

**A Generalization of the Utility Theory
using a Hybrid Idempotent–Probabilistic Measure**

Endre Pap

Vienna, Preprint ESI 1449 (2004)

January 28, 2004

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via <http://www.esi.ac.at>

A generalization of the utility theory using a hybrid idempotent-probabilistic measure

Endre Pap

ABSTRACT. There is given an overview on the results related to the generalization of the decision theory to non probabilistic uncertainty based on the characterization of the families of operations involved in generalized mixtures. It turns out that this is based on a previous result on the characterization of the pair of continuous t-norm T and t-conorm S such that the former is conditionally distributive over the latter. What is obtained is a family of mixtures that combine probabilistic and idempotent (possibilistic) mixtures via a threshold and the corresponding pseudo-additive hybrid idempotent-probabilistic measure satisfies also all other required conditions.

1. Introduction

Utility theory is based on the notion of mathematical expectation. Its axiomatic foundations, following von Neumann and Morgenstern [16] is based on the notion of probabilistic mixtures. It has been recently shown by Dubois et al. [5] that the notion of mixtures can be extended to pseudo-additive measures. Cox's well-known theorem [3] (see [19]), which justify the use of probability for treating uncertainty, was discussed in many papers. Recently, there are given some critics on it as well some relaxation on the conditions, which imply that also some non-additive measures can satisfy the required conditions, see [11]. Relaxing the condition on strict monotonicity to monotonicity on the function which occurs in the conditioning requirement, then the pair of t-norm S and t-norm T which satisfies (CD) (see Definition 2.8) and the corresponding pseudo-additive S -measure satisfy also all other required conditions. Triangular norms were first introduced in the context of probabilistic metric spaces by K. Menger [15] in more general form and by B. Schweizer and A. Sklar [21] in a today form, see also [13].

The aim of this paper is to present the answer to the following question: *what else remains possible beyond idempotent (possibilistic) and probabilistic mixtures?*

The solution obtained in [7] takes the advantage of a result obtained in [13] on the relaxed distributivity of triangular norm over a triangular conorm (called conditional distributivity). This result has a drastic consequence on the notion of

1991 *Mathematics Subject Classification.* Primary 90A10; Secondary 62C05, 90A05, 28E10.

Key words and phrases. Triangular conorm, triangular norm, hybrid idempotent-probabilistic measure, hybrid utility.

Authors want to thank to the partial financial support of of the Project MNTRS-1866.

mixtures. Beyond possibilistic and probabilistic mixtures, only a form of hybridization is possible such that the mixture is possibilistic under a certain threshold, and probabilistic above. The same distributivity property must be satisfied between the t-conorm characterizing the pseudo-additive measure, and the triangular norm expressing separability (independence), see [7, 8].

We remark that the pair of t-norm S and t-norm T which satisfies (CD) and the corresponding S -measure give a basis for an integration theory, see [13, 14].

2. Conditional distributive pairs of triangular conorms and norms

In this section we will use results from [13, 21].

DEFINITION 2.1. A triangular conorm (t-conorm for short) is a binary operation on the unit interval $[0, 1]$, i.e., a function $S : [0, 1]^2 \rightarrow [0, 1]$ such that for all $x, y, z \in [0, 1]$ the following four axioms are satisfied:

- (S1) *Commutativity* $S(x, y) = S(y, x)$,
- (S2) *Associativity* $S(x, S(y, z)) = S(S(x, y), z)$,
- (S3) *Monotonicity*
 $S(x, y) \leq S(x, z)$ whenever $y \leq z$,
- (S4) *Boundary Condition* $S(x, 0) = x$.

If S is a t-conorm, then its dual t-norm $T : [0, 1]^2 \rightarrow [0, 1]$ is given by

$$T(x, y) = 1 - S(1 - x, 1 - y).$$

EXAMPLE 2.2. The following are the three basic t-norms together with their dual t-conorms, which we shall use:

- (i) Minimum T_M and maximum S_M given by

$$T_M(x, y) = \min(x, y), \quad S_M(x, y) = \max(x, y),$$

- (ii) Product T_P and probabilistic sum S_P given by

$$T_P(x, y) = x \cdot y, \quad S_P(x, y) = x + y - x \cdot y,$$

- (iii) Lukasiewicz t-norm T_L and Lukasiewicz t-conorm S_L given by

$$T_L(x, y) = \max(x + y - 1, 0), \quad S_L(x, y) = \min(x + y, 1),$$

The following representations hold:

THEOREM 2.3. A function $S : [0, 1]^2 \rightarrow [0, 1]$ is a continuous Archimedean triangular conorm, i.e., for all $x \in]0, 1[$ we have $S(x, x) > x$, if and only if there exists a continuous, strictly increasing function $s : [0, 1] \rightarrow [0, +\infty]$ with $s(0) = 0$ such that for all $x, y \in [0, 1]$

$$S(x, y) = s^{-1}(\min(s(x) + s(y), s(1))).$$

THEOREM 2.4. A function $T : [0, 1]^2 \rightarrow [0, 1]$ is a continuous Archimedean triangular norm, i.e., for all $x \in]0, 1[$ we have $T(x, x) < x$, if and only if there exists a continuous, strictly decreasing function $t : [0, 1] \rightarrow [0, +\infty]$ with $t(1) = 0$ such that for all $x, y \in [0, 1]$

$$T(x, y) = t^{-1}(\min(t(x) + t(y), t(0))).$$

The functions s and t from Theorems 2.3 and 2.4 are then called additive generators of S and T , respectively. They are uniquely determined by S and T , respectively, up to a positive multiplicative constant.

We have the following representation for arbitrary continuous t-conorm and t-norm, see [13].

THEOREM 2.5. *A function $S : [0, 1]^2 \rightarrow [0, 1]$ (a function $T : [0, 1]^2 \rightarrow [0, 1]$) is a continuous t-conorm (t-norm) if and only if S (T) is an ordinal sum whose summands are continuous Archimedean t-conorms (t-norms).*

The t-conorm S (t-norm T) is called strict if it is continuous and strictly monotone on the open square $]0, 1[^2$. The continuous t-conorm S (t-norm T) is called nilpotent if each $a \in]0, 1[$ is a nilpotent element of S (of T), i.e., for every $a \in]0, 1[$ there exists $n \in \mathbb{N}$ such that $a_S^{(n)} = 1$ (respectively $a_T^{(n)} = 0$), where $a_S^{(n)}$ is the n -th power of a given by $S(a, \dots, a)$ (respectively $T(a, \dots, a)$) repeated a n -times. The class of continuous Archimedean t-conorms (t-norms) consists of two disjoint classes: strict and nilpotent.

The following important characterization of a strict t-norm and a nilpotent t-conorm will enable us to simplify the approach in this paper.

THEOREM 2.6. *A function $T : [0, 1]^2 \rightarrow [0, 1]$ is isomorphic to T_P , i.e., there is a strictly increasing bijection $\varphi : [0, 1] \rightarrow [0, 1]$ such that for all $x, y \in [0, 1]$ we have*

$$T(x, y) = \varphi^{-1}(T_P(\varphi(x), \varphi(y)))$$

if and only if it is a strict t-norm.

THEOREM 2.7. *A function $S : [0, 1]^2 \rightarrow [0, 1]$ is isomorphic to S_L , i.e., there is a strictly increasing bijection $\varphi : [0, 1] \rightarrow [0, 1]$ such that for all $x, y \in [0, 1]$ we have*

$$S(x, y) = \varphi^{-1}(S_L(\varphi(x), \varphi(y)))$$

if and only if it is a nilpotent t-conorm.

Continuous Archimedean t-conorms and t-norms are isomorphic to S_L and T_P , respectively.

DEFINITION 2.8. A t-norm T is conditionally distributive over a t-conorm S if for all $x, y, z \in [0, 1]$ we have

$$(CD) \quad T(x, S(y, z)) = S(T(x, y), T(x, z)),$$

whenever $S(y, z) < 1$.

The continuity of T and S implies that distributivity can be extended for a wider domain.

PROPOSITION 2.9. *Let a continuous t-norm T be conditionally distributive over a continuous t-conorm S . Let $x, y, z \in [0, 1]$ and y and z are such that $S(y, z) = 1$ and for every $b < y$ we have $S(b, z) < 1$ or for every $c < z$ we have $S(y, c) < 1$, then the distributivity $T(x, S(y, z)) = S(T(x, y), T(x, z))$ holds.*

We shall need the following theorem from [13] which gives the complete characterization of the family of continuous pairs (S, T) which satisfy the condition (CD). The proof is recalled due to the importance of the result for the paper.

THEOREM 2.10. *A continuous t-norm T is conditionally distributive over a continuous t-conorm S if and only if there exists a value $a \in [0, 1]$, a strict t-norm T^* and a nilpotent t-conorm S^* such that the additive generator s^* of S^* satisfying $s^*(1) = 1$ is also a multiplicative generator of T^* such that*

$$T = (\langle 0, a, T_1 \rangle, \langle a, 1, T^* \rangle)$$

where T_1 is an arbitrary continuous t-norm and

$$S = (\langle a, 1, S^* \rangle).$$

Proof. Suppose that a continuous t-norm T is conditionally distributive over a continuous t-conorm S . If $b \in [0, 1[$ is an idempotent element of S , then, for each $x \in [0, 1]$, $T(x, b)$ is an idempotent element of S . Therefore by the continuity of T each element in $[0, b]$ is an idempotent element of S . Hence, from Theorem 2.5 it follows that either $S = S_M$ (in which case we have proved the theorem for $a = 0$) or $S = (\langle a, 1, S^* \rangle)$, where S^* is a continuous Archimedean t-conorm and $a \in [0, 1[$.

If $S = (\langle a, 1, S^* \rangle)$ for some continuous Archimedean t-conorm S^* and some $a \in [0, 1[$, then a is an idempotent element also of T . This follows from the fact that for all $x \in]a, 1[$ with $S(x, x) < 1$ we have $T(a, S(x, x)) = S(T(a, x), T(a, x)) = T(a, x)$, i.e., $T(a, x) = T(a, x_S^{(2)})$, and consequently, for all $x \in]a, 1[$ we have $T(a, x) = T(a, x_S^{(\frac{1}{2})})$ (see Remark 3.5 in [13]), implying $a = T(a, 1) = T(a, x_S^{(\frac{1}{2})})$ for each $x \in]a, 1[$. Therefore, T can be written as an ordinal sum (see Theorem 2.5), one of its summands being $\langle a, 1, T^* \rangle$, where T^* is some continuous t-norm. Since T is conditionally distributive over S , T^* must be conditionally distributive over the Archimedean t-conorm S^* .

We will show that T^* is also Archimedean. Namely, the existence of a non-trivial idempotent element c of T^* would imply the existence of $x \in [0, 1]$ with $x < c < S^*(x, x) < 1$, leading to the contradiction

$$c = T^*(c, S^*(x, x)) = S^*(T^*(c, x), T^*(c, x)) = S^*(x, x).$$

Moreover, T^* cannot be nilpotent: if $0 < d = \sup\{x \in [0, 1] \mid T(x, x) = 0\}$ then there exists $y \in [0, 1]$ with $y < d < S^*(y, y) < 1$, leading to the contradiction

$$0 < T^*(d, S^*(y, y)) = S^*(T^*(d, y), T^*(d, y)) = 0.$$

Let θ be an arbitrary but fixed multiplicative generator of the strict t-norm T^* and s an additive generator of the continuous Archimedean t-conorm S^* . Remark that we have $S(y, z) = s^{-1}(s(x) + s(y))$ for all $(y, z) \in [0, 1]^2$ with $S(y, z) < 1$. We define the continuous, strictly increasing function $f : [0, s(1)] \rightarrow [0, 1]$ by $f = \theta \circ s^{-1}$ and note that $f(s(1)) = 1$. Taking $u = s(x)$, $v = s(y)$ and $w = s(z)$, the conditional distributivity of T^* over S^* can be rewritten as

$$(2.1) \quad f(u) \cdot f(v + w) = f(f^{-1}(f(u) \cdot f(v)) + f^{-1}(f(u) \cdot f(w)))$$

for all $u, v, w \in [0, s(1)]$ and $v + w < s(1)$. For a fixed $u \in]0, s(1)[$, we define the continuous, strictly increasing function $g_u : [0, s(1)] \rightarrow [0, s(1)]$ by $g_u(x) = f^{-1}(f(u) \cdot f(x))$ and observe that $g_u(0) = 0$ and $g_u(s(1)) = u$. Then (2.1) transforms into the Cauchy equation

$$g_u(v + w) = g_u(v) + g_u(w)$$

for all $v, w \in [0, s(1)]$ and $v + w \leq s(1)$, where the case $v + w = s(1)$ follows from the continuity of g_u . If $s(1) = \infty$ then this equation has no solution, so S^* must be a

nilpotent t-conorm. From [1], Section 2.1, Theorem 3, it follows that $g_u(x) = \frac{u}{s(1)}x$, i.e.,

$$f\left(\frac{u}{s(1)}x\right) = f(u) \cdot f(x)$$

for all $x, u \in [0, s(1)]$ (the case $u = 0$ follows from $f(0) = 0$). The only solutions of this modified Cauchy equation are given by $f(x) = \left(\frac{x}{s(1)}\right)^c$, where $c \in]0, \infty[$. Since $s(1)$ can be chosen arbitrarily in $]0, \infty[$ we may take $s(1) = 1$, leading to $f(x) = x^c$ and, consequently, we have $\theta = s^c$, i.e., s is a multiplicative generator of T^* .

Conversely, assume that a continuous t-norm T and a continuous t-conorm S have the forms from theorem, then T is conditionally distributive over S . \square

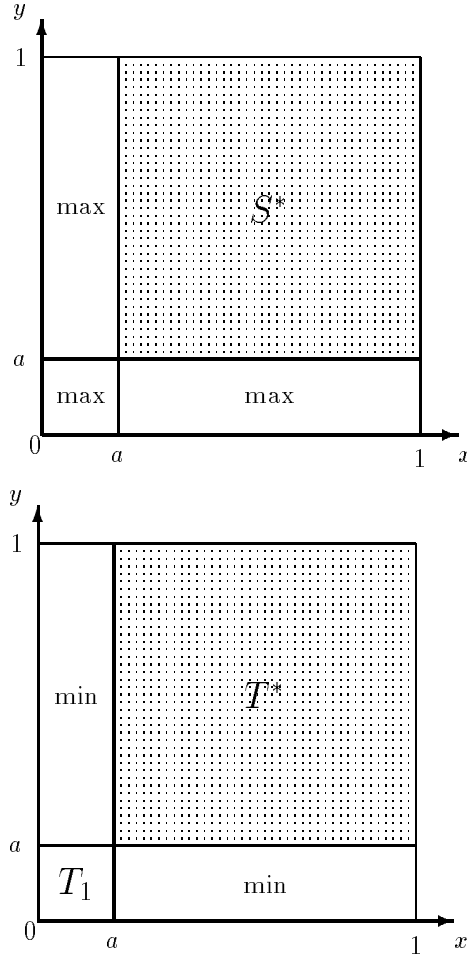


FIGURE 1. $(\langle S_M, S^* \rangle, \langle T_1, T^* \rangle)_a$ for $0 < a < 1$

We denote by $(\langle S_M, S^* \rangle, \langle T_1, T^* \rangle)_a$ the pair of continuous t-conorm S and t-norm T from Theorem 2.10, see the Figure 1.

- EXAMPLE 2.11. (i) The extreme case $a = 0$ for the pair $(\langle S_{\mathbf{M}}, S_{\mathbf{L}} \rangle, \langle T_1, T_{\mathbf{P}} \rangle)_0$, reduces on the pair $S_{\mathbf{L}}$ and $T_{\mathbf{P}}$.
- (ii) The other extreme case $a = 1$ for the pair $(\langle S_{\mathbf{M}}, S_{\mathbf{L}} \rangle, \langle T_1, T_{\mathbf{P}} \rangle)_1$, reduces on the pair $S_{\mathbf{M}}$ and an arbitrary continuous t-norm T_1 .
- (iii) For $0 < a < 1$ the pair $(\langle S_{\mathbf{M}}, S_{\mathbf{L}} \rangle, \langle T_1, T_{\mathbf{P}} \rangle)_a$ gives us the hybrid idempotent-probabilistic case.

3. S -measures, separable events, decomposition of S -measures

Let X be a fixed non-empty finite set.

DEFINITION 3.1. Let S be a t-conorm and let \mathcal{A} be a σ -algebra of subsets of X . A mapping $m : \mathcal{A} \rightarrow [0, 1]$ is called (pseudo-additive) S -measure if $m(\emptyset) = 0$, $m(X) = 1$ and if m is S -measure, i.e., for all $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$ we have

$$m(A \cup B) = S(m(A), m(B)).$$

Each S -measure $m : \mathcal{P}(X) \rightarrow [0, 1]$ is uniquely determined by the values $m(\{x\})$ with $x \in X$.

REMARK 3.2. In the general case when X is an arbitrary non-empty set (also infinite) there is an additional condition on m in Definition 3.1 namely that it is continuous from below. In this case, if S is a left continuous t-conorm, then a set function $m : \mathcal{A} \rightarrow [0, 1]$ satisfying $m(\emptyset) = 0$ is an S -measure if and only if for each sequence (A_n) of pairwise disjoint elements of \mathcal{A} we have

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n).$$

For some basic properties of S -measures see [13, 17, 24].

EXAMPLE 3.3. A set function $m : \mathcal{P}(X) \rightarrow [0, 1]$ is $S_{\mathbf{M}}$ -measure if and only if for all $A, B \in \mathcal{A}$ we have

$$m(A \cup B) = S_{\mathbf{M}}(m(A), m(B)).$$

Usually it is called idempotent (possibility) measure, denoted by Π and the corresponding distribution by π . Namely for an arbitrary function $\pi : X \rightarrow [0, 1]$, the set function $\Pi : \mathcal{P}(X) \rightarrow [0, 1]$ defined by $\Pi(A) = \sup\{\pi(x) \mid x \in A\}$ is an $S_{\mathbf{M}}$ -measure. We remark that only for X finite the notions of $S_{\mathbf{M}}$ -measure and possibility measures coincide.

REMARK 3.4. If $m : \mathcal{A} \rightarrow [0, 1]$ is an S -measure and $\psi : [0, 1] \rightarrow [0, 1]$ an increasing bijection, then $\psi^{-1} \circ m : \mathcal{A} \rightarrow [0, 1]$ is an S_{ψ} -measure, where S_{ψ} is defined by

$$S_{\psi}(x, y) = \psi^{-1}(S(\psi(x), \psi(y))).$$

It was investigated in [7] the problem which triangular norms $*$ (commutativity and associativity of $*$ reflect the corresponding properties for conjunctions and it is natural that $*$ be non-decreasing in each place and continuous) can be used for extending the notion of independence for S -measures. Since the term independence has a precise meaning in probability theory, we shall speak of separability instead.

DEFINITION 3.5. Two events A and B are said to be $*$ -separable if and only if

$$m(A \cap B) = m(A) * m(B)$$

for a triangular norm $*$.

Let A, B, C be three events such that $B \cap C = \emptyset$ and that the pairs (A, B) and (A, C) are made of separable events. Suppose m is an S -measure. Then only the equality

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

induces strong constraints on the choice of $*$ when S is fixed. Namely, we have

$$\begin{aligned} m(A \cap (B \cup C)) &= S(m(A \cap B), m(A \cap C)) \\ &= S(m(A) * m(B), m(A) * m(C)). \end{aligned}$$

Now, in probability theory, if disjoint sets B and C are independent from A , then $B \cup C$ is independent of A as well. Here again, in order to make the separability a computationally attractive property, the same property is taken for granted, i.e.,

$$m(A \cap (B \cup C)) = m(A) * m(B \cup C).$$

Hence the following identity must hold

$$\begin{aligned} m(A) * S(m(B), m(C)) \\ (3.1) \qquad \qquad \qquad &= S(m(A) * m(B), m(A) * m(C)) \end{aligned}$$

for all (A, B, C) such that $B \cap C = \emptyset$, (A, B) separable, (A, C) separable. Note that if $m(A) = 1$ or 0 , this property is always satisfied.

Assume $m(B \cup C) < 1$ in (3.1). Then the above equation coincides the condition of conditional distributivity property. It must hold for all S -measures. Then Theorem 2.10 using Theorems 2.6, 2.7 which reduce (up to isomorphism) all situations to the case in Example 2.11 for some $a \in [0, 1]$ provides the only possible choices for the pair $(S, *)$. Therefore we will consider only the case $\langle S_{\mathbf{M}}, S_{\mathbf{L}} \rangle, \langle T_{\mathbf{1}}, T_{\mathbf{P}} \rangle_a$. Then we have for some $a \in [0, 1]$

- a) for S the ordinal sum of \max on $[0, a]$ and $S_{\mathbf{L}}$ on $[a, 1]$ for some $a \in [0, 1]$;
- b) for $*$ the ordinal sum of any continuous t-norm $T_{\mathbf{1}}$ on $[0, a]$ and the product $T_{\mathbf{P}}$ on $[a, 1]$.

Hence the S -measure is an hybrid set function $m : \mathcal{P}(X) \rightarrow [0, 1]$ such that for $A \cap B = \emptyset$ we have

$$m(A \cup B) = \begin{cases} a + (1 - a) \min \left(\frac{m(A) - a}{1 - a} + \frac{m(B) - a}{1 - a}, 1 \right) & \text{if } m(A) > a, m(B) > a \\ \max(m(A), m(B)) & \text{otherwise.} \end{cases}$$

There is no way of satisfying (3.1) if m is such that $m(B) + m(C) > 1$ for some B, C that are independent. So the only reasonable S -measures that prevent this situation from happening while ensuring the normalization, and admitting of an independence concept, are

- (i) probability measures (and $*$ = product);
- (ii) possibility measures (and $*$ is any t-norm);
- (iii) pure hybrid measures m such that there is $a \in]0, 1[$ which gives for A and B disjoint

$$m(A \cup B) = \begin{cases} m(A) + m(B) - a & \text{if } m(A) > a, m(B) > a \\ \max(m(A), m(B)) & \text{otherwise,} \end{cases}$$

and for independence:

$$m(A \cap B) = \begin{cases} a + \frac{(m(A)-a)(m(B)-a)}{1-a} & \text{if } m(A) > a, m(B) > a \\ \frac{1}{a}T_1(am(A), am(B)) & \text{if } m(A) \leq a, m(B) \leq a \\ \min(m(A), m(B)) & \text{otherwise,} \end{cases}$$

and the normalization condition reads

$$\sum_{\{x\}, m(\{x\}) > a} m(\{x\}) = 1 + (\text{card}(\{x, m(\{x\}) > a\}) - 1)a.$$

Any probability distribution on a finite set X can be represented as a sequence of binary lotteries. A binary lottery is 4-uple (A, α, x, y) where $A \subset X$ and $\alpha \in [0, 1]$ such that $P(A) = \alpha$, and it represents the random event that yields x if A occurs and y otherwise, see [22]. More generally, suppose m is a S -measure on $X = \{x_1, x_2, x_3\}$ and $m_i = m(\{x_i\})$. Suppose we want to decompose the ternary tree on the left side of the Figure 2 into the binary tree of the right side so that they are equivalent.

We follow the calculations from [7]. Then the reduction of lottery property enforces the following equations

$$S(v_1, v_2) = 1, \quad T(\mu, v_1) = m_2, \quad T(\mu, v_2) = m_3,$$

where T is the triangular norm that expresses separability for S -measures. The first condition expresses that (v_1, v_2) is in the mixture set (with no truncation for t-conorm S allowed). If these equations have unique solutions, then by iterating this construction, any distribution of a S -measure can be decomposed into a sequence of binary lotteries.

The problem of normalization takes us to the following system of equations

$$(3.2) \quad \alpha_1 = T(\mu, v_1), \quad \alpha_2 = T(\mu, v_2), \quad S(v_1, v_2) = 1$$

for given α_1 and α_2 . We know that there always exists an unique solution (μ, v_1, v_2) . We are interested in the analytical forms of (μ, v_1, v_2) . We suppose without loss of generality that $\alpha_1 > \alpha_2$. Then we have the following cases:

Case I. Let $\alpha_1 > a, \alpha_2 > a$. Then (3.2) reduces on

$$\alpha_1 = a + \frac{(\mu - a)(v_1 - a)}{1 - a}, \quad \alpha_2 = a + \frac{(\mu - a)(v_2 - a)}{1 - a}, \quad 1 = v_1 + v_2 - a.$$

We obtain the unique solution

$$\mu = \alpha_1 + \alpha_2 - a, \quad v_1 = a + \frac{(1 - a)(\alpha_1 - a)}{\alpha_1 + \alpha_2 - 2a}, \quad v_2 = a + \frac{(1 - a)(\alpha_2 - a)}{\alpha_1 + \alpha_2 - 2a}.$$

Case II. Assume $\alpha_1 > a \geq \alpha_2$. Then $S = \max$, and $\mu \geq a, v_1 \geq a$ and $v_2 \leq \alpha_2 < a$. Hence assuming $T_1 = \min$ (we shall only deal with this case) the equations (3.2) write:

$$\max(v_1, v_2) = 1, \quad \alpha_1 = a + \frac{(\mu - a)(1 - a)}{1 - a} = \mu, \quad \alpha_2 = \min(\mu, v_2).$$

Assume $v_1 = 1$. Then $\mu = \alpha_1$ and $v_2 = \alpha_2$. Note that this solution is unique, since assuming $v_2 = 1$ leads to $\mu = \alpha_2 < a$, which is a contradiction.

Case III. Assume $\max(\alpha_1, \alpha_2) \leq a$. Then $S = \max$. Assume again $v_1 = 1$. Then the first equation in (3.2) yields $\mu = v_1$. Assuming $T = T_1 = \min$ the second equation of (3.2) leads to $v_2 = \alpha_2$. Hence the same solution $(\alpha_1, 1, \alpha_2)$ as in case II. Note that assuming $v_2 = 1$ again leads to a contradiction since then $\mu = \alpha_2$ and equation $\alpha_1 = T(\alpha_2, v_1)$ has no solution.

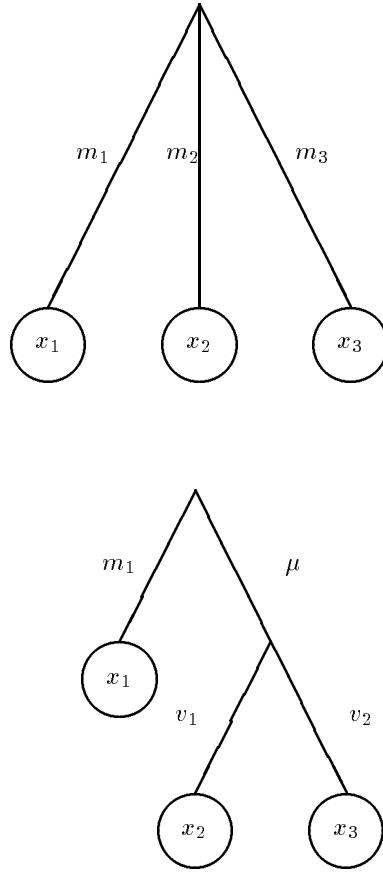


FIGURE 2. S -measure tree and the corresponding binary trees

$\max(1, \alpha_2) = 1$ the other two equations reduces on $\alpha_1 = \min(\alpha_1, 1)$, (or it can be considered the case with T_1 (we shall not examine this case) where we can take specially $T_1 = \min$) and so $\alpha_2 = \min(\alpha_1, \alpha_2)$. We have $v_1 = 1$ and $v_2 = \alpha_2$.

4. Hybrid mixtures

Let (S, T) be a pair of continuous t-conorm and t-norm, respectively, which satisfy the condition (CD). Then by Theorem 2.10 they are of the form

$$(\langle S_{\mathbf{M}}, S^* \rangle, \langle T_1, T^* \rangle)_a,$$

where S^* is a nilpotent t-conorm, T_1 an arbitrary t-norm and T^* a strict t-norm. We define the set Φ_S of ordered pairs (α, β) in the following way

$$\Phi_S = \{(\alpha, \beta) \mid (\alpha, \beta) \in [0, 1], S(\alpha, \beta) = 1\}.$$

DEFINITION 4.1. An extended mixture set is a quadruple (\mathcal{G}, M, T, S) where \mathcal{G} is a set and $M : \mathcal{G}^2 \times \Phi_S \rightarrow \mathcal{G}$ is a function (extended mixture operation) such that the following conditions are satisfied:

- M1.** $M(x, y; 1, 0) = x$;
- M2.** $M(x, y; \alpha, \beta) = M(y, x; \beta, \alpha)$;
- M3.** $M(M(x, y; \alpha, \beta), y; \gamma, \delta) = M(x, y; T(\alpha, \gamma), S(T(\beta, \gamma), T(\delta, 1)))$.

The conditions **M1-M3** imply the following condition

- M4.** $M(x, x; \alpha, \beta) = x$.

The following lemma is proved in [5].

LEMMA 4.2. Suppose M is an extended mixture, i.e., **M1.-M3.** holds for M . Then **M5.**

$M(M(x, y; \alpha, \beta), M(x, y; \gamma, \delta); \lambda, \mu) = M(x, y; S(T(\alpha, \lambda), T(\gamma, \mu)), S(T(\beta, \lambda), T(\delta, \mu)))$ holds for all $x, y \in \mathcal{G}$ and all $(\alpha, \beta), (\gamma, \delta), (\lambda, \mu) \in \Phi'$, Φ' is a non-empty subset of Φ_S , if and only if

$$T(\gamma, S(\alpha, \beta)) = S(T(\gamma, \alpha), T(\gamma, \beta)),$$

i.e., T is distributive over S on Φ' .

Let (S, T) be a pair of continuous t-conorm and t-norm, respectively, such that they satisfy the condition (CD) with $a \in [0, 1]$. We restrict ourselves on the situation $(\langle S_M, S_L \rangle, \langle T_1, T_P \rangle)_a$, since this is the most important case and other cases can be obtained by isomorphisms (see Theorems 2.6, 2.7).

We define the set $\Phi' = \Phi_{S,a}$ of ordered pairs (α, β) in the following way

$$\Phi_{S,a} = \{(\alpha, \beta) \mid (\alpha, \beta) \in]a, 1[, \alpha + \beta = 1 + a\} \cup \{(\alpha, \beta) \mid \min(\alpha, \beta) \leq a, \max(\alpha, \beta) = 1\}.$$

We have $\Phi_{S,a} \subset \Phi_S$. We have that for every $\alpha, \beta, \gamma \in \Phi_{S,a}$ the distributivity holds.

We define now the hybrid mixture set

DEFINITION 4.3. A hybrid mixture set is a quadruple (\mathcal{G}, M, T, S) where \mathcal{G} is a set, (S, T) is a pair of continuous t-conorm and t-norm, respectively, such that they satisfy the condition (CD) with $a \in [0, 1]$ and $M : \mathcal{G}^2 \times \Phi_S \rightarrow \mathcal{G}$ is a function (hybrid mixture operation) given by

$$M(x, y; \alpha, \beta) = S(T(\alpha, x), T(\beta, y)).$$

As we told we shall restrict to the case

$(\langle S_M, S_L \rangle, \langle T_1, T_P \rangle)_a$. Then it is easy to verify that M satisfies the axioms **M1-M5** on $\Phi_{S,a}$. This kind of mixtures exhaust the possible solutions to **M1-M5**.

Let (S, T) be a pair of continuous t-conorm and t-norm, respectively, of the form $(\langle S_M, S_L \rangle, \langle T_1, T_P \rangle)_a$. Let u_1, u_2 be two utilities taking values in the unit interval $[0, 1]$ and let μ_1, μ_2 be two degrees of plausibility from $\Phi_{S,a}$. Then we define the *optimistic hybrid utility function* by means of the hybrid mixture as

$$U(u_1, u_2; \mu_1, \mu_2) = S(T(u_1, \mu_1), T(u_2, \mu_2)).$$

We shall examine in details this utility function.

Case I. Let $\mu_1 > a, \mu_2 > a$, i.e., $\mu_1 + \mu_2 = 1 + a$. Then we have the following subcases:

(a) Let $u_1 > a, u_2 > a$. Then we have

$$(4.1) \quad U(u_1, u_2; \mu_1, \mu_2) = S\left(a + \frac{(u_1 - a)(\mu_1 - a)}{1 - a}, a + \frac{(u_2 - a)(\mu_2 - a)}{1 - a}\right).$$

Then $a + \frac{(u_i - a)(\mu_i - a)}{1 - a} > a$ for all $i = 1, 2$. Hence by (4.1)

$$U(u_1, u_2; \mu_1, \mu_2) = \frac{u_1(\mu_1 - a) + u_2(1 - \mu_1)}{1 - a}$$

(b) Let $u_1 \leq a, u_2 > a$. Then we have

$$\begin{aligned} U(u_1, u_2; \mu_1, \mu_2) &= S\left(u_1, a + \frac{(u_2 - a)(\mu_2 - a)}{1 - a}\right) \\ &= a + \frac{(u_2 - a)(\mu_2 - a)}{1 - a}. \end{aligned}$$

In a quite analogous way it follows for $u_1 > a, u_2 \leq a$ that

$$U(u_1, u_2; \mu_1, \mu_2) = a + \frac{(u_1 - a)(\mu_1 - a)}{1 - a}.$$

(c) Let $u_1 \leq a, u_2 \leq a$. Then

$$U(u_1, u_2; \mu_1, \mu_2) = \max(u_1, u_2).$$

Case II. Let $\mu_1 \leq a, \mu_2 = 1$ (in a quite analogous way we can consider the case $\mu_2 \leq a, \mu_1 = 1$). Then we have the following subcases, where $S = \max$:

(a) Let $u_1 > a, u_2 > a$. Then we have

$$U(u_1, u_2; \mu_1, \mu_2) = S(\mu_1, u_2) = u_2.$$

(b) Let $u_1 \leq a, u_2 > a$. Then we have

$$U(u_1, u_2; \mu_1, \mu_2) = S\left(aT_1\left(\frac{u_1}{a}, \frac{\mu_1}{a}\right), u_2\right) = u_2.$$

(c) Let $u_1 > a, u_2 \leq a$. Then we have

$$U(u_1, u_2; \mu_1, \mu_2) = S(\mu_1, u_2) = \max(\mu_1, u_2).$$

(d) Let $u_1 \leq a, u_2 \leq a$. Then we have

$$U(u_1, u_2; \mu_1, \mu_2) = \max\left(aT_1\left(\frac{u_1}{a}, \frac{\mu_1}{a}\right), u_2\right).$$

For $T_1 = \min$ the case II and case Ic are exactly idempotent (possibilistic) utility.

Although the above description of optimistic hybrid utility is rather complex, it can be easily explained, including the name optimistic.

Case I is when the decision-maker is very uncertain about the state of nature: both μ_1 and μ_2 are high and the two involved states have high plausibility. Case Ia is when the reward is high in both states- then the behavior of utility is probabilistic. Case Ib is when the reward is low in state $x_1 (u_1 \leq a)$, but high on the other state. Then the decision-maker looks forward to the best outcome and the utility is a function of u_2 and μ_2 only. In case Ic when both rewards are low, the decision-maker is possibilistic and again focuses on the best outcome. Case II is when state x_1 , is unlikely. In case IIa,b when the plausible reward is good, then the decision-maker looks forward to this reward. In case IIc where the most plausible reward is low then the decision maker still keeps some hope that state x_1 , will prevail if u_2 is really bad, but weakens the utility of state x_1 , because of it lack of plausibility. This phenomena subsides when the least plausible outcome is also bad, but the (bad) utility of x_1 , participates in the calculation of the resulting utility, by discounting μ_1 , even further. From the analysis, the optimistic attitude of an agent ranking decisions using the hybrid utility is patent.

We introduce the *pessimistic hybrid utility function* \bar{U} using the utility function U in the following way

$$\bar{U}(u_1, u_2; \mu_1, \mu_2) = 1 - U(1 - u_1, 1 - u_2; \mu_1, \mu_2).$$

Then we can give the corresponding interpretations of \bar{U} as dual interpretations to the preceding cases of U . Just go again through the above behavior analysis, interpreting u_1 and u_2 and \bar{U} as disutilities instead of utilities. For instance, in case IIa,b, the decision-maker is afraid that the worst outcome occurs ($u_2 > a$ is interpreted as penalty).

The following example from [7] of S -measure will be related to the construction of hybrid idempotent-probabilistic mixture.

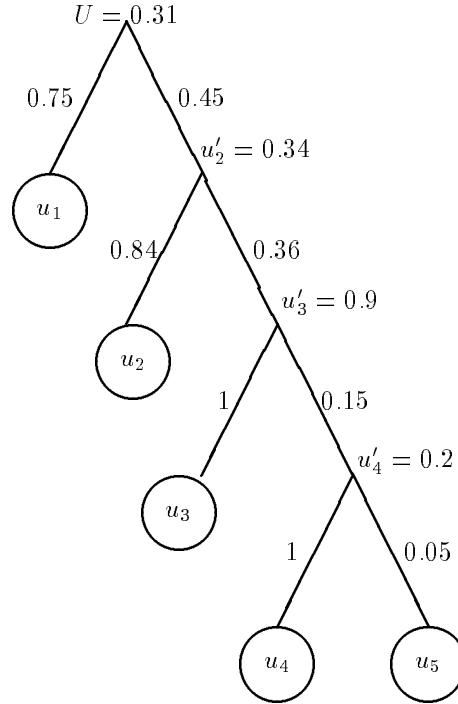


FIGURE 3. Utility binary trees from Example 4.4

EXAMPLE 4.4. We take $(\langle S_M, S_L \rangle, \langle T_1, T_P \rangle)_{0.2}$ and define an S -measure $m : \mathcal{P}(\{1, 2, 3, 4, 5\}) \rightarrow [0, 1]$ for the one point sets as:

$$m_1 = 0.75, m_2 = 0.4, m_3 = 0.25, m_4 = 0.15, m_5 = 0.05.$$

We see that $m_1 + m_2 + m_3 = 1 + 2a = 1.4$. Then the other values of S -measure m on $\mathcal{P}(\{1, 2, 3, 4, 5\})$ can be easily calculated. We construct the corresponding binary trees. It can be checked for instance that the weight of x_2 is $m_2 = T(0.45, 0.84)$.

For five utilities

$$u_1 = 0.3, u_2 = 0.1, u_3 = 0.9, u_4 = 0.2, u_5 = 0.1,$$

we can calculate the corresponding utility using the binary tree and we represent the procedure in Figure 3.

5. Axiomatization

It remains as a problem to find a corresponding axiomatization for hybrid utility as was done for classical utility theory and possibility utility theory. We shall give here an axiomatization obtained in [18].

We propose the following set of axioms for a preference relation \preceq defined over the set of all S -measures $\Delta(X)$, where X is the finite set of outcomes, to represent the optimistic utility:

H1. $\Delta(X)$ is equipped with a complete preordering structure, i.e., \preceq is reflexive, transitive and complete.

H2. (Continuity) If $m \prec m' \prec m''$ then

(i) $\exists \alpha \in]a, 1[$: $m' \sim M(m, m''; 1 + a - \alpha, \alpha)$, if $m, m', m'' > a$;

(ii) $\exists \alpha \in]0, 1[$: $m' \sim M(m, m''; 1, \alpha)$, otherwise.

H3. (Independence) For every $m, m', m'' \in \Delta(X)$ and for every $\alpha, \beta \in \Psi_{S,a}$ we have that $m' \preceq m''$ is equivalent with $M(m', m; \alpha, \beta) \preceq M(m'', m; \alpha, \beta)$.

H4. (Uncertainty prone) If $m, m' > a$ then

$m \preceq m'$ implies $m \preceq M(m, m'; \alpha, 1 + -\alpha) \preceq m'$ for $\alpha \in]a, 1[$;

otherwise $m < m'$ implies $m \prec m'$.

There was proved in [18] a representation theorem for the preference relation.

References

- [1] J. Aczél, *Lectures on Functional Equations and their Applications*, Academic Press, New York, 1969.
- [2] C. Bertoluzza, On the distributivity between t-norms and t-conorms, In Proc. Second Internat. Conference on Fuzzy Systems, FUZZ-IEEE '93, San Francisco, volume I, 140-147.
- [3] R. Cox, Probability, frequency, and reasonable expectation, *American Journal of Physics* **14** (1) (1946), 1-13.
- [4] D. Dubois, Generalized probabilistic independence and its implications for utility, *Operations Res. Letters* **5** (1985), 255-260.
- [5] D. Dubois, J. C. Fodor, H. Prade, M. Roubens, Aggregation of decomposable measures with applications to utility theory, *Theory and Decision* **41** (1996), 59-95.
- [6] D. Dubois, L. Godo, H. Prade, A. Zapico, Making Decision in a Qualitative Setting: from Decision under Uncertainty to Case-based Decision, Proceedings of the Sixth International Conference (Eds. A.G. Cohn, L. Schubert, S.C. Shapiro) "Principles of Knowledge Representation and Reasoning" (KR '98), Morgan Kaufman Publishers, Inc, San Francisco, 594-605.
- [7] D. Dubois, E. Pap, H. Prade, Hybrid probabilistic-possibilistic mixtures and utility functions, (Eds. J. Fodor, B. de Baets, P. Perny) "Preferences and Decisions under Incomplete Knowledge", volume 51 of *Studies in Fuzziness and Soft Computing*, Springer-Verlag, 2000, 51-73.
- [8] D. Dubois, E. Pap, H. Prade, Pseudo-additive measures and the independence of events, (Eds. B. Bouchon-Meuner, J. Gutiérrez-Rios, L. Magdalena, R.R. Yager) "Technologies for Constructing Intelligent Systems 1" volume 89 of *Studies in Fuzziness and Soft Computing*, Springer-Verlag, 2001, 179-191.
- [9] D. Dubois, H. Prade, The logical view of conditioning and its application to possibility and evidence theories, *Internat. J. of Approximate Reasoning* **4** (1) (1990), 23-46.
- [10] D. Dubois, H. Prade, Possibility Theory: Qualitative and Quantitative Aspects, Chapter in the Handbook of Defeasible Reasoning and Uncertainty Management Systems Vol I (Ed. Ph. Smets), Kluwer, Dordrecht, 1998, 169-226.
- [11] J. Y. Halpern, A Counterexample to Theorems of Cox and Fine, *Journal of Artif. Int. Research* **10** (1999), 67-85.

- [12] I. N. Herstein, J. Milnor, An axiomatic approach to measurable utility, *Econometrica* **21** (1953), 291-297.
- [13] E. P. Klement, R. Mesiar, E. Pap, *Triangular Norms*, Kluwer Academic Publishers, Dordrecht, 2000.
- [14] E. P. Klement, R. Mesiar, E. Pap, Integration with respect to decomposable measures, based on a conditionally distributive semiring on the unit interval, *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems* **8** (2000), 701-717.
- [15] K. Menger, Statistical metrics, *Proc. Nat. Acad. Sci. U.S.A.* **8** (1942), 535-537.
- [16] J. von Neumann, O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton Univ. Press, Princeton, NJ, 1944.
- [17] E. Pap, *Null-Additive Set Functions*, Kluwer Academic Publishers, Dordrecht and Ister Science, Bratislava, 1995.
- [18] E. Pap, M. Roca, An axiomatization of the hybrid probabilistic-possibilistic utility theory, 2002, preprint.
- [19] J. B. Paris, *The Uncertain Reasoner's Companion, A Mathematical Perspective*, Cambridge, U.K.: Cambridge University Press, 1994.
- [20] I. Rudas, Evolutionary operators: new parametric type operator families, *Int. J. of Fuzzy Systems* **23** (1999), 149-166.
- [21] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, North-Holland, Amsterdam, 1983.
- [22] G. Shafer, *The Art of Causal Conjecture*, The MIT Press, Cambridge, Massachusetts, London, 1996.
- [23] M. Sugeno, T. Murofushi, Pseudo-additive measures and integrals, *J. Math. Anal. Appl.* **122** (1987), 197-222.
- [24] S. Weber, \perp -decomposable measure and integrals for Archimedean t -conorms, *J. Math. Anal. Appl.* **101** (1984), 114-138.

DEPARTMENT OF MATHEMATICS AND INFORMATICS, UNIVERSITY OF NOVI SAD, 21000 NOVI SAD, SERBIA AND MONTENEGRO e-mail: pap@im.ns.ac.yu , pape@eunet.yu