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AUTOMORPHISM GROUPS OF PARABOLIC GEOMETRIES

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Abstract. We show that elementary algebraic techniques lead to surprising results on automorphism groups of Cartan geometries and especially parabolic geometries. The example of three–dimensional CR structures is discussed in detail.

1. Introduction

The aim of this article is to show how rather elementary algebra can be used to obtain surprising information on the automorphism groups of Cartan geometries and, more specifically, parabolic geometries. On the way, we review several basic facts about Cartan geometries, so this article can also be considered as a short introduction to some basic ideas of the theory. A detailed introduction to Cartan geometries can be found in the book [6].

Any Cartan geometry comes with a homogeneous model G/H. The crucial point for our purposes, is that the autmorphism group of any Cartan geometry can be made into a Lie group, and the Lie algebra of this group can be described explicitly in terms of the Lie algebra $\mathfrak{g}$ of G and the curvature of the geometry. This description can be improved considerably in the special case of parabolic geometries, in which $\mathfrak{g}$ is semisimple and $P \subset G$ is a parabolic subgroup. In this case, one can obtain information on possible autmorphism groups by studying certain Lie subalgebras of $\mathfrak{g}$. We work this out explicitly in the case of three–dimensional CR structures of hypersurface type, in which the algebraic problems become particularly simple. In particular, we show that the classification of homogeneous three–dimensional CR structures reduces to purely algebraic problems.

Except for the presentation, nothing in this article is really original. The proof of Corollary 2.2 sketched here can be found in the book [3]. The basic results for parabolic geometries in 2.5 can be found (in the special case of CR structures) in [7]. The results on three–dimensional CR structures go back to E. Cartan, see [1] and [4] for a modern presentation.

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2. Cartan geometries and their automorphism groups

2.1. **Cartan geometries.** Let G be a Lie group, $H \subset G$ a closed subgroup such that G/H is connected, and let $\mathfrak{h} \subset \mathfrak{g}$ be the corresponding Lie algebras. The basic idea behind Cartan geometries is to view this homogeneous space as a particularly nice instance of a differential geometric structure. Manifolds endowed with the corresponding structure can then be thought of as "curved analogs" of the homogeneous space G/H.

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The main requirement on this structure is that the automorphisms on G/H should be exactly the left actions of elements of G.

The natural projection $G \to G/H$ is an H–principal bundle, and left multiplication by $g \in G$ lifts the left action of g on G/H to an automorphism of this principal bundle. The left multiplications by elements of G can be characterized within the (infinite dimensional) space of principal bundle automorphisms of $G \to G/H$ by the fact that they preserve the left Maurer–Cartan form. This motivates the definition of "curved analogs" as general principal H–bundles endowed with a $\mathfrak{g}$–valued one form, which has all properties of the left Maurer Cartan form that do make sense in the more general context:

Definition. (1) A *Cartan geometry* of type (G, H) on a smooth manifold M is a principal H–bundle $p : \mathcal{G} \to M$ together with a one form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ such that

- $(r^h)^*\omega = \mathrm{Ad}(h)^{-1} \circ \omega$ for all $h \in H$, where r^h denotes the principal right action of h.
- $\omega(\zeta_A) = A$ for all $A \in \mathfrak{h}$, where ζ_A denotes the fundamental vector field with generator A.
- $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$.

(2) A *morphism* between two Cartan geometries $(\mathcal{G} \to M, \omega)$ and $(\tilde{\mathcal{G}} \to \tilde{M}, \tilde{\omega})$ is a principal bundle homomorphism $\Phi : \mathcal{G} \to \tilde{\mathcal{G}}$ such that $\Phi^*\tilde{\omega} = \omega$. Note that since both ω and $\tilde{\omega}$ are bijective on each tangent space, this implies that Φ is a local diffeomorphism.
(3) The *homogeneous model* of the geometry is the principal bundle $G \to G/H$ together with the left Maurer–Cartan form ω^{MC}.

The fact that interesting geometric structures can be described as Cartan geometries usually is the result of a theorem rather than a definition. In most cases of interest, the principal bundle and the Cartan connection are obtained by fairly involved constructions from some underlying data. These underlying data may for example be a geometric strcuture (a Riemannian metric, a conformal structure, a CR structure, etc.) or a differential equation of a certain type. Then one proves existence of a unique Cartan connection (with certain properties), which leads to an equivalence of the category under consideration with a category of Cartan geometries.

In this paper, we will mostly view Cartan geometries as the basic input, and not care about the equivalence to some underlying structure. Let us only describe the equivalence briefly in the case of Riemannian metrics. This simple example was one of the basic motivations for the development of the general concept of Cartan geometries.

Example. Let G be the group of rigid motions of $\mathbb{R}^n$ and $H \subset G$ the subgroup of motions fixing $0 \in \mathbb{R}$. Then $H = O(n)$ and G/H is Euclidean space $\mathbb{R}^n$. For an n–dimensional Riemannian manifold M let $\mathcal{G}$ be the orthonormal frame bundle, which is a principal $O(n)$–bundle. The bundle carries a canonical $\mathbb{R}^n$–valued one–form θ called the soldering form. On the other hand, the Levi–Civita connection of M induces a principal connection γ on $\mathcal{G}$. Then $\theta + \gamma$ can be viewed as a $\mathfrak{g}$–valued one form on $\mathcal{G}$, and is elementary to verify that this is a Cartan connection. Any isometry between Riemannian manifolds lifts to the orthonormal frame bundle and such a lift preserves and the soldering form and the Levi–Civita connection. Hence any isometry defines a morphism of Cartan geometries, and it is easy to see that conversely for any such morphism the underlying map between the bases is isometry. Thus we have obtained an equivalence of categories between n–dimensional Riemannian manifolds and a subcategory of Cartan geometries of type (G, H).

2.2. **Automorphisms.** For a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of some fixed type (G, H) let $\mathrm{Aut}(\mathcal{G}, \omega)$ be the group of automorphisms. Note that for a category of Cartan geometries which is equivalent to some category of underlying structures, this group is naturally isomorphic to the automorphism group of the underlying structure. The infinitesimal version of an automorphism $\Phi : \mathcal{G} \to \mathcal{G}$ is a vector field ξ on $\mathcal{G}$ such that $(r^h)^*\xi = \xi$ for all $h \in H$ and such that $\mathcal{L}_\xi \omega = 0$. The space $\mathfrak{inf}(\mathcal{G}, \omega)$ of all these *infinitesimal automorphisms* evidently is a Lie subalgebra of $\mathfrak{X}(\mathcal{G})$.

For $A \in \mathfrak{g}$ let $\tilde{A} \in \mathfrak{X}(\mathcal{G})$ be the "constant vector field" characterized by $\omega(\tilde{A}) = A$. In particular, $\tilde{A} = \zeta_A$ for $A \in \mathfrak{h} \subset \mathfrak{g}$. For $\xi \in \mathfrak{inf}(\mathcal{G}, \omega)$ the equation $0 = (\mathcal{L}_\xi \omega)(\tilde{A})$ immediately implies $[\xi, \tilde{A}] = 0$. Hence the flows of ξ and $\tilde{A}$ commute and denoting by $\mathrm{Fl}^{\tilde{A}}_t$ the flow of $\tilde{A}$ up to time t, we obtain $\xi(\mathrm{Fl}^{\tilde{A}}_t(u)) = T_u \mathrm{Fl}^{\tilde{A}}_t(\xi(u))$ for all $u \in \mathcal{G}$ and all $t \in \mathbb{R}$ for which the flow is defined. Since the fields $\tilde{A}$ with $A \in \mathfrak{g}$ span each tangent space, we conclude that the value of $\xi \in \mathfrak{inf}(\mathcal{G}, \omega)$ in a point $u \in \mathcal{G}$ uniquely determines ξ locally around u. By H–invariance of ξ, the value in one point determines the values along the fiber through that point, and we obtain

Proposition. *If M is connected, then for any point $u_0 \in \mathcal{G}$ the map $\xi \mapsto \omega(\xi(u_0))$ defines a linear isomorphism from $\mathfrak{inf}(\mathcal{G}, \omega)$ onto a linear subspace $\mathfrak{a} \subset \mathfrak{g}$.*

Now we have to invoke a characterization of Lie transformation groups due to R. Palais, see [5, 3]:

Theorem. *Let S be a group of diffeomorphisms of a smooth manifold N and let $\mathfrak{s} \subset \mathfrak{X}(N)$ be the space of those vector fields for which the flow is defined for all times and lies in the group S. If the Lie subalgebra of $\mathfrak{X}(N)$ generated by $\mathfrak{s}$ is finite dimensional, then it coincides with $\mathfrak{s}$ and S can be made into a Lie group with Lie algebra $\mathfrak{s}$, which acts smoothly on N.*

This result can be directly applied to our situation: If ξ is a complete vector field on $\mathcal{G}$ then the corresponding one–parameter group of diffeomorphisms is contained in $\mathrm{Aut}(\mathcal{G}, \omega)$ if and only if ξ lies in $\mathfrak{inf}(\mathcal{G}, \omega)$. By the Proposition, $\mathfrak{inf}(\mathcal{G}, \omega)$ is a finite dimensional Lie subalgebra (of dimension $\leq \dim(\mathfrak{g})$) of $\mathfrak{X}(\mathcal{G})$, so we get

Corollary. *The group $\mathrm{Aut}(\mathcal{G}, \omega)$ is a Lie group with Lie algebra given by all complete vector fields contained in $\mathfrak{inf}(\mathcal{G}, \omega)$. For connected M, one has $\dim(\mathrm{Aut}(\mathcal{G}, \omega)) \leq \dim(G)$.*

Applied to the case of Riemannian manifolds discussed in 2.1, this result shows that the isometry group of a connected n–dimensional Riemannian manifold is a Lie group of dimension at most $\frac{n(n+1)}{2}$. This bound is attained for the homogeneous model $\mathbb{R}^n$ but also for S^n, which has isometry group $SO(n+1)$. This shows that there may be non–flat manifolds, for which the automorphism group has the maximal possible dimension.

2.3. **Curvature.** Let us look more closely at the relation between infinitesimal automorphisms and curvature. There is a general notion of the curvature of a Cartan geometry $(\mathcal{G},\omega)$ for which there are two equivalent descriptions. The *curvature form* $K\in\Omega^2(\mathcal{G},\mathfrak{g})$ and the *curvature function* $\kappa:\mathcal{G}\to L(\Lambda^2\mathfrak{g},\mathfrak{g})$. They are defined by

$$K(\xi,\eta)=d\omega(\xi,\eta)+[\omega(\xi),\omega(\eta)]$$
$$\kappa(u)(X,Y)=K(u)(\tilde{X},\tilde{Y}),$$

where $\tilde{X},\tilde{Y}\in\mathfrak{X}(\mathcal{G})$ are the constant vector fields corresponding to $X,Y\in\mathfrak{g}$.

The defining properties of the Cartan connection ω imply that K is H–equivariant and horizontal. Correspondingly, the function κ is H–equivariant (for the action of H on $L(\Lambda^2\mathfrak{g},\mathfrak{g})$ induced from the adjoint action of G) and has values in $L(\Lambda^2(\mathfrak{g}/\mathfrak{h}),\mathfrak{g})$. The curvature turns out to be a complete obstruction to local isomorphism of the Cartan geometry $(\mathcal{G},\omega)$ with the homogeneous model G/H.

Let $\xi\in\mathfrak{X}(\mathcal{G})$ be a vector field such that $\mathcal{L}_\xi\omega=0$. From the definitions one easily concludes that $\mathcal{L}_\xi K=0$ and $\xi\cdot\kappa=0$. If in addition $\xi(u)$ is vertical, and $A=\omega(\xi(u))$, then $\xi(u)=\zeta_A(u)$ and equivariancy of κ implies that $(\zeta_A\cdot\kappa)(u)$ coincides with the algebraic action of $A\in\mathfrak{h}$ on $\kappa(u)\in L(\Lambda^2(\mathfrak{g}/\mathfrak{h}),\mathfrak{g})$. Hence for $\mathfrak{a}=\{\omega(\xi(u_0)):\xi\in\mathfrak{inf}(\mathcal{G},\omega)\}\subset\mathfrak{g}$ we see that all elements of $\mathfrak{a}\cap\mathfrak{h}$ annihilate $\kappa(u_0)\in L(\Lambda^2(\mathfrak{g}/\mathfrak{h}),\mathfrak{g})$.

For the Cartan geometry associated to a Riemannian manifold as in 2.1, the curvature defined above euqals the usual Riemann curvature. It is well know that this splits into the Weyl curvature, the tracefree part of the Ricci curvature and the scalar curvature. While the Weyl curvature and the tracefree part of the Ricci curvature have values in a non–trivial representation of $O(n)$, the scalar curvature has values in a trivial representation. Hence from above we conclude that any Riemannian n–manifold whose isometry group has dimension $\frac{n(n+1)}{2}$ must have trivial Weyl curvature and its Ricci curvature must be pure trace, so it must be conformally flat and Einstein. As the example of S^n shows, the scalar curvature may indeed be nontrivial.

2.4. **The Lie bracket on $\mathfrak{inf}(\mathcal{G},\omega)$.** The bracket on the Lie algebra $\mathfrak{aut}(\mathcal{G},\omega)$ of $\mathrm{Aut}(\mathcal{G},\omega)$ is induced by the negative of the Lie bracket of vector fields on $\mathcal{G}$, which also makes sense on $\mathfrak{inf}(\mathcal{G},\omega)$. For $\xi\in\mathfrak{inf}(\mathcal{G},\omega)$ and $\eta\in\mathfrak{X}(\mathcal{G})$ we compute

$$\begin{aligned}0=&(\mathcal{L}_\xi\omega)(\eta)=\xi\cdot\omega(\eta)-\omega([\xi,\eta])\\=&d\omega(\xi,\eta)+\eta\cdot\omega(\xi)\\=&\kappa(\omega(\xi),\omega(\eta))-[\omega(\xi),\omega(\eta)]+\eta\cdot\omega(\xi).\end{aligned}$$

If both ξ and η are infinitesimal automorphisms, we may combine the first and last line to obtain an expression for $-\omega([\xi,\eta])$. This shows that for fixed $u_0\in\mathcal{G}$, the above bracket on $\mathfrak{inf}(\mathcal{G},\omega)$ corresponds to the operation

$$(A,B)\mapsto[A,B]-\kappa(u_0)(A,B)\qquad(*)$$

on $\mathfrak{a}=\{\omega(\xi(u_0)):\xi\in\mathfrak{inf}(\mathcal{G},\omega)\}\subset\mathfrak{g}$.

This concludes our discussion of $\mathfrak{inf}(\mathcal{G},\omega)$ for general Cartan geometries. Let us collect the results:

- Choosing a point $u_0\in\mathcal{G}$ identifies $\mathfrak{inf}(\mathcal{G},\omega)$ with a linear subspace $\mathfrak{a}\subset\mathfrak{g}$ endowed with Lie bracket given by $(*)$.
- Any element $A\in\mathfrak{a}\cap\mathfrak{h}$ annihilates the value $\kappa(u_0)\in L(\Lambda^2(\mathfrak{g}/\mathfrak{h}),\mathfrak{h})$ of the curvature function in u_0.

2.5. **The case of parabolic geometries.** Parabolic geometries are Cartan geometries corresponding to parabolic subgroups in (real or complex) semisimple Lie groups. There is a simple way to characterize these: Let $\mathfrak{g}$ be a semisimple Lie algebra endowed with a grading of the form $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$, such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and such that the nilpotent subalgebra $\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$. Put $\mathfrak{h} := \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$. For a Lie group G with Lie algebra $\mathfrak{g}$ let H be the normalizer of $\mathfrak{h}$ in G. It turns out that H has Lie algebra $\mathfrak{h}$, and this definition is equivalent to H being a parabolic subgroup of G in the sense of representation theory. A *parabolic geometry* of type (G, H) is then defined as a Cartan geometry of that type.

Putting $\mathfrak{g}^i = \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$ defines an H–invariant filtration $\mathfrak{g} = \mathfrak{g}^{-k} \supset \cdots \supset \mathfrak{g}^k$, which makes $\mathfrak{g}$ into a filtered Lie algebra such that $\mathfrak{h} = \mathfrak{g}^0$. A parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, H) is called *regular*, if its curvature function κ satisfies $\kappa(u)(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1}$ for all $u \in \mathcal{G}$ and all $i, j = -k, \ldots, -1$.

There are general results showing that regular parabolic geometries, whose curavature satisfies an additional normalization condition, are equivalent (in the categorical sense) to certain underlying structures, see e.g. [2]. These underlying structures include conformal, almost quaternionic, non–degenerate hypersurface type CR, and quaternionic CR. Hence together with many others, these strcutures can be identified with subclasses of regular normal parabolic geometries of some type.

The first important information for our purposes concerns the curvature of parabolic geometries.

Proposition. *Let $(\mathcal{G} \to M, \omega)$ be a regular normal parabolic geometry with curvature function κ. If $\kappa \neq 0$, then the lowest homogeneous component of κ has values in a nontrivial, completely reducible representation of H.*

This representation can be computed explicitly for any given type. Since it is always nontrivial, $\mathrm{Aut}(\mathcal{G}, \omega)$ may have the maximal possible dimension $\dim(G)$ only if $\kappa = 0$ and thus the parabolic geometry is locally isomorphic to the homogeneous model.

For a parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, H) fix a point $u_0 \in \mathcal{G}$ and consider the subsapce $\mathfrak{a} = \{\omega(\xi(u_0)) : \xi \in \mathfrak{inf}(\mathcal{G}, \omega)\}$ as before. Define a filtration on $\mathfrak{a}$ by $\mathfrak{a}^i := \mathfrak{a} \cap \mathfrak{g}^i$ for $i = -k, \ldots, k$. The bracket $(*)$ from 2.4 makes $\mathfrak{a}$ into a filtered Lie algebra by regularity. It is worth noticing that the filtration can be pulled back to $\mathfrak{inf}(\mathcal{G}, \omega)$ and the result does not depend on the choice of the point $u_0 \in \mathcal{G}$ but only on $p(u_0) \in M$, since different choices of u_0 are related by the action of an element of H.

The inclusion $\mathfrak{a} \hookrightarrow \mathfrak{g}$ is filtration preserving so it induces a linear map $\mathrm{gr}(\mathfrak{a}) \to \mathrm{gr}(\mathfrak{g})$ between the associated graded vector spaces. The associated graded of a filtered Lie algebra canonically inherits a Lie bracket, and by regularity the map $\mathrm{gr}(\mathfrak{a}) \to \mathrm{gr}(\mathfrak{g})$ is a Lie algebra homomorphism. Since the filtration of $\mathfrak{g}$ is derived from a grading, we conclude that $\mathrm{gr}(\mathfrak{g}) = \mathfrak{g}$ as a Lie algebra. Thus we conclude that $\mathrm{gr}(\mathfrak{a})$ (which has the same dimension as $\mathfrak{a}$) is (isomorphic to) a graded Lie subalgebra of $\mathfrak{g}$.

3. Example: 3–dimensional CR structures

These are 3–dimensional contact manifolds together with a complex structure on the contact subbundle. The prototypical examples of such manifolds arise as follows: For a smooth real hypersurface $M \subset \mathbb{C}^2$, each tangent space of M is a real subspace in $\mathbb{C}^2$ of real dimension 3. The maximal complex subspace contained in such a tangent space has to be of complex dimension one, so we obtain a complex line bundle sitting inside the tangent bundle of M. Generically, this subbundle will be non–integrable, and thus define a contact structure on M. In this case, the hypersurface M is called non–degenerate. A local CR diffeomorphism is defined as a local diffeomorphism whose tangent maps preserve the contact subbundle and such that the restriction to the contact subbundle is complex linear.

In [1], E. Cartan shows that these structures admit a canonical normal Cartan connection of type (G, H), where $G = PSU(2,1)$ and $H \subset G$ is a Borel subgroup. This construction identifies the category of 3–dimensional CR manifolds and local CR diffeomorphisms with the category of regular normal parabolic geometries of type (G, H).

The homogeneous model in this case is $S^3 \subset \mathbb{C}^2$. Therefore, CR–manifolds which are locally isomorphic to the homogeneous model are called *spherical*.

The general results on Cartan geometries imply that the group $\mathrm{Aut}(M)$ of CR automorphisms of a 3–dimensional CR manifold M is a Lie group of dimension at most $\dim(G) = 8$. We now claim:

Theorem. *(1) If* $\dim(\mathrm{Aut}(M)) < 8$*, then* $\dim(\mathrm{Aut}(M)) \leq 5$*.*
(2) If M is not spherical, then $\dim(\mathrm{Aut}(M)) \leq 3$*.*

Proof. The grading of $\mathfrak{g} = \mathfrak{su}(2,1)$ has the form $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$ with $\mathfrak{g}_{\pm 2} \cong \mathbb{R}$, $\mathfrak{g}_{\pm 1} \cong \mathbb{C}$ and $\mathfrak{g}_0 \cong \mathbb{C}$. The Lie algebra of $\mathrm{Aut}(M)$ must be contained in $\mathfrak{inf}(\mathcal{G}, \omega)$, which gives rise to a graded Lie subalgebra $\mathrm{gr}(\mathfrak{a})$ of $\mathfrak{g}$. Hence we can prove (1) by showing that any proper graded Lie subalgebra of $\mathfrak{g}$ has dimension at most 5.

For (2) one verifies that the representation of $\mathfrak{h}$, in which the lowest nonzero homogeneous component of the curvature has its values, comes from a faithful representation of $\mathfrak{g}_0 \cong \mathbb{C}$. Thus we can prove (2) by showing that any graded Lie subalgebra of $\mathfrak{g}$ which has a trivial component in degree 0 has dimension at most 3.

For an appropriate choice of Hermitian metric on $\mathbb{C}^2$ we have

$$\mathfrak{g} = \left\{ \begin{pmatrix} \alpha + i\beta & z & i\psi \\ x & -2i\beta & -\bar{z} \\ i\varphi & -\bar{x} & -\alpha + i\beta \end{pmatrix} \right\}$$

with $\alpha, \beta, \varphi, \psi \in \mathbb{R}$ and $x, z \in \mathbb{C}$. The grading is given by the diagonals, i.e. the component $i\varphi$ lies in $\mathfrak{g}_{-2}$, the component x in $\mathfrak{g}_{-1}$, and so on. From this, one immediately reads off that the brackets between the various grading components. The main point is that the brackets $\mathfrak{g}_{\pm 1} \times \mathfrak{g}_{\pm 1} \to \mathfrak{g}_{\pm 2}$ are given by the standard symplectic form on $\mathbb{C}$, while the other brackets are essentially induced by complex multiplications.

Suppose that $\mathfrak{b} = \mathfrak{b}_{-2} \oplus \cdots \oplus \mathfrak{b}_2$ is a graded Lie subalgebra of $\mathfrak{g}$, put $n_i = \dim(\mathfrak{b}_i)$ and $n = \dim(\mathfrak{b})$, where all dimensions are over $\mathbb{R}$.

Case 1: $n_{-1} = 2$. This means that $\mathfrak{b}_{-1} = \mathfrak{g}_{-1}$ and then $[\mathfrak{b}_{-1}, \mathfrak{b}_{-1}] = \mathfrak{g}_{-2} \subset \mathfrak{b}$. Suppose there is a nonzero element $z \in \mathfrak{b}_1$. Then $[z, \mathfrak{b}_{-1}] = \mathfrak{g}_0$ and hence $[z, \mathfrak{g}_0] = \mathfrak{g}_1$ are contained in $\mathfrak{b}$, which immediately implies $\mathfrak{b} = \mathfrak{g}$. Hence we conclude that $\mathfrak{b} \neq \mathfrak{g}$ is only possible if $n_1 = 0$. This implies $n_2 = 0$, since for a nonzero element $i\psi \in \mathfrak{g}_2$ the map $\mathrm{ad}_{i\psi} : \mathfrak{g}_{-1} \to \mathfrak{g}_1$ is surjective. Hence $\mathfrak{b} \subset \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$, and we get (1) and (2).

Case 2: $n_{-1} = 1$. For $0 \neq x \in \mathfrak{b}_{-1}$ the map ad_x is a linear isomorphism $\mathfrak{g}_0 \to \mathfrak{g}_{-1}$ and $\mathfrak{g}_1 \to \mathfrak{g}_0$, so we conclude that $n_0 \leq 1$ and then $n_1 \leq 1$, which implies (1). For $n_0 = 0$ we also must have $n_1 = 0$, which implies (2).

Case 3: $n_{-1} = 0$. Since the bracket induces a linear isomorphism $\mathfrak{g}_{-2} \otimes \mathfrak{g}_1 \to \mathfrak{g}_{-1}$ we conclude that either $n_{-2} = 0$ or $n_1 = 0$, which completes the proof. □

This theorem reduces the classification of homogeneous 3–dimensional CR manifolds to pure algebra: In the spherical case, the Lie algebra of the automorphism group is a subalgebra of $\mathfrak{g} = \mathfrak{su}(2,1)$, and one can work in the homogeneous model. If M is not spherical, then $\dim(\mathrm{Aut}(M)) = 3$ and fixing a point $x_0 \in M$ the map $f \mapsto f(x_0)$ is a covering $\mathrm{Aut}(M) \to M$. The CR structure on M lifts to a left invariant structure on $\mathrm{Aut}(M)$. Hence any non–spherical homogeneous 3–dimensional CR structure is covered by a left invariant structure on a Lie group. Determining such left invariant structures is a rather simple algebraic problem.

For higher dimensional CR structures, similar methods are used in [7] to determine the second largest possible dimension for the automorphism group. In that paper, Yamaguchi completely classified the CR structures with automorphism group of this second largest dimension.

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