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Solutions of the Einstein Kaluza-Klein Equations

R. Bartnik

In this talk I reported on progress in obtaining numerical solutions for a system of equations of Kaluza-Klein type, with non-Abelian ($SU(2)$) symmetry group, under the Ansätze of staticity and spherical symmetry. The system considered, is obtained by imposing the 7-dimensional vacuum Einstein equations on a 7-manifold modelled on an $SU(2)$ principle bundle over $\mathbb{R}^{31|}$. The fibres are given a left-invariant (*not* bi-invariant) metric which varies in $\mathbb{R}^{31|}$, and the orthogonal complement of the vertical subspace determines a Yang-Mills connection, so the metric is of the form

$$g_P = \pi^* g_M + \varphi(\omega(\cdot), \omega(\cdot))$$

where P is the bundle, M is the base, ω is the $su(2)$-valued connection and φ is a metric on $su(2)$. A conformal change in g_M can be used to bring the 7-dimensional Einstein-Hilbert action for g_P into the form of a 4-dimensional E-H action coupled to (Kaluza-Klein) matter sources. Imposing the conditions of time-independence (staticity) and spherical symmetry on γ_M, ω and φ, reduces the equations to a system of radial ODE's. Some numerical solutions of this system were displayed, determined by boundary conditions ensuring regularity at $r = 0$. It has not been possible to find asymptotically flat solutions, primarily because of a term arising from the scalar curvature of the left-invariant metric φ. Interestingly, and unlike more simple-minded KK computations, this term is no longer necessarily positive (where it contributes negative energy density, or a cosmological 'constant') - but we find it changes sign only just prior to ODE blow up. It remains to be seen whether asymptotically flat solutions are possible within this ansatz.

Gravitating Solitons and Hairy Black Holes

P. Bizoń

In my talk I have outlined the recent research on stationary soliton and black hole solutions of Einstein's equations with nonlinear field sources. First I considered the globally regular solutions in the region of weak coupling and reviewed the standard existence proofs based on the implicit function theorem. I pointed out that, even for weak coupling, gravity may lead to essentially nonperturbative effects. Then the region of strong coupling was discussed, in particular it was argued that there is a critical strength of the coupling beyond which there exist no globally regular static solutions. Next, the idea of gravitational desingularization was described and illustrated by several examples, the most prominent one being the Bartnik-McKinnon solutions of the Einstein-Yang-Mills equations.

The second part of my talk was devoted to black hole solutions with nonlinear hair. It was shown that, in analogy to solitons, the existence of hairy black holes can be understood by perturbative arguments using the implicit function theorem.

Finally, different versions of the no-hair conjecture were considered and shown to be violated by explicit examples.

Self-Similar Scalar Field Collapse: Naked Singularities and Critical Behaviour

P. R. Brady

Homothetic solutions to the Einstein Scalar field equations were discussed. The solutions belong to two different classes. The first have regular origin and represent dispersal, although there is a singularity at one point. The second class includes both black holes and naked singularities with a critical evolution (which is neither) interpolating between these two extremes. Some speculation was also made about the significance of self-similarity in the recent work of Choptuik.

Testing Cosmic Censorship with Black Hole Collisions

D. Brill

Solutions of the Einstein-Maxwell equations with a cosmological constant have recently been found that represent any number of arbitrarily placed charged black holes in motion in a de Sitter space background. We discuss the global structure of these solutions, with particular regard to their physical properties.

The background de Sitter space is presented in two coordinates. Scwarzschild type coordinates cover only a static portion of the spacetime, but can easily be analytically extended. Isotropic, or cosmological coordinates cover a different half of the spacetime; the spacelike sections appear asymptotically flat because at large distances they all converge to the large, late expansion phase of de Sitter space.

De Sitter space with 'one' charged black hole also can be presented in a static frame. The analytic continuation shows that the metric really describes two black holes with opposite charges at antipodal points of the background de Sitter space. By identifying the throats of the two black holes one obtains a model with compact space sections of topology $S^2 \times S^1$. In isotropic coordinates this geometry appears to have the exponential time dependence characteristic of de Sitter space, and this can be chosen to be either expanding or contracting. To cover this spacetime completely a contracting and an expanding patch have to be joined at the cosmological horizon. There is an upper limit on the mass of the black hole, beyond which the solution shows a naked singularity.

The multi-mass solutions due to Kastor and Traschen do not admit a static frame (in general they have no continuous symmetries); they are presented in the cosmological coordinates. As in the previous, simpler examples, there is an expanding and a contracting version, and neither of these coordinates covers a complete spacetime. Because the charge of each black hole equals its mass, the holes only take part in the common cosmological motion; the pairwise gravitational and electromagnetic forces cancel, as in the Majumdar-Papetrou solution. In each coordinate patch one can identify the (outer) black hole horizons. In the expanding part the black holes move apart and are always separate. In the contracting part the black holes approach each other and eventually merge. The total mass parameter after merger is the sum of the mass parameters before the merger; in this sense there is no gravitational energy emitted during the merger.

One can join an expanding and a contracting part of Kastor-Traschen spacetime together at their cosmological horizons; however, this continuation is not analytic because the solutions are generically only C^2 at the cosmological horizon, and C^1 at the inner black hole horizon. For certain discrete values of the total mass the continuation is smooth at the cosmological horizon, and by careful arrangement of the other holes around a given black hole, its inner horizon can be made more differentiable, so that the curvature does not diverge there.

If the individual masses are less than the upper limit, the data on a Cauchy surface at early times near the holes is nonsingular; but if the total mass exceeds the upper limit, a naked singularity will form to the future of a Cauchy horizon. The implications of these results for cosmic censorship are discussed.

D.R. Brill, S.A. Hayward, *Class. Quantum Grav.* **11** (1994), 359-370.

D.R. Brill, G.T. Horowitz, D. Kastor and J. Traschen, Phys. Rev. **D49** (1994), 840.

Non Abelian Relativistic Fluids

Y. Choquet-Bruhat

Classical models for matter with Yang-Mills change density, and the gauge and gravitational fields it generates.

I. Kinetic models. The matter is described by a distribution function $f(\kappa, p, q)$ where q, a Lie algebra element, is the charge of a particle at the point x with momentum p. The spacetime metric and the Yang-Mills fields satisfy the Einstein and Yang-Mills equations with stress energy and current generated by f, while f satisfies a Boltzmann equation. Local existence for a solution is proved. It extends to global existence for small data if the particles have zero mass and the spacetime metric is assumed to be strongly asymptotically Minkowskian.

II. Fluid models. For finite conductivity the results are analogous to those for electric charge, but the vorticity is not conserved. For infinite conductivity the wave fronts do not split in general into Alfven and magneto-acoustic, but are tangent to an, in general irreducible, 6^{th} order cone.

For 'small' data the local-in-time existence theorem extends to global existence when the fluid is ultrarelativistic and the metric is strongly asymptotically Minkowskian.

Critical Phenomena in Gravitational Collapse

Matt Choptuik

I discuss results of a detailed numerical study of the gravitational collapse of massless scalar fields in spherical symmetry. This study has revealed a variety of non-linear phenomena which arise in the strong-field regime where black holes form or 'almost form'. These phenomena - which include scaling, universality, and a type of self-similarity (scale-periodicity) - appear to be analogous to standard (statistical-mechanical) critical phenomena. In particular, black hole formation may be viewed as a phase transition with black hole mass playing the role of order parameter. Key features of the near-critical, strong-field dynamics will be illustrated using video footage.

Strong Cosmic Censorship in Vacuum Space-Times with Compact, Locally Homogeneous Cauchy Surfaces.

Piotr T. Chruściel

It is widely believed that an important question in classical general relativity is that of *strong cosmic censorship* (SCC), due to Penrose. A mathematical formulation thereof, essentially due to Moncrief and Eardley, is the following:

Consider the collection of initial data for, say, vacuum or electro–vacuum space–times, with the initial data surface Σ *being compact, or with the initial data* (Σ, γ, K) *asymptotically flat. For generic such data the maximal globally hyperbolic development thereof is inextendible*

The failure of the above would mean a serious lack of predictability of Einstein's equations, an unacceptable feature of a physical theory.

Because of the difficulty of the strong cosmic censorship problem, a full understanding of the issues which arise in this context seems to be completely out of reach at this stage. For this reason there is some interest in trying to understand that question under various restrictive hypotheses, *e.g.*, under symmetry hypotheses. It turns out that even in the class of homogeneous space–times the question of SCC has not been answered until the work described here, because of the difficulties in understanding the global dynamical behaviour of the Bianchi IX models. The results have been obtained in collaboration with Alan Rendall. The first main result is the following:

0.1. Theorem. *Strong cosmic censorship holds in the class of vacuum Bianchi IX space–times with* $L(p, 1)$, $p = 1, 2$ *spatial topology.*

The result is wrong for spatial topology $L(p, 1)$, $p > 2$.

Recall that the issue of strong cosmic censorship arises because of possible non–uniqueness of solutions of Einstein equations beyond Cauchy horizons. This is because any Cauchy data (Σ, γ, K) define a unique (up to isometry) maximal globally hyperbolic development (M, g), but whenever (M, g) is extendible uniqueness of the extensions is lost, at least if one does not impose some further restrictions. In the talk I present two constructions which show that this occurs in any space–time with a Cauchy horizon, even when analyticity conditions are imposed. The Taub–NUT space–times are a well known example with the property that every Taub–NUT initial data admit at least two maximal developments. Now the essential difference between those two space–times is in the way the boundaries $\partial\mathcal{D}(\Sigma)$ are "glued" to the globally hyperbolic Taub region, and it is natural to ask how many ways of doing this glueing exist. I present a theorem which shows that the standard Taub–NUT space–times exhaust all the possibilities. Another result that we can prove is that the only maximal Bianchi IX space–times on which $G = SO(3)$ or $G = SU(2)$ acts by isometries are the standard Taub–NUT space–times. It would be of interest to find some other conditions, weaker than the above, which single out the standard Taub–NUT space–times in the collection of all the extensions of the globally hyperbolic region of the Taub–NUT space–times.

An interesting class of space–times in which to study the question of strong cosmic censorship is that of space–times evolving out of *locally homogeneous* initial data. It turns out that the methods needed to analyze the Bianchi IX case carry over without any essential modifications to the case of locally homogeneous initial data. In that class of space–times the SCC question turns out to depend upon the topology of the partial Cauchy surface. This is due to the fact that some topologies allow only those locally homogeneous initial data for which the resulting maximal globally hyperbolic space–time is extendible. It should be stressed that the number of parameters needed to parameterize the spatially homogeneous metrics on a given three dimensional manifold will in general be larger than the number of local parameters, because further global parameters are needed.

0.2. Theorem. *SCC holds in the class of spatially compact, (spatially) locally homogeneous vacuum space-times with Bianchi symmetry type, in the following sense: the number of local parameters describing solutions with Cauchy horizons is strictly smaller than the number of local parameters allowed by the corresponding simply–connected models.*

When considering the exact known solutions which contain Cauchy horizons, it is striking that they seem to display more symmetries then the typical representatives of the families they belong to. We have the following rather elegant formulation of SCC in this class of space–times:

0.3. Theorem. *Let (M, g) be a vacuum space–time with a compact locally homogeneous partial Cauchy surface Σ such that $D(\Sigma)$ is maximal globally hyperbolic and such that $\partial D(\Sigma) \neq \emptyset$. Then the local Killing vector algebra is at least four–dimensional.*

Classical and Conformal Superspace

Arthur E. Fischer

Let M be a compact connected n-manifold, $\mathcal{M} =$ Riem (M) the space of smooth

Riemannian metrics on $M, \mathcal{D} =$ Diff (M) the group (under composition) of smooth diffeomorphisms of M, $\mathcal{D}_0$ the connected component of the identity of $\mathcal{D}, \mathcal{P} =$ Pos (M) the abelian group (under pointwise multiplication) of smooth positive functions on M, and $\mathcal{D} \times \mathcal{P}$ the semi-direct product of $\mathcal{D}$ and $\mathcal{P}$. *Classical superspace* is defined as the space of Riemannian geometries

$$S \equiv \mathcal{M}/\mathcal{D};$$

pointwise conformal superspace is defined as the space of pointwise conformal structures

$$\mathcal{P}C \equiv \mathcal{M}/\mathcal{P};$$

and *conformal superspace* is defined as the space of conformal geometries

$$CS \equiv \frac{\mathcal{M}/\mathcal{P}}{\mathcal{D}} \approx \frac{\mathcal{M}}{\mathcal{D} \times \mathcal{P}}.$$

As has been emphasized in previous studies of superspace, the structure of superspace and conformal superspace is linked to the topology of M. This linkage has resulted in the stratification theorem for classical and conformal superspace. In this report we discuss other aspects of this linkage. In particular, we define the degree of symmetry of a manifold M as the maximum dimension of the isometry group for any Riemannian metric on M, and show that the stratifications of classical and conformal superspace and their $\mathcal{D}_0$-restricted counterparts *restricted superspace* $S_0 = \mathcal{M}/\mathcal{D}_0$ and *restricted conformal superspace* $(CS)_0 = \frac{\mathcal{M}/\mathcal{P}}{\mathcal{D}_0}$ are orbifolds if and only if the degree of symmetry of M is zero.

For $n \geq 3$, we define the Yamabe type of a manifold, and show that if M is of Yamabe type -1, then conformal superspace is homeomorphic to $\mathcal{M}_{-1}/\mathcal{D}$ (where $\mathcal{M}_{-1}$ are those metrics with scalar curvature =-1). If, additionally, deg $M = 0$, then this homeomorphism is a homeomorphism of orbifolds.

The case $n = 3$ is of the most importance for applications to general relativity. In this case we generate a list that classifies compact connected orientable 3-manifolds by their Yamabe type and their degree of symmetry. For example, assume that the *extended Poincaré conjecture* is true, namely, *that every compact connected 3-manifold with finite fundamental group is diffeomorphic to a Clifford-Klein spherical space form.* Then, *M is a prime compact connected orientable 3-manifold with deg $M = 0$ if and only if M is diffeomorphic to either a manifold of flat type F_6 (Yamabe type 0), or to a compact orientable $K(\pi, 1)$-manifold of non-flat type whose fundamental group π does not have an infinite cyclic center (Yamabe type -1).* Moreover, in either case, *$\mathcal{D}_0$ acts freely on $\mathcal{M}$ and $\mathcal{M}/\mathcal{P}$, and $\mathcal{S}_0$ and $(\mathcal{CS})_0$ are ILH-manifolds.*

That $(\mathcal{CS})_0$ is an (infinite-dimensional) manifold under these topological conditions on an orientable M is a generalization of the case $n = 2$, genus $M \geq 2$, for in that case $(\mathcal{CS})_0 \approx \mathcal{T}(M)$ is the Teichmüller space of M, a finite-dimensional manifold diffeomorphic to $\mathbf{R}^{6 \text{ genus } M - 6}$.

Boundary Conditions for Anti-Ele Sitter-Type Space-Times

Helmut Friedrich

The existence of asymptotically simple solutions to Einstein's field equations $Ric(\tilde{g}) = \lambda \tilde{g}$ with positive cosmological constant λ is discussed. It is shown that smooth standard Cauchy data on a space-like slice satisfying the λ-constraints and the conditions of asymptotic simplicity and a smooth conformal (Lorentz) structure on a cylinder representing the conformal boundary at space-like and null infinity which satisfies the 'corner conditions' determine a unique asymptotically simple solution. The discussion is based on a new conformal representation of the Einstein equations.

Solitons and Gravity

G W. Gibbons

In this talk I intend to overview the contribution to general relativity coming from the general set of ideas associated with the concept of a soliton and in particular to emphasize the special importance of *supersymmetry*. Roughly speaking one means by a soliton a stable time-independent solution of the classical held equations having least energy for some fixed (often topological) charge, and having a mass fixed in terms of that charge. In pure gravity the soliton idea does not seem to be very fruitful but in $N \geq 2$ supersymmetry it leads one to consider extreme Reissner-Nordstrom Black Holes as solitons, the role of the conceived charge being played by Wheele's 'charge without charge' concept of flux trapped in topology. The ideas generalize to various dilatons and to higher dimensions and lead to solutions representing various p-branes Some solutions are completely non-singular, others related by various duality symmetries and possessing as much supersymmetry have singularities. An outstanding problem is to what extent these singular solutions may be regarded as legitimate solitons. A possible resolutions of this problem may emerge in string theory.

The Existence of Hypersurfaces of Constant Mean Curvature in Asymptotically Schwarzschild spacetimes

M. Iriondo

We prove the existence of spacelike hypersurfaces with constant mean curvature (CMC) in asymptotically Schwarzschild spacetimes at null infinity and the uniform elipticity of the mean curvature operator, seen as a differential operator in the

hyperbolic geometry, which makes possible a careful analysis of the behaviour of the hypersurface near null infinity.

By definition, the spacetimes which are asymptotically flat at future null infinity admit a compactification, i.e. there is a conformal factor Ω such that $\Omega = 0$ defines the null boundary, diffeomorphic to $S^2 \times \mathbf{R}$ and $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ is a smooth Lorentzian metric.

For those spacetimes, one may easily prove the existence of coordinates $\{u, x, y^A\}$, where u is the affine parameter of null geodesics on $\mathcal{I}^+$, x are integral curves of null vectors satisfying $\tilde{\nabla} u_{|\mathcal{I}^+} = \partial_x, y^A$ are the usual spherical coordinates and the conformal factor $\Omega = x + \mathcal{O}(x^3)$.

In other words we obtain a foliation of null hypersurfaces near null infinity, which intersect $\mathcal{I}^+$ at two dimensional manifolds diffeomorphic to S^2. We call the asymptotically flat spacetimes that have such foliation and where the metric satisfies a decay condition *asymptotically Schwarzschild.*

¿From this foliation we can construct a new foliation of spacelike hypersurfaces, *the hyperboloidal foliation*, which intersects $\mathcal{I}^+$ at the same cut.

In the neighbourhoods of $\mathcal{I}^+$ given by small (V_E^-) and τ large (V_E^+), we prove that this foliation has properties similar to those by the hyperboloids.

More precisely the spacetime together with this foliation satisfies in (V_E^+) the following condition

$$\max(||Ric||, ||\alpha||, ||\alpha^{-1}\nabla\alpha||, ||\hat{T}||_2) \leq C,$$

where α is the lapse, $\hat{T}$ is the normal of the foliation, and the norms are taken with respect to a four dimensional positive definite metric.

In (V_E^+) we use this foliation as local upper barriers. Imposing an *interior condition*, this allows the choice of a test function leading to an a priori estimate of the supremum of the height functions of hypersurfaces satisfying the Dirichlet problem in an increasing sequence of domains with boundary data approaching null infinity at a prescribed cut.

Using this global upper barrier and causality arguments we obtain the existence of a spacelike hypersurface which is C^∞ on each compact set.

Having the existence and local regularity of the CMC hypersurface, we need to prove that this hypersurface becomes null only at null infinity. Therefore using again this special foliation as lower barrier in V_E^-, we can estimate the angle between the foliation and the CMC hypersurface, which in turn gives the uniform ellipticity with respect to the hyperbolic geometry.

Effect on Radiative Wave Tails on Black Hole Interiors

W. Israel

There is currently a divergence of views concerning the structure of the singularity inside a black hole.

It is generally agreed, and in accord with the strong cosmic censorship hypothesis, that, generically, the final singularity is probably spacelike and (classically) of mixmaster type.

Some recent studies of how the internal geometry is affected by gravitational wave-tails left in the wake of a collapse suggest the presence, in addition, of a milder, precursory singularity which is light like and coincident with the inner (Cauchy) horizon (corresponding to infinite external advanced time). It extends an infinite affine distance into the past inside the hole, and tapers to the strong final singularity at its future end.

The opposing view, based on general stability arguments and numerical integrations of spherical models, is that this light like segment of the singularity does not survive generically, but is pre-empted by some kind of space like singularity.

This report offers a perspective, and, hopefully, some clarification of the issues, by recourse to analytical approximations and physical arguments.

Analytical and Numerical Methods for EYM and Related Systems

D. Maison

Beginning with the work of Bartnik and McKinnon a number of numerical investigations revealed several interesting non-perturbative phenomena of the Einstein-Yang-Mills (EYM) system with or without additional fields. Among these are the existence of discrete families of regular static, spherically symmetric solutions and non-abelian black holes.

Due to the high symmetry the field equations can be reduced to a system of 1^{st} order ODE's. It turns out that the physically relevant boundary points (e.g. $r = 0$ resp. $r = \infty$ for regular solutions) are singular points of these equations. Numerical as well as analytical methods to obtain global solutions require as a first step the construction of local solutions near the singular boundary points. This can be done by a suitable linearisation at the singular points, a second step is to connect these local solutions. Already in the simplest case of the pure EYM theory the dynamical system to be studied is 4-dimensional. Unfortunately no general theory seems to be available to control the global behaviour of the solutions of such systems. Fortunately it turns out that the system at hand seems to behave essentially like a 2-dimensional one with a rather regular manifold of orbits. Exploiting this fact it becomes possible to prove the global existence of regular and black hole solutions for the massless EYM system as well as for a massive EYM model. The latter may be obtained from an EYM-Higgs theory in the limit of infinite Higgs-mass, freezing the Higgs field at its symmetry breaking vacuum value. Due to the singularity of the Higgs field the metric develops a conical singularity at the origin in this massive model.

The results described in the talk are contained in two papers:

P. Breitenlohner, P. Forgács, D. Maison, Comm. Math. Phys. **163** (1994), 141.

P. Breitenlohner, P. Forgács, D. Maison, *Gravity Monopole Solutions II*, in preparation.

Analytical and Numerical Studies of Spacetime Singularities

Vincent Moncrief

This talk surveyed several different analytical and numerical approaches to the study of (cosmological) spacetime singularities. A (rigorous) analytical approach, using light-cone and/or energy estimates, is currently applicable only to very special (highly symmetric) families of solutions such as Gowdy metrics. A (non-rigorous) 'multiple-scales' expansion technique is applicable to much more general classes of spacetimes but, at present, gives useful results only for asymptotically velocity dominated solutions of Einstein's equations.

A numerical approach based on a higher order accurate, symplectic integration scheme has been developed in collaboration with Beverly Berger (Oakland University) and applied to the Gowdy metrics as a test case. It is currently being extended to treat general $U(1)$-symmetric metrics on $T^3 \times R$. In principle it could be applied to metrics having no symmetries at all. The method seems ideally suited to reveal asymptotic velocity dominance or mixmaster behaviour when these arise in particular solutions.

The Structure of Conformal Singularities

Kerri Newman

In 1967 Roger Penrose first made the suggestion that, if the Weyl tensor were to tend to zero at the Big Bang, a condition known as the Weyl Curvature Hypothesis, then Einstein's equations would force space-time to be Robertson-Walker. The underlying idea here is that, if the Weyl tensor could be related to some form of gravitational entropy, then a condition of zero initial entropy for the Universe would presumably imply a zero initial Weyl tensor, and that this would in turn imply spatial isotropy of the evolving space-time. This would provide at least a basis for an explanation of the high degree of spatial isotropy exhibited by the physical Universe.

Only recently has it been possible to establish Penrose's original suggestion as a theorem. Part of the difficulty came in the identification of a suitable mathematical framework within which to work. Such a framework has turned out to be provided by that of a conformal singularity, a concept dual to that of conformal infinity. Roughly speaking, one assumes that the 'physical' space-time metric may be conformally transformed such that the singular Big Bang becomes a smooth space-like 'initial hypersurface'. Within this framework, the Weyl Curvature Hypothesis turns out to be equivalent to the initial 3-metric on the initial hypersurface being spatially isotropic. From any such initial 3-metric one can evolve a (conformally transformed) Robertson-Walker space-time. However one needs a uniqueness theorem to show that no other type of space-time can evolve from such initial data. More generally, one seeks a uniqueness theorem for the conformally transformed

Einstein equations, for any initial 3-metric on the initial hypersurface. Indeed one would also like to have an existence theorem which shows that any (sufficiently smooth) initial 3-metric can be evolved into some space-time.

In order to proceed it is necessary to improve some condition on the energy tensor of the physical space-time. It may be observed that the Robertson-Walker space-times are all perfect fluid and, of those with constant adiabatic index γ, only those with $\gamma = 4/3$ evolve from a Big Bang of conformal type. It is therefore appropriate to work within the class of $\gamma = 4/3$ perfect fluid space-times.

The major step in the establishing of the required existence and uniqueness theorems was to formulate the conformally transformed Einstein equations, together with subsidiary gauge-fixing conditions, as a first order, symmetric hyperbolic system. Unfortunately this system turned out to be singular, and new techniques had to be developed to handle it. It was then possible to conclude that any initial 3-metric, whose first ten derivatives are locally square-integrable, may be evolved by the conformally transformed Einstein equations. Furthermore, the conformally transformed $\gamma = 4/3$ Robertson-Walker space-times are the unique C^9 evolution of C^3 Weyl Curvature Hypothesis initial data.

Spherical Gravitational Collapse

Niall Ó Murchadha

The trapped surface conjecture was posed by H.-J. Seifert as a statement that if a sufficient amount of matter was placed into a small enough volume, a trapped surface would form around the matter. This can be viewed as an explicit version of the widely-held belief that if enough matter was squeezed into a small enough volume the system would gravitationally collapse. The major advantage of the trapped surface conjecture is that one need not consider the dynamics of collapse, the equation of state of the matter plays no role and only the constraint equations of general relativity need be considered.

In this lecture a precise version of the trapped surface conjecture for spherically symmetric systems is derived. Edward Malec and the speaker have considered a maximal, spherically symmetric slice through an asymptotically flat spacetime. On this slice one has a matter - field with density ρ and current - density j_r. Take a two - sphere of radius R_o and define the mass - content $M(R_o) = \int \rho dv$, the radial momentum $P(R_o) = \int j_r dv$. If

$$M(R_o) - P(R_o) > L(R_o)$$

then the surface at R_o is trapped.

The constraint equations of general relativity can be written as first - order equations for the optical scalars θ, θ', the expansions in the future - pointing and past - pointing null directions. These equations can be used to derive bounds on the optical scalars. These bounds are used to derive the spherical version of the trapped surface conjecture above and to illuminate the singularity - avoidance property of regular slices.

Dirichlet Problem for Stationary Einstein Equations with Applications to Stability Limits of Rotating Stars

H. Pfister

For stationary systems, Einstein's equations are (in appropriate coordinates) elliptic, and therefore a boundary value problem is the natural one (boundary e.g. surface of a star, horizon of a black hole, mass shell, cosmic void etc.). Problem: Existence, uniqueness and regularity of Dirichlet solutions for the nonlinear system of Einstein equations. For small data there is a proof by Reula (1989). For generic data the nonlinearity of Einstein equations (quadratic in first derivatives $|\nabla U|^2$) is the limiting case between generic existence and nonexistence. For not too big a factor of the $|\nabla U|^2$-term, regularity of Dirichlet solutions is proven (Hildebrandt et al. 1975), and an existence proof seems possible. These mathematical facts are not directly applicable to the axisymmetric vacuum Einstein equations, and an extension to the general stationary case and to ideal fluid matter seems possible. For Schwarzschild and Kerr the limitation on the factor a is intimately connected with stability limits (horizon, ergosphere). Presupposing unique solvability of the Dirichlet problem for generic data, one can make arguments concerning the solution manifolds for the exterior and interior of rotating stars, possibly also for global solutions.

On the Spherically Symmetric Vlasov-Einstein System

G. Rein

In Newtonian astrophysics large, collisionless, self gravitating ensembles of mass points can be described by the Vlasov-Poisson system. Since for this system global existence and uniqueness of solutions to the corresponding initial value problem is established, the Vlasov equation, i.e., a collisionless gas, may be a suitable way to describe matter in general relativity, which motivates the study of the Vlasov-Einstein system. In the first part of my talk I considered the spherically symmetric, asymptotically flat case. After a review of some joint results with A. D. Rendall such as local existence and continuation of solutions$_+$ and global existence for small data, a more recent joint result with A.D.R. and J. Schaeffer was presented in more detail, which says that if singularities ever develop then the first one has to be at the centre of symmetry.

The techniques used there can also be used in a cosmological situation to show the existence of solutions of the Vlasov-Einstein system with spherical, plane, or hyperbolic symmetry which have a curvature and crushing singularity.

Crushing Singularities in Spacteimes with Spherical or Plane Symmetry

A. D. Rendall

Perhaps the simplest inhomogeneous cosmological models are the spherically symmetric models on $S^2 \times S^1$ and the plane symmetric models on the three-dimensional torus. The work reported here is part of an effort to understand the nature of singularities in space-times of this type. The main result is to show that under certain assumptions, including that of a sufficiently well-behaved matter model, the singularities in these spacetimes are crushing, i.e. that there exists a foliation by compact spacelike hypersurfaces whose mean curvature blows up as the singularity is approached. An example of such matter model is the collisionless gas, where the matter is described by the Vlasov equation.

It is always possible to prove the existence of a crushing singularity in at least one time direction in spacetimes which develop from constant mean curvature initial data with the above symmetry. When the initial mean curvature is zero (maximal initial data) crushing singularities are obtained in both time directions. The proof uses a continuation argument. Using the mean curvature as a time coordinate transforms the question of the presence of a crushing singularity into a global existence problem. Local existence for this problem is guaranteed by standard theorems. Thus it makes sense to talk about the maximal interval of existence. The strategy is to bound enough quantities on this interval to show that if the interval were finite the solution could be extended, contradicting maximality. A key step in the argument uses (a slight generalisation of) recent inequalities of Malec and O Murchadha on the expansions of null geodesics in spherically symmetric spacetimes.

An interesting aspect of the proof is that a significant part of it can be done without specifying the matter model beyond requiring it to satisfy the dominant energy and non-negative pressures conditions. Thus estimates are obtained which show that even if the density blows up at the end of the maximal interval of existence many geometrical quantities remain bounded. These estimates apply in particular to dust spacetimes where shell-crossing singularities are likely to occur frequently. The property of a matter model which qualifies it as well-behaved in the sense used above is, neglecting questions of detailed differentiablity properties, that solutions of the equations describing this matter model should not develop singularities in a given regular spacetime. Thus these results provide support for the intuitive picture that non-crushing singularities in solutions of the Einstein equations arise directly from properties of the matter model chosen.

The Newtonian Limit of Einstein's Equations of Gravity

B. Schmidt

The first part of the talk reviewed the 4-dimensional formulation of Newton's theory and the 'frame theory', a combined formulation of Newton's and Einstein's theories. Thus the relations between the structures and laws of both theories are explained in a satisfactory way. (Ehlers 1981).

The aim of the second part was to demonstrate that this frame theory does not only resolve some philosophical issues but is also a tool which can be used to obtain results within Einstein's theory. To make this point I considered the following four examples:

1) Studying post-Newtonian expansions from the point of view of the frame theory clarifies conceptual problems and demonstrates the limitations of such expansions. (A. Rendall 1991).

2) The existence of time dependent solutions of the Einstein-Vlasov system has been demonstrated which have a Newtonian limit. Furthermore any solution of the Newtonian-Vlasov system can appear as the limit of such a relativistic family (A. Rendall 1993).

3) Perturbing away from a Newtonian rigidly rotating fluid solution by an implicit function theorem argument the first existence proof of such solutions within Einsteins's theory was given recently. (U. Heilig 1993)

4) Families of relativistic stellar modes with a Newtonian limits can be constructed. Linearisation on such models describe stellar oscillations. Each Newtonian normal mode gives rise to a 1-parameter family of relativistic quasinormal modes. Thus the existence of relativistic quasinormal modes can be proved. (B. Schmidt 1994).

On the Einstein-Yang-Mills System for Arbitrary Gauge Groups

Norbert Straumann

In an introduction several solved and open problems for the EYM system have been mentioned. Still unsolved are, for instance, the staticity problem for rotating black holes, the generalization of the Israel theorem for static black holes, or the question whether the 2-planes orthogonal to the two independent Killing fields for a stationary rotating EYM black hole are integrable. From past experience one may doubt, whether all these properties for the Einstein-Maxwell system generalize to the non-abelian case.

In the main part of the talk we introduced first the general set up for spherically symmetric EYM field configurations and gave a group and bundle theoretic classification of solitons and black holes in terms of integral lattice points, which are at the same time in the closed fundamental Weyl chamber and on certain hyperplanes of the Stiefel diagram. This machinery is used to set up the EYM field equations in adapted form. We then showed in some detail that generic solitons and black holes (corresponding to points in the *open* Weyl chamber) are *always unstable*. Details of the argument, together with references to the work mentioned above, can be found in the recent preprint.*

*A slightly revised version, including the black hole case, will soon be available

O. Brodbeck and N. Straumann, ZU-TH 19/94.

An Astrophysical Shock-Wave Solution of the Einstein Equations Modelling an Explosion

Blake Temple

In the recent paper, *Shock wave solutions of the Einstein equations: The Oppenheimer-Snyder model of gravitational collapse extended to the case of non-zero pressure*, (all joint work with Joel Smoller), the authors constructed a class of exact, spherically symmetric shock-wave solutions of the Einstein equations of general relativity. This paper concluded with the derivation of a set of ordinary differential equations (ODE's) that describe the matching of a Friedmann-Robertson-Walker (which we refer to as (R-W)) type metric to an Oppenheimer-Tolman, (or (O-T), our name for static, spherically symmetric metrics that solve the Einstein equations for a perfect fluid), such that the interface between the two metrics defines a spherically symmetric, fluid dynamical shock-wave. In this talk we describe an explicit solution of these equations that models a shock-wave exploding outward into a *static, singular, isothermal sphere.* We show that the density, pressure and sound speed are larger behind the shock, and the Lax characteristic condition holds at the shock so long as the sound speed is not too close to the speed of light. In particular, this implies that the shock-wave moves everywhere at less than the speed of light. Since the Big Bang occurs in the inner R-W solution at the same instant that the shock-wave emerges from r=0 in the outer O-T solution, this explicit solution provides a scenario by which the Big Bang begins with a shock-wave explosion. In the classical theory of shock waves, the interface in the well-known Oppenheimer-Snyder solution is a contact discontinuity, and this means that the solution is time-reversible. In contrast, our shock-wave solutions in which $p \neq 0$ are irreversible solutions of the Einstein gravitational field equations in which the irreversibility, loss of information, and increase of entropy in the fluids puts time-irreversibility into the dynamics of the gravitational field. In the limit of small sound speeds and weak gravitational fields our model recovers the Newtonian limit.

The First Law of Black Hole Mechanics in an Arbitrary Lagrangian Theory of Gravity

R. M. Wald

A Lagrangian formulation of a field theory provides that theory with a wealth of auxiliary structure, which underlines many of its fundamental properties. For any Lagrangian theory, one can define a symplectic structure, and for a diffeomorphism invariant theory, one can define a Noether current and Noether charge associated with any choice of vector field on spacetime. These quantities give rise to natural definition of energy and angular momentum, and a general identity relates the variations of these quantities to the symplectic structure. We show that in the case of a spacetime with a black hole possessing a bifurcate Killing horizon, this identity gives rise to the first law of black hole mechanics. As a consequence (i) we prove the first law of black hole mechanics in an arbitrary Lagrangian theory of gravity, (ii) black hole entropy is identified as the Noether charge of the horizon with respect to the horizon Killing field and (iii) a single, local geometrical formula for black hole entropy is obtained.

N-Black Hole Stationary and Axially Symmetric Solutions of the Einstein-Maxwell Equations

Gilbert Weinstein

It is well known that the Einstein-Maxwell equations reduce in the stationary and axially symmetric case to an axially symmetric harmonic map $\varphi : \mathbb{R}^3 \backslash \Sigma \to HC$, where Σ is a subset of the axis of symmetry, and HC is the complex hyperbolic space. Using results from [Weinstein '94], we construct solutions of these equations which can be interpreted as equilibrium configurations of multiple co-axially rotating charged black holes held apart by a singular strut. We prove the metrics constructed from these solutions are smooth across the two unbounded components of the axis of symmetry and are asymptotically flat. We also obtain an expression for the angle deficiencies across the unbounded components. These results generalize our previous work where we studied the vacuum case.