

**Cosmological Solutions of the Vlasov–Einstein System  
with Spherical, Plane, and Hyperbolic Symmetry****Gerhard Rein**

Vienna, Preprint ESI 147 (1994)

October 24, 1994

Supported by Federal Ministry of Science and Research, Austria  
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# Cosmological solutions of the Vlasov-Einstein system with spherical, plane, and hyperbolic symmetry

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## Abstract

The Vlasov-Einstein system describes a self-gravitating, collisionless gas within the framework of general relativity. We investigate the initial value problem in a cosmological setting with spherical, plane, or hyperbolic symmetry and prove that for small initial data solutions exist up to a spacetime singularity which is a curvature and a crushing singularity. An important tool in the analysis is a local existence result with a continuation criterion saying that solutions can be extended as long as the momenta in the support of the phase-space distribution of the matter remain bounded.

## 1 Introduction

When describing the evolution of self-gravitating matter fields within the context of general relativity, the choice of the matter model is crucial. One can, for example, describe the matter as a perfect fluid, as dust, or as a collisionless gas. In the latter case the matter is represented by a number density  $f$  on phase-space, i. e., on the tangent bundle  $TM$  of the spacetime manifold  $M$ . The phase-space density  $f$  satisfies a continuity equation, the so-called Vlasov equation, which says that  $f$  is constant along the geodesics of the spacetime metric. Taking the energy-momentum tensor  $T^{\alpha\beta}$  generated by  $f$  as the source term in Einstein's field equations, one obtains the Vlasov-Einstein system for a self-gravitating, collisionless gas:

$$p^\alpha \partial_{x^\alpha} f - \Gamma_{\beta\gamma}^\alpha p^\beta p^\gamma \partial_{p^\alpha} f = 0,$$

$$G^{\alpha\beta} = 8\pi T^{\alpha\beta},$$

$$T^{\alpha\beta} = \int p^\alpha p^\beta f |g|^{1/2} \frac{d^4p}{m},$$

where  $\Gamma_{\beta\gamma}^\alpha$  denote the Christoffel symbols of the spacetime metric  $g_{\alpha\beta}$ ,  $|g|$  denotes its determinant,  $G^{\alpha\beta}$  the Einstein tensor,  $x^\alpha$  are coordinates on  $M$ ,  $p^\alpha$  the corresponding coordinates on the tangent space, greek indices always run from 0 to 3, and

$$m = \left| g_{\alpha\beta} p^\alpha p^\beta \right|^{1/2}$$

is the rest mass of a particle at the corresponding phase-space point.

In [5] this system was investigated in the asymptotically flat, spherically symmetric case, i. e., with a metric of the form

$$ds^2 = -e^{2\mu} dt^2 + e^{2\lambda} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

where  $(t, r, \theta, \varphi)$  are the usual Schwarzschild coordinates and  $\lambda$  and  $\mu$  depend only on  $t$  and  $r$  and vanish at  $r = \infty$ . The main result was that small initial data lead to global, geodesically complete solutions, a result which has no analogue for perfect fluids or dust. Indeed, Christodoulou has shown that in the gravitational collapse of a dust cloud naked singularities can develop even for small data [1]. Therefore, the Vlasov model seems to be particularly suited to describe the behaviour of matter in general relativity. This is further substantiated by the fact that for the Vlasov-Poisson system, which is the Newtonian analogue of the Vlasov-Einstein system, there are global existence results both in the case of an isolated system, which corresponds to the asymptotically flat case in the relativistic problem, and in the cosmological case, cf. [3, 12] and [8].

In the present paper we investigate the Vlasov-Einstein system in a cosmological setting. In order to simplify the problem we assume that the system has a high degree of symmetry and take the metric to be of the form

$$ds^2 = -e^{2\mu} dt^2 + e^{2\lambda} dr^2 + t^2(d\theta^2 + \sin_\epsilon^2\theta d\varphi^2), \quad (1.1)$$

where

$$\sin_\epsilon \theta = \begin{cases} \sin \theta & \text{for } \epsilon = 1, \\ 1 & \text{for } \epsilon = 0, \\ \sinh \theta & \text{for } \epsilon = -1, \end{cases}$$

$t > 0$  denotes a timelike coordinate,  $r \in [0, 1]$ , and the functions  $\lambda$  and  $\mu$  depend only on  $t$  and  $r$  and are periodic in  $r$ . The angular coordinates  $\theta$

and  $\varphi$  parametrize the surfaces of constant  $t$  and  $r$ , which are the orbits of the symmetry action and which are spheres in the case of spherical symmetry  $\epsilon = 1$ , tori in the case of plane symmetry  $\epsilon = 0$ , and hyperbolic planes in the case of hyperbolic symmetry  $\epsilon = -1$ . They range in the domains  $[0, \pi] \times [0, 2\pi]$ ,  $[0, 2\pi] \times [0, 2\pi]$ , or  $[0, \infty[ \times [0, 2\pi]$  respectively. It should be pointed out that the coordinates  $(t, r, \theta, \varphi)$  will in general not cover the whole spacetime manifold, but they do cover a neighborhood of the singularity at  $t = 0$  which will allow us to investigate the nature of this singularity. One way to think of the above metric is to consider the Schwarzschild metric

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

If one passes through the event horizon at  $r = 2M$  then the  $(0,0)$ - and  $(1,1)$ -components of the metric change sign so that the Schwarzschild radius  $r$  becomes the timelike coordinate. If one interchanges the notation for  $t$  and  $r$  and compactifies the hypersurfaces of constant  $t$  by making the components of the metric periodic in  $r$  one obtains a metric of the type (1.1) with  $\epsilon = 1$ .

As in [5] we restrict ourselves to the case where all particles have the same rest mass, normalized to 1, and move forward in time, i. e.,  $f$  is supported on the submanifold

$$PM := \left\{ g_{\alpha\beta} p^\alpha p^\beta = -1, p^0 > 0 \right\}$$

of the tangent bundle, which is invariant under the geodesic flow. Due to the symmetry the distribution function  $f$  can be written as a function of  $t, r, w := e^\lambda p^1$ , and  $F := t^4(p^2)^2 + t^4 \sin^2\theta (p^3)^2$ . After calculating the Vlasov equation in these variables and the non-trivial components of the Einstein tensor and the energy-momentum tensor and denoting by  $\dot{\phantom{x}}$  and  $\prime$  the derivatives of the metric components with respect to  $t$  or  $r$  respectively, the complete Vlasov-Einstein system reads as follows:

$$\partial_t f + \frac{e^{\mu-\lambda} w}{\sqrt{1+w^2+F/t^2}} \partial_r f - \left( \dot{\lambda} w + e^{\mu-\lambda} \mu' \sqrt{1+w^2+F/t^2} \right) \partial_w f = 0, \quad (1.2)$$

$$e^{-2\mu} (2t\dot{\lambda} + 1) + \epsilon = 8\pi t^2 \rho, \quad (1.3)$$

$$e^{-2\mu} (2t\dot{\mu} - 1) - \epsilon = 8\pi t^2 p, \quad (1.4)$$

$$\mu' = -4\pi t e^{\mu+\lambda} j, \quad (1.5)$$

$$e^{-2\lambda}(\mu'' + \mu'(\mu' - \lambda')) - e^{-2\mu}(\ddot{\lambda} + (\dot{\lambda} + 1/t)(\dot{\lambda} - \dot{\mu})) = 8\pi q, \quad (1.6)$$

where

$$\rho(t, r) := T_0^0(t, r) = \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1 + w^2 + F/t^2} f(t, r, w, F) dF dw, \quad (1.7)$$

$$p(t, r) := T_1^1(t, r) = \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{w^2}{\sqrt{1 + w^2 + F/t^2}} f(t, r, w, F) dF dw, \quad (1.8)$$

$$j(t, r) := -e^{\lambda - \mu} T_0^1(t, r) = \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} w f(t, r, w, F) dF dw, \quad (1.9)$$

$$q(t, r) := T_2^2(t, r) = \frac{\pi}{2t^4} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{F}{\sqrt{1 + w^2 + F/t^2}} f(t, r, w, F) dF dw; \quad (1.10)$$

the (3,3)-component of the field equations which is also non-trivial coincides with the (2,2)-component due to the symmetry. Note that on  $PM$  we can express  $p^0$  by the other coordinates and in the above new variables obtain

$$p^0 = e^{-\mu} \sqrt{1 + w^2 + F/t^2}.$$

Note also that  $F$  is a conserved quantity of the geodesic flow, the modulus of the angular momentum of particles, and thus there is no  $F$ -derivative in the Vlasov equation. Furthermore, the latter equation does not depend on  $\epsilon$ .

We are going to study the initial value problem corresponding to this system and prescribe initial data at time  $t=1$ ,

$$f(1, r, w, F) = \mathring{f}(r, w, F), \quad \lambda(1, r) = \mathring{\lambda}(r), \quad \mu(1, r) = \mathring{\mu}(r).$$

Our main result is that the solutions to this initial value problem exist on the time interval  $]0, 1]$  provided the data satisfy a certain smallness assumption, and we prove that the singularity at  $t=0$  is not just a coordinate singularity, but a “real” spacetime singularity, a curvature and a crushing singularity.

The paper proceeds as follows: In the next section we extract a certain subsystem from the full Vlasov-Einstein system (1.2)–(1.6) and show that this subsystem is equivalent to the full system. In Section 3 we prove a local-in-time existence and uniqueness result for classical solutions, together with a continuation criterion which says that when going backward in time, i. e., towards the singularity a solution can be extended as long as the support of  $f$  remains bounded with respect to  $w$ . The main difficulty here is to show that a solution cannot break down due to a blow-up of a derivative of

$f$  (or one of its moments), that is to say, there is no formation of shocks. This continuation criterion is used in Section 4 to prove that solutions exist on  $]0, 1]$  for sufficiently small initial data, the support of  $f$  with respect to  $w$  is shown to decay like  $t^c$  for some  $c > 0$  as  $t \rightarrow 0$ . The structure of the singularity at  $t=0$  is analyzed in Section 5. In the last section we briefly investigate the behaviour of the solutions for  $t > 1$ . In order to extend a solution forward in time one needs to bound the support of  $f$  with respect to  $w$  and the metric component  $e^{2\mu}$ . A bound on the former quantity can be established regardless of the size of the initial data, but for  $\epsilon = 1$  it can be shown that  $e^{2\mu}$  blows up in finite coordinate time.

As to the physical relevance of the situation studied here our point of view is that some results in general relativity are intended to describe concrete, real-world phenomena while others are intended to elucidate general features of the theory like for example the structure of possible singularities. The present paper belongs to the second category.

An approach which may generalize to other situations more easily than the present one but gives less information on the structure of the singularity and on questions of existence of solutions is taken in [11], where the system with spherical and plane symmetry is analyzed using constant mean curvature slicing. As is shown in [10] homogeneous solutions, i. e., solutions which are independent of  $r$  have a curvature singularity provided  $f$  is not identically zero.

To conclude this introduction we mention some further results on the asymptotically flat, spherically symmetric Vlasov-Einstein system: In [6] it is shown that solutions of this system converge to solutions of the Vlasov-Poisson system in the Newtonian limit. In [9] it is shown that if a singularity forms the first one has to form at the centre of symmetry. The existence of static solutions is established in [4, 7].

**Acknowledgements:** I would like to thank A. D. Rendall for helpful discussions and comments. The present investigation was completed during a stay at the Erwin Schrödinger International Institute for Mathematical Physics in Vienna. I would like to thank the Institute and in particular P. Aichelburg and R. Beig for the kind invitation.

## 2 Equivalent subsystems

Let us first make precise the regularity properties which we require of a solution:

**Definition 2.1** Let  $I \subset \mathbb{R}^+$  be an interval.

- (a)  $f \in C^1(I \times \mathbb{R}^2 \times \mathbb{R}_0^+)$  is regular, if  $f(t, r+1, w, F) = f(t, r, w, F)$  for  $(t, r, w, F) \in I \times \mathbb{R}^2 \times \mathbb{R}_0^+$ ,  $f \geq 0$ , and  $\text{supp} f(t, r, \cdot, \cdot)$  is compact, uniformly in  $r$  and locally uniformly in  $t$ .
- (b)  $\rho$  (or  $p, j, q$ )  $\in C^1(I \times \mathbb{R})$  is regular, if  $\rho(t, r+1) = \rho(t, r)$  for  $(t, r) \in I \times \mathbb{R}$ .
- (c)  $\lambda \in C^1(I \times \mathbb{R})$  is regular, if  $\dot{\lambda} \in C^1(I \times \mathbb{R})$  and  $\lambda(t, r+1) = \lambda(t, r)$  for  $(t, r) \in I \times \mathbb{R}$ .
- (d)  $\mu \in C^1(I \times \mathbb{R})$  is regular, if  $\mu' \in C^1(I \times \mathbb{R})$  and  $\mu(t, r+1) = \mu(t, r)$  for  $(t, r) \in I \times \mathbb{R}$ .

We identify such functions with their restrictions to the interval  $[0, 1]$  with respect to  $r$ . The fact that regularity means different things for different objects will cause no ambiguities.

Let  $(f, \lambda, \mu)$  be a regular solution of the subsystem (1.2), (1.3), (1.3) on an interval  $I$  with  $1 \in I$ . We want to show that (1.5) and (1.6) hold as well. Integrating (1.3) we obtain

$$te^{-2\mu(t,r)} = e^{-2\mu^\circ(r)} - \epsilon(t-1) - 8\pi \int_1^t p(s,r) s^2 ds, \quad (2.1)$$

and

$$-2t\mu'(t,r)e^{-2\mu(t,r)} = -2\mu^\circ(r)e^{-2\mu^\circ(r)} - 8\pi \int_1^t p'(s,r) s^2 ds.$$

From (1.2) and integration by parts it follows that

$$\begin{aligned} \int_1^t p'(s,r) s^2 ds &= \pi \int_1^t \int_{-\infty}^{\infty} \int_0^{\infty} \frac{w^2}{\sqrt{1+w^2+F/s^2}} \partial_r f(s,r,w,F) dF dw ds \\ &= \int_1^t (\dot{\lambda} - \dot{\mu}) e^{\lambda-\mu} j(s,r) s^2 ds - e^{\lambda-\mu} j(s,r) s^2 \Big|_{s=1}^{s=t} \\ &\quad - \int_1^t \mu'(s,r) (\rho(s,r) + p(s,r)) s^2 ds \\ &\quad - 2 \int_1^t \dot{\lambda}(s,r) e^{\lambda-\mu} j(s,r) s^2 ds. \end{aligned}$$

Adding (1.3) and (1.3) yields

$$\dot{\lambda} + \dot{\mu} = 4\pi t e^{2\mu} (\rho + p), \quad (2.2)$$

and if we assume that the constraint equation (1.5) holds at time  $t=1$  then these identities imply

$$te^{-2\mu} (\mu' + 4\pi te^{\lambda+\mu} j) = -4\pi \int_1^t (\rho + p) (\mu' + 4\pi se^{\lambda+\mu} j) s^2 ds$$

so that

$$\mu' + 4\pi te^{\lambda+\mu} j = 0$$

on  $I$ , i. e., (1.5) holds for all  $t \in I$ . The latter equation can be differentiated with respect to  $r$  to yield

$$\mu'' = (\lambda' + \mu')\mu' - 4\pi te^{\lambda+\mu} j'.$$

From (1.2) we obtain by integration by parts the identity

$$\begin{aligned} j'(t, r) &= \frac{\pi}{t^2} e^{\lambda-\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \left[ -\sqrt{1+w^2+F/t^2} \partial_t f \right. \\ &\quad \left. + \left( \dot{\lambda} w \sqrt{1+w^2+F/t^2} + e^{\mu-\lambda} \mu' (1+w^2+F/t^2) \right) \partial_w f \right] dF dw \\ &= -\frac{\pi}{t^2} e^{\lambda-\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} \partial_t f dF dw - e^{\lambda-\mu} \dot{\lambda} (\rho + p) - 2\mu' j. \end{aligned}$$

Equation (1.3) can be rewritten in the form

$$\dot{\lambda} = 4\pi te^{2\mu} \rho - \frac{1 + \epsilon e^{2\mu}}{2t}. \quad (2.3)$$

Since

$$\begin{aligned} \dot{\rho}(t, r) &= -\frac{2\rho(t, r)}{t} - \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{F/t^3}{\sqrt{1+w^2+F/t^2}} f dF dw \\ &\quad + \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} \partial_t f dF dw \\ &= -\frac{2\rho(t, r)}{t} - \frac{2q(t, r)}{t} + \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} \sqrt{1+w^2+F/t^2} \partial_t f dF dw, \end{aligned}$$

differentiating (2.3) with respect to  $t$  yields

$$\begin{aligned} \ddot{\lambda} &= -4\pi e^{2\mu} \rho + 2\dot{\lambda}\dot{\mu} + \frac{\dot{\mu}}{t} - 8\pi e^{2\mu} q \\ &\quad + \frac{4\pi^2}{t} e^{2\mu} \int \sqrt{1+w^2+F/t^2} \partial_t f dF dw + \frac{1 + \epsilon e^{2\mu}}{2t^2}. \end{aligned}$$

Combining these identities implies the remaining field equation (1.6). Thus we have established the following result:

**Proposition 2.2** *Let  $(f, \lambda, \mu)$  be a regular solution of (1.2), (1.3), (1.3) on some time interval  $I \subset \mathbb{R}^+$  with  $1 \in I$ , and let the initial data satisfy (1.5) for  $t = 1$ . Then (1.5) and (1.6) hold for all  $t \in I$ .*

When proving local existence it now suffices to consider the subsystem (1.2), (1.3), (1.3). However, it would then become technically very unpleasant to control  $\mu'$ . It is more convenient to consider an auxiliary system, which consists of the modified Vlasov equation

$$\partial_t f + \frac{e^{\mu-\lambda} w}{\sqrt{1+w^2+F/t^2}} \partial_r f - \left( \dot{\lambda} w + e^{\mu-\lambda} \tilde{\mu} \sqrt{1+w^2+F/t^2} \right) \partial_w f = 0, \quad (2.4)$$

together with (1.3), (1.3), and

$$\tilde{\mu} = -4\pi t e^{\lambda+\mu} j. \quad (2.5)$$

Assume that we have a regular solution  $(f, \lambda, \mu, \tilde{\mu})$  of this system, where regularity for  $\tilde{\mu}$  means that this function has the same properties as  $\mu'$  for  $\mu$  regular. We want to show that  $\tilde{\mu}$  is nothing else than  $\mu'$  so that by Proposition 2.2  $(f, \lambda, \mu)$  solves the full system. As above,

$$t\mu'(t, r)e^{-2\mu} = \overset{\circ}{\mu}'(r)e^{-2\overset{\circ}{\mu}} + 4\pi \int_1^t p'(s, r) s^2 ds$$

and

$$\int_1^t p' s^2 ds = - \int_1^t (\dot{\lambda} + \dot{\mu}) e^{\lambda-\mu} j s^2 ds - e^{\lambda-\mu} j s^2 \Big|_{s=1}^{s=t} - \int_1^t \tilde{\mu} (\rho + p) s^2 ds.$$

Using (2.2) and (2.5) we obtain

$$\mu' t e^{-2\mu} = \overset{\circ}{\mu}' e^{-2\overset{\circ}{\mu}} - 4\pi t^2 e^{\lambda-\mu} j + 4\pi e^{\overset{\circ}{\lambda}-\overset{\circ}{\mu}} \overset{\circ}{j}$$

so that (1.5) holds for all  $t \in I$  if it holds for  $t = 1$ .

**Proposition 2.3** *Let  $(f, \lambda, \mu, \tilde{\mu})$  be a regular solution of (2.4), (1.3), (1.3), (2.5). Then  $(f, \lambda, \mu)$  solves (1.2)–(1.6).*

We conclude this section with a result which reflects the conservation of the number of particles in our system and is an immediate consequence of the Vlasov equation.

**Proposition 2.4** *Assume that  $f$  is regular and satisfies (1.2) with regular coefficients  $\lambda$  and  $\mu$ . Then*

$$\partial_t \left( e^\lambda \int_{-\infty}^{\infty} \int_0^{\infty} f dF dw \right) + \partial_r \left( e^\mu \int_{-\infty}^{\infty} \int_0^{\infty} \frac{w}{\sqrt{1+w^2+F/t^2}} f dF dw \right) = 0$$

and

$$\int_0^1 \int_{-\infty}^{\infty} \int_0^{\infty} e^{\lambda(t,r)} f(t,r,w,F) dF dw dr$$

is conserved.

### 3 Local existence and continuation of solutions

In this section we prove the following local existence and uniqueness result with the continuation criterion described in the introduction:

**Theorem 3.1** *Let  $\mathring{f} \in C^1(\mathbb{R}^2 \times \mathbb{R}_0^+)$  with  $\mathring{f}(r+1, w, F) = \mathring{f}(r, w, F)$  for  $(r, w, F) \in \mathbb{R}^2 \times \mathbb{R}_0^+$ ,  $\mathring{f} \geq 0$ , and*

$$\begin{aligned} w_0 &:= \sup \left\{ |w| \mid (r, w, F) \in \text{supp } \mathring{f} \right\} < \infty, \\ F_0 &:= \sup \left\{ F \mid (r, w, F) \in \text{supp } \mathring{f} \right\} < \infty. \end{aligned}$$

Let  $\mathring{\lambda}, \mathring{\mu} \in C^1(\mathbb{R})$  with  $\mathring{\lambda}(r) = \mathring{\lambda}(r+1)$ ,  $\mathring{\mu}(r+1) = \mathring{\mu}(r)$  for  $r \in \mathbb{R}$  and

$$\mathring{\mu}'(r) = -4\pi e^{\mathring{\lambda} + \mathring{\mu}} \mathring{j}(r), \quad r \in \mathbb{R}.$$

*In the case of hyperbolic symmetry  $\epsilon = -1$  assume in addition that  $\mathring{\mu}(r) < 0$  for  $r \in \mathbb{R}$ . Then there exists a unique, left maximal, regular solution  $(f, \lambda, \mu)$  of (1.2)–(1.6) with  $(f, \lambda, \mu)(1) = (\mathring{f}, \mathring{\lambda}, \mathring{\mu})$  on a time interval  $]T, 1]$  with  $T \in [0, 1[$ . If*

$$\sup \left\{ |w| \mid (t, r, w, F) \in \text{supp } f \right\} < \infty$$

then  $T = 0$ .

**Remark:** To motivate the restriction on  $\mathring{\mu}$  in the case  $\epsilon = -1$  consider the following “pseudo-Schwarzschild” solution

$$ds^2 = - \left( 1 + \frac{2M}{t} \right)^{-1} dt^2 + \left( 1 + \frac{2M}{t} \right) dr^2 + t^2 (d\theta^2 + \sinh^2 \theta d\varphi^2)$$

with  $M \in \mathbb{R}$  which is a vacuum solution of our system. The restriction  $\mathring{\mu} < 0$  is equivalent to  $M > 0$  so that this case is contained in the present investigation. For  $M = 0$  the spacetime is flat, and for  $M < 0$  it has a coordinate-singularity at  $t = -2M$ , but the spacetime can be extended through this singularity, something we want to exclude from our investigation.

*Proof of Theorem 3.1:* Define

$$\mathring{\tilde{\mu}} := \mathring{\mu}' ,$$

and consider the auxiliary system (2.4), (1.3), (1.3), (2.5). We construct a sequence of iterative solutions in the following way:

*Iterative scheme:* Let  $\lambda_0(t, r) := \mathring{\lambda}(r)$ ,  $\mu_0(t, r) := \mathring{\mu}(r)$ ,  $\tilde{\mu}_0(t, r) := \mathring{\tilde{\mu}}(r)$  for  $t \in ]0, 1]$ ,  $r \in \mathbb{R}$ . If  $\lambda_{n-1}$ ,  $\mu_{n-1}$ ,  $\tilde{\mu}_{n-1}$  are already defined and regular on  $]0, 1] \times \mathbb{R}$  then let

$$G_{n-1}(t, r, w, F) := \left( \frac{e^{\mu_{n-1} - \lambda_{n-1}} w}{\sqrt{1 + w^2 + F/t^2}}, -\dot{\lambda}_{n-1} w - e^{\mu_{n-1} - \lambda_{n-1}} \tilde{\mu}_{n-1} \sqrt{1 + w^2 + F/t^2} \right)$$

and denote by  $(R_n, W_n)(s, t, r, w, F)$  the solution of the characteristic system

$$\frac{d}{ds}(R, W) = G_{n-1}(s, R, W, F)$$

with initial data

$$(R_n, W_n)(t, t, r, w, F) = (r, w), (t, r, w, F) \in ]0, 1] \times \mathbb{R}^2 \times \mathbb{R}_0^+;$$

note that  $F$  is constant along characteristics. Define

$$f_n(t, r, w, F) := \mathring{f}((R_n, W_n)(1, t, r, w, F), F),$$

that is,  $f_n$  is the solution of

$$\begin{aligned} \partial_t f_n + \frac{e^{\mu_{n-1} - \lambda_{n-1}} w}{\sqrt{1 + w^2 + F/t^2}} \partial_r f_n \\ - \left( \dot{\lambda}_{n-1} w + e^{\mu_{n-1} - \lambda_{n-1}} \tilde{\mu}_{n-1} \sqrt{1 + w^2 + F/t^2} \right) \partial_w f_n = 0 \end{aligned}$$

with  $f_n(1) = \mathring{f}$ , and define  $\rho_n$ ,  $p_n$ ,  $j_n$ ,  $q_n$  by the integrals (1.7)–(1.7) with  $f$  replaced by  $f_n$ . Recall that the solution of (1.3) is given by (2.1) so we define  $\mu_n$  by

$$e^{-2\mu_n(t, r)} = \frac{e^{-2\mathring{\mu}(r)} + \epsilon}{t} - \epsilon - \frac{8\pi}{t} \int_1^t p_n(s, r) s^2 ds; \quad (3.1)$$

note that the right hand side of this equation is positive on  $]0, 1] \times \mathbb{R}$ . It is at this point that we need our additional assumption on the initial data in the case  $\epsilon = -1$ . Finally define

$$\dot{\lambda}_n(t, r) := 4\pi t e^{2\mu_n} \rho_n(t, r) - \frac{1 + \epsilon e^{2\mu_n}}{2t} \quad (3.2)$$

cf. (2.3),

$$\begin{aligned} \lambda_n(t, r) &:= \dot{\lambda}(r) + \int_1^t \dot{\lambda}_n(s, r) ds, \\ \tilde{\mu}_n(t, r) &:= -4\pi t e^{\mu_n + \lambda_n} j_n(t, r). \end{aligned} \quad (3.3)$$

These iterates are regular on the time interval  $]0, 1]$ , in particular, since  $\lambda_{n-1}$ ,  $\mu_{n-1}$ ,  $\dot{\lambda}_{n-1}$ ,  $\tilde{\mu}_{n-1}$  are continuous on  $]0, 1] \times \mathbb{R}$  and periodic in  $r$ , these functions are bounded on compact subintervals of  $]0, 1]$ , uniformly in  $r$ , and since  $G_{n-1}$  is linearly bounded with respect to  $w$  the characteristics  $R_n$ ,  $W_n$  exist on the time interval  $]0, 1]$ .

The proof of Theorem 3.1 now consists in showing in a number of steps that the iterates constructed above converge in a sufficiently strong sense.

*Step 1:* As a first step we establish a uniform bound on the momenta in the support of the distribution functions  $f_n$ , more precisely we want to bound the quantities

$$P_n(t) := \sup \left\{ |w| \mid (r, w, F) \in \text{supp } f_n(t) \right\}$$

uniformly in  $n$ . On  $\text{supp } f_n(t)$  we have

$$\sqrt{1 + w^2 + F/t^2} \leq \sqrt{1 + P_n(t)^2 + F_0/t^2} \leq \frac{1 + F_0}{t} (1 + P_n(t)),$$

and thus

$$\|\rho_n(t)\| \leq c \frac{(1 + F_0)^2}{t^3} \|f\| (1 + P_n(t))^2$$

and

$$\|p_n(t)\|, \|j_n(t)\| \leq c \frac{F_0}{t^2} \|f\| P_n(t)^2.$$

Throughout the paper  $\|\cdot\|$  denotes the  $L^\infty$ -norm on the function space in question; we have used the fact that  $\|f_n(t)\| = \|f\|$  for  $n \in \mathbb{N}$  and  $t \in ]0, 1]$ . The numerical constant  $c$  may change from line to line and does not depend on  $n$  or  $t$  or on the initial data. In view of the continuation criterion it is important to keep track of any dependence on the latter. From (3.1) it follows that

$$e^{-2\mu_n(t, r)} \geq \frac{c_1}{t}$$

where

$$c_1 = c_1(\mathring{\mu}) := \begin{cases} \inf e^{-2\mathring{\mu}} & \text{for } \epsilon = 1 \text{ or } \epsilon = 0, \\ \inf e^{-2\mathring{\mu}} - 1 & \text{for } \epsilon = -1. \end{cases}$$

By (3.2), (3.3), and the above estimates on  $\rho_n$  and  $j_n$  we get

$$\left| e^{\mu_n - \lambda_n} \tilde{\mu}_n(s, r) \right| \leq 4\pi s e^{2\mu_n} |j_n(s, r)| \leq c \frac{F_0}{c_1} \|f\mathring{\circ}\| P_n(s)^2$$

and

$$\begin{aligned} \left| \dot{\lambda}_n(s, r) \right| &\leq 4\pi s e^{2\mu_n} \rho_n(s, r) + \frac{1 + e^{2\mu_n}}{2s} \\ &\leq \frac{c}{c_1} (1 + F_0)^2 \|f\mathring{\circ}\| \frac{(1 + P_n(s))^2}{s} + \frac{1 + 1/c_1}{2s}. \end{aligned}$$

Thus

$$\left| \dot{W}_{n+1}(s) \right| \leq \frac{c_2}{s} (1 + P_n(s))^2 (1 + |W_{n+1}(s)|),$$

where

$$c_2 = c_2(f, F_0, \mathring{\mu}) := c(1 + 1/c_1)(1 + F_0)^2(1 + \|f\mathring{\circ}\|).$$

This implies that

$$P_{n+1}(t) \leq w_0 + c_2 \int_t^1 \frac{1}{s} (1 + P_n(s))^2 (1 + P_{n+1}(s)) ds.$$

Let  $z_1$  be the left maximal solution of the equation

$$z_1(t) = w_0 + c_2 \int_t^1 \frac{1}{s} (1 + z_1(s))^3 ds,$$

which exists on some interval  $]T_1, 1]$  with  $T_1 \in [0, 1[$ . By induction

$$P_n(t) \leq z_1(t), \quad t \in ]T_1, 1], \quad n \in \mathbb{N},$$

and all the quantities which were estimated against  $P_n$  in the above argument are bounded by certain powers of  $z_1$  on  $]T_1, 1]$ .

*Step 2:* Here we establish bounds on certain derivatives of the iterates. In particular we need a uniform bound on the Lipschitz-constant of the right hand side  $G_n$  of the characteristic system in order to prove convergence in

the next step. Differentiating (3.1) and (3.2) with respect to  $r$  one obtains the identities

$$\begin{aligned}\mu'_n(t, r) &= \frac{e^{2\mu_n}}{t} \left( \dot{\mu}'(r) e^{-2\dot{\mu}} + 4\pi \int_1^t p'_n(s, r) s^2 ds \right), \\ \dot{\lambda}'_n(t, r) &= e^{2\mu_n} \left( 8\pi t \mu'_n(t, r) \rho_n(t, r) + 4\pi t \rho'_n(t, r) - \frac{\epsilon}{t} \mu'_n(t, r) \right), \\ \lambda'_n(t, r) &= \dot{\lambda}'(r) + \int_1^t \dot{\lambda}'_n(s, r) ds.\end{aligned}$$

In the following  $C_1$  denotes a continuous function on  $]T_1, 1]$  which depends only on  $z_1$ . By Step 1,

$$\|\rho'_n(t)\|, \|p'_n(t)\|, \|j'_n(t)\| \leq C_1(t) \|\partial_r f_n(t)\|.$$

Define

$$D_n(t) := \sup \left\{ \|\partial_r f_n(s)\| \mid t \leq s \leq 1 \right\}.$$

Then the above estimates and the formulas for the derivatives of the metric components show that

$$\|\mu'_n(t)\|, \|\lambda'_n(t)\|, \|\dot{\lambda}'_n(t)\| \leq C_1(t)(c_3 + D_n(t)),$$

where  $c_3 := \|e^{-2\dot{\mu}} \dot{\mu}'\| + \|\dot{\lambda}'\| + 1$ . From (3.3) it follows that

$$e^{\mu_n - \lambda_n} \tilde{\mu}_n = -4\pi t e^{2\mu_n} j_n,$$

and

$$\left| \left( e^{\mu_n - \lambda_n} \tilde{\mu}_n \right)'(t, r) \right| \leq C_1(t)(c_3 + D_n(t)).$$

We are now in the position to estimate the derivatives of  $G_n$  with respect to  $r$  and  $w$ :

$$\begin{aligned}\partial_r G_n(t, r, w, F) &= \left( (\mu_n - \lambda_n)' e^{\mu_n - \lambda_n} \frac{w}{\sqrt{1 + w^2 + F/t^2}}, \right. \\ &\quad \left. - (e^{\mu_n - \lambda_n} \tilde{\mu}_n)' \sqrt{1 + w^2 + F/t^2} - \dot{\lambda}'_n w \right), \\ \partial_w G_n(t, r, w, F) &= \left( e^{\mu_n - \lambda_n} \frac{1 + F/t^2}{\sqrt{1 + w^2 + F/t^2}^3}, \right. \\ &\quad \left. - e^{\mu_n - \lambda_n} \tilde{\mu}_n \frac{w}{\sqrt{1 + w^2 + F/t^2}} - \dot{\lambda}'_n \right),\end{aligned}$$

and thus

$$\begin{aligned} |\partial_r G_n(t, r, w, F)| &\leq C_1(t)(c_3 + D_n(t)), \\ |\partial_w G_n(t, r, w, F)| &\leq C_1(t) \end{aligned}$$

for  $t \in ]T_1, 1]$ ,  $r \in \mathbb{R}$ ,  $F \in [0, F_0]$ , and  $|w| \leq z_1(t)$ . Differentiating the characteristic system we obtain

$$\left| \frac{d}{ds} \partial_r (R_{n+1}, W_{n+1})(s, t, r, w, F) \right| \leq C_1(s)(c_3 + D_n(s)) |\partial_r (R_{n+1}, W_{n+1})(s, t, r, w, F)|,$$

and thus for  $(r, w, F) \in \text{supp} f_{n+1}(t) \cup \text{supp} f_n(t)$

$$|\partial_r (R_{n+1}, W_{n+1})(1, t, r, w, F)| \leq \exp \left( \int_t^1 C_1(s)(c_3 + D_n(s)) ds \right).$$

By definition of  $D_n$  this implies that

$$D_{n+1}(t) \leq \|\partial_{(r,w)} \mathring{f}\| \exp \left( \int_t^1 C_1(s)(c_3 + D_n(s)) ds \right).$$

Let  $z_2$  be the left maximal solution of

$$z_2(t) = \|\partial_{(r,w)} \mathring{f}\| \exp \left( \int_t^1 C_1(s)(c_3 + z_2(s)) ds \right),$$

which exists on an interval  $]T_2, 1] \subset ]T_1, 1]$ . Then by induction,

$$D_n(t) \leq z_2(t), \quad t \in ]T_2, 1], \quad n \in \mathbb{N},$$

and all the quantities estimated against  $D_n$  above can be bounded in terms of  $z_2$  on  $]T_2, 1]$ .

*Step 3:* Let  $[\delta, 1] \subset ]T_2, 1]$  be an arbitrary compact subset on which the estimates of Steps 1 and 2 hold. We will show that on such an interval the iterates converge uniformly. Define

$$\alpha_n(t) := \sup \left\{ \|f_{n+1}(\tau) - f_n(\tau)\| \mid \tau \in [t, 1] \right\},$$

and let  $C$  denote a constant which may depend on the functions  $z_1$  and  $z_2$  introduced in the previous two steps. Then

$$\|\rho_{n+1}(t) - \rho_n(t)\|, \|p_{n+1}(t) - p_n(t)\|, \|j_{n+1}(t) - j_n(t)\| \leq C \alpha_n(t),$$

and thus

$$\begin{aligned} & \|\lambda_{n+1}(t) - \lambda_n(t)\|, \|\dot{\lambda}_{n+1}(t) - \dot{\lambda}_n(t)\|, \\ & \|\mu_{n+1}(t) - \mu_n(t)\|, \|\tilde{\mu}_{n+1}(t) - \tilde{\mu}_n(t)\| \leq C\alpha_n(t). \end{aligned}$$

Therefore,

$$|G_{n+1} - G_n|(s, r, w, F) \leq C\alpha_{n-1}(s).$$

By Step 2

$$\left| \partial_{(r,w)} G_n(s, r, w, F) \right| \leq C$$

for all  $s \in [\delta, 1]$ ,  $n \in \mathbb{N}$ , and  $(r, w, F)$  with  $|w| \leq z_1(s)$ . For characteristics which start in  $\text{supp} \mathring{f}$  this implies

$$\begin{aligned} \left| \frac{d}{ds}(R, W)_{n+1} - \frac{d}{ds}(R, W)_n \right|(s, t, r, w, F) \leq \\ C \left| (R, W)_{n+1} - (R, W)_n \right|(s, t, r, w, F) + C\alpha_{n-1}(s), \end{aligned}$$

and by Gronwall's inequality

$$\left| (R, W)_{n+1} - (R, W)_n \right|(1, t, r, w, F) \leq C \int_t^1 \alpha_{n-1}(s) ds.$$

If we recall how  $f_n$  was defined in terms of the characteristics this implies

$$\alpha_n(t) \leq C \int_t^1 \alpha_{n-1}(s) ds, \quad n \geq 1.$$

By induction we obtain

$$\alpha_n(t) \leq C \frac{C^n (1-t)^n}{n!} \leq \frac{C^{n+1}}{n!}$$

for  $n \in \mathbb{N}$  and  $t \in [\delta, 1]$ . This implies that  $f_n$  and all the other quantities whose differences were estimated in terms of  $\alpha_n$  converge on  $[\delta, 1]$ , uniformly with respect to all their arguments. These quantities therefore have continuous limits, but the established convergence is not yet strong enough to conclude the differentiability of, say,  $f := \lim_{n \rightarrow \infty} f_n$ . In order to achieve the latter we need the following lemma:

**Lemma 3.2** *Let  $(\lambda, \mu, \tilde{\mu})$  be regular on some interval  $I \subset \mathbb{R}^+$ , and let  $(R, W)(\cdot, t, r, w, F)$  be the solution of*

$$\dot{r} = \frac{e^{\mu-\lambda} w}{\sqrt{1+w^2+F/s^2}}, \quad \dot{w} = -\dot{\lambda}(s, r)w - e^{\mu-\lambda} \tilde{\mu}(s, r) \sqrt{1+w^2+F/s^2}$$

with  $(R, W)(t, t, r, w, F) = (r, w)$  for  $(t, r, w, F) \in I \times \mathbb{R}^2 \times \mathbb{R}_0^+$ . Define

$$\begin{aligned}\xi(s) &:= e^{(\lambda-\mu)(s, R)} \partial R(s, t, r, w, F), \\ \eta(s) &:= \partial W(s, t, r, w, F) \\ &\quad + \left( \sqrt{1+w^2+F/s^2} e^{\lambda-\mu} \dot{\lambda} \right) \Big|_{(s, (R, W)(s, t, r, w, F))} \partial R(s, t, r, w, F)\end{aligned}$$

for  $\partial \in \{\partial_r, \partial_w\}$ . Then these quantities satisfy the following system of differential equations

$$\begin{aligned}\dot{\xi}(s) &= a_1(s, R(s), W(s), F) \xi(s) + a_2(s, R(s), W(s), F) \eta(s), \\ \dot{\eta}(s) &= (a_3 + a_5)(s, R(s), W(s))(s, F) \xi(s) + a_4(s, R(s), W(s), F) \eta(s),\end{aligned}$$

where

$$\begin{aligned}a_1(s, r, w, F) &:= \frac{w^2}{1+w^2+F/s^2} \dot{\lambda} - \dot{\mu}, \\ a_2(s, r, w, F) &:= \frac{1+F/s^2}{(1+w^2+F/s^2)^{3/2}}, \\ a_3(s, r, w, F) &:= -\frac{1}{s} \sqrt{1+w^2+F/s^2} \left( \dot{\lambda} - \dot{\mu} + \frac{F/s^2}{1+w^2+F/s^2} \dot{\lambda} \right), \\ a_4(s, r, w, F) &:= -\frac{w}{\sqrt{1+w^2+F/s^2}} \left( e^{\mu-\lambda} \tilde{\mu} + \frac{w}{\sqrt{1+w^2+F/s^2}} \dot{\lambda} \right), \\ a_5(s, r, w, F) &:= -\sqrt{1+w^2+F/s^2} e^{2\mu} \\ &\quad \left( e^{-2\lambda} (\tilde{\mu}' + \tilde{\mu}(\mu' - \lambda')) - e^{-2\mu} (\ddot{\lambda} + (\dot{\lambda} + 1/s)(\dot{\lambda} - \dot{\mu})) \right).\end{aligned}$$

In particular, if  $\tilde{\mu} = \mu'$  and  $(\lambda, \mu)$  solves the field equations (1.3)–(1.6) then

$$a_5(s, r, w, F) = -\sqrt{1+w^2+F/s^2} e^{2\mu(s, r)} 8\pi q(s, r).$$

The proof is only a lengthy calculation and therefore omitted. The lemma will be used twice in the further argument: In the next step it will be used to prove that also certain derivatives of the converging sequences obtained in the previous step converge, thus obtaining a regular, local solution. Then it will be used in Step 6 to show that control on the support of the solution with respect to  $w$  suffices to extend the solution to the interval  $]0, 1]$ .

*Step 4:* Fix  $\delta \in ]T_2, 1]$  and  $U > 0$ , and consider the system derived in Lemma 3.2 with  $(\lambda_n, \mu_n, \tilde{\mu}_n)$  instead of  $(\lambda, \mu, \tilde{\mu})$ , and call the corresponding coefficients  $a_{n,i}$ ,  $i = 1, \dots, 5$ . By Steps 1 and 2 we have the estimates

$$|a_{n,i}(t, r, w, F)| + \left| \partial_{(r,w)} a_{n,i}(t, r, w, F) \right| \leq C \quad (3.4)$$

for  $n \in \mathbb{N}$ ,  $i = 1, \dots, 4$ ,  $0 \leq F \leq F_0$ ,  $|w| \leq U$ , and  $t \in [\delta, 1]$ . The only new terms to estimate here are  $\dot{\mu}_n$  and  $\dot{\mu}'_n$ , but from (3.1) we obtain

$$\dot{\mu}_n = 4\pi t e^{2\mu_n} p_n + \frac{1 + \epsilon e^{2\mu_n}}{2t},$$

and

$$\dot{\mu}'_n = 2\mu'_n \left( \dot{\mu}_n - \frac{1}{2t} \right) + 4\pi t e^{2\mu_n} p'_n,$$

so both of these terms are bounded by Steps 1 and 2. The convergence established in Step 3 shows that

$$a_{n,i}(t, r, w, F) - a_{m,i}(t, r, w, F) \rightarrow 0, \quad n, m \rightarrow \infty, \quad i = 1, \dots, 4,$$

uniformly on  $[\delta, 1] \times \mathbb{R} \times [-U, U] \times [0, F_0]$ . The crucial term in the present argument is  $a_{n,5}$ , more precisely the expression

$$g_n := e^{-2\lambda_n} (\tilde{\mu}'_n + \tilde{\mu}_n (\mu'_n - \lambda'_n)) - e^{-2\mu_n} \left( \ddot{\lambda}_n + (\dot{\lambda}_n + 1/t)(\dot{\lambda}_n - \dot{\mu}_n) \right).$$

If the iterates solved the field equation (1.6) then this term would equal  $8\pi q_n$  and would also converge. The idea how to treat  $a_{n,5}$  is to show that  $g_n - 8\pi q_n \rightarrow 0$  for  $n \rightarrow \infty$  and then use the fact that  $q_n$  converges and has uniformly bounded  $r$ -derivative. Now

$$\tilde{\mu}'_n = (\mu'_n + \lambda'_n) \tilde{\mu}_n - 4\pi t e^{\mu_n + \lambda_n} j'_n$$

and

$$\ddot{\lambda}_n = 2\dot{\lambda}_n \dot{\mu}_n + \frac{\dot{\mu}_n}{t} + 4\pi t e^{2\mu_n} \left( \dot{\rho}_n + \frac{\rho_n}{t} \right) + \frac{1 + \epsilon e^{2\mu_n}}{2t^2}.$$

From the definition of  $\rho_n$  we obtain

$$\begin{aligned} \dot{\rho}_n &= -\frac{2}{t} \rho_n - \frac{2}{t} q_n \\ &\quad - e^{\mu_{n-1} - \lambda_{n-1}} j'_n - 2\tilde{\mu}_{n-1} e^{\mu_{n-1} - \lambda_{n-1}} j_n - \dot{\lambda}_{n-1} (\rho_n + p_n), \end{aligned}$$

where we used the Vlasov equation to express  $\partial_t f_n$  and integrated by parts; note that the coefficients in that equation have index  $n-1$ . Inserting all this into the expression for  $g_n$  yields, after cancelling a number of terms,

$$\begin{aligned} g_n &= 2e^{-2\lambda_n} \tilde{\mu}_n \left( \mu'_n - e^{\mu_{n-1} - \mu_n + \lambda_n - \lambda_{n-1}} \tilde{\mu}_{n-1} \right) \\ &\quad + 4\pi t j'_n \left( e^{\mu_{n-1} - \lambda_{n-1}} - e^{\mu_n - \lambda_n} \right) \\ &\quad + e^{-2\mu_n} (\dot{\lambda}_n + \dot{\mu}_n) (\dot{\lambda}_{n-1} - \dot{\lambda}_n) + 8\pi q_n. \end{aligned}$$

By Steps 1, 2, and 3 it remains to show that  $\mu'_n - \tilde{\mu}_{n-1} \rightarrow 0$  for  $n \rightarrow \infty$  in order to conclude that  $g_n \rightarrow 8\pi q_n$ . To see the former differentiate (3.1) with respect to  $r$  to obtain

$$\mu'_n = \frac{e^{2\mu_n}}{t} \left( -\dot{\mu}' e^{-2\dot{\mu}} + 4\pi \int_1^t p'_n(s, r) s^2 ds \right).$$

Differentiating the defining integral of  $p_n$ , using the Vlasov equation for  $f_n$  to express  $\partial_r f_n$ , and integrating by parts with respect to  $w$  and  $s$  results in the relation

$$\begin{aligned} \mu'_n &= \frac{e^{2\mu_n}}{t} \left( -\dot{\mu}' e^{-2\dot{\mu}} + 4\pi e^{\dot{\lambda} - \dot{\mu}} \dot{j} \right) + t e^{\mu_n - \mu_{n-1} - \lambda_n + \lambda_{n-1}} \tilde{\mu}_n \\ &\quad + \frac{e^{2\mu_n}}{t} \int_1^t s e^{-2\mu_n} \left[ e^{\mu_n - \mu_{n-1} - \lambda_n + \lambda_{n-1}} (\dot{\lambda}_{n-1} + \dot{\mu}_{n-1}) \tilde{\mu}_n - (\dot{\lambda}_n + \dot{\mu}_n) \tilde{\mu}_{n-1} \right] ds, \end{aligned}$$

and since the initial data satisfy the constraint (1.5),  $\mu'_n \rightarrow \dot{\mu}$  for  $n \rightarrow \infty$ , in particular,  $\mu'_n - \tilde{\mu}_{n-1} \rightarrow 0$  for  $n \rightarrow \infty$ .

In Step 3 we have shown that among other quantities the characteristics  $(R_n, W_n)(1, t, r, w, F)$  converge. Now for any  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that for  $n, m \geq N$ ,  $s \in [\delta, 1]$ ,  $r \in \mathbb{R}$ ,  $|w| \leq U$ , and  $F \in [0, F_0]$  we have

$$\left| a_{n,5}(s, r, w, F) - (-8\pi) e^{2\mu_n} \sqrt{1 + w^2 + F/s^2} q_n(s, r) \right| \leq \varepsilon$$

and

$$|\partial_r q_n(s, r)| \leq C, \quad |q_n(s, r) - q_m(s, r)| \leq \varepsilon,$$

which together with the estimates (3.4) implies that

$$\left| \dot{\xi}_n - \dot{\xi}_m \right| (s) + |\dot{\eta}_n - \dot{\eta}_m| (s) \leq C\varepsilon + C|\xi_n - \xi_m|(s) + C|\eta_n - \eta_m|(s).$$

This implies the convergence of  $\partial_{(r,w)}(R_n, W_n)(1, t, r, w, F)$ ; note that the transformation from  $\partial(R, W)$  to  $(\xi, \eta)$  in Lemma 3.2 is invertible, and the

coefficients in the transformation are convergent in the present situation. Thus the limiting characteristic  $(R, W)(1, t, r, w, F)$  and therefore also  $f$  are continuously differentiable with respect to  $r$  and  $w$ . This in turn implies that all the moments calculated from  $f$  are differentiable with respect to  $r$ , the coefficients in the limiting characteristic system are continuously differentiable with respect to  $r, w$ , and  $F$ , and thus  $(R, W)(1, t, r, w, F)$  is also differentiable with respect to  $F$  and  $t$ , and the regularity of the limit  $(f, \lambda, \mu)$  is established.

*Step 5:* The estimates on the difference of two consecutive iterates derived in Step 3 can also be used on the difference of two solutions  $f$  and  $g$  with the same initial data. This results in the estimate

$$\sup\{\|f(\tau) - g(\tau)\| \mid \tau \in [t, 1]\} \leq C \int_t^1 \sup\{\|f(\tau) - g(\tau)\| \mid \tau \in [s, 1]\} ds$$

on any compact subinterval of  $]0, 1[$  on which both solutions exist, and thus  $f = g$  there.

*Step 6:* To conclude the proof of Theorem 3.1 it remains to establish the continuation criterion. Let  $(f, \lambda, \mu, \tilde{\mu})$  be a left maximal solution of the auxiliary system (2.4), (1.3), (1.3), (2.5) with existence interval  $]T, 1[$ . By Proposition 2.2  $(f, \lambda, \mu)$  solves (1.2)–(1.6). Now assume that

$$P^* := \sup\{|w| \mid (r, w, F) \in \text{supp} f(t), t \in ]T, 1[\} < \infty.$$

We want to show that  $T = 0$  so let us assume that  $T > 0$  and take  $t_0 \in ]T, 1[$ . We will show that the system has a solution with initial data  $(f(t_0), \lambda(t_0), \mu(t_0))$  prescribed at  $t = t_0$  which exists on an interval  $[t_0 - \delta, t_0]$  with  $\delta > 0$  independent of  $t_0$ . By moving  $t_0$  close enough to  $T$  this would extend our initial solution beyond  $T$ , a contradiction to the initial solution being left maximal.

Steps 1–5 have shown that such a solution exists at least on the left maximal existence interval of the solutions  $(z_1, z_2)$  of

$$\begin{aligned} z_1(t) &= W_0 + c_2 \int_t^{t_0} \frac{1}{s} (1 + z_1(s))^3 ds, \\ z_2(t) &= \|\partial_{(r,w)} f(t_0)\| \exp\left(\int_t^{t_0} C_1(s)(c_3 + z_2(s)) ds\right), \end{aligned}$$

where

$$W_0 := \sup\{|w| \mid (r, w, F) \in \text{supp} f(t_0)\},$$

$$c_1 = c_1(\mu(t_0)) := \begin{cases} \inf e^{-2\mu(t_0)} & \text{for } \epsilon = 1 \text{ or } \epsilon = 0, \\ \inf e^{-2\mu(t_0)} - 1 & \text{for } \epsilon = -1, \end{cases}$$

$$c_2 = c_2(f(t_0), F_0, \mu(t_0)) := c(1 + 1/c_1)(1 + F_0)^2(1 + \|f(t_0)\|),$$

$$c_3 = \|e^{-2\mu(t_0)} \mu'(t_0)\| + \|\lambda'(t_0)\| + 1,$$

and the function  $C_1$  depends on  $z_1$ . Now  $W_0 \leq P^*$ ,  $\|f(t_0)\| = \|\mathring{f}\|$ ,  $F_0$  is unchanged since  $F$  is constant along characteristics, and (2.1) shows that  $c_1(\mu(t_0)) \geq c_1(\mathring{\mu})$ . Thus there exists a constant  $c_2^* > 0$  such that  $c_2(f(t_0), F_0, \mu(t_0))/s \leq c_2^*$  for  $t_0 \in ]T, 1]$  and  $s \in [T/2, 1]$ . Let  $z_1^*$  denote the left maximal solution of

$$z_1^*(t) = P^* + c_2^* \int_t^{t_0} (1 + z_1^*(s))^3 ds.$$

Next observe that the coefficients  $a_1, \dots, a_5$  in Lemma 3.2 are uniformly bounded  $]T, 1]$  along characteristics in  $\text{supp} f$  if we let  $\tilde{\mu} = \mu'$  and use the field equation (1.6). The lemma then shows that

$$D^* := \sup \left\{ \|\partial_{(r,w)} f(t)\| \mid T < t \leq 1 \right\} < \infty.$$

From

$$\begin{aligned} \mu'(t, r) &= \frac{e^{2\mu}}{t} \left( \mathring{\mu}'(r) e^{-2\mathring{\mu}} + 4\pi \int_1^t p'(s, r) s^2 ds \right), \\ \dot{\lambda}'(t, r) &= e^{2\mu} \left( 8\pi t \mu'(t, r) \rho(t, r) + 4\pi t \rho'(t, r) - \frac{\epsilon}{t} \mu'(t, r) \right), \\ \lambda'(t, r) &= \mathring{\lambda}'(r) + \int_1^t \dot{\lambda}'(s, r) ds \end{aligned}$$

we obtain a uniform bound  $c_3(\mu(t_0), \lambda(t_0)) \leq c_3^*$ . Let  $z_2^*$  be the left maximal solution of

$$z_2^*(t) = D^* \exp \left( \int_t^{t_0} C_1^*(s) (c_3^* + z_2^*(s)) ds \right),$$

where  $C_1^*$  depends on  $z_1^*$  in the same way as  $C_1$  depends on  $z_1$ . Clearly,  $z_1^*$  and  $z_2^*$  exist on an interval  $[t_0 - \delta, t_0]$  with  $\delta > 0$  independent of  $t_0$ . If we choose  $\delta < T/2$  then  $z_1 \leq z_1^*$  and  $z_2 \leq z_2^*$  by construction, in particular,  $z_1$  and  $z_2$  exist on  $[t_0 - \delta, t_0]$ , and the proof of the continuation criterion and of Theorem 3.1 is complete.  $\square$

## 4 Existence up to $t=0$

In this section we show that the solutions obtained in the previous section exist on the interval  $]0, 1]$  provided the initial data are sufficiently small.

**Theorem 4.1** *Let  $(\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{\mu})$  be initial data as in Theorem 3.1, and assume that*

$$c := \frac{1}{2} - 10\pi^2 w_0 F_0 \sqrt{1 + w_0^2 + F_0} \|e^{2\overset{\circ}{\mu}}\| \|\overset{\circ}{f}\| > 0$$

*in case  $\epsilon = 0$  or  $\epsilon = 1$ , and*

$$c := \frac{1}{2} (1 - \|e^{2\overset{\circ}{\mu}}\|) - 10\pi^2 w_0 F_0 \sqrt{1 + w_0^2 + F_0} \frac{\|e^{2\overset{\circ}{\mu}}\|}{1 - \|e^{2\overset{\circ}{\mu}}\|} \|\overset{\circ}{f}\| > 0$$

*in case  $\epsilon = -1$ —note that  $\|e^{2\overset{\circ}{\mu}}\| < 1$  in this case by the assumption in Theorem 3.1. Then the corresponding solution exists on the interval  $]0, 1]$ , and*

$$|w| \leq w_0 t^c, \quad (r, w, F) \in \text{supp } f(t), \quad t \in ]0, 1].$$

*Proof :* Let  $(\overset{\circ}{f}, \overset{\circ}{\mu}, \overset{\circ}{\lambda})$  be initial data satisfying the smallness assumption, and define

$$P(t) := \sup \left\{ |w| \mid (r, w, F) \in \text{supp } f(t) \right\}, \quad t \in ]T, 1].$$

The characteristic system for (1.2) and the field equations (1.3) and (1.5) imply that

$$\begin{aligned} \dot{w} &= -\dot{\lambda} w - e^{\mu - \lambda} \mu' \sqrt{1 + w^2 + F/t^2} \\ &= 4\pi t e^{2\mu} \left( j \sqrt{1 + w^2 + F/t^2} - \rho w \right) + \frac{1 + \epsilon e^{2\mu}}{2t} w. \end{aligned}$$

Assume that  $P(t) \leq w_0$  for some  $t \in ]T, 1]$ , which is true at least for  $t = 1$ . Then

$$\begin{aligned} 0 \leq \rho(t, r) &\leq \frac{\pi}{t^2} \int_{-w_0}^{w_0} \int_0^{F_0} \sqrt{1 + w^2 + F/t^2} f(t, r, w, F) dF dw \\ &\leq 2\pi w_0 F_0 \sqrt{1 + w_0^2 + F_0} \|\overset{\circ}{f}\| t^{-3}, \end{aligned}$$

and

$$\begin{aligned} j(t, r) &\leq \frac{\pi}{t^2} \int_0^{P(t)} \int_0^{F_0} w f(t, r, w, F) dF dw \leq \frac{\pi}{2} w_0 F_0 \|\overset{\circ}{f}\| P(t) t^{-2}, \\ j(t, r) &\geq \frac{\pi}{t^2} \int_{-P(t)}^0 \int_0^{F_0} w f(t, r, w, F) dF dw \geq -\frac{\pi}{2} w_0 F_0 \|\overset{\circ}{f}\| P(t) t^{-2}. \end{aligned}$$

Next we have the estimate

$$e^{-2\mu(t,r)} = \frac{e^{-2\mu^\circ(r)} + \epsilon}{t} - \epsilon - 8\pi \int_1^t s^2 p(s,r) ds \geq \frac{e^{-2\mu^\circ(r)} + \epsilon}{t} - \epsilon$$

so that

$$e^{-2\mu(t,r)} \geq \frac{c_1}{t}$$

where

$$c_1 := \begin{cases} \inf e^{-2\mu^\circ} & \text{for } \epsilon = 0 \text{ or } \epsilon = 1, \\ \inf e^{-2\mu^\circ} - 1 & \text{for } \epsilon = -1. \end{cases}$$

Thus  $e^{2\mu(t,r)} \leq c_1^{-1}t$ , and

$$\frac{1 + \epsilon e^{2\mu}}{2t} \geq \frac{1}{2t} =: \frac{c_2}{t}$$

for  $\epsilon = 0$  or  $\epsilon = 1$ ,

$$\frac{1 + \epsilon e^{2\mu}}{2t} \geq \frac{1}{2t} \left(1 - \frac{1}{1 + c_1 t^{-1}}\right) \geq \frac{1}{2} \frac{c_1}{1 + c_1} \frac{1}{t} =: \frac{c_2}{t}$$

for  $\epsilon = -1$ . Assume that  $w(t) > 0$ . Then

$$\begin{aligned} \dot{w}(t) &\geq \left( c_2 - 8\pi^2 w_0 F_0 \sqrt{1 + w_0^2 + F_0} \frac{1}{c_1} \|f^\circ\| \right) \frac{w(t)}{t} \\ &\quad - 2\pi^2 w_0 F_0 \sqrt{1 + w_0^2 + F_0} \frac{1}{c_1} \|f^\circ\| \frac{P(t)}{t}, \end{aligned} \quad (4.1)$$

whereas if  $w(t) < 0$  we obtain the estimate

$$\begin{aligned} \dot{w}(t) &\leq \left( c_2 - 8\pi^2 w_0 F_0 \sqrt{1 + w_0^2 + F_0} \frac{1}{c_1} \|f^\circ\| \right) \frac{w(t)}{t}, \\ &\quad + 2\pi^2 w_0 F_0 \sqrt{1 + w_0^2 + F_0} \frac{1}{c_1} \|f^\circ\| \frac{P(t)}{t}. \end{aligned} \quad (4.2)$$

If we let  $t=1$  in (4.1) and (4.2) our smallness condition on the initial data implies that there exists a small constant  $\delta > 0$  such that  $\dot{w}(1) > 0$  if  $w_0/(1+\delta) \leq w(1) \leq w_0$ , and  $\dot{w}(1) < 0$  if  $-w_0 \leq w(1) \leq -w_0/(1+\delta)$ . This implies that  $P(t) < w_0$  on some interval  $]t_0, 1[$  which we choose maximal with this property. On the interval  $]t_0, 1[$  the estimates (4.1) and (4.2) hold for any characteristic which runs in  $\text{supp} f$  and for which  $w(t) > 0$  or  $w(t) < 0$  respectively.

Let  $t \in ]t_0, 1]$  be such that  $w(t) > 0$  for a characteristic in  $\text{supp} f$ , and choose  $t_1 > t$  maximal with  $w(s) > 0$  for  $s \in [t, t_1[$ . Then (4.1) holds on  $[t, t_1[$  which by Gronwall's inequality implies that

$$\begin{aligned} w(t) &\leq \exp\left(c_3 \int_{t_1}^t \frac{ds}{s}\right) \left[ w(t_1) - c_4 \int_{t_1}^t \exp\left(-c_3 \int_{t_1}^s \frac{d\tau}{\tau}\right) \frac{P(s)}{s} ds \right] \\ &= (t/t_1)^{c_3} \left[ w(t_1) + c_4 t_1^{c_3} \int_t^{t_1} s^{-1-c_3} P(s) ds \right], \end{aligned}$$

where

$$\begin{aligned} c_3 &:= c_2 - 8\pi^2 w_0 F_0 \sqrt{1 + w_0^2 + F_0} \frac{1}{c_1} \|f\|, \\ c_4 &:= 2\pi^2 w_0 F_0 \sqrt{1 + w_0^2 + F_0} \frac{1}{c_1} \|f\|. \end{aligned}$$

If  $t_1 = 1$  then

$$w(t) \leq t^{c_3} \left( w_0 + c_4 \int_t^1 s^{-1-c_3} P(s) ds \right).$$

If  $t_1 < 1$  then  $w(t_1) = 0$ , and

$$\begin{aligned} w(t) &\leq (t/t_1)^{c_3} c_4 t_1^{c_3} \int_t^{t_1} s^{-1-c_3} P(s) ds \\ &\leq t^{c_3} \left( w_0 + c_4 \int_t^1 s^{-1-c_3} P(s) ds \right). \end{aligned}$$

Consider now  $t \in ]t_0, 1]$  such that  $w(t) < 0$ , and choose  $t_1 > t$  maximal with  $w(s) < 0$  for  $s \in [t, t_1[$ . Repeating the above argument, but now using (4.2) instead of (4.1), yields the estimate

$$w(t) \geq (t/t_1)^{c_3} \left( w(t_1) - c_4 t_1^{c_3} \int_t^{t_1} s^{-1-c_3} P(s) ds \right),$$

and distinguishing the cases  $t_1 = 1$  and  $t_1 < 1$  as above implies that

$$-w(t) \leq t^{c_3} \left( w_0 + c_4 \int_t^1 s^{-1-c_3} P(s) ds \right).$$

Therefore,

$$P(t) \leq t^{c_3} \left( w_0 + c_4 \int_t^1 s^{-1-c_3} P(s) ds \right)$$

for all  $t \in ]t_0, 1]$ . Applying Gronwall's inequality again yields the estimate

$$P(t) \leq w_0 t^{c_3 - c_4}, \quad t \in ]t_0, 1],$$

since  $z(t) := w_0 t^{c_3 - c_4}$  is the solution of the integral equation

$$z(t) = t^{c_3} \left( w_0 + c_4 \int_t^1 s^{-1-c_3} z(s) ds \right)$$

which is equivalent to the initial value problem

$$\dot{z}(t) = (c_3 - c_4) \frac{z(t)}{t}, \quad z(1) = w_0.$$

The estimate on  $P(t)$  implies in particular that  $t_0 = T$ , i. e., it holds on the whole existence interval of the solution, which by Theorem 3.1 implies  $T = 0$ , and the proof is complete.  $\square$

## 5 The nature of the singularity at $t = 0$

In this section we investigate the behaviour of solutions as  $t \rightarrow 0$ . Theorem 4.1 shows that there are solutions which exist on  $]0, 1]$  so that we do not investigate the empty set. However, the results which follow do for the most part not depend on the smallness assumption which we used in the previous section. First we show that solutions which exist on  $]0, 1]$  have a curvature singularity at  $t = 0$ , more precisely:

**Theorem 5.1** *Let  $(f, \lambda, \mu)$  be a regular solution of (1.2)–(1.6) on the interval  $]0, 1]$ . Then*

$$\left( R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \right) (t, r) \geq 6 \left( \inf e^{-2\mu} + \epsilon \right) t^{-6}, \quad t \in ]0, 1], \quad r \in \mathbb{R},$$

where  $R_{\alpha\beta\gamma\delta}$  denotes the Riemann curvature tensor corresponding to the metric given by  $\lambda$  and  $\mu$ .

The curvature scalar considered here is sometimes called Kretschmann scalar.

*Proof :* It can be shown that

$$\begin{aligned} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} &= 4 \left( e^{-2\lambda} (\mu'' + \mu'(\mu' - \lambda')) - e^{-2\mu} (\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\mu})) \right)^2 \\ &\quad + \frac{8}{t^2} \left( e^{-4\mu} \dot{\lambda}^2 + e^{-4\mu} \dot{\mu}^2 - 2e^{-2(\lambda+\mu)} (\mu')^2 \right) \\ &\quad + \frac{4}{t^4} \left( e^{-2\mu} + \epsilon \right)^2 \\ &=: K_1 + K_2 + K_3. \end{aligned}$$

One way to see this is to use a computer algebra system like Maple, another is to exploit the symmetry of the metric and split the summation

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = R_{abcd}R^{abcd} + R_{ABCD}R^{ABCD} + 4R_{AbCd}R^{AbCd},$$

where lower case Latin indices take the values 0 and 1, and upper case Latin indices take the values 2 and 3. Now

$$\begin{aligned} R_{abcd} &= \left( e^{-2\lambda} (\mu'' + \mu'(\mu' - \lambda')) - e^{-2\mu} (\ddot{\lambda} + \dot{\lambda}(\lambda - \dot{\mu})) \right) (g_{ac}g_{bd} - g_{bc}g_{ad}), \\ R_{ABCD} &= t^{-2} (e^{-2\mu} + \epsilon) (g_{AC}g_{BD} - g_{BC}g_{AD}), \\ R_{AbCd} &= t^{-1} g_{AC}\Gamma_{bd}^0, \end{aligned}$$

which can be seen after a lengthy calculation. Inserting these expressions into the above summations yields the formula for the Kretschmann scalar. The first term  $K_1$  is clearly nonnegative and can be dropped. In order to estimate  $K_2$  we insert the expressions

$$\begin{aligned} e^{-2\mu}\dot{\lambda} &= 4\pi t\rho - \frac{\epsilon + e^{-2\mu}}{2t}, \\ e^{-2\mu}\dot{\mu} &= 4\pi tp + \frac{\epsilon + e^{-2\mu}}{2t}, \\ e^{-\mu-\lambda}\mu' &= -4\pi tj \end{aligned}$$

into the formula for  $K_2$  and obtain

$$\frac{t^2}{8}K_2 = 16\pi^2 t^2 (\rho^2 + p^2 - 2j^2) - 4\pi t(\rho - p) \frac{\epsilon + e^{-2\mu}}{t} + \frac{(\epsilon + e^{-2\mu})^2}{2t^2}.$$

Now

$$\begin{aligned} |j(t, r)| &\leq \frac{\pi}{t^2} \int_{-\infty}^{\infty} \int_0^{\infty} (1 + w^2 + F/t^2)^{1/4} f^{1/2} \frac{|w|}{(1 + w^2 + F/t^2)^{1/4}} f^{1/2} dF dw \\ &\leq \rho(t, r)^{1/2} p(t, r)^{1/2} \end{aligned}$$

by the Cauchy-Schwarz inequality. Therefore

$$\rho^2 + p^2 - 2j^2 \geq \rho^2 + p^2 - 2\rho p = (\rho - p)^2$$

and

$$\begin{aligned} \frac{t^2}{8}K_2 &\geq (4\pi t(\rho - p))^2 - 4\pi t(\rho - p) \frac{\epsilon + e^{-2\mu}}{t} + \frac{(\epsilon + e^{-2\mu})^2}{2t^2} \\ &= \left( 4\pi t(\rho - p) - \frac{\epsilon + e^{-2\mu}}{2t} \right)^2 + \frac{1}{4} \frac{(\epsilon + e^{-2\mu})^2}{t^2}. \end{aligned}$$

Thus

$$\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\right)(t,r) \geq 6 \frac{\left(\epsilon + e^{-2\mu(t,r)}\right)^2}{t^4}.$$

Recalling (2.1) we obtain

$$e^{-2\mu} + \epsilon = \frac{e^{-2\mu^\circ} + \epsilon}{t} - \frac{8\pi}{t} \int_1^t s^2 p(s,r) ds \geq \frac{e^{-2\mu^\circ} + \epsilon}{t},$$

and inserting this into the previous estimate completes the proof.  $\square$

Next we show that the singularity at  $t=0$  is also a crushing singularity in the sense that the mean curvature of the surfaces of constant  $t$  blows up as  $t \rightarrow 0$ .

**Theorem 5.2** *Let  $(f, \lambda, \mu)$  be a solution of (1.2)–(1.6) on  $]0, 1]$  and define  $c := \inf e^{-\mu^\circ}$  for  $\epsilon = 0$  or  $\epsilon = 1$ , and  $c := \frac{3}{2} \inf (e^{-2\mu^\circ} - 1)^{1/2}$  for  $\epsilon = -1$ . Let  $k(t, r)$  denote the mean curvature of the surfaces of constant  $t$ . Then*

$$k(t, r) \leq -ct^{-3/2}.$$

*Proof :* For a metric of the form

$$ds^2 = -\varphi^2(t, x) dt^2 + g_{ij} dx^i dx^j$$

where  $i, j$  run from 1 to 3 the second fundamental form is given by

$$k_{ij} = -(2\varphi)^{-1} \dot{g}_{ij},$$

and its trace  $k(t, x) = k_i^i(t, x)$  is the mean curvature of the surface of constant  $t$ , at the point  $x$ . For the metric in our present situation we obtain

$$\begin{aligned} k_{11} &= -\frac{1}{2} e^{-\mu} \partial_t (e^{2\lambda}) = -e^{2\lambda - \mu} \dot{\lambda}, \\ k_{22} &= -te^{-\mu}, \\ k_{33} &= -te^{-\mu} \sin_\epsilon \theta, \end{aligned}$$

and

$$k(t, r) = -e^{-\mu} \left( \dot{\lambda} + \frac{2}{t} \right),$$

cf. (1.1). Recalling (2.3) we obtain

$$\dot{\lambda} = e^{2\mu} \left( 4\pi t \rho - \frac{\epsilon + e^{-2\mu}}{2t} \right) \geq -e^{2\mu} \frac{\epsilon + e^{-2\mu}}{2t},$$

and

$$k(t, r) \leq e^\mu \frac{\epsilon - 3e^{-2\mu}}{2t}.$$

If  $\epsilon = 0$  or  $\epsilon = -1$  then

$$k(t, r) \leq -\frac{3}{2t}e^{-\mu},$$

and the estimate

$$e^{-2\mu} \geq \frac{e^{-2\mu} + \epsilon}{t}$$

completes the proof of our assertion in these two cases. For  $\epsilon = 1$  the estimate

$$e^{-2\mu} \geq \frac{e^{-2\mu}}{t} > 1 = \epsilon,$$

which holds for  $t$  small, shows that

$$k(t, r) \leq -\frac{e^{-\mu}}{t} \leq -\frac{e^{-\mu}}{t^{3/2}},$$

and the proof is complete also in this case.  $\square$

Since we have already determined the second fundamental form it requires little additional effort to investigate the quotients

$$\frac{k_1^1}{k}, \frac{k_2^2}{k}, \frac{k_3^3}{k}$$

in the limit  $t \rightarrow 0$ . The question whether these quotients have limits is connected to the concept of a velocity dominated singularity, cf. [2]. It turns out that these limits exist and can be determined in the case of small initial data, more precisely:

**Theorem 5.3** *Let  $(f, \lambda, \mu)$  be a solution of (1.2)–(1.6) with small initial data as described in Theorem 4.1. Then*

$$\lim_{t \rightarrow 0} \frac{k_1^1(t, r)}{k(t, r)} = -\frac{1}{3},$$

$$\lim_{t \rightarrow 0} \frac{k_2^2(t, r)}{k(t, r)} = \lim_{t \rightarrow 0} \frac{k_3^3(t, r)}{k(t, r)} = \frac{2}{3},$$

*uniformly in  $r \in \mathbb{R}$ .*

*Proof :* We have

$$\begin{aligned}\frac{k_1^1(t,r)}{k(t,r)} &= \frac{t\dot{\lambda}(t,r)}{t\dot{\lambda}(t,r)+2}, \\ \frac{k_2^2(t,r)}{k(t,r)} &= \frac{k_3^3(t,r)}{k(t,r)} = \frac{1}{t\dot{\lambda}(t,r)+2}.\end{aligned}$$

From (2.3) we have

$$t\dot{\lambda} = 4\pi t^2 e^{2\mu} \rho - \epsilon \frac{e^{2\mu}}{2} - \frac{1}{2}.$$

As we have seen in the proof of Theorem 4.1

$$e^{2\mu(t,r)} \leq Ct,$$

and the estimate on the  $w$ -support of  $f$  in that theorem shows that

$$\rho(t,r) = \frac{\pi}{t^2} \int_{-P(t)}^{P(t)} \int_0^{F_0} \sqrt{1+w^2+F/t^2} f(t,r,w,F) dF dw \leq Ct^{-3+c}$$

so that

$$4\pi t^2 e^{2\mu} \rho(t,r) \leq Ct^c \rightarrow 0, \quad t \rightarrow 0.$$

Thus

$$t\dot{\lambda}(t,r) \rightarrow -\frac{1}{2}, \quad t \rightarrow 0,$$

uniformly in  $r$ , and the proof is complete.  $\square$

**Remark:** As is easily checked a homogeneous and isotropic solution in the case  $\epsilon = 0$  is given by

$$f(t,w,F) := \varphi(t^2 w^2 + F), \quad \lambda(t) := \ln t, \quad e^{-2\mu(t)} = \frac{8\pi}{3} t^2 \rho(t),$$

where  $\varphi \in C^1([0, \infty[)$  is an arbitrary, nonnegative function with compact support. For this solution, the limits considered in Theorem 5.3 are  $1/3, 1/3, 1/3$  which shows that the result in that theorem need not be true for solutions whose initial data violate the smallness assumption.

## 6 Going forward in time

Our main interest in this paper lies the behaviour of solutions as  $t$  approaches the singularity at  $t=0$ . Nevertheless, it is possible to consider the solutions

for  $t \geq 1$ . Our main result here is that one can establish a bound on the  $w$ -support of  $f$  and in spite of that the solutions need not exist for all  $t \geq 1$ . Since this is in sharp contrast to the spherically symmetric, asymptotically flat case, cf. [5], and also to well known results for the Vlasov-Poisson and Vlasov-Maxwell system we include these considerations.

If one goes through the proof of Theorem 3.1, but now for  $t \geq 1$ , one immediate problem is that (3.1) need not define  $\mu_n$  for all  $t \geq 1$  but only on some finite time interval. This results in a more restrictive continuation criterion in the corresponding local existence theorem the proof of which is omitted:

**Theorem 6.1** *Let  $(\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{\mu})$  be initial data as in Theorem 3.1. Then there exists a unique, right maximal, regular solution  $(f, \lambda, \mu)$  of (1.2)–(1.6) with  $(f, \lambda, \mu)(1) = (\overset{\circ}{f}, \overset{\circ}{\lambda}, \overset{\circ}{\mu})$  on a time interval  $[1, T[$  with  $T \in ]1, \infty]$ . If*

$$\sup\{|w| \mid (t, r, w, F) \in \text{supp } f\} < \infty$$

and

$$\sup\{e^{2\mu(t,r)} \mid r \in \mathbb{R}, t \in [1, T[ \} < \infty$$

then  $T = \infty$ .

The main result of this section is the following theorem:

**Theorem 6.2** *Assume that  $(f, \lambda, \mu)$  is a solution of (1.2)–(1.6) on a right maximal interval of existence  $[1, T[$ , and*

$$\sup\{e^{2\mu(t,r)} \mid r \in \mathbb{R}, t \in [1, T[ \} < \infty.$$

Then  $T = \infty$ .

*Proof* : Assume that  $T < \infty$ . We show that under the assumption on  $e^{2\mu}$  we obtain the bound

$$\sup\{|w| \mid (t, r, w, F) \in \text{supp } f\} < \infty,$$

which is a contradiction to Theorem 6.1. The proof of the bound on  $w$  is similar in spirit to the proof of [9, Theorem 2.1]. Define

$$\begin{aligned} P_+(t) &:= \sup\{w \mid (r, w, F) \in \text{supp } f(t)\}, \\ P_-(t) &:= \inf\{w \mid (r, w, F) \in \text{supp } f(t)\}, \end{aligned}$$

assume that  $P_+(t) > 0$  for some  $t \in [1, T[$ , and let  $w(t) = w > 0$  denote the  $w$ -component of a characteristic in  $\text{supp}f$ . We have

$$\begin{aligned} \dot{w} &= \frac{4\pi^2}{t} e^{2\mu} \int_{-\infty}^{\infty} \int_0^{\infty} \left( \tilde{w} \sqrt{1 + w^2 + F/t^2} - w \sqrt{1 + \tilde{w}^2 + \tilde{F}/t^2} \right) f d\tilde{F} d\tilde{w} \\ &\quad + \frac{1 + \epsilon e^{2\mu}}{2t} w. \end{aligned}$$

Let us abbreviate

$$\xi = \tilde{w} \sqrt{1 + w^2 + F/t^2} - w \sqrt{1 + \tilde{w}^2 + \tilde{F}/t^2}.$$

As long as  $w(s) > 0$  we have the following estimates: If  $\tilde{w} \leq 0$  then  $\xi \leq 0$ . If  $\tilde{w} > 0$  then

$$\begin{aligned} \xi &= \frac{\tilde{w}^2(1 + w^2 + F/s^2) - w^2(1 + \tilde{w}^2 + \tilde{F}/s^2)}{\tilde{w} \sqrt{1 + w^2 + F/s^2} + w \sqrt{1 + \tilde{w}^2 + \tilde{F}/s^2}} \\ &= \frac{\tilde{w}^2(1 + F/s^2) - w^2(1 + \tilde{F}/s^2)}{\tilde{w} \sqrt{1 + w^2 + F/s^2} + w \sqrt{1 + \tilde{w}^2 + \tilde{F}/s^2}} \\ &\leq C \frac{\tilde{w}}{w(s)}, \end{aligned}$$

and thus

$$\begin{aligned} \dot{w}(s) &\leq C \frac{1}{s w(s)} \int_0^{\tilde{P}_+(s)} \int_0^{F_0} \tilde{w} f(s, r, \tilde{w}, \tilde{F}) d\tilde{F} d\tilde{w} + \frac{C}{s} w(s) \\ &\leq \frac{C}{s} \left( \frac{\tilde{P}_+(s)^2}{w(s)} + w(s) \right), \end{aligned}$$

where  $\tilde{P}_+ := \max\{P_+, 0\}$ . Thus

$$\frac{d}{ds} w^2(s) \leq \frac{C}{s} \tilde{P}_+(s)^2$$

as long as  $w(s) > 0$ . Let  $t_0 \in [1, t[$  be defined minimal such that  $w(s) > 0$  for  $s \in ]t_0, t[$ . Then

$$w^2(t) \leq w^2(t_0) + C \int_{t_0}^t \frac{1}{s} \tilde{P}_+(s)^2 ds.$$

Now either  $t_0 > 1$  and  $w(t_0) = 0$  or  $t_0 = 1$  and  $w(t_0) \leq w_0$ . Thus

$$\tilde{P}_+(t)^2 \leq w_0^2 + C \int_1^t \frac{1}{s} \tilde{P}_+(s)^2 ds$$

for all  $t \in [1, T[$ , since this estimate is trivial if  $P_+(t) \leq 0$ . If  $T < \infty$  this estimate implies that  $P_+$  is bounded on  $[1, T[$ . Estimating  $\dot{w}(s)$  from below in case  $w(s) < 0$  along the same lines shows that  $P_-$  is bounded as well, and the proof is complete.  $\square$

**Remark:** In the case of spherical symmetry it is easy to see that a solution cannot exist for all  $t \geq 1$ , regardless of the size of initial data. For such a solution  $(f, \lambda, \mu)$  the estimate

$$\begin{aligned} e^{-2\mu(t,r)} &= \frac{e^{-\dot{\mu}(r)} + 1}{t} - 1 - \frac{8\pi}{t} \int_1^t s^2 p(s,r) ds \\ &\leq \frac{e^{-\dot{\mu}(r)} + 1}{t} - 1 \end{aligned}$$

has to hold on the interval of existence  $[1, T[$ . Since the right hand side of this estimate tends to  $-1$  for  $t \rightarrow \infty$  it follows that  $T < \infty$  and

$$\|e^{2\mu(t)}\| \rightarrow \infty, \quad t \rightarrow T,$$

by Theorem 6.2. This should be compared with the behaviour of the Schwarzschild metric in the limit  $r \nearrow 2M$ , i. e., approaching the event horizon from inside of the black hole; cf. our interpretation of the metric (1.1) in the introduction.

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