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ALMOST BLOCK INDEPENDENCE AND BERNOULLICITY OF $\mathbb{Z}^d$-ACTIONS BY AUTOMORPHISMS OF COMPACT ABELIAN GROUPS

Daniel J. Rudolph and Klaus Schmidt

Abstract. We prove that a $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group is Bernoulli if and only if it has completely positive entropy. The key ingredients of the proof are the extension of certain notions of asymptotic block independence from $\mathbb{Z}$-actions to $\mathbb{Z}^d$-action and their equivalence with Bernoullicity, and a surprisingly close link between one of these asymptotic block independence properties for $\mathbb{Z}^d$-actions by automorphisms of compact, abelian groups and the product formula for valuations on global fields.

1. Introduction

Let $d \geq 1$. A measure-preserving $\mathbb{Z}^d$-action T on a probability space $(X, \mathfrak{S}, \mu)$ is *Bernoulli* if there exists a standard probability space $(Y, \mathfrak{T}, \nu)$ such that T is measurably conjugate to the shift-action σ of $\mathbb{Z}^d$ on $(Y^{\mathbb{Z}^d}, \mathfrak{T}^{\mathbb{Z}^d}, \nu^{\mathbb{Z}^d})$, where $\mathfrak{T}^{\mathbb{Z}^d}$ is the product Borel field on $Y^{\mathbb{Z}^d}$, and where σ is defined by

$$(\sigma_{\mathbf{m}}(y))_{\mathbf{n}} = y_{\mathbf{m}+\mathbf{n}} \tag{1.1}$$

for every $\mathbf{m} \in \mathbb{Z}^d$ and $y = (y_{\mathbf{n}}) = (y_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}^d) \in Y^{\mathbb{Z}^d}$. In particular T is Bernoulli if and only if there exists a countably generated sigma-algebra $\mathfrak{U} \subset \mathfrak{S}$ with the following properties:

(1) $\mathfrak{U}$ is *independent* under T, i.e.

$$\nu(B_0 \cap T_{-\mathbf{n}_1}(B_1) \cap \dots \cap T_{-\mathbf{n}_k}(B_k)) = \nu(B_0) \cdot \ldots \cdot \nu(B_k)$$

whenever $k \geq 1$, $B_0, \dots, B_k$ lie in $\mathfrak{U}$, and $\mathbf{0}, \mathbf{n}_1, \dots, \mathbf{n}_k$ are distinct elements in $\mathbb{Z}^d$,

(2) $\Sigma(\bigcup_{\mathbf{n}\in\mathbb{Z}^d} T_{-\mathbf{n}}(\mathfrak{U})) = \mathfrak{S} \pmod{\nu}$, where $\Sigma(\mathcal{C})$ is the sigma-algebra generated by a collection of sets $\mathcal{C} \subset \mathfrak{S}$.

If a (countably generated) sigma-algebra $\mathfrak{U} \subset \mathfrak{S}$ satisfies (1), but not necessarily (2), then $\mathfrak{V} = \Sigma(\bigcup_{\mathbf{n}\in\mathbb{Z}^d} T_{-\mathbf{n}}(\mathfrak{U}))$ is a *Bernoulli factor* of T.

Since Bernoulli actions of $\mathbb{Z}^d$ are measurably conjugate if and only if they have the same entropy ([OW]), the Bernoulli property plays an important rôle in the isomorphism theory of $\mathbb{Z}^d$-actions. There is, however, a major difference between the cases where $d = 1$ and where $d > 1$: there exist many 'natural' examples of Bernoulli transformations (such as hyperbolic diffeomorphisms of compact manifolds), but natural Bernoulli actions of $\mathbb{Z}^d$ with $d > 1$ are much harder to come by. One obvious reason for this difference is that Bernoulli actions (or even actions with positive entropy) of $\mathbb{Z}^d$, $d > 1$, cannot be realized smoothly on connected, finite-dimensional manifolds, so that such actions are less 'geometric' than for $d = 1$. The second reason is that, for a single measure preserving automorphism T of a probability space, Bernoullicity is characterized by certain independence properties of the 'future' and 'past' time

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evolutions defined by T. Appropriate notions of independence of 'future' and 'past' for $\mathbb{Z}^d$-actions are necessarily less intuitive and more difficult to verify (cf. e.g. [Kam]).

In this paper we consider a particular class of $\mathbb{Z}^d$-actions: the $\mathbb{Z}^d$-actions by automorphisms of compact, abelian groups. Although positive entropy again forces the abelian groups to be large (e.g. infinite-dimensional if they are connected), these actions can be analyzed effectively by replacing the geometric tools available on finite dimensional manifolds with methods from harmonic analysis and commutative algebra (cf. [KS], [LSW], [Sc1]).

For $d = 1$, the Bernoullicity of ergodic group automorphisms was established in a series of papers by Katznelson, Lind, Aoki, Thomas, and others (cf. e.g. [Aok], [AT], [Kat], [Li1], [Li2], [MT]). Whereas every ergodic automorphism of a compact group automatically has completely positive entropy ([Rok]), ergodic $\mathbb{Z}^d$-actions by automorphisms of compact, abelian groups with $d > 1$ need not have positive entropy, and one requires at least completely positive entropy to conjecture Bernoullicity. For $d = 2$, the Bernoullicity of a particular $\mathbb{Z}^2$-action with completely positive entropy was stated in [Wa1], and of arbitrary expansive $\mathbb{Z}^2$-actions by automorphisms of compact, abelian groups with completely positive entropy in [Wa2]. In this paper we prove the following general result conjectured in [LSW] (Conjecture 6.8).

1.1. Theorem. *Let $d \geq 1$, and let α be a $\mathbb{Z}^d$-action by automorphism of a compact, abelian group X. Then α is Bernoulli if and only if it has completely positive entropy with respect to the normalized Haar measure λ_X of X.*

As pointed out in [Sc2] and [Wa1], Theorem 1.1 implies the measurable conjugacy of certain topologically non-conjugate $\mathbb{Z}^d$-actions. For example, if $d > 1$, and if α is a $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X with completely positive entropy, then Theorem 1.1 guarantees that α is measurably conjugate to every $\mathbb{Z}^d$-action of the form $\alpha_A\colon \mathbf{n} \mapsto \alpha_{A\mathbf{n}}$ with $A \in \mathrm{GL}(d, \mathbb{Z})$, but α_A is not in general topologically conjugate to α (cf. [Sc1]).

The general approach of this paper follows the strategy of [Kat], adapted to $\mathbb{Z}^d$-actions along the lines of [Wa1] and [Wa2]. In Section 2 we present a characterization of Bernoullicity in terms of various asymptotic independence properties related to P. Shields' work on $\mathbb{Z}$-actions in [Shi] and prove their equivalence with Bernoullicity (Theorem 2.3). In Theorem 2.14 we show that Bernoullicity is also implied by another asymptotic independence condition associated with a former conjecture by A. Vershik (the summable Vershik condition), which appears particularly suited for symbolic and algebraic $\mathbb{Z}^d$-actions.

By using standard machinery from [KS] and [Sc1], the proof of Theorem 1.1 can be reduced to shift-actions of $\mathbb{Z}^d$ on closed, shift-invariant subgroups of $(\mathbb{T}^m)^{\mathbb{Z}^d}$ for some $m \geq 1$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. If $X \subset (\mathbb{T}^m)^{\mathbb{Z}^d}$ is such a subgroup, then λ_X can be viewed as a shift-invariant measure on $(\mathbb{T}^m)^{\mathbb{Z}^d}$. In Section 3 we deal with the case where $m = 1$ and λ_X is a shift-invariant measure on $\mathbb{T}^{\mathbb{Z}^d}$, and prove that under the assumption of completely positive entropy the projections of λ_X onto disjoint d-dimensional cubes in $\mathbb{Z}^d$ become rapidly independent as these cubes move apart (Lemma 3.6). Lemma 3.7 shows that this asymptotic independence of d-dimensional cubes implies the hypothesis of Theorem 2.14, so that λ_X is Bernoulli under the shift-action of $\mathbb{Z}^d$ on $\mathbb{T}^{\mathbb{Z}^d}$. There is an interesting connection between Lemma 3.6 and the product formula for valuations on algebraic number fields, which first emerged for $d = 1$ in discussions between D. Lind and the second author (KS) during the Symbolic Dynamics Programme at the MSRI, Berkeley, in 1992 (cf. [LS]—an early precursor of this idea appears in an unpublished note by Y. Katznelson on the Bernoullicity of ergodic automorphisms of certain solenoids). The main difference in the proofs of the relevant asymptotic independence in [Wa2] and Lemma 3.6 could be expressed (somewhat cryptically) by saying that the use of *all* places (not just the infinite ones) of certain algebraic number fields arising in the proof of that lemma obviates the need for assuming expansiveness. In addition the papers [Wa1] and [Wa2] contain a gap in the derivation of Bernoullicity from this asymptotic independence result which is filled by Section 2 in this paper.

Section 4 is also quite close to the relevant part of [Wa2], except that we use a more direct approach to derive the Bernoullicity of an arbitrary $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group from that of its basic constituent parts. The avoidance of some of the more sophisticated aspects of relative Bernoulli theory in this derivation is based on Theorem 2.11. There is a small amount of duplication of results from [Wa2] in Section 4 in order to make this exposition more self-contained.

2. Almost block independence and Bernoullicity of $\mathbb{Z}^d$-actions

In this section we extend a slightly weakened form of P. Shields' notion of 'almost block independence' to $\mathbb{Z}^d$-actions (cf. [Shi]). In order to put the essential ingredients of the following argument into clearer focus we present it in a slightly more general context which allows the reader to obtain with no extra effort the analogous result for those discrete, amenable groups which can be tiled by Følner sets. We begin with a few definitions.

2.1. Definition. For every $k \geq 1$ we set $B_k = \{-k, \dots, k\}^d \subset \mathbb{Z}^d$.

(1) Let $F \subset \mathbb{Z}^d$ be a non-empty, finite subset, and let $k \geq 1$ and $\varepsilon > 0$. The set F is (k, ε)-*invariant* if $|(F + \mathbf{n}) \cap F| \geq (1 - \varepsilon)|F|$ for every $\mathbf{n} \in B_k$.

(2) A sequence $(F_n,\ n \geq 1)$ of finite subsets of $\mathbb{Z}^d$ is a *Følner sequence* if $F_n \nearrow \mathbb{Z}^d$ as $n \to \infty$, and if there exists, for every $k \geq 1$ and $\varepsilon > 0$, an integer $N = N(k, \varepsilon)$ such that F_n is (k, ε)-invariant for every $n \geq N$.

(3) Let $F \subset \mathbb{Z}^d$ be a finite, non-empty subset, $\varepsilon > 0$ and $1 \leq k \leq K$. A collection $\mathcal{A}$ of disjoint, non-empty subsets of F is called a *partial cover* of F. If $\mathcal{A}$ is a partial cover of F we set $[\mathcal{A}] = \bigcup_{A \in \mathcal{A}} A$ and call $\mathcal{A}$ a (k, K, ε)-*cover* of F if the following conditions hold.

   (i) For every $A \in \mathcal{A}$ there exists an $\mathbf{n} \in \mathbb{Z}^d$ with $A \subset B_K + \mathbf{n}$,

   (ii) each $A \in \mathcal{A}$ is (k, ε)-invariant,

   (iii) $|[\mathcal{A}]| \geq (1 - \varepsilon)|F|$.

The simplest example of a Følner sequence in $\mathbb{Z}^d$ is obviously the sequence $(B_n,\ n \geq 1)$ itself. If a finite set $F \subset \mathbb{Z}^d$ is (k, ε)-invariant, then it is clear that

$$\begin{aligned} \bigl|\{\mathbf{m} \in F : B_k + \mathbf{m} \subset F\}\bigr| &\geq (1 - |B_k|\varepsilon)|F|, \\ |F + B_k| &\leq (1 + |B_k|\varepsilon)|F|. \end{aligned} \tag{2.1}$$

Let (Y,δ) be a compact, metric space with $\operatorname{diam}(Y)=\sup_{y,y'\in Y}\delta(y,y')=1$ and with Borel field $\mathfrak{B}_Y$, and let, for every non-empty set $F\subset\mathbb{Z}^d$, Y^F be the compact, metrizable space of all maps $x\colon F\longmapsto Y$, furnished with the product topology. If $\varnothing\neq F'\subset F\subset\mathbb{Z}^d$ then $\pi_{F'}\colon Y^F\longmapsto Y^{F'}$ denotes the restriction to F' of each element of Y^F (or, equivalently, the projection of each $x\in Y^F$ onto its coordinates in F').

For every non-empty set $F\subset\mathbb{Z}^d$ we write $M_1(Y^F)$ for the space of Borel probability measures on Y^F and note that $M_1(Y^F)$ is compact and metrizable in the weak*-topology. If $F=\mathbb{Z}^d$ and σ is the shift-action (1.1) of $\mathbb{Z}^d$ on $Y^{\mathbb{Z}^d}$ then we denote by $M_1(Y^{\mathbb{Z}^d})^\sigma\subset M_1(Y^{\mathbb{Z}^d})$ the set of shift-invariant probability measures on $Y^{\mathbb{Z}^d}$, which is again compact in the weak*-topology.

If $\varnothing\neq F'\subset F\subset\mathbb{Z}^d$ then the coordinate projection $\pi_{F'}\colon Y^F\longmapsto Y^{F'}$ induces a continuous, surjective map $\mu\mapsto\mu_{F'}=\mu\pi_{F'}^{-1}$ from $M_1(Y^F)$ to $M_1(Y^{F'})$, defined by

$$\mu_{F'}(B)=\mu\pi_{F'}^{-1}(B)=\mu\pi_{F'}^{-1}(B) \tag{2.2}$$

for every $\mu\in M_1(Y^F)$ and every $B\in\mathfrak{B}_{Y^{F'}}$, where $\mathfrak{B}_{Y^{F'}}$ is the product Borel field of $Y^{F'}$.

For every finite or countably infinite partition $\mathcal{A}$ of a set $F\subset\mathbb{Z}^d$ and every $\mu\in M_1(Y^F)$ we define a probability measure $\mu^{\mathcal{A}}\in M_1(Y^F)$ by

$$\mu^{\mathcal{A}}=\prod_{A\in\mathcal{A}}\mu_A, \tag{2.3}$$

where μ_A is described in (2.2), and where $\prod_{A\in\mathcal{A}}\mu_A$ is the product measure on $Y^F\cong\prod_{A\in\mathcal{A}}Y^A$ of the measures μ_A, $A\in\mathcal{A}$. If $\mathcal{A}'$ is a partition of F which refines $\mathcal{A}$, then

$$(\mu^{\mathcal{A}})^{\mathcal{A}'}=(\mu^{\mathcal{A}'})^{\mathcal{A}}=\mu^{\mathcal{A}'}.$$

If $\mathcal{A}$ is a partial cover of F we set $A_\infty=F\smallsetminus[\mathcal{A}]$, denote by $\mathcal{A}_\infty=\mathcal{A}\cup\{A_\infty\}$ the partition of F obtained by adding A_∞ to $\mathcal{A}$, and define $\mu^{\mathcal{A}}$ by

$$\mu^{\mathcal{A}}=\mu^{\mathcal{A}_\infty}\in M_1(Y^F) \tag{2.4}$$

for every $\mu\in M_1(Y^F)$.

Motivated by P. Shields we write, for every $F\subset\mathbb{Z}^d$ and $\mu_1,\mu_2\in M_1(Y^F)$,

$$C(\mu_1,\mu_2)=\{\nu\in M_1(Y^F\times Y^F):\nu(B\times Y^F)=\mu_1(B)\ \text{and}\ \nu(Y^F\times B)=\mu_2(B)\\ \text{for every Borel set}\ B\subset Y^F\}$$

for the set of *couplings* of μ_1 and μ_2. Note that $C(\mu_1,\mu_2)$ is a closed (and hence compact) subset of $M_1(Y^F\times Y^F)$ which consists of all probability measures on $Y^F\times Y^F$ whose projections onto the two coordinates coincide with μ_1 and μ_2, respectively. For $F=\mathbb{Z}^d$ and $\mu_1,\mu_2\in M_1(Y^{\mathbb{Z}^d})^\sigma$ we denote by

$$J(\mu_1,\mu_2)=C(\mu_1,\mu_2)\cap M_1(Y^{\mathbb{Z}^d}\times Y^{\mathbb{Z}^d})^{\sigma\times\sigma}$$

for the set of *joinings* (or $(\sigma\times\sigma)$-invariant couplings) of measures in $M_1(Y^{\mathbb{Z}^d})^\sigma$.

In order to define the $\bar{d}$-metric on $M_1(Y^F)$, $F \subset \mathbb{Z}^d$, we write a typical element $z \in Y^F \times Y^F \cong (Y \times Y)^F$ as $z = ((z_{\mathbf{n}}^{(1)}, z_{\mathbf{n}}^{(2)}), \mathbf{n} \in F)$ with $z_{\mathbf{n}}^{(i)} \in Y$ for every $\mathbf{n} \in F$ and $i = 1, 2$, and set $\pi_{\{\mathbf{n}\}}^{(i)} \colon z \mapsto z_{\mathbf{n}}^{(i)}$ for every $\mathbf{n} \in F$, $i = 1, 2$. If F is finite and $\mu_1, \mu_2 \in M_1(Y^F)$, put

$$\bar{d}_F(\mu_1, \mu_2) = \inf_{\nu \in C(\mu_1, \mu_2)} \frac{1}{|F|} \sum_{\mathbf{n} \in F} \int_{Y^F \times Y^F} \delta\big(\pi_{\{\mathbf{n}\}}^{(1)}, \pi_{\{\mathbf{n}\}}^{(2)}\big)\, d\nu. \tag{2.5}$$

Since $C(\mu_1, \mu_2)$ is compact, the infimum in (2.5) is actually a minimum. If F_1, F_2 are finite subsets of $\mathbb{Z}^d$, $F \subset F_1 \cap F_2$, and $\mu_i \in M_1(Y^{F_i})$, then we set

$$\bar{d}_F(\mu_1, \mu_2) = \bar{d}_F(\mu_1 \pi_F^{-1}, \mu_2 \pi_F^{-1}).$$

For $F = \mathbb{Z}^d$ and $\mu_1, \mu_2 \in M_1(Y^{\mathbb{Z}^d})^\sigma$ we put

$$\bar{d}(\mu_1, \mu_2) = \limsup_{M \to \infty} \bar{d}_{B_M}(\mu_1, \mu_2) = \min_{\nu \in J(\mu_1, \mu_2)} \int \delta(\pi_{\{\mathbf{0}\}}^{(1)}, \pi_{\{\mathbf{0}\}}^{(2)})\, d\nu, \tag{2.6}$$

where the last identity follows from the ergodic theorem. If μ_1 and μ_2 are ergodic under σ the minimum in (2.6) can be replaced by the minimum over all $(\sigma \times \sigma)$-ergodic elements of $J(\mu_1, \mu_2)$.

For every non-empty, finite subset $F \subset \mathbb{Z}^d$ the map $\bar{d}_F \colon M_1(Y^F) \times M_1(Y^F) \longmapsto \mathbb{R}$ has the following properties.

(1) $\bar{d}_F$ is a metric on $M_1(Y^F)$ which induces the weak*-topology on $M_1(Y^F)$;

(2) If $F' \subset F$ then

$$\frac{|F'|}{|F|} \bar{d}_{F'}(\mu_1, \mu_2) \leq \bar{d}_F(\mu_1, \mu_2) \leq \frac{|F'|}{|F|} \bar{d}_{F'}(\mu_1, \mu_2) + \left(1 - \frac{|F'|}{|F|}\right) \tag{2.7}$$

for all μ_1, μ_2 in $M_1(Y^F)$;

(3) If Y is finite and $\mu_1, \mu_2 \in M_1(Y^F)$ then there exists an element $\nu \in C(\mu_1, \mu_2)$ with $\nu(\{(z, z)\}) = \min\{\mu_1(\{z\}), \mu_2(\{z\})\}$ for every $z \in Y^F$, and

$$\bar{d}_F(\mu_1, \mu_2) = \frac{1}{2} \sum_{z \in Y^F} |\mu_1(\{z\}) - \mu_2(\{z\})|. \tag{2.8}$$

For every $N \geq 0$ we denote by

$$\mathcal{B}_N = \{B_N + \mathbf{n} : \mathbf{n} \in (2N+1)\mathbb{Z}^d\} \tag{2.9}$$

the tiling of $\mathbb{Z}^d$ by translates of B_N. We are ready for our basic definitions.

2.2. Definition. Let $\mu \in M_1(Y^{\mathbb{Z}^d})^\sigma$.

(1) μ is *sporadically almost block independent* (*sporadically a.b.i.*) if there exist, for every $\varepsilon > 0$, an integer $K \geq 0$, a Følner sequence $(F_n, n \geq 1)$ in $\mathbb{Z}^d$ and, for each $n \geq 1$, a partition $\mathcal{A}^{(n)}$ of F_n such that each $A \in \mathcal{A}^{(n)}$ is contained in some translate of B_K and

$$\bar{d}_{F_n}(\mu, \mu^{\mathcal{A}^{(n)}}) < \varepsilon.$$

(2) μ is *universally almost block independent* (*universally a.b.i.*) if there exist, for every $\varepsilon > 0$, an integer $k \geq 0$ and an $\varepsilon' > 0$ with

$$\bar{d}_{B_N}(\mu, \mu^{\mathcal{A}}) < \varepsilon$$

for every $K, N \geq 1$, and for every (k, K, ε')-cover $\mathcal{A}$ of B_N.

(3) μ is *almost box independent* if

$$\lim_{N\to\infty} \limsup_{M\to\infty} \bar{d}_{B_M}(\mu, \mu^{\mathcal{B}_N}) = 0.$$

(4) μ is *Bernoulli* if the $\mathbb{Z}^d$-action σ on $(Y^{\mathbb{Z}^d}, \mathfrak{B}_{Y^{\mathbb{Z}^d}}, \mu)$ is Bernoulli.

We remark in passing that almost box independence is very closely related to Shields' one-dimensional concept of almost block independence in [Shi].

2.3. Theorem. *The following conditions are equivalent for every shift-invariant probability measure μ on $Y^{\mathbb{Z}^d}$.*

(1) *μ is almost box independent;*
(2) *μ is sporadically a.b.i.;*
(3) *μ is universally a.b.i.;*
(4) *μ is Bernoulli.*

We begin the proof of Theorem 2.3 by collecting a few basic facts about Bernoulli measures from [OW].

2.4. Lemma ([OW]). *If Y is finite then the Bernoulli measures are $\bar{d}$-closed in $M_1(Y^{\mathbb{Z}^d})^\sigma$.*

2.5. Lemma ([OW]). *Let T be a measure preserving $\mathbb{Z}^d$-action on a probability space $(X, \mathfrak{S}, \mu)$ which is Bernoulli. Then every T-invariant sigma-algebra in $\mathfrak{S}$ is a Bernoulli factor of T.*

2.6. Lemma ([OW]). *Let T be a measure preserving $\mathbb{Z}^d$-action on a probability space $(X, \mathfrak{S}, \mu)$. If there exists a sequence $(\mathfrak{V}_n,\, n \geq 1)$ of T-invariant sigma-algebras in $\mathfrak{S}$ such that $\mathfrak{V}_n \nearrow \mathfrak{S}$ as $n \to \infty$ and each $\mathfrak{V}_n$ is a Bernoulli factor of μ, then μ is Bernoulli.*

2.7. Lemma. *The set of Bernoulli measures in $M_1(Y^{\mathbb{Z}^d})^\sigma$ is $\bar{d}$-closed (cf. (2.6)).*

Proof. Suppose that μ is the $\bar{d}$-limit of a sequence (μ_j) of Bernoulli measures in $M_1(Y^{\mathbb{Z}^d})^\sigma$. We choose an increasing sequence $(\mathcal{P}_n,\, n \geq 1)$ of finite Borel partitions of Y such that, for every $n \geq 1$ and $P \in \mathcal{P}_n$, $\operatorname{diam}(P) = \sup_{y,y' \in P} \delta(y, y') \leq 1/n$ and $\mu(\partial P) = 0$, where ∂P is the boundary of P. For each $n \geq 1$ we choose a finite set $H_n \subset Y$ which intersects each $P \in \mathcal{P}_n$ in exactly one point and define a map $\eta_n \colon Y \longmapsto H_n$ by demanding that $\eta_n(y) \in P \cap H_n$ for every $y \in P \in \mathcal{P}_n$. Denote by $\boldsymbol{\eta}_n \colon Y^{\mathbb{Z}^d} \longmapsto H_n^{\mathbb{Z}^d}$ the map obtained by setting, for every $x = (x_{\mathbf{m}}) \in Y^{\mathbb{Z}^d}$, $\boldsymbol{\eta}_n(x) = (\eta_n(x_{\mathbf{m}}),\, \mathbf{m} \in \mathbb{Z}^d)$. Our choice of $\mathcal{P}_n$ implies that $\lim_{j\to\infty} \mu_j(P) = \mu(P)$ for every $n \geq 1$ and every $P \in \mathcal{P}_n$, and that $\lim_{j\to\infty} \bar{d}(\mu_j \boldsymbol{\eta}_n^{-1}, \mu \boldsymbol{\eta}_n^{-1}) = 0$ for every $n \geq 1$. By applying the Lemmas 2.4–2.5 we see that $\mu \boldsymbol{\eta}_n^{-1}$ is Bernoulli for every $n \geq 1$, and Lemma 2.6 allows us to conclude that μ is Bernoulli. □

2.8. Lemma. *If $\mu \in M_1(Y^{\mathbb{Z}^d})^\sigma$ is almost box independent then it is Bernoulli.*

Proof. We prove that μ is Bernoulli by finding a sequence $(\mu_j,\, j \geq 1)$ of Bernoulli measures in $M_1(Y^{\mathbb{Z}^d})^\sigma$ with $\bar{d}(\mu_j, \mu) < \frac{1}{j}$ for every $j \geq 1$, and by applying Lemma 2.7.

In order to construct the measure μ_j for a given $j \geq 1$ we use the almost box independence of μ to find a $K \geq 1$ with

$$\limsup_{L \to \infty} \bar{d}_{B_L}(\mu, \mu^{\mathcal{B}_K}) < \frac{1}{4j},$$

where $\mu^{\mathcal{B}_K}$ is defined in (2.3) and (2.9). Next we set $M = L + K$, where $L > 4j$ satisfies that B_L is a union of boxes in $\mathcal{B}_K$, $\bar{d}_{B_L}(\mu, \mu^{\mathcal{B}_K}) < \frac{1}{4j}$ and $|B_{L+K}| < (1 + \frac{1}{4j})|B_L|$ (cf. (2.1)).

Put $\Xi = \{0,1\}^{\mathbb{Z}^d}$, denote by ν the equidistributed Bernoulli measure on Ξ, and write σ' for the shift-action (1.1) of $\mathbb{Z}^d$ on Ξ. We choose a Borel set $C \subset \Xi$ such that $\nu(\bigcup_{\mathbf{n} \in B_M} \sigma'_{\mathbf{n}}(C)) > 1 - \frac{1}{4j}$, and $\sigma'_{\mathbf{n}}(C) \cap \sigma'_{\mathbf{n}'}(C) = \varnothing$ whenever $\mathbf{n}, \mathbf{n}'$ are distinct elements of B_M (in other words, C is the base of a Rokhlin tower of size B_M for σ' which covers $1 - \frac{1}{4j}$ of the space Ξ). Let $Z = Y^{B_M}$, $\mu' = (\mu_{B_M})^{\mathbb{Z}^d} \in M_1(Z^{\mathbb{Z}^d})$, and let σ'' be the shift-action (1.1) of $\mathbb{Z}^d$ on $Z^{\mathbb{Z}^d}$. The product-action $T\colon \mathbf{n} \mapsto \sigma'_{\mathbf{n}} \times \sigma''_{\mathbf{n}}$ of $\mathbb{Z}^d$ on the product space $(\Xi \times Z^{\mathbb{Z}^d}, \mathfrak{B}_\Xi \otimes \mathfrak{B}_{Z^{\mathbb{Z}^d}}, \nu \times \mu')$ is obviously Bernoulli, since it is the cartesian product of two Bernoulli actions of $\mathbb{Z}^d$. The proof of the lemma will be completed by finding a Borel map $\boldsymbol{\phi}\colon \Xi \times Z^{\mathbb{Z}^d} \longmapsto Y^{\mathbb{Z}^d}$ with $\boldsymbol{\phi} \cdot T_{\mathbf{n}} = \sigma_{\mathbf{n}} \cdot \boldsymbol{\phi}$ for every $\mathbf{n} \in \mathbb{Z}^d$, and such that $\mu_j = (\nu \times \mu')\boldsymbol{\phi}^{-1}$ satisfies that $\bar{d}(\mu, \mu_j) < \frac{1}{j}$. According to Lemma 2.5, μ_j is Bernoulli.

In order to construct the map $\boldsymbol{\phi}$ we write a typical element $w \in W = \Xi \times Z^{\mathbb{Z}^d}$ as $w = (x, v)$ with $x = (x_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}^d) \in \Xi$ and $v = (v_{(\mathbf{m},\mathbf{n})}, \mathbf{m} \in B_M, \mathbf{n} \in \mathbb{Z}^d) \in Z^{\mathbb{Z}^d}$, where $x_{\mathbf{n}} \in \{0,1\}$ and $v_{(\mathbf{m},\mathbf{n})} \in Y$ for every $\mathbf{m} \in B_M$ and $\mathbf{n} \in \mathbb{Z}^d$. Fix a point $\bar{y} \in Y$ and define a map $\phi\colon W \longmapsto Y$ by setting, for every $w = (x, v) \in W$,

$$\phi(w) = \begin{cases} \bar{y} & \text{if } x \in \Xi \smallsetminus \bigcup_{\mathbf{m} \in B_M} \sigma'_{\mathbf{m}}(C), \\ v_{(\mathbf{m},-\mathbf{m})} & \text{if } x \in \sigma'_{\mathbf{m}}(C) \text{ for some } \mathbf{m} \in B_M, \end{cases}$$

and denote by $\boldsymbol{\phi}\colon W \longmapsto Y^{\mathbb{Z}^d}$ the map $\boldsymbol{\phi}(w) = (\boldsymbol{\phi}(w)_{\mathbf{n}}) \in Y^{\mathbb{Z}^d}$ with

$$\boldsymbol{\phi}(w)_{\mathbf{n}} = \phi(T_{\mathbf{n}}(w))$$

for every $w \in W$ and $\mathbf{n} \in \mathbb{Z}^d$.

We can interpret $\boldsymbol{\phi}$ geometrically by considering, for each $x \in \Xi$, an array of windows of size B_M in $\mathbb{Z}^d$, whose centres coincide with the set of all $\mathbf{n} \in \mathbb{Z}^d$ for which $\sigma'_{\mathbf{n}}(x) \in C$. Inside each window we see a copy of Y^{B_M} on which $(\nu \times \mu')\boldsymbol{\phi}^{-1}$ induces the measure μ_{B_M}. The distributions in different windows are independent of each other, and are independent of the placement of these windows. The coordinates of $\boldsymbol{\phi}(w)$ which don't lie in any of these windows are all equal to $\bar{y}$.

For every $x \in \Xi$ we set

$$\begin{aligned} R(x) &= \{\mathbf{m} \in \mathbb{Z}^d : \sigma'_{\mathbf{m}}(x) \in C\}, \\ \bar{R}(x) &= \bigcup_{\mathbf{m} \in R(x)} B_M + \mathbf{m}, \\ \Omega(x) &= \{(y_{\mathbf{n}}) \in Y^{\mathbb{Z}^d} : y_{\mathbf{n}} = \bar{y} \text{ for every } \mathbf{n} \in \mathbb{Z}^d \smallsetminus \bar{R}(x)\} \\ &\cong \prod_{\mathbf{m} \in R(x)} Y^{B_M + \mathbf{m}} \times \prod_{\mathbf{m} \in \mathbb{Z}^d \smallsetminus \bar{R}(x)} \{\bar{y}\}, \end{aligned}$$

and define a continuous, injective map $\tau_x\colon \Omega(x) \longmapsto W$ by setting, for every $v \in \Omega(x)$, $\tau_x(v) = (x, (\pi_{B_M}(\sigma_{\mathbf{n}}(v)), \mathbf{n} \in \mathbb{Z}^d))$. The composition $\boldsymbol{\phi} \cdot \tau_x\colon \Omega(x) \longmapsto Y^{\mathbb{Z}^d}$ is injective for every $x \in X$.

Let $\nu^{(x)}$ be the probability measure on $\Omega(x) \cong \prod_{\mathbf{m}\in R(x)} Y^{B_M+\mathbf{m}} \times \prod_{\mathbf{m}\in\mathbb{Z}^d\smallsetminus\bar{R}(x)}\{\bar{y}\}$ corresponding to the product measure obtained by placing a copy of μ_{B_M} on each $Y^{B_M+\mathbf{m}}$ with $\mathbf{m}\in R(x)$ and the point mass at $\bar{y}$ on each coordinate $\mathbf{m}\in\mathbb{Z}^d\smallsetminus\bar{R}(x)$, and denote by $\mu^{(x)}\in M_1(Y^{\mathbb{Z}^d})$ the image of $\nu^{(x)}$ under the map $\boldsymbol{\phi}\cdot\tau_x\colon\Omega(x)\longmapsto Y^{\mathbb{Z}^d}$. Then

$$\mu_j(B) = (\nu\times\mu')\boldsymbol{\phi}^{-1}(B) = \int_{x\in\Xi}\mu^{(x)}(B)\,d\nu(x) \tag{2.10}$$

for every Borel set $B\subset Y^{\mathbb{Z}^d}$.

In every translate $B_M+\mathbf{m}$, $\mathbf{m}\in\mathbb{Z}^d$, of B_M we can find a unique element $\mathbf{n}(\mathbf{m})\in\mathbb{Z}^d$ such that $B_L+\mathbf{n}(\mathbf{m})\subset B_M+\mathbf{m}$ and $B_L+\mathbf{n}(\mathbf{m})$ is a precise union of tiles in $\mathcal{B}_K$. As μ is σ-invariant,

$$\bar{d}_{B_L+\mathbf{n}(\mathbf{m})}(\mu,\mu^{\mathcal{B}_K}) < \frac{1}{4j}.$$

We fix a point $x\in\Xi$, consider in each of the windows $B_M+\mathbf{m}$, $\mathbf{m}\in R(x)$, the corresponding precise union $B_L+\mathbf{n}(\mathbf{m})$ of tiles in $\mathcal{B}_K$, and set

$$\bar{R}(x)' = \bigcup_{\mathbf{m}\in R(x)} B_L+\mathbf{n}(\mathbf{m}).$$

Since what appears in each of the windows $B_L+\mathbf{n}(\mathbf{m})$, $\mathbf{m}\in R(x)$, is independent both for $\mu^{\mathcal{B}_K}$ and $\mu^{(x)}$,

$$\limsup_{N\to\infty}\bar{d}_{\bar{R}(x)'\cap B_N}(\mu^{\mathcal{B}_K},\mu^{(x)}) < \frac{1}{4j}.$$

Birkhoff's individual ergodic theorem, applied to σ' on $(\Xi,\mathfrak{B}_\Xi,\nu)$, shows that, for ν-a.e. $x\in\Xi$,

$$\limsup_{N\to\infty}\frac{|\bar{R}(x)'\cap B_N|}{|B_N|}\ge 1-\frac{1}{2j}$$

and (2.7) yields that

$$\limsup_{N\to\infty}\bar{d}_{B_N}(\mu^{\mathcal{B}_K},\mu^{(x)}) < \frac{3}{4j}.$$

By integrating over $x\in\Xi$ in (2.10) we see that

$$\bar{d}(\mu,\mu_j) = \lim_{N\to\infty}\bar{d}_{B_N}(\mu,\mu_j) < \frac{1}{j}. \quad\square$$

2.9. Lemma. *If $\mu\in M_1(Y^{\mathbb{Z}^d})^\sigma$ is sporadically a.b.i. then it is universally a.b.i.*

Proof. We fix $\varepsilon>0$. As μ is sporadically a.b.i., there exist a $K_0\in\mathbb{N}$, a Følner sequence $(F_n,\ n\ge 1)$ in $\mathbb{Z}^d$ and, for each $n\ge 1$, a partition $\mathcal{A}^{(n)}$ of F_n such that each $A\in\mathcal{A}^{(n)}$ is contained in some translate of B_{K_0} and $\bar{d}_{F_n}(\mu,\mu^{\mathcal{A}^{(n)}})<\frac{\varepsilon}{10}$. Let K,N be arbitrary elements of $\mathbb{N}$, and let $\mathcal{B}$ be a $(2K_0,K,\frac{\varepsilon}{10|B_{2K_0}|})$-cover of B_N (for such a cover to exist, K and N obviously have to be sufficiently large). We wish to show that

$$\bar{d}_{B_N}(\mu,\mu^{\mathcal{B}})<\varepsilon.$$

Fix an integer n which is so large that F_n is $(2N,\frac{\varepsilon}{10|B_{2N}|})$-invariant. Then (2.1) shows that there exists a subset $F\subset F_n$ which is a union of boxes in $\mathcal{B}_N$, and which satisfies that

$$|F| > \left(1-\frac{\varepsilon}{10}\right)|F_n|.$$

We write F as

$$F = \bigcup_{\mathbf{n} \in E} B_N + \mathbf{n} \tag{2.11}$$

with $E \subset (2N+1)\mathbb{Z}^d$ and define a partial cover

$$\mathcal{C} = \{B + \mathbf{n} : B \in \mathcal{B}, \, \mathbf{n} \in E\}$$

of F_n consisting of all translates of the sets in $\mathcal{B}$ by elements of E. Since each $C \in \mathcal{C}$ is $(2K_0, \frac{\varepsilon}{10|B_{2K_0}|})$-invariant, (2.1) shows that

$$\left|\{\mathbf{n} \in C : B_{K_0} + \mathbf{n} \not\subset C\}\right| < \frac{\varepsilon}{10}|C|$$

for every $C \in \mathcal{C}$.

By deleting any set in the partial cover $\mathcal{A}^{(n)}$ which is not completely contained in some $C \in \mathcal{C}$ we obtain a new partial cover $\mathcal{A}'$ of F_n, and we set $\mathcal{C}' = \{C \cap [\mathcal{A}'] : C \in \mathcal{C}\}$. Since each $A \in \mathcal{A}^{(n)}$ is contained in a translate of B_{K_0}, (2.1) and the $(2K_0, \frac{\varepsilon}{10|B_{K_0}|})$-invariance of each $C \in \mathcal{C}$ guarantee that

$$|[\mathcal{A}']| > \left(1 - \frac{\varepsilon}{10}\right)|F_n|,$$

$$\bar{d}_{F_n}(\mu, \mu^{\mathcal{A}'}) < \frac{2\varepsilon}{10}, \qquad \bar{d}_{[\mathcal{A}']}(\mu, \mu^{\mathcal{A}'}) < \frac{4\varepsilon}{10},$$

and hence, since $\mathcal{A}'$ refines $\mathcal{C}'$ as a partition of $[\mathcal{A}']$,

$$\bar{d}_{[\mathcal{A}']}(\mu^{\mathcal{C}'}, \mu^{\mathcal{A}'}) = \bar{d}_{[\mathcal{A}']}(\mu^{\mathcal{C}'}, (\mu^{\mathcal{A}'})^{\mathcal{C}'}) < \frac{4\varepsilon}{10}$$

and

$$\bar{d}_{[\mathcal{A}']}(\mu, \mu^{\mathcal{C}'}) < \frac{8\varepsilon}{10}.$$

Furthermore, since $|[\mathcal{A}']| > (1 - \frac{\varepsilon}{10})|F|$, (2.7) shows that

$$\bar{d}_F(\mu, \mu^{\mathcal{C}}) < \frac{9\varepsilon}{10} < \varepsilon,$$

and (2.11) implies that there exists an $\mathbf{n} \in E$ with

$$\bar{d}_{B_N + \mathbf{n}}(\mu, \mu^{\mathcal{B}+\mathbf{n}}) < \varepsilon,$$

where $\mathcal{B} + \mathbf{n} = \{B + \mathbf{n} : B \in \mathcal{B}\} = \{C \cap (B_N + \mathbf{n}) : C \in \mathcal{C}\}$. As μ is σ-invariant, translation by $\mathbf{n} \in E$ does not affect the last equation, so that

$$\bar{d}_{B_N}(\mu, \mu^{\mathcal{B}}) < \varepsilon.$$

By setting $\varepsilon' = \frac{\varepsilon}{10|B_{2K_0}|}$ we obtain that μ is universally a.b.i., as claimed. $\square$

Proof of Theorem 2.3. The implications (3)⇒(1) and (3)⇒(2) are obvious from Definition 2.2. From Lemma 2.8 we know that (1)⇒(4) and Lemma 2.9 yields that (2)⇒(3). If Y is finite, then the remaining implication (4)⇒(2) is an elementary coding exercise; for infinite Y one needs in addition the approximation argument from the proof of Lemma 2.7. □

All three definitions (sporadic a.b.i., universal a.b.i. and almost box independence) make sense for arbitrary measures $\mu \in M_1(Y^{\mathbb{Z}^d})$. In this generality, however, only almost box independence has good dynamical implications: it allows us to formulate a version of 'relative Bernoullicity' for not necessarily shift-invariant measures.

Suppose that (Z, δ') is a compact, metric space, and let $\sigma = \sigma^{(Y)}$, $\sigma' = \sigma^{(Z)}$ and $T = \sigma \times \sigma'$ be the shift-actions (1.1) of $\mathbb{Z}^d$ on $Y^{\mathbb{Z}^d}$, $Z^{\mathbb{Z}^d}$ and $(Y \times Z)^{\mathbb{Z}^d}$. We write a typical element of $(Y \times Z)^{\mathbb{Z}^d} \cong Y^{\mathbb{Z}^d} \times Z^{\mathbb{Z}^d}$ as (y, z) with $y = (y_{\mathbf{n}}) \in Y^{\mathbb{Z}^d}$ and $z = (z_{\mathbf{n}}) \in Z^{\mathbb{Z}^d}$ and denote by $\pi^{(Y)}(y, z) = y$ and $\pi^{(Z)}(y, z) = z$ the projections of a point in $(Y \times Z)^{\mathbb{Z}^d}$ onto its 'coordinates' in $Y^{\mathbb{Z}^d}$ and $Z^{\mathbb{Z}^d}$. Fix a measure $\mu \in M_1((Y \times Z)^{\mathbb{Z}^d})^T$, set $\mu^{(Z)} = \mu\,(\pi^{(Z)})^{-1} \in M_1(Z^{\mathbb{Z}^d})^{\sigma'}$, and apply standard decomposition theory to find a Borel map $z \mapsto \mu_z$ from $Z^{\mathbb{Z}^d}$ to $M_1(Y^{\mathbb{Z}^d})$ such that

$$\int h\, d\mu = \int_{Z^{\mathbb{Z}^d}} \int_{Y^{\mathbb{Z}^d}} h(y, z)\, d\mu_z(y)\, d\mu^{(Z)}(z) \tag{2.12}$$

for every continuous map $h: Y^{\mathbb{Z}^d} \times Z^{\mathbb{Z}^d} \longmapsto \mathbb{R}$, and

$$\mu_{\sigma_{\mathbf{n}}^{(Z)}(z)} = \mu_z \sigma_{-\mathbf{n}}^{(Y)} \tag{2.13}$$

for every $z \in Z^{\mathbb{Z}^d}$ and $\mathbf{n} \in \mathbb{Z}^d$.

2.10. Definition. The measure $\mu \in M_1((Y \times Z)^{\mathbb{Z}^d})^T$ is *relatively almost box independent (relatively almost box independent) with respect to* $Z^{\mathbb{Z}^d}$ if, for $\mu^{(Z)}$-a.e. $z \in Z^{\mathbb{Z}^d}$,

$$\lim_{N \to \infty} \limsup_{M \to \infty} \bar{d}_{B_M}(\mu_z, \mu_z^{\mathcal{B}_N}) = 0.$$

Relative Bernoulli theory leads one to expect that the $\mathbb{Z}^d$-action T on $((Y \times Z)^{\mathbb{Z}^d}, \mathfrak{B}_{(Y \times Z)^{\mathbb{Z}^d}}, \mu)$ is measurably isomorphic to the Cartesian product of $\sigma^{(Z)}$ on $(Z^{\mathbb{Z}^d}, \mathfrak{B}_{Z^{\mathbb{Z}^d}}, \mu^{(Z)})$ with some Bernoulli action of $\mathbb{Z}^d$ (cf. [Tho]). Since a proof of this would take us too far afield we settle for the following more modest statement which will suffice for our purposes in Section 4.

2.11. Theorem. *Suppose that $\mu \in M_1((Y \times Z)^{\mathbb{Z}^d})^T$ satisfies the following conditions.*

(1) *$\mu^{(Z)} \in M_1(Z^{\mathbb{Z}^d})^{\sigma^{(Z)}}$ is Bernoulli;*

(2) *μ is relatively almost box independent with respect to $Z^{\mathbb{Z}^d}$.*

Then μ is Bernoulli.

The proof of Theorem 2.11 depends on the following lemma.

2.12. Lemma. *Suppose that $\mu \in M_1((Y \times Z)^{\mathbb{Z}^d})^T$ is relatively almost box independent with respect to $Z^{\mathbb{Z}^d}$, and that $\mu^{(Z)}$ is ergodic under $\sigma^{(Z)}$. Then there exists, for every $\varepsilon > 0$, an $\varepsilon' > 0$ and a Borel set $V \subset Z^{\mathbb{Z}^d}$ with $\mu(V) = 1$ such that*

$$\liminf_{M \to \infty} \bar{d}_{B_M}(\mu_z, \mu_{z'}) < \varepsilon$$

whenever $z, z' \in V$ *and*

$$\limsup_{M\to\infty} \frac{1}{|B_M|} \sum_{\mathbf{n}\in B_M} \delta'(z_{\mathbf{n}}, z'_{\mathbf{n}}) < \varepsilon'. \tag{2.14}$$

Proof. Lusin's theorem implies the existence of a closed set $C \subset Z^{\mathbb{Z}^d}$ such that $\mu^{(Z)}(C) > 1 - \varepsilon/100$ and the map $z \mapsto \mu_z$ is continuous on C. We use the relative box independence of μ to find an integer $N_0 \geq 0$ such that

$$\mu^{(Z)}(\Omega) > 1 - \varepsilon/100,$$

where

$$\Omega = \left\{ z \in Z^{\mathbb{Z}^d} : \limsup_{M\to\infty} \bar{d}_{B_M}(\mu_z, \mu_z^{\mathcal{B}_{N_0}}) < \varepsilon/100 \right\} \subset Z^{\mathbb{Z}^d}.$$

Let $\boldsymbol{\delta}'$ be some metric inducing the product topology on $Z^{\mathbb{Z}^d}$. The (uniform) continuity of the map $z \mapsto \mu_z$ on C allows us to find an $\varepsilon'' > 0$ such that

$$\varepsilon'' < \left(\frac{\varepsilon}{100}\right)^2 \frac{1}{|B_{N_0}|} \tag{2.15}$$

and

$$\bar{d}_{B_{N_0}}(\mu_z, \mu_{z'}) < \varepsilon/100 \tag{2.16}$$

whenever $z, z' \in C$ and

$$\boldsymbol{\delta}'(z, z') < \sqrt{\varepsilon'' |B_{N_0}|}.$$

We choose an $\varepsilon' > 0$ with

$$\limsup_{M\to\infty} \frac{1}{|B_M|} \sum_{\mathbf{n}\in B_M} \boldsymbol{\delta}'(\sigma'_{\mathbf{n}}(z), \sigma'_{\mathbf{n}}(z')) < \varepsilon'' \tag{2.17}$$

for every $(z, z') \in Z^{\mathbb{Z}^d}$ satisfying (2.14). As

$$\mu^{(Z)}(\Omega \cap C) > 1 - \varepsilon/50,$$

the ergodicity of $\mu^{(Z)} \in M_1(Z^{\mathbb{Z}^d})^{\sigma^{(Z)}}$ guarantees that there exists a Borel set $V \subset Z^{\mathbb{Z}^d}$ with $\mu^{(Z)}(V) = 1$ and

$$\lim_{M\to\infty} \frac{1}{|B_M|} \sum_{\mathbf{n}\in B_M} 1_{\Omega\cap C}(\sigma^{(Z)}_{\mathbf{n}}(z)) > 1 - \varepsilon/50$$

for every $z \in V$, where $1_{\Omega\cap C}$ is the indicator function of $\Omega \cap C$. By considering, for every $z \in V$, the density of the set $\{\mathbf{n} \in \mathbb{Z}^d : \sigma^{(Z)}_{\mathbf{n}}(z) \notin \Omega \cap C\}$ in $\mathbb{Z}^d$ we see that

$$\limsup_{M\to\infty} \frac{1}{|B_M|} \sum_{\mathbf{n}\in B_M} 1_{\Omega\cap C}(\sigma^{(Z)}_{\mathbf{n}}(z)) \cdot 1_{\Omega\cap C}(\sigma^{(Z)}_{\mathbf{n}}(z')) > 1 - \varepsilon/25$$

for every $(z, z') \in (\Omega \cap C)^2$. Hence there exists, for every (sufficiently large) $M \geq 1$ and every $(z, z') \in (\Omega \cap C)^2$, an element $\mathbf{m}(z, z', M) \in B_{N_0}$ with

$$\limsup_{M\to\infty} \frac{|B_{N_0}|}{|B_M|} \sum_{\mathbf{n}\in B_M \cap (\Gamma_{N_0} + \mathbf{m}(z,z',M))} 1_{\Omega\cap C}(\sigma^{(Z)}_{\mathbf{n}}(z)) \cdot 1_{\Omega\cap C}(\sigma^{(Z)}_{\mathbf{n}}(z')) > 1 - \varepsilon/25, \tag{2.18}$$

where

$$\Gamma_{N_0} = (2N_0+1)\mathbb{Z}^d.$$

We fix temporarily $(z,z')\in V^2$ satisfying (2.14) and hence (2.17), and observe that

$$\limsup_{M\to\infty}\frac{|B_{N_0}|}{|B_M|}\sum_{\mathbf{n}\in B_M\cap(\Gamma_{N_0}+\mathbf{m}(z,z',M))}\boldsymbol{\delta}'(\sigma_{\mathbf{n}}^{(Z)}(z),\sigma_{\mathbf{n}}^{(Z)}(z'))<\varepsilon''|B_{N_0}|. \tag{2.19}$$

Let

$$\Delta_M=\bigl\{\mathbf{n}\in B_M\cap(\Gamma_{N_0}+\mathbf{m}(z,z')) : (\sigma_{\mathbf{n}}^{(Z)}(z),\sigma_{\mathbf{n}}^{(Z)}(z'))\in(\Omega\cap C)^2 \\ \text{and } \boldsymbol{\delta}'(\sigma_{\mathbf{n}}^{(Z)}(z),\sigma_{\mathbf{n}}^{(Z)}(z'))<\sqrt{\varepsilon''|B_{N_0}|}\bigr\}.$$

According to (2.18)–(2.19) and (2.15),

$$\limsup_{M\to\infty}\frac{|B_{N_0}||\Delta_M|}{|B_M|}\ge 1-\varepsilon/25-\varepsilon/100=1-\varepsilon/20, \tag{2.20}$$

and (2.16) guarantees that

$$\bar d_{B_{N_0}}(\mu_{\sigma_{\mathbf{n}}^{(Z)}(z)},\mu_{\sigma_{\mathbf{n}}^{(Z)}(z')})<1-\varepsilon/100 \tag{2.21}$$

for every $\mathbf{n}\in\Delta_M$.

Let

$$\mathcal{B}_{N_0}^{(M)}=\{(B+\mathbf{m}(z,z',M))\cap B_M : B\in\mathcal{B}_{N_0}\}$$

be the tiling of B_M induced by the tiling $\mathcal{B}_{N_0}+\mathbf{m}(z,z',M)=\{B+\mathbf{n}(z,z',M) : B\in\mathcal{B}_{N_0}\}$ of $\mathbb{Z}^d$. From (2.13) and (2.20)–(2.21) it is clear that

$$\begin{aligned}\bar d_{B_M}(\mu_z^{\mathcal{B}_{N_0}^{(M)}},\mu_{z'}^{\mathcal{B}_{N_0}^{(M)}}) &\le 1-\frac{|\Delta_M|}{|B_M|}+\frac{1}{|B_M|}\sum_{\mathbf{n}\in\Delta_M}\bar d_{B_{N_0}}(\mu_{\sigma_{\mathbf{n}}(z)},\mu_{\sigma_{\mathbf{n}}(z')})\\ &\le 1-\frac{|\Delta_M|}{|B_M|}+\frac{1}{|\Delta_M|}\sum_{\mathbf{n}\in\Delta_M}\bar d_{B_{N_0}}(\mu_{\sigma_{\mathbf{n}}(z)},\mu_{\sigma_{\mathbf{n}}(z')})\end{aligned}$$

for every M, and that

$$\limsup_{M\to\infty}\bar d(\mu_z^{\mathcal{B}_{N_0}^{(M)}},\mu_{z'}^{\mathcal{B}_{N_0}^{(M)}})\le\varepsilon/20+\varepsilon/100=3\varepsilon/50.$$

Since B_{N_0} is finite there exists an element $\mathbf{m}\in B_{N_0}$ with $\mathbf{m}(z,z',M)=\mathbf{m}$ for infinitely many M. We fix such an $\mathbf{m}\in B_{N_0}$, put $\mathbf{M}=\{M\ge 1 : \mathbf{m}(z,z',M)=\mathbf{m}\}$, and note that $(\sigma_{\mathbf{n}}^{(Z)}(z),\sigma_{\mathbf{n}}^{(Z)}(z'))\in\Omega^2$ for every $M\in\mathbf{M}$ and $\mathbf{n}\in\Delta_M$, and that

$$\begin{aligned}\liminf_{M\to\infty}\bar d_{B_M}(\mu_z,\mu_{z'}^{\mathcal{B}_{N_0}^{(M)}}) &=\liminf_{M\to\infty}\bar d_{B_M-\mathbf{m}}(\mu_{\sigma_{\mathbf{m}}(z)},(\mu_{\sigma_{\mathbf{m}}(z')})^{\mathcal{B}_{N_0}})\\ &\le\liminf_{\substack{M\to\infty\\ M\in\mathbf{M}}}\bar d_{B_M-\mathbf{m}}(\mu_{\sigma_{\mathbf{m}}(z)},(\mu_{\sigma_{\mathbf{m}}(z')})^{\mathcal{B}_{N_0}})<\varepsilon/100.\end{aligned}$$

Since the same argument holds for z' we obtain that

$$\begin{aligned}\liminf_{M\to\infty}\bar d_{B_M}(\mu_z,\mu_{z'}) &\le\liminf_{M\to\infty}\bigl(\bar d_{B_M}(\mu_z,\mu_z^{\mathcal{B}_{N_0}^{(M)}})+\bar d_{B_M}(\mu_z^{\mathcal{B}_{N_0}^{(M)}},\mu_{z'}^{\mathcal{B}_{N_0}^{(M)}})+\bar d_{B_M}(\mu_{z'},\mu_{z'}^{\mathcal{B}_{N_0}^{(M)}})\bigr)\\ &\le\varepsilon/100+3\varepsilon/50+\varepsilon/100<\varepsilon. \quad\square\end{aligned}$$

Proof of Theorem 2.11. Suppose that $\mu \in M_1((Y \times Z)^{\mathbb{Z}^d})^T$ is relatively almost box independent with respect to $Z^{\mathbb{Z}^d}$, and that $\mu^{(Z)}$ is almost box independent Then

$$\lim_{N\to\infty} \limsup_{M\to\infty} \bar{d}_{B_M}(\mu^{(Z)}, (\mu^{(Z)})^{\mathcal{B}_N}) = 0. \tag{2.22}$$

Consider the decomposition $\{\mu_z : z \in Z^{\mathbb{Z}^d}\}$ of μ in (2.12)–(2.13), and define $\mu_N \in M_1(Y^{\mathbb{Z}^d} \times Z^{\mathbb{Z}^d})$ by

$$\int h \, d\mu_N = \int_{Z^{\mathbb{Z}^d}} \int_{Y^{\mathbb{Z}^d}} h(y,z) \, d\mu_z^{\mathcal{B}_N}(y) \, d\mu^{(Z)}(z)$$

for every continuous map $h: Y^{\mathbb{Z}^d} \times Z^{\mathbb{Z}^d} \longmapsto \mathbb{R}$ and every $N \geq 1$. Then

$$\int h \, d\mu^{\mathcal{B}_N} = \int_{Z^{\mathbb{Z}^d}} \int_{Y^{\mathbb{Z}^d}} h(y,z) \, d\mu_z^{\mathcal{B}_N}(y) \, d(\mu^{(Z)})^{\mathcal{B}_N}(z),$$

and (2.22) and Lemma 2.12 imply that

$$\lim_{N\to\infty} \liminf_{M\to\infty} \bar{d}_{B_M}(\mu_N, \mu^{\mathcal{B}_N}) = 0.$$

The relative almost box independence of μ guarantees that

$$\lim_{N\to\infty} \limsup_{M\to\infty} \bar{d}_{B_M}(\mu, \mu_N) \leq \lim_{N\to\infty} \limsup_{M\to\infty} \int \bar{d}_{B_M}(\mu_z, \mu_z^{\mathcal{B}_N}) \, d\mu^{(Z)}(z) = 0$$

and hence that

$$\lim_{N\to\infty} \liminf_{M\to\infty} \bar{d}_{B_M}(\mu, \mu^{\mathcal{B}_N}) = 0.$$

As the measures μ and $\mu^{\mathcal{B}_N}$ are invariant under the $\mathbb{Z}^d$-action $\mathbf{m} \mapsto T_{N\mathbf{m}}$ on $(Y \times Z)^{\mathbb{Z}^d}$ we can apply (2.6) to see that

$$\liminf_{M\to\infty} \bar{d}_{B_M}(\mu, \mu^{\mathcal{B}_N}) = \limsup_{M\to\infty} \bar{d}_{B_M}(\mu, \mu^{\mathcal{B}_N})$$

for every $N \geq 1$, and hence that

$$\lim_{N\to\infty} \limsup_{M\to\infty} \bar{d}_{B_M}(\mu, \mu^{\mathcal{B}_N}) = 0.$$

This shows that μ is almost box independent, and Theorem 2.3 completes the proof of Theorem 2.11. $\square$

In order to apply the criteria for Bernoullicity in Theorem 2.3 to $\mathbb{Z}^d$-actions by automorphisms of compact, abelian groups we need to introduce a further condition which is very close to a property originally introduced by A. Vershik for $d = 1$. Although Vershik's conjecture that this property implies Bernoullicity has been shown to be wrong (cf. [Rot]), the following Theorem 2.14, which strengthens some of the results in [DDP] for $d = 1$, is an indication that Vershik's conjecture was correct in spirit.

2.13. Definition. A measure $\mu \in M_1(Y^{\mathbb{Z}^d})^\sigma$ is *summably Vershik* if there exists a Følner sequence $(F_n, n \geq 1)$ in $\mathbb{Z}^d$ and, for each $n \geq 2$, a partial cover $\mathcal{A}^{(n)}$ of F_n by translates of F_{n-1} with the following properties.

(1) $\sum_{n\geq 2}(1 - |[\mathcal{A}^{(n)}]|/|F_n|) < \infty$;
(2) $\sum_{n\geq 2} \bar{d}_{F_n}(\mu, \mu^{\mathcal{A}^{(n)}}) < \infty$.

2.14. Theorem. *If a measure $\mu \in M_1(Y^{\mathbb{Z}^d})^\sigma$ is summably Vershik then it is sporadically a.b.i. and hence Bernoulli.*

Proof. For every $n \geq 1$ and $m > n$ we define inductively a partial cover $\mathcal{A}^{(m,n)}$ of F_m by translates of F_n. Put $\mathcal{A}^{(m,m-1)} = \mathcal{A}^{(m)}$. If $\mathcal{A}^{(m,j)}$ has been defined for $j = n+1, \dots, m-1$, and if $\mathcal{A}^{(m,n+1)} = \{F_{n+1} + \mathbf{n} : \mathbf{n} \in E^{(m,n+1)}\}$, say, then

$$\mathcal{A}^{(m,n)} = \{A + \mathbf{n} : A \in \mathcal{A}^{(n+1)} \text{ and } \mathbf{n} \in E^{(m,n+1)}\}.$$

From this construction it is clear that

$$|[\mathcal{A}^{(m,n)}]| \geq |F_m| \cdot \prod_{i=n+1}^{m} c_i,$$

where

$$c_i = \frac{|[\mathcal{A}^{(i)}]|}{|F_i|}$$

for every $i \geq 2$. An inductive calculation shows that

$$\bar{d}_{F_m}(\mu, \mu^{\mathcal{A}^{(m,n)}}) \leq \sum_{i=n+1}^{m} \bar{d}_{F_i}(\mu, \mu^{\mathcal{A}^{(i)}}) + \sum_{i=n+1}^{m} (1 - c_i),$$

and by setting

$$c'_j = \sum_{i=j+1}^{\infty} \bar{d}_{F_i}(\mu, \mu^{\mathcal{A}^{(i)}}) + \sum_{i=j+1}^{\infty} (1 - c_i)$$

we obtain that, for every $n \geq 2$ and $m > n$,

$$\bar{d}_{F_m}(\mu, \mu^{\mathcal{A}(m,n)}) \leq c'_n,$$

and that $\lim_{n\to\infty} c'_n = 0$.

Given any $\varepsilon > 0$ we choose $n \geq 2$ such that $c'_n < \varepsilon$. Then

$$|[\mathcal{A}^{(m,n)}]| \geq (1 - \varepsilon)|F_m|$$

and

$$\bar{d}_{F_m}(\mu, \mu^{\mathcal{A}^{(m,n)}}) < \varepsilon$$

for every $k > j$, and by choosing an integer $K \geq 0$ such that $F_n \subset B_K$ we see that μ is sporadically a.b.i. $\square$

3. The proof of Theorem 1.1 in a special case

We recall the algebraic description of $\mathbb{Z}^d$-actions by automorphisms of compact, abelian groups in [KS], [Sc1] and [LSW]. Let $\mathfrak{R}_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ be the ring of Laurent polynomials with integral coefficients in the commuting variables $u_1, \dots, u_d$. We write a typical element $f \in \mathfrak{R}_d$ as $f = \sum_{\mathbf{m} \in \mathbb{Z}^d} c_f(\mathbf{m}) u^{\mathbf{m}}$ with $u^{\mathbf{m}} = u_1^{m_1} \cdot \ldots \cdot u_d^{m_d}$ and $c_f(\mathbf{m}) \in \mathbb{Z}$ for every $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$, where $\sum_{\mathbf{m} \in \mathbb{Z}^d} |c_f(\mathbf{m})| < \infty$. If α is a $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group

X (which is always assumed to be metrizable), then the additively written dual group $\mathfrak{M} = \hat{X}$ is a countable module over the ring $\mathfrak{R}_d$ with operation

$$f \cdot a = \sum_{\mathbf{m} \in \mathbb{Z}^d} c_f(\mathbf{m}) \widehat{\alpha_{\mathbf{m}}}(a) \tag{3.1}$$

for every $f \in \mathfrak{R}_d$ and $a \in \mathfrak{M}$; here $\widehat{\alpha_{\mathbf{m}}}$ is the automorphism of $\mathfrak{M} = \hat{X}$ dual to $\alpha_{\mathbf{m}}$. In particular,

$$\widehat{\alpha_{\mathbf{m}}}(a) = u^{\mathbf{m}} \cdot a \tag{3.2}$$

for every $\mathbf{m} \in \mathbb{Z}^d$ and $a \in \mathfrak{M}$. Conversely, if $\mathfrak{M}$ is a countable $\mathfrak{R}_d$-module, and if

$$\widehat{\alpha_{\mathbf{m}}^{\mathfrak{M}}}(a) = u^{\mathbf{m}} \cdot a \tag{3.3}$$

for every $\mathbf{m} \in \mathbb{Z}^d$ and $a \in \mathfrak{M}$, then we obtain a $\mathbb{Z}^d$-action

$$\alpha^{\mathfrak{M}} \colon \mathbf{m} \mapsto \alpha_{\mathbf{m}}^{\mathfrak{M}} \tag{3.4}$$

on the compact, abelian group

$$X^{\mathfrak{M}} = \widehat{\mathfrak{M}} \tag{3.5}$$

dual to the $\mathbb{Z}^d$-action $\widehat{\alpha^{\mathfrak{M}}} \colon \mathbf{m} \mapsto \widehat{\alpha_{\mathbf{m}}^{\mathfrak{M}}}$ on $\mathfrak{M}$.

The simplest $\mathfrak{R}_d$-modules are the *cyclic* ones, i.e. those of the form $\mathfrak{R}_d/\mathfrak{a}$, where $\mathfrak{a}$ is an ideal in $\mathfrak{R}_d$. In this section we prove Theorem 1.1 for $\mathbb{Z}^d$-actions α for which the corresponding $\mathfrak{R}_d$-module $\mathfrak{M}$ defined by (3.1)–(3.2) is restricted even further: we assume that $\mathfrak{M} = \mathfrak{R}_d/\mathfrak{p}$ for some prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$.

3.1. Proposition. *Let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal. Then the $\mathbb{Z}^d$-action $\alpha = \alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is Bernoulli if and only if it has positive entropy.*

Theorem 4.2 in [LSW] shows that, if $\mathfrak{p} \subset R_d$ is a prime ideal, then the entropy $h(\alpha^{\mathfrak{R}_d/\mathfrak{p}})$ of $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ is given by

$$h(\alpha^{\mathfrak{R}_d/\mathfrak{p}}) = \begin{cases} \infty & \text{if } \mathfrak{p} = \{0\}, \\ \log \mathbb{M}(f) & \text{if } \mathfrak{p} = (f) = f\mathfrak{R}_d \text{ for some } 0 \neq f \in \mathfrak{R}_d, \\ 0 & \text{if } \mathfrak{p} \text{ is not principal,} \end{cases} \tag{3.6}$$

where

$$\mathbb{M}(f) = \exp\left(\int_{\mathbb{S}^d} \log |f(\mathbf{s})| \, d\lambda_{\mathbb{S}^d}(\mathbf{s}) \right) \tag{3.7}$$

is the *Mahler measure* of the Laurent polynomial $f \in \mathfrak{R}_d$ ($\lambda_{\mathbb{S}^d}$ is the normalized Haar measure of the multiplicative group $\mathbb{S}^d = \{\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{C}^d : |s_1| = \dots = |s_d| = 1\}$). Furthermore, if $0 \neq f \in \mathfrak{R}_d$, then $\log \mathbb{M}(f) = h(\alpha^{\mathfrak{R}_d/(f)}) = 0$ if and only if f is a *generalized cyclotomic polynomial,* i.e. if f is of the form $f(u) = a u^{\mathbf{m}} c(u^{\mathbf{n}})$, where $a \in \{1, -1\}$, $\mathbf{m} \in \mathbb{Z}^d$, $\mathbf{0} \neq \mathbf{n} \in \mathbb{Z}^d$, and where $c(\cdot)$ is a cyclotomic polynomial in a single variable. Finally, Theorem 6.5 in [LSW] implies that, if $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal, then $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ has completely positive entropy if and only if $h(\alpha^{\mathfrak{R}_d/\mathfrak{p}}) > 0$.

For later use we require a little bit of insight into the Mahler measures of polynomials in a single variable. Let $\mathbb{K}$ be an algebraic number field, denote by $P^{\mathbb{K}}$, $P_{\mathsf{f}}^{\mathbb{K}}$ and $P_{\infty}^{\mathbb{K}}$ the sets of places, finite places and infinite places of $\mathbb{K}$, and consider, for every $v \in P^{\mathbb{K}}$, the locally compact completion $\mathbb{K}_v$ of $\mathbb{K}$ with respect to v. The *valuation* $|\cdot|_v$ of v is obtained by fixing a compact set $C \subset \mathbb{K}_v$ with non-empty interior and by setting, for every $a \in \mathbb{K}$,

$$|a|_v = \frac{\lambda_{\mathbb{K}}(aC)}{\lambda_{\mathbb{K}}(C)},$$

where $\lambda_{\mathbb{K}}$ is a Haar measure on the additive group $\mathbb{K}_v$, and where $aC = \{ab : b \in C\}$ (cf. [Cas], [Wei]). The *product formula* states that, with this choice of $|\cdot|_v$, $v \in P^{\mathbb{K}}$,

$$\prod_{v \in P^{\mathbb{K}}} |a|_v = \begin{cases} 1 & \text{if } 0 \neq a \in \mathbb{K}, \\ 0 & \text{if } a = 0. \end{cases} \tag{3.8}$$

If $f = c_0 + c_1 u_1 + \dots + c_k u_1^k \in \mathfrak{R}_1$ is an irreducible polynomial with $c_0 c_k \neq 0$, and with roots $\zeta_1, \dots, \zeta_k$, and if $\mathbb{K} = \mathbb{Q}[\zeta_1]$ is the algebraic number field generated by ζ_1, then Theorem 1 in [LW] and Theorem 3.1 in [LSW] imply that

$$\begin{aligned} \log \mathbb{M}(f) &= \int_{\mathbb{S}} \log |f(s)| \, d\lambda_{\mathbb{S}}(s) = \log |c_k| + \sum_{v \in P^{\mathbb{K}}_{\infty}} \log^+ |\zeta_1|_v \\ &= \sum_{v \in P^{\mathbb{K}}} \log^+ |\zeta_1|_v, \end{aligned} \tag{3.9}$$

where $\log^+(t) = \max\{0, \log t\}$ for every $t > 0$.

We begin the proof of Proposition 3.1 by recalling two lemmas from [Kat].

Let $(Y, \mathfrak{T}, \mu)$ be a probability space, and let $\varepsilon > 0$. Two finite partitions $\mathcal{P}, \mathcal{Q}$ in $\mathfrak{T}$ are *ε-independent* (with respect to μ) if

$$\sum_{P \in \mathcal{P},\, Q \in \mathcal{Q}} |\mu(P \cap Q) - \mu(P)\mu(Q)| < \varepsilon. \tag{3.10}$$

3.2. Lemma. *Let $(Y, \mathfrak{T}, \mu)$ be a probability space, $\mathcal{P}, \mathcal{Q}$ finite partitions in $\mathfrak{T}$, and $\varepsilon > 0$. Suppose that there exist a set $E \in \mathfrak{T}$ and non-negative, measurable maps $f_P, g_Q \colon Y \longmapsto \mathbb{R}$, $P \in \mathcal{P}$, $Q \in \mathcal{Q}$, such that the following conditions are satisfied.*

(1) *$\mu(E) < \varepsilon^2$;*
(2) *$f_P(y) \geq 1$ for every $P \in \mathcal{P}$ and $y \in P \smallsetminus E$, and $g_Q(y) \geq 1$ for every $Q \in \mathcal{Q}$ and $y \in Q \smallsetminus E$;*
(3) *$\sum_{P \in \mathcal{P}} \int f_P \, d\mu < 1 + \varepsilon^2$ and $\sum_{Q \in \mathcal{Q}} \int g_Q \, d\mu < 1 + \varepsilon^2$;*
(4) *$\int f_P g_Q \, d\mu = \int f_P \, d\mu \int g_Q \, d\mu$ for every $P \in \mathcal{P}, Q \in \mathcal{Q}$.*

Then $\mathcal{P}$ and $\mathcal{Q}$ are 30ε-independent.

In [Kat] the partitions $\mathcal{P}$ and $\mathcal{Q}$ are, in fact, stated to be 11ε-independent, but this difference is of no importance. For every $k \geq 2$ we write $\mathcal{P}^{(k)} \subset \mathfrak{B}_{\mathbb{T}}$ for the partition of $\mathbb{T}$ into k intervals of equal length: $\mathcal{P}^{(k)} = \{P_0^{(k)}, \dots, P_{k-1}^{(k)}\}$ with $P_j^{(k)} = [\frac{j}{k}, \frac{j+1}{k}) + \mathbb{Z} \subset \mathbb{T}$ for $j = 0, \dots, k-1$.

3.3. Lemma. *Let $k \geq 2$ and $m \geq 1$. There exists a Borel set $E^{(k,m)} \subset \mathbb{T}$ and continuous, non-negative functions $h_j^{(k,m)} \colon \mathbb{T} \longmapsto \mathbb{R}$, $j = 0, \dots, k-1$, with the following properties.*

(1) *$\lambda_{\mathbb{T}}(E^{(k,m)}) < \frac{1}{m^2}$, and $\sum_{j=0}^{k-1} h_j^{(k,m)} \leq 1 + km^2$ for every $t \in \mathbb{T}$;*

(2) *For every* $j=0,\dots,k-1$, $t_1 \in P_j^{(k)} \smallsetminus E^{(k,m)}$, $t_2 \in \mathbb{T} \smallsetminus (P_j^{(k)} \cup E^{(k,m)})$,

$$h_j^{(k,m)}(t_1) \geq 1, \quad h_j^{(k,m)}(t_2) \leq \frac{1}{m^2};$$

(3) *For every* $j=0,\dots,k-1$,

$$\int h_j^{(k,m)}(t) e^{-2\pi i n t}\, d\lambda_{\mathbb{T}}(t) = 0$$

whenever $|n| > 8k^3m^6$, *i.e.* $h_j^{(k,m)}$ *is a trigonometric polynomial involving only trigonometric functions* $e^{2\pi i n t}$ *with* $|n| \leq 8k^3 m^6$.

Proof. For every $n \geq 64$ and $t \in \mathbb{T}$ we set

$$a_n(t) = \frac{1}{\sqrt{n}} \cdot \sum_{l=0}^{n-1} e^{2\pi i l x} = \frac{1}{\sqrt{n}} \cdot \frac{e^{2\pi i n t}-1}{e^{2\pi i t}-1},$$

$$b_n(t) = |a_n(t)|^2 = \sum_{l=-n+1}^{n-1} \frac{n-|l|}{n} \cdot e^{2\pi i l t}.$$

Then $b_n \geq 0$, $\int b_n \, d\lambda_{\mathbb{T}} = 1$ and, for every $t \in \mathbb{T}$ with $\|t\| = \min\{|t-m| : m \in \mathbb{Z}\} \geq n^{-1/3}$,

$$|a_n(t)| \leq \frac{1}{\sqrt{n}} \cdot \frac{1}{|\sin \pi t|} \leq \frac{1}{4n^{1/6}}, \quad b_n(t) \leq \frac{1}{16 n^{1/3}} \tag{3.11}$$

Put $F^{(n)} = \{t \in \mathbb{T} : |t - \frac{l}{n}| < \frac{1}{n^{1/3}} \text{ for some } l \in \mathbb{Z}\}$ and set, for every $j \in \{0,\dots,k-1\}$, $g_j^{(n)} = 1_{P_j^{(k)}} * b_n + \frac{1}{16n^{1/3}}$, where $*$ denotes convolution, and where $1_{P_j^{(k)}}$ is the indicator function of $P_j^{(k)}$. Since b_n is non-negative, $g_j^{(n)}$ is non-negative, $g_j^{(n)} \leq 1 + \frac{1}{16n^{1/3}}$,

$$\sum_{j=0}^{k-1} g_j^{(n)} = b_n * \left(\sum_{j=0}^{k-1} 1_{P_j^{(k)}}\right) + \frac{k}{16n^{1/3}} = 1 + \frac{k}{16n^{1/3}},$$

and the inequality (3.11) implies that $g_j^{(n)}(t) \geq 1$ for every $t \in P_j^{(k)} \smallsetminus F^{(n)}$. The proof is completed by setting $n = 8k^3m^6$, $E^{(k,m)} = F^{(n)}$ and $h_j^{(k,m)} = g_j^{(n)}$ for every $j = 0,\dots,k-1$. $\square$

In order to proceed further we require an explicit realization of $(X^{\mathfrak{R}_d/\mathfrak{p}}, \alpha^{\mathfrak{R}_d/\mathfrak{p}})$, where $\mathfrak{p} \subset \mathfrak{R}_d$ is a principal prime ideal (cf. [Sc1]). Let $d \geq 1$, $f \in \mathfrak{R}_d$, and let $(f) = f\mathfrak{R}_d$ be the principal ideal generated by f. We define the shift-action σ of $\mathbb{Z}^d$ on $\mathbb{T}^{\mathbb{Z}^d}$ by (1.1), put

$$g(\sigma)(x) = \sum_{\mathbf{m} \in \mathbb{Z}^d} c_g(\mathbf{m}) \sigma_{\mathbf{m}}(x) \in \mathbb{T}^{\mathbb{Z}^d} \tag{3.12}$$

for every $g = \sum_{\mathbf{m} \in \mathbb{Z}^d} c_g(\mathbf{m}) u^{\mathbf{m}} \in \mathfrak{R}_d$ and $x = (x_{\mathbf{n}}) \in \mathbb{T}^{\mathbb{Z}^d}$, and identify $\mathfrak{R}_d$ with the dual group $\widehat{\mathbb{T}^{\mathbb{Z}^d}}$ of $\mathbb{T}^{\mathbb{Z}^d}$ by setting

$$\langle g, x \rangle = e^{2\pi i (g(\sigma)(x))_{\mathbf{0}}}$$

for every $g \in \mathfrak{R}_d$ and $x \in \mathbb{T}^{\mathbb{Z}^d}$. With this identification we obtain that

$$X^{\mathfrak{R}_d/(f)} = \{x \in \mathbb{T}^{\mathbb{Z}^d} : f(\sigma)(x) = 0 \in \mathbb{T}^{\mathbb{Z}^d}\}, \tag{3.13}$$

and that $\alpha^{\mathfrak{R}_d/(f)}$ is the restriction of σ to $X^{\mathfrak{R}_d/(f)} \subset \mathbb{T}^{\mathbb{Z}^d}$. In particular, if $f = 0$, then $\alpha^{\mathfrak{R}_d/(f)}$ is the shift-action of $\mathbb{Z}^d$ on $\mathbb{T}^{\mathbb{Z}^d}$ and hence Bernoulli with infinite entropy. If $f = cu^{\mathbf{m}}$ for some non-zero $c \in \mathbb{Z}$ and some $\mathbf{m} \in \mathbb{Z}^d$, then we may obviously assume that $\mathbf{m} = \mathbf{0}$, and that $c = p$ is a rational prime. In this case

$$\begin{aligned} X^{\mathfrak{R}_d/(f)} &= \{x = (x_{\mathbf{n}}) \in \mathbb{T}^{\mathbb{Z}^d} : px_{\mathbf{n}} = 0 \pmod 1 \text{ for every } \mathbf{n} \in \mathbb{Z}^d\} \\ &\cong \left\{\frac{k}{p} : 0 \le k < p\right\}^{\mathbb{Z}^d}, \end{aligned} \tag{3.14}$$

and $\alpha^{\mathfrak{R}_d/(f)}$ corresponds to the shift-action of $\mathbb{Z}^d$ on $\{\frac{k}{p} : 0 \le k < p\}^{\mathbb{Z}^d}$ and is thus Bernoulli with entropy $\log p$. We are left with the case where $f = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n})u^{\mathbf{n}}$ is a non-zero, irreducible Laurent polynomial which is not generalized cyclotomic (in order to ensure that $h(\alpha^{\mathfrak{R}_d/(f)}) > 0$). The following observation helps to overcome a minor technical difficulty in proving that $\alpha^{\mathfrak{R}_d/(f)}$ is Bernoulli.

3.4. Lemma. *Let $A \in \mathrm{GL}(d,\mathbb{Z})$, and let $f^A = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n})u^{A\mathbf{n}} \in \mathfrak{R}_d$. Then $\alpha = \alpha^{\mathfrak{R}_d/(f)}$ is Bernoulli if and only if $\alpha^{\mathfrak{R}_d/(f^A)}$ is Bernoulli.*

Proof. We define a continuous group isomorphism $\psi_A \colon \mathbb{T}^{\mathbb{Z}^d} \longmapsto \mathbb{T}^{\mathbb{Z}^d}$ by setting $(\psi_A(x))_{\mathbf{n}} = x_{A\mathbf{n}}$ for every $x = (x_{\mathbf{n}}) \in \mathbb{T}^{\mathbb{Z}^d}$ and $\mathbf{n} \in \mathbb{Z}^d$ and note that $\psi_A(X^{\mathfrak{R}_d/(f)}) = X^{\mathfrak{R}_d/(f^A)}$ and $\psi_A \cdot \alpha_{\mathbf{n}}^{\mathfrak{R}_d/(f)} = \alpha_{A\mathbf{n}}^{\mathfrak{R}_d/(f^A)} \cdot \psi_A$ for every $\mathbf{n} \in \mathbb{Z}^d$. In particular, $\alpha = \alpha^{\mathfrak{R}_d/(f)}$ is algebraically conjugate to the $\mathbb{Z}^d$-action $\alpha_A^{\mathfrak{R}_d/(f^A)} \colon \mathbf{n} \mapsto \alpha_{A\mathbf{n}}^{\mathfrak{R}_d/(f^A)}$, and the definition of Bernoullicity shows that $\alpha^{\mathfrak{R}_d/(f^A)}$ is Bernoulli if and only if $\alpha \cong \alpha_A^{\mathfrak{R}_d/(f^A)}$ is Bernoulli. □

Motivated by Lemma 3.4 we make the following *ad hoc* definition.

3.5. Definition. An irreducible element $f = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n})u^{\mathbf{n}} \in \mathfrak{R}_d$ is *nice* if $c_f(\mathbf{0}) \ne 0$, and if there exists an element $\mathbf{r}^* = (r_1^*, \dots, r_d^*) \in \mathbb{Z}^d$ such that $r_i^* > 0$ for $i = 1, \dots, d$, $c_f(\mathbf{r}^*) \ne 0$, and $c_f(\mathbf{n}) = 0$ for every $\mathbf{n} \in \mathbb{Z}^d \smallsetminus Q_{\mathbf{r}^*}^*$, where

$$Q_{\mathbf{r}^*}^* = \{\mathbf{0}, \mathbf{r}^*\} \cup \{\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d : 0 < n_i < r_i^* \text{ for all } i = 1, \dots, d\}.$$

If $d = 1$ every non-constant, irreducible polynomial $f \in \mathfrak{R}_1$ with non-zero constant term is nice; if $d \ge 2$, and if $f \in \mathfrak{R}_d$ is irreducible and has at least two non-zero coefficients, then we can find $\mathbf{m} \in \mathbb{Z}^d$ and $A \in \mathrm{GL}(d,\mathbb{Z})$ such that $u^{\mathbf{m}} f^A$ is nice.

Until further notice we assume that $d \ge 1$, $f \in \mathfrak{R}_d$ is a nice, irreducible polynomial which is not generalized cyclotomic, and that $\alpha = \alpha^{\mathfrak{R}_d/(f)}$ is the shift-action of $\mathbb{Z}^d$ on the group $X = X^{\mathfrak{R}_d/(f)} \subset \mathbb{T}^{\mathbb{Z}^d}$ in (3.13). Let $k \ge 2$, define the partition $\mathcal{P}^{(k)} \subset \mathfrak{B}_{\mathbb{T}}$ as in the paragraph preceding Lemma 3.3, and put

$$\mathcal{P}_{\mathbf{n}}^{(k)} = \pi_{\{\mathbf{n}\}}^{-1}(\mathcal{P}^{(k)}) \subset \mathfrak{B}_X \tag{3.15}$$

for every $\mathbf{n} \in \mathbb{Z}^d$, where $\pi_{\{\mathbf{n}\}} \colon X \longmapsto \mathbb{T}$ is the projection onto the $\mathbf{n}$-th coordinate. For every $n \ge 1$ we consider the finite partition

$$\mathcal{Q}_n^{(k)} = \bigvee_{\mathbf{n}\in B_n} \mathcal{P}_{\mathbf{n}}^{(k)}. \tag{3.16}$$

Our first task consists of showing that, for $j = 1, \dots, d$, the partitions

$$\begin{aligned} \mathcal{A}(k,n,j)^- &= \alpha_{(2^n+n^2)\mathbf{e}^{(j)}}\bigl(\mathcal{Q}_{2^n}^{(k)}\bigr), \\ \mathcal{A}(k,n,j)^+ &= \alpha_{-(2^n+n^2)\mathbf{e}^{(j)}}\bigl(\mathcal{Q}_{2^n}^{(k)}\bigr), \end{aligned} \tag{3.17}$$

become rapidly independent as $n \to \infty$, where $\mathbf{e}^{(j)}$ is the j-th unit vector in $\mathbb{Z}^d$.

In order to quantify this claim we set, for every $j = 1, \dots, d$,

$$Q(2^n, j)^{\pm} = B_{2^n} \pm (2^n + n^2)\mathbf{e}^{(j)}, \quad E(n,j)^{\pm} = \bigcup_{\mathbf{r} \in Q(2^n,j)^{\pm}} \alpha_{-\mathbf{r}}(E^{(k,|\mathbf{r}|^{2d})}),$$

where $|\mathbf{r}| = \max\{|r_1|, \dots, |r_d|\}$ for every $\mathbf{r} \in \mathbb{Z}^d$. For all atoms

$$A^- = \bigcap_{\mathbf{r} \in Q(2^n,j)^-} \alpha_{-\mathbf{r}}(P_{j_\mathbf{r}}^{(k)}) \in \mathcal{A}(k,n,j)^-, \quad A^+ = \bigcap_{\mathbf{r} \in Q(2^n,j)^+} \alpha_{-\mathbf{r}}(P_{j_\mathbf{r}}^{(k)}) \in \mathcal{A}(k,n,j)^+$$

with $j_\mathbf{r} \in \{0, \dots, k-1\}$ for every $\mathbf{r}$ we let

$$\begin{aligned} g_{A^-} &= \prod_{\mathbf{r} \in Q(2^n,j)^-} h_{j_\mathbf{r}}^{(k,|\mathbf{r}|^{2d})} \cdot \pi_{\{\mathbf{0}\}} \cdot \alpha_\mathbf{r} = \prod_{\mathbf{r} \in Q(2^n,j)^-} h_{j_\mathbf{r}}^{(k,|\mathbf{r}|^{2d})} \cdot \pi_{\{\mathbf{r}\}}, \\ g_{A^+} &= \prod_{\mathbf{r} \in Q(2^n,j)^+} h_{j_\mathbf{r}}^{(k,|\mathbf{r}|^{2d})} \cdot \pi_{\{\mathbf{0}\}} \cdot \alpha_\mathbf{r} = \prod_{\mathbf{r} \in Q(2^n,j)^+} h_{j_\mathbf{r}}^{(k,|\mathbf{r}|^{2d})} \cdot \pi_{\{\mathbf{r}\}}, \end{aligned}$$

and apply Lemma 3.3 to obtain the following.

(i) The sets $E(n,j)^{\pm} \subset X$ satisfy that

$$\begin{aligned} \lambda_X(E(n,j)^{\pm}) &\leq \sum_{\mathbf{r} \in Q(2^n,j)^{\pm}} \frac{1}{|\mathbf{r}|^{4d}} \\ &< (2d-1) \sum_{r=n^2}^{\infty} (2r+1)^{d-1} r^{-4d} < 2^d n^{-6d}; \end{aligned}$$

(ii) For every $A^{\pm} \in \mathcal{A}(k,n,j)^{\pm}$, $g_{A^{\pm}} \geq 0$;

(iii) The continuous maps $g_{A^{\pm}} \colon X \longmapsto \mathbb{R}$, $A^{\pm} \in \mathcal{A}(k,n,j)^{\pm}$, satisfy that

$$\begin{aligned} \sum_{A^+ \in \mathcal{A}(k,n,j)^+} g_{A^+} &\leq \prod_{\mathbf{r} \in Q(2^n,j)^+} \left(1 + \frac{k}{|\mathbf{r}|^{4d}}\right) \\ &< \prod_{r=n^2}^{\infty} \left(1 + \frac{k}{r^{4d}}\right)^{(2d-1)(2r+1)^{d-1}}, \\ \sum_{A^- \in \mathcal{A}(k,n,j)^-} g_{A^-} &\leq \prod_{\mathbf{r} \in Q(2^n,j)^-} \left(1 + \frac{k}{|\mathbf{r}|^{4d}}\right) \\ &< \prod_{r=n^2}^{\infty} \left(1 + \frac{k}{r^{4d}}\right)^{(2d-1)(2r+1)^{d-1}} \end{aligned}$$

(iv) If $A^{\pm} \in \mathcal{A}(k,n,j)^{\pm}$, and if

$$g_{A^{\pm}}(x) = \sum_{a \in \mathfrak{R}_d/(f)} \hat{g}_{A^{\pm}}(a) \cdot \langle x, a \rangle$$

is the Fourier series of $g_{A^\pm}$, then every $a \in \mathfrak{R}_d/(f)$ with $\hat{g}_{A^-}(a) \neq 0$ for some $A^- \in \mathcal{A}(k,n,j)^-$ is of the form $a = \phi + (f)$, where

$$\phi = \sum_{\mathbf{r} \in Q(2^n,j)^-} c_\phi(\mathbf{r}) u^{\mathbf{r}} \in \mathfrak{R}_d$$

with

$$|c_\phi(\mathbf{r})| \leq 8k^3 |\mathbf{r}|^{12d}$$

for every $\mathbf{r} \in Q(2^n,j)^-$; similarly, if $\hat{g}_{A^+}(a) \neq 0$ for some $A^+ \in \mathcal{A}(k,n,j)^+$ and $a \in \mathfrak{R}_d/(f)$, then $a = \psi + (f)$, where

$$\psi = \sum_{\mathbf{r} \in Q(2^n,j)^+} c_\phi(\mathbf{r}) u^{\mathbf{r}} \in \mathfrak{R}_d$$

with

$$|c_\phi(\mathbf{r})| \leq 8k^3 |\mathbf{r}|^{12d}$$

for every $\mathbf{r} \in Q(2^n,j)^+$.

In the next lemma r_j^* is the degree of f in the variable u_j (cf. Definition 3.5).

3.6. Lemma. *For $j = 1, \dots, d$, and for every integer $n \geq 2kd$ with*

$$e^{-2n^2 h(\alpha)} \cdot (k^3 2^{14dn})^{r_j^*} < 1, \tag{3.18}$$

the partitions $\mathcal{A}(k,n,j)^-$ and $\mathcal{A}(k,n,j)^+$ in (3.17) are $30k2^{d/2}n^{-3d}$-independent.

Proof. Since all the definitions involved are invariant under permutations of coordinates it will suffice to prove the lemma for $j = 1$. Since f is nice (Definition 3.5), we can write it in the form $f = \sum_{j=0}^{r_1^*} g_j u_1^j$ with $g_j \in \mathbb{Z}[u_2, \dots, u_d]$ for every $j = 0, \dots, r_1^*$, where $g_0 = c_f(\mathbf{0}) \neq 0$ and $g_{r_1^*} = c_f(\mathbf{r}^*) u_2^{r_2^*} \cdot \dots \cdot u_d^{r_d^*} \neq 0$.

Suppose that we can prove the following for every integer $n \geq 2kd$ satisfying (3.18): if $\psi^-, \psi^+ \in \mathfrak{R}_d$ are of the form

$$\psi^- = \sum_{\mathbf{r} \in Q(2^n,1)^-} c_{\psi^-}(\mathbf{r}) u^{\mathbf{r}}, \quad \psi^+ = \sum_{\mathbf{r} \in Q(2^n,1)^+} c_{\psi^+}(\mathbf{r}) u^{\mathbf{r}} \tag{3.19}$$

with

$$|c_{\psi^-}(\mathbf{r})| \leq 8k^3 |\mathbf{r}|^{12d}, \quad |c_{\psi^+}(\mathbf{r}')| \leq 8k^3 |\mathbf{r}'|^{12d} \tag{3.20}$$

for every $\mathbf{r} \in Q(2^n,1)^-$, $\mathbf{r}' \in Q(2^n,1)^+$, and if

$$\psi^- + (f) = \psi^+ + (f), \tag{3.21}$$

then

$$\psi^- \in (f), \quad \psi^+ \in (f). \tag{3.22}$$

If $A^- \in \mathcal{A}(k,n,1)^-$, $A^+ \in \mathcal{A}(k,n,1)^+$, then condition (iv) preceding the statement of this lemma and (3.19)–(3.22) together show that no non-trivial character $a \in \hat{X} = \mathfrak{M}$ has the property that a occurs in the Fourier series of g_{A^-} and $-a$ in the Fourier series of g_{A^+}. Hence

$\int g_{A^-} g_{A^+} \, d\lambda_X = \int g_{A^-} \, d\lambda_X \cdot \int g_{A^-} \, d\lambda_X$ for every $A^- \in \mathcal{A}(k,n,1)^-$, $A^+ \in \mathcal{A}(k,n,1)^+$. From the conditions (i)–(iii) above we obtain that $\lambda_X(E(n,1)^{\pm}) < 2^d n^{-6d}$, and that

$$\begin{aligned}\log\Bigl(\sum_{A^{\pm}\in\mathcal{A}(k,n,1)^{\pm}} g_{A^{\pm}}(x)\Bigr) &< \sum_{r\geq n^2}(2d-1)(2r+1)^{d-1}\log\Bigl(1+\frac{k}{r^{4d}}\Bigr)\\ &< (2d-1)\cdot\sum_{r\geq n^2}(2r+1)^{d-1}\frac{k}{r^{4d}}\\ &< 2^d(2d-1)\cdot\sum_{r\geq n^2}\frac{k}{r^{3d+1}} < 2^d\frac{k}{n^{6d}}\end{aligned}$$

for every $x \in X$. Since $\log t \geq \frac{1}{2}(t-1) \geq \frac{1}{k}(t-1)$ whenever $1 \leq t \leq 2$ we see that

$$\sum_{A^{\pm}\in\mathcal{A}(k,n,1)^{\pm}} g_{A^{\pm}}(x) \leq 2^d \frac{k^2}{n^{6d}}$$

whenever $n \geq 2kd$, and Lemma 3.2 yields that $\mathcal{A}(k,n,1)^-$ and $\mathcal{A}(k,n,1)^+$ are $30k2^{d/2}n^{-3d}$-independent.

In order to prove that (3.19)–(3.21) imply (3.22) whenever $n \geq 2kd$ satisfies (3.18) we let $p_2, \dots, p_d$ be distinct rational primes with $p_l > r_l^*$ for every $l = 2, \dots, d$, choose primitive p_l-th unit roots ω_l, $l = 2, \dots, d$, and set

$$\mathbf{p} = (p_2, \dots, p_d), \quad [\mathbf{p}] = \prod_{l=2}^{d}(p_l - 1), \quad \boldsymbol{\omega} = (\omega_2, \dots, \omega_d).$$

Let $\mathbb{L} = \mathbb{Q}[\omega_2, \dots, \omega_d]$, put $h(u_1) = f(u_1, \omega_2, \dots, \omega_d)/c_{\mathbf{0}}(f) \in \mathbb{L}[u_1]$, and consider the decomposition $h = h_1 \cdot \ldots \cdot h_q$ of h into irreducible elements of $\mathbb{L}[u_1]$ with constant terms 1. By applying the Galois group $\mathrm{Gal}[\mathbb{L}:\mathbb{Q}]$ of $\mathbb{L}$ over $\mathbb{Q}$ to $h = h_1 \cdot \ldots \cdot h_q$ we obtain, for every $\kappa \in \mathrm{Gal}[\mathbb{L}:\mathbb{Q}]$, a corresponding decomposition $h^\kappa = h_1^\kappa \cdot \ldots \cdot h_q^\kappa$, where h^κ and h_i^κ are the images of h and h_i under κ. Put, for every $i = 1, \dots, q$,

$$\begin{aligned}H_i &= \prod_{\kappa\in\mathrm{Gal}[\mathbb{L}:\mathbb{Q}]} h_i^\kappa \in \mathbb{Q}[u_1],\\ H(u_1) = \prod_{i=1}^{q} H_i(u_1) &= c_f(\mathbf{0})^{-[\mathbf{p}]}\cdot\prod_{j_2=1}^{p_2-1}\cdots\prod_{j_d=1}^{p_d-1} f(u_1, \omega_2^{j_2}, \dots, \omega_d^{j_d}).\end{aligned}$$

Elementary Galois theory implies that there exists, for every $i \in \{1, \dots, q\}$, an integer $t_i \geq 1$ and an irreducible polynomial $G_i \in \mathbb{Q}[u_1]$ with constant term 1 such that

$$H_i = G_i^{t_i}. \tag{3.23}$$

Let $\zeta(i) = \zeta(\mathbf{p}, \boldsymbol{\omega}, i)$ be a root of h_i, put $\mathbb{K}_i = \mathbb{Q}[\zeta(i)]$, $\mathbb{L}_i = \mathbb{L}[\zeta(i)]$, and recall that, if v is a place of $\mathbb{K}_i$, then

$$|a|_v^{m_i} = \prod_{\{w\in P^{\mathbb{L}_i}: w \text{ lies above } v\}} |a|_w \tag{3.24}$$

for every $a \in \mathbb{K}_i$, where $m_i = [\mathbb{L}_i : \mathbb{K}_i]$.

With this notation at hand we return to the assertion of the lemma and assume that $n \geq 2kd$ obeys (3.18), and that $\psi^-, \psi^+ \in \mathfrak{R}_d$ satisfy (3.19)–(3.21). Since the ideal $(f) = f\mathfrak{R}_d \subset \mathfrak{R}_d$ is prime, (3.21) is equivalent to the condition that

$$\begin{aligned} \psi^-(\xi) = \sum_{\mathbf{r} \in Q(2^n,1)^-} c_{\psi^-}(\mathbf{r})\xi^{\mathbf{r}} &= \sum_{\mathbf{r} \in Q(2^n,1)^+} c_{\psi^+}(\mathbf{r})\xi^{\mathbf{r}} = \psi^+(\xi) \\ &= \psi(\xi), \ \text{ say,} \end{aligned} \tag{3.25}$$

for every $\xi = (\xi_1, \dots, \xi_d) \in V_{\mathbb{C}}(f) = \{\xi \in (\mathbb{C} \smallsetminus \{0\})^d : f(\xi) = 0\}$, where $\xi^{\mathbf{r}} = \xi_1^{r_1} \cdot \ldots \cdot \xi_d^{r_d}$ for every $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{Z}^d$ and $\xi \in V_{\mathbb{C}}(f)$, and (3.22) amounts to saying that

$$\psi(\xi) = 0 \tag{3.26}$$

for every $\xi \in V_{\mathbb{C}}(f)$. In particular, if $\zeta(i) = \zeta(\mathbf{p}, \boldsymbol{\omega}, i)$ is the root of h_i chosen above, then $f(\zeta(i), \omega_2, \dots, \omega_d) = 0$, i.e. $\xi(i) = \xi(\mathbf{p}, \boldsymbol{\omega}, i) = (\zeta(i), \omega_2, \dots, \omega_d) \in V_{\mathbb{C}}(f)$.

In order to investigate the behaviour of $\psi(\xi(i))$ we observe that

$$|\psi(\xi(i))|_w = \Bigl| \sum_{\mathbf{r} \in Q(2^n,1)^-} c_{\psi^-}(\mathbf{r})\xi(i)^{\mathbf{r}} \Bigr|_w = \Bigl| \sum_{\mathbf{r} \in Q(2^n,1)^+} c_{\psi^+}(\mathbf{r})\xi(i)^{\mathbf{r}} \Bigr|_w \tag{3.27}$$

for every $w \in P^{\mathbb{L}_i}$. Put $|a|_w^- = \min\{|a|_w, |a|_w^{-1}\}$ for every non-zero element $a \in \mathbb{L}$ and conclude from (3.20) and (3.27) that

$$|\psi(\xi(i))|_w \leq \begin{cases} (|\zeta(i)|_w^-)^{n^2} & \text{if } w \text{ is finite,} \\ (|\zeta(i)|_w^-)^{n^2} \cdot \bigl|\sum_{\mathbf{r} \in Q(2^n,1)^+} 8k^3 |\mathbf{r}|^{12d}\bigr|_w & \text{if } w \text{ is infinite,} \end{cases}$$

since $|\omega_j|_w = 1$ for all $j = 2, \dots, d$ and $w \in P^{\mathbb{L}_i}$. An elementary calculation shows that

$$\begin{aligned} \sum_{\mathbf{r} \in Q(2^n,1)^+} 8k^3 |\mathbf{r}|^{12d} &< \sum_{j=n^2}^{2^n+n^2} 8k^3(2d-1)(2j+1)^{d-1} j^{12d} \\ &< 8k^3(2d-1)2^{d-1} \cdot \sum_{j=n^2}^{2^n+n^2} (j+1)^{13d-1} \\ &< 2^d k^3 (2^n + n^2 + 1)^{13d} < k^3 2^{14dn} \end{aligned}$$

so that

$$|\psi(\xi(i))|_w \leq \begin{cases} (|\zeta(i)|_w^-)^{n^2} & \text{if } w \text{ is finite,} \\ (|\zeta(i)|_w^-)^{n^2} \cdot |k^3 2^{14dn}|_w & \text{if } w \text{ is infinite.} \end{cases}$$

According to (3.24),

$$\prod_{w \in P^{\mathbb{L}_i} \text{ lies above } v} |\psi(\xi(i))|_w \leq \begin{cases} |\zeta(i)^{m_i n^2}|_v^- & \text{if } v \text{ is finite,} \\ |\zeta(i)^{m_i n^2}|_v^- \cdot |k^3 2^{14dn}|_v^{m_i} & \text{if } v \text{ is infinite} \end{cases}$$

for every $v \in P^{\mathbb{K}_i}$, and

$$\prod_{w \in P^{\mathbb{L}_i}} |\psi(\xi(i))|_w \leq \Bigl(\prod_{v \in P^{\mathbb{K}_i}} |\zeta(i)^{n^2}|_v^- \Bigr)^{m_i} \cdot (k^3 2^{14dn})^{m_i [\mathbb{K}_i : \mathbb{Q}]}.$$

By varying $i \in \{1, \dots, q\}$ and using (3.23) we obtain that

$$\begin{aligned} &\prod_{i=1}^{q} \biggl(\prod_{w \in P^{\mathbb{L}_i}} |\psi(\xi(i))|_w \biggr)^{t_i/m_i[\mathbf{p}]} \\ &\le \prod_{i=1}^{q} \biggl(\prod_{v \in P^{\mathbb{K}_i}} |\zeta(i)|_v^- \biggr)^{t_i n^2/[\mathbf{p}]} \cdot \prod_{i=1}^{q} (k^3 2^{14dn})^{t_i[\mathbb{K}_i:\mathbb{Q}]/[\mathbf{p}]}. \end{aligned} \tag{3.28}$$

If

$$\begin{aligned} F^{(\mathbf{p})}(u_1) &= \prod_{j_2=1}^{p_2-1} \cdots \prod_{j_d=1}^{p_d-1} f(u_1, \omega_2^{j_2}, \dots, \omega_d^{j_d}) \\ &= H(u_1) \cdot c_f(\mathbf{0})^{[\mathbf{p}]} = \prod_{i=1}^{q} G_i(u_1)^{t_i} \cdot c_f(\mathbf{0})^{[\mathbf{p}]}, \end{aligned}$$

then we claim that

$$\prod_{i=1}^{q} \biggl(\prod_{v \in P^{\mathbb{K}_i}} |\zeta(i)|_v^- \biggr)^{t_i n^2/[\mathbf{p}]} = \mathbb{M}(F^{(\mathbf{p})})^{-2n^2/[\mathbf{p}]}. \tag{3.29}$$

Indeed, apply (3.9) and (3.8) to each of the polynomials G_i to obtain that

$$\mathbb{M}(G_i)^{-2} = \biggl(\prod_{v \in P^{\mathbb{K}_i}} \exp(\log^+ |\zeta(i)|_v) \biggr)^{-2} = \prod_{v \in P^{\mathbb{K}_i}} |\zeta(i)|_v^- .$$

By taking the product over $i = 1, \dots, q$ and taking into consideration (3.24) we have proved (3.29).

In order to see that

$$\mathbb{M}(F^{(\mathbf{p})})^{-2n^2/[\mathbf{p}]} \longrightarrow e^{-2n^2 h(\alpha)} \tag{3.30}$$

as the distinct primes $(p_2, \dots, p_d)$ tend to ∞ we conclude from Lemma 3.5 in [LSW] that the map

$$\mathbf{s} = (s_2, \dots, s_d) \mapsto \int_{\mathbb{S}} \log |f(s, s_2, \dots, s_d)| \, d\lambda_{\mathbb{S}}(s) = \int_{\mathbb{S}} \log |f(s, \mathbf{s})| \, d\lambda_{\mathbb{S}}(s)$$

from $\mathbb{S}^{d-1}$ to $\mathbb{R}$ is continuous. As

$$\frac{1}{[\mathbf{p}]} \log \mathbb{M}(F^{(\mathbf{p})})$$

is a Riemann sum approximation of the integral

$$\int_{\mathbb{S}^{d-1}} \biggl(\int_{\mathbb{S}} \log |f(s, \mathbf{s})| \, d\lambda_{\mathbb{S}}(s) \biggr) \, d\lambda_{\mathbb{S}^{d-1}}(\mathbf{s})$$

we obtain (3.30) by letting $p_i \to \infty$ for $i = 2, \dots, d$.

By counting degrees we see that

$$\prod_{i=1}^{q} (k^3 2^{14dn})^{t_i[\mathbb{K}_i:\mathbb{Q}]/[\mathbf{p}]} = (k^3 2^{14dn})^{r_1^*} \tag{3.31}$$

for every $\mathbf{p} = (p_2, \dots, p_d)$, and (3.28)–(3.31) show that

$$\begin{aligned} \prod_{i=1}^{q} \biggl(\prod_{w \in P^{\mathbb{L}_i}} |\psi(\xi(i))|_w \biggr)^{t_i/m_i[\mathbf{p}]} &\le \mathbb{M}(F(\mathbf{p}))^{-2n^2/[\mathbf{p}]} \cdot (k^3 2^{14dn})^{r_1^*} \\ &\longrightarrow e^{-2n^2 h(\alpha)} \cdot (k^3 2^{14dn})^{r_1^*} \end{aligned}$$

as the primes p_i tend to ∞. In particular, as n satisfies (3.18),

$$\prod_{i=1}^{q}\Bigl(\prod_{w\in P^{\mathbb{L}_i}}|\psi(\xi(i))|_w\Bigr)^{t_i/m_i[\mathbf{p}]}<1$$

whenever the p_i are sufficiently large. However, the product formula (3.8) implies that

$$\prod_{i=1}^{q}\Bigl(\prod_{w\in P^{\mathbb{L}_i}}|\psi(\xi(i))|_w\Bigr)^{t_i/m_i[\mathbf{p}]}=\begin{cases}0 & \text{if } \prod_{i=1}^{q}\psi(\xi(i))=0,\\ 1 & \text{otherwise,}\end{cases}$$

and hence that

$$\prod_{i=1}^{q}\psi(\xi(\mathbf{p},\boldsymbol{\omega},i))=0 \tag{3.32}$$

for every vector $\mathbf{p}=(p_2,\dots,p_d)$ consisting of distinct and sufficiently large rational primes, and for all corresponding primitive unit roots $\boldsymbol{\omega}=(\omega_2,\dots,\omega_d)$. For every $\mathbf{s}=(s_2,\dots,s_d)\in\mathbb{S}^{d-1}$ we denote by $\zeta(\mathbf{s},1),\dots,\zeta(\mathbf{s},r_1^*)$ the roots of $f(u_1,\mathbf{s})=f(u_1,s_2,\dots,s_d)$. By (3.25), (3.32) and continuity,

$$\prod_{i=1}^{r_1^*}\psi^-(\zeta(\mathbf{s},i),\mathbf{s})=\prod_{i=1}^{r_1^*}\psi^+(\zeta(\mathbf{s},i),\mathbf{s})=0$$

for every $\mathbf{s}\in\mathbb{S}^{d-1}$. In particular, if the coordinates $s_2,\dots,s_d$ of $\mathbf{s}$ are transcendental and algebraically independent, then Galois theory shows that

$$\psi(\zeta(\mathbf{s},i),\mathbf{s})=\psi^+(\zeta(\mathbf{s},i),\mathbf{s})=\psi^-(\zeta(\mathbf{s},i),\mathbf{s})=0$$

for $i=1,\dots,r_1^*$. The complex varieties $V_{\mathbb{C}}(\psi^-)=\{c\in(\mathbb{C}\smallsetminus\{0\})^d:\psi^-(c)=0\}$ and $V_{\mathbb{C}}(\psi^+)=\{c\in(\mathbb{C}\smallsetminus\{0\})^d:\psi^+(c)=0\}$ of ψ^- and ψ^+ thus contain generic points of $V_{\mathbb{C}}(f)$, so that $\psi^-,\psi^+\in(f)$ (cf. [Rei]). This shows that (3.19)–(3.21) imply (3.26) and hence (3.22) and completes the proof of the lemma. □

We define a Borel map $\boldsymbol{\phi}^{(k)}\colon X\longmapsto\{0,\dots,k-1\}^{\mathbb{Z}^d}$ by setting

$$(\boldsymbol{\phi}^{(k)}(x))_{\mathbf{n}}=\sum_{j=0}^{k-1}k\cdot 1_{P_j^{(k)}}(x_{\mathbf{n}})$$

for every $x=(x_{\mathbf{n}})\in X\subset\mathbb{T}^{\mathbb{Z}^d}$ and $\mathbf{n}\in\mathbb{Z}^d$, where $\mathcal{P}^{(k)}=\{P_0^{(k)},\dots,P_{k-1}^{(k)}\}$ is the partition of $\mathbb{T}$ defined before the statement of Lemma 3.3, denote by $T^{(k)}$ the shift-action (1.1) of $\mathbb{Z}^d$ on $\{0,\dots,k-1\}^{\mathbb{Z}^d}$, and set $\mu^{(k)}=\lambda_X(\boldsymbol{\phi}^{(k)})^{-1}$. From the definition of $\boldsymbol{\phi}^{(k)}$ it is clear that

$$\boldsymbol{\phi}^{(k)}\cdot\alpha_{\mathbf{n}}=T_{\mathbf{n}}^{(k)}\cdot\boldsymbol{\phi}^{(k)}$$

for every $\mathbf{n}\in\mathbb{Z}^d$, and that $\mu^{(k)}\in M_1(\{0,\dots,k-1\}^{\mathbb{Z}^d})^{T^{(k)}}$.

3.7. Lemma. *The measure $\mu=\mu^{(k)}\in M_1(\{0,\dots,k-1\}^{\mathbb{Z}^d})^{T^{(k)}}$ is summably Vershik and hence Bernoulli.*

Proof. For every $n \geq 1$ we put $m_n = \sum_{l=1}^{n}(l+d+1)^2$, $F_n = B_{2^n+2m_n}$, and define a partial cover

$$\mathcal{A}^{(n)} = \left\{ F_{n-1} + \sum_{j=1}^{d} i_j((n+d+1)^2 + 2^{n-1} + m_{n-1})\mathbf{e}^{(j)} : (i_1, \dots, i_d) \in \{1, -1\}^d \right\}$$

of F_n, $n \geq 2$, by translates of F_{n-1} with

$$\sum_{n \geq 2} \left(1 - \frac{|[\mathcal{A}^{(n)}]|}{|F_n|} \right) < \infty. \tag{3.33}$$

From (2.6) we know that

$$\bar{d}_{F_n}(\mu, \mu^{\mathcal{A}^{(n)}}) \leq \bar{d}_{[\mathcal{A}^{(n)}]}(\mu, \mu^{\mathcal{A}^{(n)}}) + \left(1 - \frac{|[\mathcal{A}^{(n)}]|}{|F_n|} \right)$$

for every $n \geq 2$, and we claim that

$$\bar{d}_{[\mathcal{A}^{(n)}]}(\mu, \mu^{\mathcal{A}^{(n)}}) < 30dk2^{d/2}n^{-3d}$$

whenever $n \geq 2kd$ satisfies (3.18). Indeed,

$$[\mathcal{A}^{(n)}] \subset Q(2^{n+1}, j)^+ \cup Q(2^{n+1}, j)^-$$

for $j = 1, \dots, d$, and Lemma 3.6 shows that the partitions

$$\bigvee_{\mathbf{n} \in [\mathcal{A}^{(n)}] \cap Q(2^{n+1}, j)^+} \mathcal{P}_{\mathbf{n}}^{(k)}, \qquad \bigvee_{\mathbf{n} \in [\mathcal{A}^{(n)}] \cap Q(2^{n+1}, j)^-} \mathcal{P}_{\mathbf{n}}^{(k)}$$

are $30k2^{d/2}n^{-3d}$-independent whenever n is sufficiently large. We define partitions $\mathcal{B}^{(n,j)}$, $j = 0, \dots, d$, of $[\mathcal{A}^{(n)}]$ by setting $\mathcal{B}^{(n,0)} = \{[\mathcal{A}^{(n)}]\}$ and

$$\mathcal{B}^{(n,j)} = \{[\mathcal{A}^{(n)}] \cap Q(2^{n+1}, 1)^{\pm} \cap \dots \cap Q(2^{n+1}, j)^{\pm}\}$$

for every $j = 1, \dots, d$. From (2.7) it is clear that

$$\bar{d}_{[\mathcal{A}^{(n)}]}(\mu^{\mathcal{B}^{(n,j-1)}}, \mu^{\mathcal{B}^{(n,j)}}) \leq 30k2^{d/2}n^{-3d}$$

for $j = 1, \dots, d$. Since $\mathcal{B}^{(n,0)} = \{[\mathcal{A}^{(n)}]\}$ and $\mathcal{B}^{(n,d)} = \mathcal{A}^{(n)}$ we conclude that

$$\bar{d}_{[\mathcal{A}^{(n)}]}(\mu, \mu^{\mathcal{A}^{(n)}}) \leq \sum_{j=1}^{d} \bar{d}_{[\mathcal{A}^{(n)}]}(\mu^{\mathcal{B}^{(n,j-1)}}, \mu^{\mathcal{B}^{(n,j)}}) \leq 30dk2^{d/2}n^{-3d}$$

and, by (2.6), that

$$\bar{d}_{F_n}(\mu, \mu^{\mathcal{A}^{(n)}}) \leq 30dk2^{d/2}n^{-3d} + \left(1 - \frac{|[\mathcal{A}^{(n)}]|}{|F_n|} \right)$$

for every sufficiently large $n \geq 2$. In conjunction with (3.33) this proves that $\mu = \mu^{(k)}$ is summably Vershik, and the final assertion follows from Theorem 2.14. □

Proof of Proposition 3.1. Let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal, and let $\alpha = \alpha^{\mathfrak{R}_d/\mathfrak{p}}$. If $h(\alpha) > 0$, then $\mathfrak{p}$ is principal by (3.6). The cases where $\mathfrak{p} = \{0\}$ or $\mathfrak{p} = (p)$ for some rational prime $p > 1$ were dealt with in the discussion preceding Lemma 3.4. In order to complete the proof of Proposition 3.1 we assume that $\mathfrak{p} = (f) = f\mathfrak{R}_d$ for some non-zero, irreducible polynomial $f = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n})u^{\mathbf{n}} \in \mathfrak{R}_d$ with at least two non-zero coefficients and apply Lemma 3.4 to assume in addition that f is nice in the sense of Definition 3.5. We realize $\alpha = \alpha^{\mathfrak{R}_d/(f)}$ as the shift-action of $\mathbb{Z}^d$ on the closed, shift-invariant subgroup $X = X^{\mathfrak{R}_d/(f)} \subset \mathbb{T}^{\mathbb{Z}^d}$ in (3.13) and observe that the Bernoullicity of α is equivalent to the assertion that the measure $\lambda_X \in M_1(\mathbb{T}^{\mathbb{Z}^d})^\sigma$ is Bernoulli, where σ is the shift-action (1.1) of $\mathbb{Z}^d$ on $\mathbb{T}^{\mathbb{Z}^d}$. If we identify each partition $\mathcal{P}_{\mathbf{n}}^{(k)}$ in (3.15) with the corresponding partition of $\mathbb{T}^{\mathbb{Z}^d} \supset X$, then Lemma 3.7 amounts to saying that the σ-invariant sigma-algebra $\mathfrak{V}^{(k)} = \bigvee_{\mathbf{n}\in\mathbb{Z}^d} \sigma_{-\mathbf{n}}(\mathcal{P}_{\mathbf{0}}^{(k)}) = \bigvee_{\mathbf{n}\in\mathbb{Z}^d} \mathcal{P}_{\mathbf{n}}^{(k)} \subset \mathfrak{B}_{\mathbb{T}^{\mathbb{Z}^d}}$ is a Bernoulli factor of σ on $(\mathbb{T}^{\mathbb{Z}^d}, \mathfrak{B}_{\mathbb{T}^{\mathbb{Z}^d}}, \lambda_X)$ for every $k \geq 2$. Since $\mathfrak{V}^{(k)} \nearrow \mathfrak{B}_{\mathbb{T}^{\mathbb{Z}^d}}$ as $k \to \infty$, Lemma 2.6 implies that λ_X is Bernoulli. □

4. The proof of Theorem 1.1

In order to deduce Theorem 1.1 from Proposition 3.1 we need a little more algebra. Let $\mathfrak{M} = \hat{X}$ be an $\mathfrak{R}_d$-module. A prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ is *associated with* $\mathfrak{M}$ if $\mathfrak{p} = \{f \in \mathfrak{R}_d : f \cdot a = 0\}$ for some $a \in \mathfrak{M}$. Conversely, if $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal and $\mathfrak{M}$ an $\mathfrak{R}_d$-module, then $\mathfrak{M}$ is *associated with* $\mathfrak{p}$ if $\mathfrak{p}$ is the only prime ideal associated with $\mathfrak{M}$, and $\mathfrak{M}$ is $\mathfrak{p}$-*elementary* if there exist finitely many submodules

$$\mathfrak{M} = \mathfrak{N}_s \supset \cdots \supset \mathfrak{N}_0 = \{0\}$$

such that $\mathfrak{N}_j/\mathfrak{N}_{j-1} \cong \mathfrak{R}_d/\mathfrak{p}$ for every $j = 1, \dots, s$. The key ingredient of the proof of Theorem 1.1 is the following result, whose verification we postpone for the moment.

4.1. Proposition. *Let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal with $h(\alpha^{\mathfrak{R}_d/\mathfrak{p}}) > 0$, and let $\mathfrak{N}$ be a $\mathfrak{p}$-elementary $\mathfrak{R}_d$-module. Then $\alpha^{\mathfrak{N}}$ is Bernoulli.*

Assuming Proposition 4.1, Theorem 1.1 is proved along exactly the same lines as the main result in [Wa2].

4.2. Lemma. ([LSW]) *Let α be a $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X, and let $\mathfrak{M} = \hat{X}$ be the $\mathfrak{R}_d$-module defined in* (3.1)–(3.2)*, The following conditions are equivalent.*

1. *For every prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ associated with $\mathfrak{M}$, the $\mathbb{Z}^d$-action $\alpha^{\mathfrak{R}_d/\mathfrak{p}}$ defined in* (3.3)–(3.4) *has positive entropy;*
2. *α has completely positive entropy.*

4.3. Lemma. ([Sc1], [ScW], [Wa2]) *Let $\mathfrak{M}$ be a Noetherian $\mathfrak{R}_d$-module with associated prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_m$. Then there exists, for each $i = 1, \dots, m$, a $\mathfrak{p}_i$-elementary $\mathfrak{R}_d$-module $\mathfrak{N}^{(i)}$, such that $\mathfrak{M}$ is isomorphic to a submodule of $\mathfrak{N} = \bigoplus_{i=1}^m \mathfrak{N}^{(i)}$.*

4.4. Lemma. *Let $(\mathfrak{M}_n,\ n \geq 1)$ be an increasing sequence of Noetherian submodules of an $\mathfrak{R}_d$-module $\mathfrak{M}$ with $\mathfrak{M} = \bigcup_{n\geq 1} \mathfrak{M}_n$. Then $\alpha^{\mathfrak{M}_n}$ is Bernoulli for every $n \geq 1$ if and only if $\alpha^{\mathfrak{M}}$ is Bernoulli.*

Proof. For every $n \geq 1$ we consider the closed, $\alpha^{\mathfrak{M}}$-invariant subgroup $Y_n = \mathfrak{M}_n^{\perp} \subset X = X^{\mathfrak{M}}$ and observe that $X/Y_n = \widehat{\mathfrak{M}_n}$, and that $\alpha^{\mathfrak{M}_n}$ is equal to the $\mathbb{Z}^d$-action α^{X/Y_n} induced by $\alpha = \alpha^{\mathfrak{M}}$ on X/Y_n. Since $\mathfrak{M}_n \nearrow \mathfrak{M}$ as $n \to \infty$, the sequence $(Y_n,\ n \geq 1)$ decreases to the trivial subgroup $\{0\} \subset X$, and the sigma-algebras $\mathfrak{B}_{X/Y_n} = \{B \in \mathfrak{B}_X : B + y = B \text{ for every } y \in Y_n\}$ increase to $\mathfrak{B}_X$. Our assumption implies that α is Bernoulli on $(X, \mathfrak{B}_{X/Y_n}, (\lambda_X)_{\mathfrak{B}_{X/Y_n}})$ for every $n \geq 1$, and [OW] guarantees that α is Bernoulli (cf. Lemma 2.6). □

Proof of Theorem 1.1. Let α be a $\mathbb{Z}^d$-action by automorphisms of a compact, abelian group X with completely positive entropy, and let $\mathfrak{M} = \hat{X}$ be the $\mathfrak{R}_d$-module defined by (3.1)–(3.1). If $\mathfrak{M}$ is not Noetherian there exists an increasing sequence of finitely generated—and hence Noetherian—submodules $(\mathfrak{N}_k,\ k \geq 1)$ with $\mathfrak{M} = \bigcup_{k\geq 1} \mathfrak{N}_k$. Lemma 4.4 implies that $\alpha = \alpha^{\mathfrak{M}}$ is Bernoulli if and only if $\alpha^{\mathfrak{N}_k}$ is Bernoulli for every $k \geq 1$. Furthermore, since every prime ideal associated with some $\mathfrak{N}_k$ is also associated with $\mathfrak{M}$, and since every prime ideal associated with $\mathfrak{M}$ is associated with some $\mathfrak{N}_k$, Lemma 4.2 guarantees that α has completely positive entropy if and only if $\alpha^{\mathfrak{N}_k}$ has completely positive entropy for every $k \geq 1$, This observation shows that it suffices to prove Theorem 1.1 under the additional assumption that the $\mathfrak{R}_d$-module $\mathfrak{M} = \hat{X}$ is Noetherian.

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the prime ideals associated with $\mathfrak{M}$, which is now assumed to be Noetherian. Then $h(\alpha^{\mathfrak{R}_d/\mathfrak{p}_i}) > 0$ for $i = 1, \dots, m$, and Lemma 4.3 allows us to find $\mathfrak{p}_i$-elementary modules $\mathfrak{N}^{(i)}$, $i = 1, \dots, m$, such that $\mathfrak{M}$ is (isomorphic to) a submodule of $\mathfrak{N} = \bigoplus_{i=1}^m \mathfrak{N}^{(i)}$. If $\bar{X} = \widehat{\mathfrak{N}}$ and $\bar{\alpha} = \alpha^{\mathfrak{N}}$, then the group homomorphism $\phi\colon \bar{X} \longmapsto X$ dual to the inclusion $\mathfrak{M} \subset \mathfrak{N}$ is continuous and surjective, and $\phi \cdot \bar{\alpha}_{\mathbf{n}} = \alpha_{\mathbf{n}} \cdot \phi$ for every $\mathbf{n} \in \mathbb{Z}^d$. Proposition 4.1 shows that $\alpha^{\mathfrak{N}^{(i)}}$ is Bernoulli for $i = 1, \dots, m$. Hence $\bar{\alpha} = \alpha^{\mathfrak{N}^{(1)}} \times \dots \times \alpha^{\mathfrak{N}^{(m)}}$ is Bernoulli, and α is Bernoulli by Lemma 2.5. □

The remainder of this section is devoted to the proof of Proposition 4.1. For every $l \geq 1$ we denote by $\sigma^{(l)}$ the shift-action of $\mathbb{Z}^d$ on $V^l = (\mathbb{T}^{\mathbb{Z}^d})^l \cong (\mathbb{T}^l)^{\mathbb{Z}^d}$. For $l = 1$ we write σ instead of $\sigma^{(1)}$.

For the next two lemmas we assume that $f = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n}) u^{\mathbf{n}} \in \mathfrak{R}_d$ is a non-zero, irreducible polynomial which is nice in the sense of Definition 3.5. We regard $Y = X^{\mathfrak{R}_d/(f)}$ as the closed, shift-invariant subgroup (3.13) of V and identify $\alpha^{\mathfrak{R}_d/(f)}$ with the restriction of σ to Y. For every $v \in V$ we define a probability measure $\lambda^{(v)} \in M_1(V)$ by setting

$$\lambda^{(v)}(B) = \lambda_Y(B + v) \tag{4.1}$$

for every $B \in \mathfrak{B}_V$.

4.5. Lemma. *The measures $\lambda^{(v)}$, $v \in V$, are uniformly almost box independent in the sense that*

$$\lim_{N\to\infty} \limsup_{M\to\infty} \biggl(\sup_{v\in V} \bar{d}_{B_M}(\lambda^{(v)}, (\lambda^{(v)})^{\mathcal{B}_N}) \biggr) = 0. \tag{4.2}$$

Proof. Proposition 3.1 and Theorem 2.3 together imply that λ_Y is Bernoulli and thus almost box independent. Hence there exists, for every $M, N \geq 1$, a probability measure $\nu^{(N)} \in C(\lambda_Y, \lambda_Y^{\mathcal{B}_N}) \subset M_1(V^2)$ which is invariant under the $\mathbb{Z}^d$-action $\mathbf{n} \mapsto \sigma^{(2)}_{(2N+1)\mathbf{n}}$ on V^2, and which satisfies that

$$\lim_{M\to\infty} \bar{d}_{B_M}(\lambda_Y, \lambda_Y^{\mathcal{B}_N}) = \lim_{M\to\infty} \frac{1}{|B_M|} \sum_{\mathbf{n}\in B_M} \int_{V^2} \delta(\pi^{(1)}_{\{\mathbf{n}\}}, \pi^{(2)}_{\{\mathbf{n}\}})\, d\nu^{(N)},$$

where $\pi^{(i)}_{\{\mathbf{n}\}}(v^{(1)}, v^{(2)}) = v^{(i)}_{\mathbf{n}}$ for every $(v^{(1)}, v^{(2)}) \in V^2$, $i = 1, 2$ and $\mathbf{n} \in \mathbb{Z}^d$. For every $v \in V$ we define a homeomorphism $R_v \colon V^2 \longmapsto V^2$ by $R_v(v^{(1)}, v^{(2)}) = (v^{(1)} + v, v^{(2)} + v)$ for every $(v^{(1)}, v^{(2)}) \in V^2$. The measure $\nu^{(N)} R_v \in M_1(V^2)$ satisfies that

$$\begin{aligned}\bar{d}_{B_M}(\lambda^{(v)}, (\lambda^{(v)})^{\mathcal{B}_N}) &= \frac{1}{|B_M|} \sum_{\mathbf{n}\in B_M} \int_{V^2} \delta(\pi^{(1)}_{\{\mathbf{n}\}}, \pi^{(2)}_{\{\mathbf{n}\}})\, d\nu^{(N)} R_v \\ &= \bar{d}_{B_M}(\lambda_Y, \lambda_Y^{\mathcal{B}_N})\end{aligned}$$

for every $M, N \geq 0$, and by letting first M and then N tend to infinity we obtain (4.2). □

4.6. Lemma. *Let $f \in \mathfrak{R}_d$ be a nice, irreducible polynomial with $h(\alpha^{\mathfrak{R}_d/(f)}) > 0$, and let $\mathfrak{N}$ be an (f)-elementary $\mathfrak{R}_d$-module. Then $\alpha^{\mathfrak{N}}$ is Bernoulli.*

Proof. Suppose that $\mathfrak{N}$ is of the form $\mathfrak{N} = \mathfrak{N}_s \supset \cdots \supset \mathfrak{N}_0 = \{0\}$ with $\mathfrak{N}_j/\mathfrak{N}_{j-1} \cong \mathfrak{R}_d/(f)$ for every $j = 1, \dots, s$. We prove the Bernoullicity of $\alpha^{\mathfrak{N}}$ by induction on s. If $s = 1$ then $\mathfrak{N} = \mathfrak{R}_d/(f)$, and $\alpha^{\mathfrak{N}}$ is Bernoulli by Proposition 3.1. Assume therefore that $s > 1$, and that we have proved the Bernoullicity of $\alpha^{\mathfrak{N}'}$ for every (f)-elementary $\mathfrak{R}_d$-module $\mathfrak{N}'$ of the form $\mathfrak{N}' = \mathfrak{N}'_{s-1} \supset \cdots \supset \mathfrak{N}'_0 = \{0\}$ with $\mathfrak{N}'_j/\mathfrak{N}'_{j-1} \cong \mathfrak{R}_d/(f)$ for every $j = 1, \dots, s-1$.

Now assume that $\mathfrak{N}$ is an (f)-elementary $\mathfrak{R}_d$-module with submodules $\mathfrak{N} = \mathfrak{N}_s \supset \cdots \supset \mathfrak{N}_0 = \{0\}$ such that$\mathfrak{N}_j/\mathfrak{N}_{j-1} \cong \mathfrak{R}_d/(f)$ for every $j = 1, \dots, s$. Choose elements $a_1, \dots, a_s$ in $\mathfrak{N} = \mathfrak{N}_s$ such that $\mathfrak{N}_j = \mathfrak{R}_d \cdot a_j + \mathfrak{N}_{j-1}$ for $j = 1, \dots, s$ and consider the corresponding surjective homomorphism $\hat{\psi} \colon \mathfrak{R}_d^s \longmapsto \mathfrak{N}$ with $\hat{\psi}(f_1, \dots, f_s) = \sum_{i=1}^s f_i \cdot a_i$ for every $(f_1, \dots, f_s) \in \mathfrak{R}_d^s$. The injective dual homomorphism $\psi \colon X^{\mathfrak{N}} \longmapsto V^s = \widehat{\mathfrak{R}_d^s}$ satisfies that $\sigma^{(s)}_{\mathbf{n}} \cdot \psi = \psi \cdot \alpha^{\mathfrak{N}}_{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^d$, and allows us to regard $X = X^{\mathfrak{N}}$ as a closed, shift-invariant subgroup of V^s. Furthermore, if $X_j = \mathfrak{N}_j^{\perp} \subset X \subset V^s$, then $X_0 = X$ and

$$\begin{aligned} X_j &= \{x \in X : \pi^{(1)}(x) = \cdots = \pi^{(j)}(x) = 0\}, \\ X_{j-1}/X_j &\cong X^{\mathfrak{R}_d/(f)}, \\ X^{\mathfrak{N}_j} = X/X_j &\cong \eta^{(j)}(X) \subset V^{s-1} \end{aligned} \tag{4.3}$$

for every $j = 1, \dots, s$.

We set $W = \eta^{(s-1)}(X) \subset V^{s-1}$ and note that $\pi^{(s)}(X_{s-1}) = Y = X^{\mathfrak{R}_d/(f)}$. According to (4.3) our induction hypothesis implies that the restriction of $\sigma^{(s-1)}$ to W is Bernoulli, and Proposition 3.1 guarantees that $\sigma = \sigma^{(1)}$ is Bernoulli on Y. As in (2.12)–(2.13) we obtain a family $\{\mu_w : w \in V^{s-1}\} \subset M_1(V)$ with

$$\int h\, d\lambda_X = \int_{V^{s-1}} \int_V h(v^{(1)}, \dots, v^{(s)})\, d\mu_{(v^{(1)}, \dots, v^{(s-1)})}(v^{(s)})\, d\lambda_W(v^{(1)}, \dots, v^{(s-1)}),$$

and (4.3) implies that there exists, for λ_W-a.e. $w \in V^{s-1}$, an element $v(w) \in V$ with

$$\mu_w = \lambda^{(v(w))}.$$

By Lemma 4.5 and Definition 2.11, λ_X is relatively almost box independent with respect to V^{s-1}, and Theorem 2.12 shows that $\lambda_X = \lambda_{X^{\mathfrak{N}}} \in M_1(V^s)^{\sigma^{(s)}}$ is Bernoulli. $\square$

Proof of Proposition 4.2. Let $\mathfrak{p} \subset \mathfrak{R}_d$ be a prime ideal with $h(\alpha^{\mathfrak{R}_d/\mathfrak{p}}) > 0$, and let $\mathfrak{N}$ be a $\mathfrak{p}$-elementary $\mathfrak{R}_d$-module. As we saw in the proof of Proposition 3.1, $\mathfrak{p}$ is principal, and the description at the beginning of the proof of Lemma 4.6 shows that there exists an $s \geq 1$ and elements $a_1, \dots, a_s$ in $\mathfrak{N}$ such that $\mathfrak{N} = \sum_{j=1}^{s} \mathfrak{R}_d \cdot a_j$ and $\{h \in \mathfrak{R}_d : h \cdot a_j \in \mathfrak{N}_{j-1}\} = \mathfrak{p}$ for every $j = 1, \dots, s$, where $\mathfrak{N}_0 = \{0\}$ and $\mathfrak{N}_j = \sum_{i=1}^{j} \mathfrak{R}_d \cdot a_i$ for $j = 1, \dots, s$.

In particular, if $\mathfrak{p} = \{0\}$, then $\{a_1, \dots, a_s\}$ is linearly independent over $\mathfrak{R}_d$, $\mathfrak{N} \cong \mathfrak{R}_d^s$, and $\alpha^{\mathfrak{N}}$ is conjugate to the shift-action (1.1) of $\mathbb{Z}^d$ on $(\mathbb{T}^s)^{\mathbb{Z}^d}$ and hence Bernoulli.

If $\mathfrak{p} = (p)$ for some rational prime $p > 1$ we set $X_j = \mathfrak{N}_j^{-} \subset X = X^{\mathfrak{N}} = \widehat{\mathfrak{N}}$ for $j = 0, \dots, s$. Then $X_s = \{0\}$, $X_{s-1} \cong X^{\mathfrak{R}_d/\mathfrak{p}}$, and (3.14) allows us to identify the $\mathbb{Z}^d$-action $\alpha^{X_{s-1}}$ induced by $\alpha = \alpha^{\mathfrak{N}}$ on X_{s-1} with the shift-action σ of $\mathbb{Z}^d$ on $(\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}^d}$. Following [Wa2] we claim that there exists a Haar measure preserving Borel isomorphism $\phi: X \longmapsto X/X_{s-1} \times (\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}^d}$ which carries α to the cartesian product $\alpha^{X/X_{s-1}} \times \sigma$, where $\alpha^{X/X_{s-1}}$ is the $\mathbb{Z}^d$-action induced by α on X/X_{s-1}.

In order to construct ϕ we set $W = X/X_{s-1}$ and choose a Borel map $\zeta: W \longmapsto X$ with $\zeta(x + X_{s-1}) + X_{s-1} = x + X_{s-1}$ for every $x \in X$ (cf. Lemma 1.5.1 in [Par]), define a Borel isomorphism $\psi: X \longmapsto W \times X_{s-1}$ by setting $\psi(x) = (x + X_{s-1}, x - \zeta(x + X_{s-1}))$ for every $x \in X$, and use the identification (3.14) of X_{s-1} with $V = (\mathbb{Z}/p\mathbb{Z})^{\mathbb{Z}^d}$ to regard ψ as an isomorphism $\psi: X \longmapsto W \times V$. The $\mathbb{Z}^d$-action α' on $W \times V$ defined by $\alpha'_{\mathbf{n}} = \psi \cdot \alpha_{\mathbf{n}} \cdot \psi^{-1}$ is of the form

$$\alpha'_{\mathbf{n}}(w, v) = (\alpha^W_{\mathbf{n}}(w), \sigma_{\mathbf{n}}(v) + c(\mathbf{n}, w)) \tag{4.4}$$

for every $\mathbf{n} \in \mathbb{Z}$, where $c: \mathbb{Z}^d \times W \longmapsto V$ is a Borel map with

$$\sigma_{\mathbf{m}}(c(\mathbf{n}, w)) + c(\mathbf{m}, \alpha^W_{\mathbf{n}}(w)) = c(\mathbf{m} + \mathbf{n}, w) \tag{4.5}$$

for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$ and $w \in W$. If c is of the form

$$c(\mathbf{n}, \cdot) = \sigma_{\mathbf{n}} \cdot \gamma - \gamma \cdot \alpha^W_{\mathbf{n}} \tag{4.6}$$

for every $\mathbf{n} \in \mathbb{Z}^d$, where $\gamma: W \longmapsto V$ is Borel, then the map $\phi(w, v) = (w, v + \gamma(w))$ from $W \times V$ to $W \times V$ carries α' to the product action $\alpha^W \times \sigma$ of $\mathbb{Z}^d$ on $W \times V$. In order to find a solution γ of (4.6) we write $\pi_{\{\mathbf{n}\}}: V \longmapsto \mathbb{Z}/p\mathbb{Z}$ for the $\mathbf{n}$-th coordinate projection and set $c_{\mathbf{n}}(\mathbf{m}, w) = \pi_{\{\mathbf{n}\}}(c(\mathbf{m}, w))$ for every $\mathbf{n} \in \mathbb{Z}^d$ and $w \in W$. Then (4.6) is equivalent to the solution of the equations

$$c_{\mathbf{m}}(\mathbf{n}, w) = \gamma_{\mathbf{m}+\mathbf{n}}(w) - \gamma_{\mathbf{m}}(\alpha^W_{\mathbf{n}}(w)) \tag{4.7}$$

for every $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$ and $w \in W$ in terms of Borel maps $\gamma_{\mathbf{m}}: W \longmapsto \mathbb{Z}/p\mathbb{Z}$, $\mathbf{m} \in \mathbb{Z}^d$; if all these equations can be solved, then $\gamma: W \longmapsto V$ is obtained by setting $\pi_{\{\mathbf{m}\}} \cdot \gamma = \gamma_{\mathbf{m}}$ for every $\mathbf{m} \in \mathbb{Z}^d$.

In order to solve (4.7) we set, for every $w \in W$, $\gamma_{\mathbf{0}}(w) = 0$ and use (4.5) to solve (4.7) inductively for $\mathbf{n} = \mathbf{0}$ and for every $\mathbf{n} \in \mathbb{Z}^d$.

This shows that α is indeed conjugate to $\alpha^{X/X_{s-1}} \times \sigma$ on $X/X^{s-1} \times V$, and by replacing $\mathfrak{N} = \mathfrak{N}_s$ with $\mathfrak{N}_{s-1}$, X with $\widehat{\mathfrak{N}_{s-1}} = X/X_{s-1}$, and α with $\alpha^{\mathfrak{N}_{s-1}}$ we see that α is conjugate to $\alpha^{X/X_{s-1}} \times \sigma \times \sigma$ on $X/X_{s-2} \times V^2$. By using induction we obtain after s steps that α is conjugate to $\sigma \times \cdots \times \sigma$ on $V \times \cdots \times V = V^s$, and hence Bernoulli.

This shows that α is conjugate to $\alpha^{X/X_{s-1}} \times \sigma$ on $X/X^{s-1} \times V$, and by replacing $\mathfrak{N} = \mathfrak{N}_s$ with $\mathfrak{N}_{s-1}$, X with $\widehat{\mathfrak{N}_{s-1}} = X/X_{s-1}$, and α with $\alpha^{\mathfrak{N}_{s-1}}$ we see that α is conjugate to $\alpha^{X/X_{s-1}} \times \sigma \times \sigma$ on $X/X_{s-2} \times V^2$. By using induction we obtain after s steps that α is conjugate to $\sigma \times \cdots \times \sigma$ on $V \times \cdots \times V = V^s$, and hence Bernoulli.

Finally we have to deal with the case where $\mathfrak{p} = (f)$ for some irreducible element $f = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_f(\mathbf{n}) u^{\mathbf{n}} \in \mathfrak{R}_d$ which has at least two non-zero coefficients. The same consideration as in Lemma 3.4 allows us to assume that f is nice (Definition 3.5), in which case the Bernoullicity of $\alpha = \alpha^{\mathfrak{N}}$ is proved in Lemma 4.6. □

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