$
$$x \cdot' 0' = 0'$$
$$x \cdot' 1 = 1$$

Axiom MV5): $\alpha(0) = \alpha(1') = 1$.
Axiom MV6): $\alpha(1) = \alpha(0') = 0$.
Axiom MV7): $\alpha(\alpha(x) \cdot' y) \cdot' y = (\alpha(\alpha(x)) \cdot \alpha(y)) \cdot' y = (x \cdot \alpha(y)) \cdot' y = \alpha(\alpha(y) \cdot' x) \cdot' x = (y \cdot \alpha(x)) \cdot' x$.

□

Proposition 3.6. *Let $\mathcal{A} = (A, \oplus, \odot, \neg, 0, 1)$ be an MV-algebra. Then the reducts $R^{\vee}_{\mathcal{A}} = (A, \vee, 0, \odot, 1)$ and $R^{\wedge}_{\mathcal{A}} = (A, \wedge, 1, \oplus, 0)$ (where $\wedge$ and $\vee$ are defined in Equations (2) and (3)), are respectively an lc-semiring and a dual lc-semiring and $(R^{\vee}_{\mathcal{A}}, R^{\wedge}_{\mathcal{A}}, \neg)$ is a coupled semiring.*

Proof. CS1) In every MV-algebra A, the reducts $(A, \vee, 0)$, $(A, \wedge, 1)$,$(A, \oplus, 0)$ and $(A, \odot, 1)$ are commutative monoids. Further, by checking on the MV-algebra $[0, 1]$ (see Theorem 2.4) it is possible to establish that in every MV-algebra it holds

$$\begin{aligned} x \oplus (y \wedge z) &= (x \oplus y) \wedge (x \oplus z) \\ x \odot (y \vee z) &= (x \odot y) \vee (x \odot z) \end{aligned}$$

hence in both R_1 and R_2 the second operation distributes over the first one. Further

$$\begin{aligned} x \odot 0 &= 0 \\ x \oplus 1 &= 1. \end{aligned}$$

CS2) By definition.

CS3) $\neg : A \to A$ is a isomorphism of monoid $(A, \odot)$ onto monoid $(A, \oplus)$ and by Axiom MV6:

$$\neg(x \oplus y) = \neg x \odot \neg y$$

hence $\neg$ is an isomorphism of semirings.

CS4) By Axiom MV4 and MV7 of MV-algebras

$$\neg\neg x = \neg(\neg x \oplus 0) \oplus 0 = \neg(\neg 0 \oplus x) \oplus x = \neg(1 \oplus x) \oplus x = \neg 1 \oplus x = 0 \oplus x = x.$$

CS5) By Equation (2).

□

Proposition 3.7. *Let $\mathcal{R} = (R, \vee, 0, \cdot, 1)$ be a lattice-ordered semiring, and $\neg$ be a unary operation over R such that*

(a) $\neg 0 = 1$

(b) $x \vee y = \neg(\neg(y \cdot \neg x) \cdot \neg x)$.

Further, let $x \cdot' y = \neg(\neg x \cdot \neg y)$ and $x \wedge y = \neg(\neg x \vee \neg y)$. Then $(R, \cdot, \cdot', \neg, 0, 1)$ is an MV-algebra and $((R, \vee, 0, \cdot, 1), (R, \wedge, 1, \cdot', 0), \neg)$ is a coupled semiring.

Example 3.8. Let G be an abelian ℓ-group and let u be a unit of G. Then the set $\Gamma(G, u) = \{g \in G \mid 0 \le g \le u\}$ can be equipped with a structure of lc-semiring $\Gamma_1 = (\Gamma(G, u), \vee, 0, \odot, u)$ by

$$g \odot g' = u \wedge (g + g' - u).$$

Further, if we set

$$\begin{aligned} g \oplus g' &= u \wedge (g + g'), \\ \neg g &= u - g \end{aligned}$$

the structure $\Gamma_2 = (\Gamma(G, u), \wedge, u, \oplus, 0)$ is a dual lc-semiring and $(\Gamma_1, \Gamma_2, \neg)$ is a coupled semiring. The MV-algebra $(\Gamma(G, u), \oplus, \odot, \neg, 0, 1)$ is the MV-algebra corresponding to the ℓ-group G with strong unit u accordingly to the notation in [2].

Example 3.9. Let G be an abelian ℓ-group and consider the set $G^* = G \cup \{-\infty\}$. Set for every $g \in G$, $-\infty \le g$ and $-\infty + g = g + (-\infty) = -\infty$. Then $(G^*, \max, +)$ is an lc-semiring.

Analogously, if we set $G^+ = G \cup \{\infty\}$ then $(G, \min, +)$ is a dual lc-semiring.

Example 3.10. Let $A = (A, \odot, \neg, 0, 1)$ be an MV-algebra and consider the set $A \cup \{-\infty\}$ where the operations $\odot$ and $\neg$ are extended as follows:

$$\begin{aligned} a \odot -\infty &= -\infty \\ \neg(-\infty) &= 1. \end{aligned}$$

Then $(A \cup \{-\infty\}, \odot, \neg, -\infty, 1)$ is a BL-algebra obtained as *ordinal sum* of the boolean algebra $\{-\infty, 0\}$ and the MV-algebra A. The BL-algebra $A \cup \{-\infty\}$ can be equipped of a structure of lc-semiring in the following way:

$$\begin{aligned} x+y &= x \vee y \\ x \cdot y &= x \odot y. \end{aligned}$$

3.1. **Semimodules.** Let $R = (R, \vee, 0, \odot, 1)$ be an lc-semiring. A left R-semimodule $(M, R, \cdot)$ is a commutative monoid $(M, +)$ with an additive identity 0 and with an external operation $\cdot : (r, m) \in R \times M \mapsto rm \in M$ called *scalar multiplication*, which satisfies the following conditions for every $r, r' \in R$ and $m, m' \in M$:

- $(r \odot r')m = r(r'm)$;
- $r(m + m') = rm + rm'$;
- $(r \vee r')m = rm + r'm$;
- $1m = m$;
- $r0 = 0 = 0m$.

The definition of right R-semimodule is the analogous where the scalar multiplication is defined as a function $M \times R \to M$. An R-bisemimodule is a both right and left R-semimodule such that $(rm)r' = r(mr')$.

Example 3.11. Let $\mathcal{A} = (A, \oplus, \odot, \neg, 0, 1)$ be an MV-algebra and let us consider the lc-semiring reduct $R^{\vee}_{\mathcal{A}} = (A, \vee, \odot, 0, 1)$ and the monoid reduct $M^{\wedge}_{\mathcal{A}} = (A, \wedge, 1)$. We can define a left A-semimodule $(R^{\vee}_{\mathcal{A}}, M^{\wedge}_{\mathcal{A}}, \cdot)$ be setting $a \cdot b = \neg a \oplus b = a \to b$. Indeed

- $(a \odot a') \cdot b = \neg(a \odot a') \oplus b = (\neg a \oplus \neg a') \oplus b = \neg a \oplus (\neg a' \oplus b) = a \cdot (a' \cdot b)$.
- $a \cdot (b \wedge b') = \neg a \oplus (b \wedge b') = (\neg a \oplus b) \wedge (\neg a \oplus b') = (a \cdot b) \wedge (a \cdot b')$
- $(a \vee a') \cdot b = \neg(a \vee a') \oplus b = (\neg a \wedge \neg a') \oplus b = (\neg a \oplus b) \wedge (\neg a' \oplus b) = (a \cdot b) \wedge (a' \cdot b)$
- $1 \cdot b = 0 \oplus b = b$
- $a \cdot 1 = \neg a \oplus 1 = 1 = 0 \cdot b$.

We can also consider the dual of the previous example. Let $\mathcal{A} = (A, \oplus, \odot, \neg, 0, 1)$ be an MV-algebra and let us consider the dual lc-semiring reduct $R^{\wedge}_{\mathcal{A}} = (A, \wedge, \oplus, 1, 0)$ and the monoid reduct $M^{\vee}_{\mathcal{A}} = (A, \vee, 0)$. We shall define a left A-semimodule $(R^{\wedge}_{\mathcal{A}}, M^{\vee}_{\mathcal{A}}, \times)$ be setting $a \times b = \neg a \odot b$.

Let $K = (M, R, \cdot)$ and $K' = (N, S, \times)$ be two left semimodules over R and S respectively.

A **linear bimorphism** is a pair of functions (φ, ψ) with $\varphi : M \to N$ and $\psi : R \to S$ such that

$$\varphi(m + m') = \varphi(m) + \varphi(m') \tag{10}$$

$$\varphi(r \cdot m) = \psi(r) \times \varphi(m). \tag{11}$$

Example 3.12. Consider the left semirings $(R^{\vee}_{\mathcal{A}}, M^{\wedge}_{\mathcal{A}}, \cdot)$ and $(R^{\wedge}_{\mathcal{A}}, M^{\vee}_{\mathcal{A}}, \times)$ as in Example 3.11. Then $(\neg, \neg)$ is a linear bimorphism. Indeed $\neg(m \vee m') = \neg m \wedge \neg m'$ and $\neg(\neg r \odot m) = r \oplus \neg m = \neg(r) \times \neg(m)$.

Example 3.13. Let us consider the MV-algebra $([0,1], \oplus, \odot, \neg, 0, 1)$, the semiring reduct $R^{\wedge} = ([0,1], \wedge, 1, \oplus, 0)$ and the monoid reduct $M^{\vee} = ([0,1], \vee, 0)$. Let X and Y be any sets and consider the sets of functions $(M^{\vee})^X$, $(M^{\vee})^Y$ that can be equipped with a structure of monoids by pointwise extension of operations, and sets $(R^{\wedge})^X$, $(R^{\wedge})^Y$ that can be analogously equipped with a structure of semirings. Then a function $H : X \times Y \to [0,1]$ can be extended to a linear bimorphism (φ, ψ)

between semimodules $K = ((M^\vee)^X, (R^\wedge)^X, \cdot)$ and $K' = ((M^\vee)^Y, (R^\wedge)^Y, \cdot)$ where $\cdot$ is the operation $s \cdot m = \neg s \odot m$, by setting

$$\varphi(f)(y) = \bigvee_{x \in X} f(x) \odot H(x,y) \tag{12}$$

and

$$\psi(f)(y) = \bigwedge_{x \in X} (f(x) \wedge H(x,y)). \tag{13}$$

Indeed it can be easily checked that $\varphi(f \vee g) = \varphi(f) \vee \varphi(g)$. Further,

$$\psi(g) \cdot \varphi(f) = (\neg \bigwedge_{x \in X} (g(x) \wedge R(x,y))) \oplus (\bigvee_{x \in X} f(x) \odot R(x,y)) =$$

$$= (\bigvee_{x \in X} \neg g(x) \vee \neg R(x,y)) \odot (\bigvee_{x \in X} f(x) \odot R(x,y)) =$$

$$= \bigvee_{x \in X} (\neg g(x) \vee \neg R(x,y)) \odot (f(x) \odot R(x,y)) =$$

$$= \bigvee_{x \in X} (\neg g(x) \odot f(x) \odot R(x,y)) \vee (\neg R(x,y) \odot f(x) \odot R(x,y)) =$$

$$= \bigvee_{x \in X} (\neg g(x) \odot f(x)) \odot R(x,y) = \varphi(g \cdot f).$$

We can proceed dually by considering reducts $R^\vee$ and $M^\wedge$.

4. Applications

In this section we shall see some applications in which semirings have a crucial role.

4.1. **Convolution.** Let $(M, *, \epsilon)$ be a monoid and $A^\vee = (A, \vee, 0, \cdot, 1)$ be a lc-semiring.

Let $A[M]$ be the family of all functions $f : M \to A$ with finite support. On $A[M]$ the following operations of supremum and convolution are defined:

$$(f \vee g)(m) = f(m) \vee g(m),$$
$$(f \circledast g)(m) = \bigvee_{m' * m'' = m} \{f(m') \odot g(m'') \mid (m', m'') \in \mathrm{supp}(f) \times \mathrm{supp}(g)\}.$$

Then $A^\vee[M] = (A[M], \vee, \circledast, \mathbf{0}, \mathbf{1})$, where $\mathbf{0}$ is the function identically equal to 0 and $\mathbf{1}$ is the function which takes value 1 over ϵ and 0 otherwise, is a commutative, idempotent semiring, called *convolution semiring*.

Dually, starting from the dual lc-semiring $A^\wedge$, we can construct the *dual convolution semiring* $A^\wedge[M]$. In this case

$$(f \circledast^D g)(m) = \bigwedge_{m' * m'' = m} \{f(m') \oplus g(m'') \mid (m', m'') \in \mathrm{supp}(f) \times \mathrm{supp}(g)\}.$$

Let $\mathcal{A}$ be an MV-algebra and consider now the semiring reducts $R^\vee_{\mathcal{A}} = (A, \vee, 0, \odot, 1)$, $R^\wedge_{\mathcal{A}} = (A, \wedge, 1, \oplus, 0)$ and the monoids reducts $M^\oplus_{\mathcal{A}} = (A, \oplus, 0)$, $M^\odot_{\mathcal{A}} = (A, \odot, 1)$.

Semirings $R^\vee_{\mathcal{A}}$ and $R^\wedge_{\mathcal{A}}$, together with negation $\neg$, form a coupled semiring. Unfortunately, the convolution semiring $R^\vee_{\mathcal{A}}[M^\odot_{\mathcal{A}}]$ and the dual convolution semiring $R^\vee_{\mathcal{A}}[M^\oplus_{\mathcal{A}}]$ cannot be canonically equipped with a unary operation in order to have a coupled semiring.

Anyway we have

Proposition 4.1. *Let $\mathcal{A}$ be an MV-algebra. Semirings $R^{\vee}_{\mathcal{A}}$ and $R^{\wedge}_{\mathcal{A}}, \neg$ are embeddable respectively in the convolution semiring $R^{\vee}_{\mathcal{A}}[M^{\odot}_{\mathcal{A}}]$ and the dual convolution semiring $R^{\wedge}_{\mathcal{A}}[M^{\oplus}_{\mathcal{A}}]$.*

Proof. The map sending each $a \in A$ into the function

$$f_a : x \in A \to \begin{cases} a & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

is a semiring monomorphism between the semiring $R^{\vee}_{\mathcal{A}} = (A, \vee, 1, \odot, 0)$ and the convolution semiring $R^{\vee}_{\mathcal{A}}[M^{\odot}_{\mathcal{A}}]$.

Indeed, if $a = 1$ then $f_a = \mathbf{1}$ and if $a = 0$ then $f_a = \mathbf{0}$. Further, if $a, b \in A$, then

$$f_a(x) \vee f_b(x) = \begin{cases} a \vee b & \text{if } x = 0 \\ 0 \vee 0 = 0 & \text{otherwise} \end{cases} = f_{a \vee b}(x)$$

and

$$f_a(x) \circledast f_b(x) = \bigvee_{x' \odot x'' = x} \{f_a(x') \odot f_b(x'') \mid (x', x'') \in \text{supp}(f_a) \times \text{supp}(f_b)\} =$$

$$= \begin{cases} 0 & \text{if } x \neq 0 \\ a \odot b & \text{if } x = 0 \end{cases} = f_{a \odot b}(x).$$

Analogously it can be proved that $R^{\wedge}_{\mathcal{A}}$ is embeddable the dual convolution semiring $R^{\wedge}_{\mathcal{A}}[M^{\oplus}_{\mathcal{A}}]$. □

We want now to define polynomials starting from an lc-semiring R. Let us fix an indeterminate t and consider the multiplicative monoid $M = \{t^i \mid i \in \mathbb{N}\}$. The semiring $R[M]$ is denoted by $R[t]$ and it is called the *semiring of polynomials in the indeterminate t*.

Elements of $R[t]$ are functions assigning for each $i \in \mathbb{N}$ a value $f(t^i) \in R$. Each $f \in R[t]$ will be denoted by the expression $\Sigma f(i) t^i$ stressing the fact that f can be indeed considered as defined on elements of $\mathbb{N}$ (and taking value greater than 0 only on finitely many i's). $f(i)$ is also said the i-th coefficient of the polynomial f. Two polynomials f and g are equal if they have the same coefficients $f(j) = g(j)$ for every j.

Convolution operation in the semiring of polynomials is given by:

$$(f \circledast g) = \sum \left(\bigvee_{h+k=j} f(h) \odot g(k) \right) t^j.$$

Consider now a set A and the free monoid A^* generated by A. We can extend the definition of convolution to the set $R\langle A^* \rangle$ of all the functions from A^* into a semiring R by setting

$$(f \circledast g)(w) = \sum \{f(w') \odot g(w'') \mid w'w'' = w\}. \tag{14}$$

Since every word w can be decomposed in finitely many ways, then the sum in (14) is finite. Then $(R\langle A^* \rangle, \vee, \circledast, \mathbf{0}, \mathbf{1})$ is the *semiring of formal series*.

4.2. **Automata.** If K is a semiring, a K-subset of a set X is a function from X in K. Let K be an lc-semiring and A be a finite nonempty set. The set of functions from $A \times A$ into K is denoted by $\mathcal{M}_A(K)$ and such functions are called $(A \times A)$-*matrices* on K. Given $f, g \in \mathcal{M}_A(K)$, we consider the operations of sum and row-column product by setting:

$$\begin{aligned} (f+g)(i,j) &= f(i) \vee g(j) \\ (f \cdot g)(i,j) &= \bigvee \{f(i,k) \cdot g(i,k) \mid k \in A\}. \end{aligned}$$

If we consider as multiplicative identity the diagonal matrix and as additive identity the matrix with all elements equal to 0, then $\mathcal{M}_A(K), +, \cdot, \mathbf{0}, \mathbf{1})$ is a semiring. Unfortunately, this is neither a commutative semiring or a lattice-ordered semiring.

Given an lc-semiring K and a set Σ, a K-Σ *automaton* is a triple

$$\mathcal{A} = (Q, I, T, E)$$

where Q is a finite set of states, $I, T : Q \to K$, are the K-subsets of Q of *initial and terminal states* and E is a function assigning to each triple $(p, \sigma, q) \in Q \times \Sigma \times Q$ a value in K.

The function E can be also be considered as a matrix $(E_{pq})_{p,q \in Q}$ where $E_{pq} \in K\langle \Sigma^* \rangle$.

The *behavior* of an automaton $\mathcal{A}$ is a function $|\mathcal{A}| : \Sigma^* \to K$ given by the following settings: to each couple of states p, q we assign the *label* $E_{pq} \in K[\Sigma^*]$ and when the automaton is in the state p and the input $\sigma \in \Sigma$ is given, then it goes to each other state q with degree $E_{pq}(\sigma)$.

A letter $\sigma \in \Sigma$ is accepted by the automaton $\mathcal{A}$ with degree

$$|\mathcal{A}|(\sigma) = \bigvee_{p,q \in Q} I(p) \odot E(p, \sigma, q) \odot T(q).$$

If p and q are elements of Q, the *n-path* $\pi = (p_1 = p, p_2 \ldots, p_n = q)$ is assigned with the label $\|\pi\| \in K[\Sigma^*]$ such that

$$\|\pi\|(\sigma_1 \ldots \sigma_n) = \bigodot_{i=1}^{n-1} E(p_i, \sigma_i, p_{i+1})$$

The set of all n-paths between p and q will be denoted by $P^n(p,q)$. Then a word $s = \sigma_1 \ldots \sigma_n \in \Sigma^*$ is accepted by automata $\mathcal{A}$ with degree

$$|\mathcal{A}|(s) = \bigvee_{p,q \in Q} \bigvee_{\pi \in P^n(p,q)} I(p) \odot \|\pi\|(s) \odot T(q).$$

When K is a lc-semiring, we say that a K-subset A of Σ^* is *recognizable* if there exists a K-Σ automaton $\mathcal{A}$ such that $A = |\mathcal{A}|$.

Many properties of automata can be repeated for K-Σ automata.

Proposition 4.2. *The class of recognizable K-subsets of Σ^* is closed under finite union, intersection and reversal.*

The following proposition is due to Schützenberger. If E is a matrix by E^+ we denote the matrix

$$E^+ = E + E^2 + \ldots + E^n + \ldots$$

Proposition 4.3. *A K-set A is recognizable if and only if there exists an integer $n > 1$ and an $n \times n$ matrix E such that $A = E^+_{1n}$.*

4.3. **Applications to Fuzziness.** One of the most important notions in fuzzy set theory is the *extension principle* [15], used to combine fuzzy sets and relations or to define operations of a mathematical function on fuzzy sets.

Consider systems of fuzzy relation equations: if $*$ is any t-norm on $[0, 1]$ and for every $i = 1, \ldots, n$, A_i are fuzzy subsets of X, B_i are fuzzy subsets of Y and R is a fuzzy subset of $X \times Y$, then the system

$$B_i(y) = A_i \circ R = \bigvee_{x \in X} A_i(x) * R(x, y) \tag{15}$$

is considered as the *composition* of the relation R with fuzzy sets A_i. Let us consider now an MV-algebra $\mathcal{A}$ and the lc-semiring reduct $R_{\mathcal{A}}^{\vee}$. Then we can generalize the above approach and consider the relation R and the composition operator $\circ$ in Equation (15) as a function $U : A^X \to A^Y$ such that

$$U(f)(y) = \bigvee_{x \in X} f(x) \odot R(x, y).$$

By Example 3.13, we know that there exists $\psi : A \to A$ (given by Equation (11) such that (U, ψ) is a linear bimorphism.

Vice-versa, it is well known (see for example [6], [12]) that for every map $U : A^X \to A^Y$ such that

$$\begin{aligned} U(\bigvee_{i \in I} f_i) &= \bigvee_{i \in I} U(f_i) \\ U(a \cdot f) &= a \odot U(f) \end{aligned}$$

there exists a function $\varphi \in [0, 1]^{X \times Y}$ such that

$$U(f)(y) = \bigvee_{x \in X} f(x) \odot \varphi(x, y).$$

We can ask if the functional U is a linear function, in the semiring A^X.

5. Conclusions

In this work we made the first steps toward the description of many linear phenomena in the framework of Łukasiewicz logic and MV-algebras, establish a link with lattice-ordered commutative semirings. In this way we also found new interesting interpretation for this kind of logic. Many results can be extended from semirings theory to MV-algebras and many problems arise in the characterization of properties of semirings in order to have a perfect correspondence. Anyway we also note that new fields like many-valued automata theory and many-valued formal languages could find in this framework the more appropriate ground where to develop.

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