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THE DIRAC OPERATOR ON PIN MANIFOLDS

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Abstract. Odd-dimensional Riemannian spaces that are non-orientable, but have a pin structure, require the consideration of the twisted adjoint representation of the corresponding pin group. It is shown here how the Dirac operator should be modified, also on even-dimensional spaces, to make it equivariant with respect to the action of that group when the twisted adjoint representation is used in the definition of the pin structure. An explicit description of a pin structure on a hypersurface, defined by its immersion in a Euclidean space, is used to derive a simple formula for the Dirac operator in that case.

1. Introduction

Most of the research on the Dirac operator on Riemannian spaces is restricted to the case of orientable manifolds. It is of some interest to treat also the non-orientable case that requires the introduction of pin structures. In physics, even in the orientable case, one considers spinor fields transforming under space and time reflections, which are covered by elements of a suitable pin group. The generalization to the non-orientable case involves interesting subtleties. First of all, for a real vector space with a quadratic form of signature (k, l), the Clifford construction yields two groups $\mathrm{Pin}_{k,l}$ and $\mathrm{Pin}_{l,k}$, which need not be isomorphic; see [Ka] and Section 3 for a precise statement. This fact is of interest also to physics [CDeW]. There are non-orientable spaces with a metric tensor field of signature (k, l) admitting either a $\mathrm{pin}_{k,l}$-structure or a $\mathrm{pin}_{l,k}$-structure. If a space admits a $\mathrm{spin}_{k,l}$-structure, then it is orientable and admits both these structures. Real projective spaces and quadrics provide the simplest examples of such situations [DaT1, CaGuT]. If the dimension $k + l$ is even, then one can use either the adjoint or the twisted adjoint representation of $\mathrm{Pin}_{k,l}$. If one uses the twisted adjoint representation, as one has to do when $k + l$ is odd, then the classical Dirac operator (see, e.g., [ABP, LM, BoWo]) needs to be modified to make it equivariant with respect to the action of the pin group [T1, T2].

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In this paper, the relation between the adjoint and the twisted adjoint representation of the pin group is considered in some detail (Section 3) and a new derivation of the modified Dirac operator is given in Section 4. A canonical pin structure on a hypersurface immersed in a Euclidean space is described in Section 5 and shown to have a trivial associated bundle of 'Dirac' or 'Pauli' spinors . A convenient formula for the Dirac operator on such hypersurfaces is derived in Section 6. As a simple illustration, the spectrum and the eigenfunctions of the Dirac operator on real projective spaces are found on the basis of the corresponding results for spheres.

2. NOTATION AND PRELIMINARIES

2.1. Clifford algebras and pin groups. Throughout this paper, by an algebra I mean an associative algebra with a unit element, which is denoted by 1. A homomorphism of algebras is understood to map one unit into another. If V is a finite-dimensional vector space, then V^* denotes its dual and the value $f(v)$ of the 1-form $f \in V^*$ on $v \in V$ is often denoted by $\langle v, f\rangle$. If $h : V \to W$ is a linear map (homomorphism of vector spaces), then its *transpose* ${}^t h : W^* \to V^*$ is defined by $\langle v, {}^t h(f)\rangle = \langle h(v), f\rangle$ for every $v \in V$ and $f \in W^*$. Let V be a real, oriented, m-dimensional vector space with an isomorphism $h : V \to V^*$ which is symmetric, $h = {}^t h$, and such that the quadratic form $V \to \mathrm{R}$, given by $v \mapsto \langle v, h(v)\rangle$ is of signature (k, l), $k + l = m$. One says that the pair (V, h) is a *quadratic space* of dimension m and signature (k, l). The corresponding *Clifford algebra* (see, e.g., [ABS, Ka, LM, BuT]),

$$\mathrm{Cl}(h) = \mathrm{Cl}^0(h) \oplus \mathrm{Cl}^1(h),$$

contains $\mathrm{R} \oplus V$ and is Z_2-graded by the *main automorphism* α characterized by $\alpha(1) = 1$ and $\alpha(v) = -v$ for every $v \in V$. Every $a \in \mathrm{Cl}(h)$ is decomposed into its even and odd components, a_0 and a_1, respectively, such that $a_\varepsilon \in \mathrm{Cl}^\varepsilon(h)$ and $a = a_0 + a_1 = \alpha(a_0 - a_1)$. For every $v \in V$, its Clifford square is $v^2 = \langle v, h(v)\rangle$. Let $(e_1, \dots, e_m)$ be an orthonormal frame in V of the preferred orientation. The matrix $(h_{\mu\nu})$, where $h_{\mu\nu} = h(e_\mu, e_\nu)$, $\mu, \nu = 1, \dots, m$ and its inverse $h^{\mu\nu}$ are symmetric; they are sometimes used to 'lower' and 'raise' indices. The square of the volume element $\mathrm{vol}(h) = e_1 \dots e_m$ is

$$\mathrm{vol}(h)^2 = i(h)^2, \quad \text{where} \quad i(h) \in \{1, \mathrm{i}\}.$$

For every $v \in V$ one has $\mathrm{vol}(h)v = (-1)^{m+1} v\,\mathrm{vol}(h)$. Therefore, if m is even, then α is an inner automorphism, $\alpha(a) = \mathrm{vol}(h) a\, \mathrm{vol}(h)^{-1}$.

It follows from the universality of Clifford algebras that the Clifford map

$$V \to \mathrm{Cl}(h), \quad v \mapsto \mathrm{vol}(h)v,$$

extends to the homomorphism of algebras,

$$j : \mathrm{Cl}((-1)^{m+1}\mathrm{vol}(h)^2 h) \to \mathrm{Cl}(h), \tag{1}$$

such that $j(1) = 1$ and $j(v) = \mathrm{vol}(h)v$ for $v \in V$. For m even, this homomorphism is bijective and respects the Z_2-grading of the algebras. If m is even and $\mathrm{vol}(h)^2 = 1$, then $j : \mathrm{Cl}(-h) \to \mathrm{Cl}(h)$ is an isomorphism of algebras. If m is even and $\mathrm{vol}(h)^2 = -1$, then the algebras $\mathrm{Cl}(h)$ and $\mathrm{Cl}(-h)$ are not isomorphic and j is an inner automorphism of $\mathrm{Cl}(h)$ given by $j(a) = \frac{1}{2}(1 + \mathrm{vol}(h))a(1 - \mathrm{vol}(h))$. If m is odd, then the homomorphism (1) is onto the even subalgebra $\mathrm{Cl}^0(h)$. In this case, the volume element corresponding to $\mathrm{vol}(h)^2 h$ has a positive square. Therefore, if m is odd and h is such that $\mathrm{vol}(h)^2 = 1$, then $j(\mathrm{vol}(h)) = 1$ and there is the exact sequence of homomorphisms of algebras with units, $0 \to \mathrm{Cl}^-(h) \to \mathrm{Cl}(h) \xrightarrow{j} \mathrm{Cl}^0(h) \to 0$, where $\mathrm{Cl}^-(h) = \{a \in \mathrm{Cl}(h) : \mathrm{vol}(h)a = -a\}$ is the subalgebra of anti-selfdual elements of $\mathrm{Cl}(h)$. There is no analogous sequence for m odd and h such that $\mathrm{vol}(h)^2 = -1$. For m odd, the algebras $\mathrm{Cl}(h)$ and $\mathrm{Cl}(-h)$ are never isomorphic. The algebras $\mathrm{Cl}^0(h)$ and $\mathrm{Cl}^0(-h)$ are isomorphic irrespective of m and h.

An element $u \in V$ is said to be a *unit vector* if either $u^2 = 1$ or $u^2 = -1$. The group $\mathrm{Pin}(h)$ is defined as the subset of $\mathrm{Cl}(h)$ consisting of products of all finite sequences of unit vectors; the group multiplication is induced by the Clifford product[1]. The spin group is

$$\mathrm{Spin}(h) = \mathrm{Pin}(h) \cap \mathrm{Cl}^0(h).$$

The Lie algebra $\mathrm{spin}(h)$ of $\mathrm{Spin}(h)$ can be identified with the subspace of $\mathrm{Cl}^0(h)$ spanned by all elements of the form $uv - vu$, where $u, v \in V$. The Lie bracket in $\mathrm{spin}(h)$ coincides with the commutator induced by the Clifford product.

If $V = \mathrm{R}^{k+l}$ and one wants to specify the signature (k, l) of h, then one writes $\mathrm{vol}_{k,l}$, $\mathrm{Cl}_{k,l}$, $\mathrm{Pin}_{k,l}$ and $\mathrm{Spin}_{k,l}$ instead of $\mathrm{vol}(h)$, $\mathrm{Cl}(h)$, $\mathrm{Pin}(h)$ and $\mathrm{Spin}(h)$, respectively; a similar notation is used for the orthogonal groups $\mathrm{O}(h)$ and $\mathrm{SO}(h)$. Since the groups $\mathrm{Spin}(h)$ and $\mathrm{Spin}(-h)$ are isomorphic, one writes Spin_m instead of $\mathrm{Spin}_{m,0} = \mathrm{Spin}_{0,m}$. Since $\mathrm{vol}^2_{2n,0} = \mathrm{vol}^2_{0,2n}$ one can also write vol_{2n} instead of $\mathrm{vol}_{2n,0}$ or $\mathrm{vol}_{0,2n}$. For every pair (k, l) of non-negative integers, there is the *isomorphism* of algebras,

$$\iota : \mathrm{Cl}_{k,l} \to \mathrm{Cl}^0_{k,l+1} \quad \text{given by} \quad \iota(a_0 + a_1) = a_0 + a_1 e_{k+l+1}.$$

By restriction, it gives rise to the *monomorphism* of groups

$$\iota : \mathrm{Pin}_{k,l} \to \mathrm{Spin}_{k,l+1}. \tag{2}$$

The corresponding monomorphism of the (pseudo)orthogonal groups,

$$\kappa : \mathrm{O}_{k,l} \to \mathrm{SO}_{k,l+1} \text{ such that } \kappa \circ \rho = \rho \circ \iota, \tag{3}$$

satisfies $\kappa(A)e_\mu = Ae_\mu$, $\mu = 1, \ldots, k + l$ and $\kappa(A)e_{k+l+1} = (\det A)e_{k+l+1}$ for every $A \in \mathrm{O}_{k,l}$.

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[1] This definition of the pin group follows [Ba, LM] and can be traced to Cartan, cf. §§12, 97 and 127 in [C]. An equivalent definition, using the notion of spinor norm and based on the simplicity of $\mathrm{Cl}(h)$ for m even and and of $\mathrm{Cl}^0(h)$ for m odd, is in [BrWe, Che, ABS]

2.2. Notation concerning smooth manifolds and bundles. In this article, most of the time, I use the standard terminology and notation of differential geometry and mathematical physics [AbMa, He, KoNo]. For the convenience of the reader, some of the notation is summarized in the following two paragraphs.

2.2.1. *Manifolds, bundles and groups.* All manifolds, maps and bundles are assumed to be smooth; manifolds are paracompact and bundles are locally trivial. If $\pi : E \to M$ and $\sigma : F \to N$ are two bundles , then the pair (f, f') of maps $f : M \to N$ and $f' : E \to F$ is a *morphism of bundles* if $\sigma \circ f' = f \circ \pi$. A bundle is trivial if it is isomorphic to a Cartesian product of its base by the typical fiber. A map $s : M \to E$ is a *section* of π if $\pi \circ s = \mathrm{id}_M$. For every manifold M, there is the *tangent bundle* $\tau_M : TM \to M$. If $f : M \to N$ is a map of manifolds, then $Tf : TM \to TN$ is the derived map of their tangent bundles and (f, Tf) is a morphism of bundles. For $x \in M$, there is the linear map $T_x f : T_x M \to T_{f(x)}N$ of the fiber $T_x M$ of the bundle $TM \to M$ into the corresponding fiber of the other bundle. Given a bundle $\sigma : F \to N$ and a map $f : M \to N$, one defines the bundle $\pi : E \to M$ *induced* by f from σ as follows: $E = \{(x, q) \in M \times F : \sigma(q) = f(x)\}$ and $\pi(x, q) = x$. There is then also a canonical map, $f' : E \to F$, given by $f'(x, q) = q$ and the pair (f, f') is a morphism of bundles. A *Riemannian space* is a connected manifold M with a metric tensor field g which need not be definite; if it is, then one refers to M as a *proper* Riemannian space. For every $x \in M$, the metric tensor defines a symmetric isomorphism $g_x : T_x M \to T_x{}^* M$. If M is a Riemannian space, then there is a quadratic space (V, h) such that, for every $x \in M$, there is a linear isometry $p : V \to T_x M$, .i.e. a linear isomorphism such that ${}^t p \circ g_x \circ p = h$. One says that (V, h) is *local model* of the Riemannian space and that p is an *orthonormal frame at* x. If (e_μ) is an orthonormal frame in V, then p can be identified with the collection of vectors (p_μ), where $p_\mu = p(e_\mu)$, $\mu = 1, \ldots, m = \dim V = \dim M$.

If ω is a differential form on a manifold, then $\mathrm{d}\omega$ is its exterior derivative. Wedge denotes the exterior product of forms. If X is a vector field on M and ω is a $(p+1)$-form, then $X \lrcorner \omega$ is the p-form such that $(X \lrcorner \omega)(X_1, \ldots, X_p) = \omega(X, X_1, \ldots, X_p)$ for every collection $(X_1, \ldots, X_p)$ of vector fields on M. In particular, if $f : M \to \mathrm{R}$ and X is a vector field, then $X \lrcorner \mathrm{d}f = \langle X, \mathrm{d}f\rangle = X(f)$ is the derivative of the function f in the direction of the vector field X.

By a group is meant here a Lie group; a subgroup is a closed Lie subgroup. Given a representation $\gamma : G \to \mathrm{GL}(S)$ of the group G in a vector space S and a homomorphism $h : G \to G'$ of groups, one says that a representation $\gamma' : G' \to \mathrm{GL}(S)$ *extends* γ (relative to h) if $\gamma' \circ h = \gamma$.

2.2.2. *Principal and associated bundles.* A *principal bundle* with structure group G ('principal G-bundle') and projection π of its total space P to the base manifold M is sometimes represented, symbolically, by the sequence $G \to P \xrightarrow{\pi} M$. The group G is assumed to act on P to the right: there is a map $\delta : P \times G \to P$ such that, if $\delta(a)(p) = \delta(p,a)$, then $\pi \circ \delta(a) = \pi$, $\delta(a) \circ \delta(b) = \delta(ba)$ and $\delta(1_G) = \mathrm{id}_P$, where $p \in P$, $a, b \in G$ and 1_G is the unit of G. One writes pa instead of $\delta(p,a)$. A principal bundle admitting a section f is trivial, i.e. isomorphic (in the category of principal bundles) to the product bundle $M \times G \to M$; a *trivializing map* (isomorphism of principal bundles) is given by $(x,a) \mapsto f(x)a$, where $x \in M$ and $a \in G$. Let there be given a left action of the group G on the manifold S, i..e. a map $\gamma : G \times S \to S$ such that, if $\gamma(a)(\varphi) = \gamma(a,\varphi)$, then $\gamma(a) \circ \gamma(b) = \gamma(ab)$ and $\gamma(1_G) = \mathrm{id}_S$ for every $a, b \in G$ and $\varphi \in S$. One then defines the bundle $\pi_E : E \to M$, *associated* with P by γ. Its typical fiber is S and its total space E, often denoted by $P \times_\gamma S$, is the set of all equivalence classes of the form $[(p,\varphi)]$, where $(p,\varphi) \in P \times S$ and $[(p',\varphi')] = [(p,\varphi)]$ if, and only if, there exists $a \in G$ such that $p' = pa$ and $\varphi = \gamma(a)\varphi'$. The projection is given by $\pi_E([(p,\varphi)]) = \pi(p)$. If S is a vector space, then the associated bundle is a *vector bundle*. A homomorphism $\iota : G \to G'$ of groups defines a left action of G on G', viz. $(a,b) \mapsto \iota(a)b$, where $a \in G$ and $b \in G'$; the corresponding bundle $P \times_\iota G' \to M$, associated with $P \to M$, is a principal G'-bundle.

3. Representations of the pin groups

3.1. The vector representations. For every invertible $u \in V$, the map $v \mapsto -uvu^{-1}$ is a reflection in the hyperplane orthogonal to the vector u; this observation leads to the definition of the *twisted adjoint* vector representation ρ of the group $\mathrm{Pin}(h)$ in V: for every $a \in \mathrm{Pin}(h)$ the map $\rho(a) : V \to V$, given by

$$\rho(a)v = \alpha(a)va^{-1} \tag{4}$$

is orthogonal,

$$^t\rho(a) \circ h \circ \rho(a) = h, \tag{5}$$

and there is the exact sequence of group homomorphisms

$$1 \to \{1,-1\} \to \mathrm{Pin}(h) \xrightarrow{\rho} \mathrm{O}(h) \to 1.$$

Replacing in (4) the vector v by the μth vector e_μ of an orthonormal frame in V, one obtains

$$e_\nu \rho^\nu{}_\mu(a) = \alpha(a)e_\mu a^{-1} \tag{6}$$

In this equation, and elsewhere in this paper, there is tacitly assumed a summation (the *Einstein convention*) over the range of repeated tensor indices.

The *adjoint* vector representation Ad is defined by

$$\mathrm{Ad}(a)v = ava^{-1}$$

and leads to the exact sequences of group homomorphisms

$$1 \to \left\{ \begin{array}{c} \{1,-1\} \\ \{1,-1,\mathrm{vol}(h),-\mathrm{vol}(h)\} \end{array} \right\} \to \mathrm{Pin}(h) \xrightarrow{\mathrm{Ad}} \left\{ \begin{array}{c} \mathrm{O}(h) \\ \mathrm{SO}(h) \end{array} \right\} \to 1 \quad \left\{ \begin{array}{l} \text{for } m \text{ even,} \\ \text{for } m \text{ odd.} \end{array} \right.$$

The homomorphisms ρ and Ad coincide when restricted to $\mathrm{Spin}(h)$. For every quadratic space (V, h), irrespective of the parity of m, there is the exact sequence

$$1 \to \mathrm{Z}_2 \to \mathrm{Spin}(h) \xrightarrow{\rho} \mathrm{SO}(h) \to 1, \tag{7}$$

where $\mathrm{Z}_2 = \{1, -1\}$.

For every *even*-dimensional quadratic space (V, h), one can consider four central extensions of $\mathrm{O}(h)$ by Z_2, associated with the groups $\mathrm{Pin}(\pm h)$, namely

$$\rho \text{ and } \mathrm{Ad} : \mathrm{Pin}(h) \to \mathrm{O}(h), \quad \text{and } \rho \text{ and } \mathrm{Ad} : \mathrm{Pin}(-h) \to \mathrm{O}(h),$$

but, in each case, only two among the four are inequivalent.[2] Indeed, if m is even, then

$$\rho = \mathrm{Ad} \circ j.$$

as may be seen by checking that both sides coincide on the generating subset V. More precisely:

(i.) if $\mathrm{vol}(h)^2 = 1$, then the extensions

$$\mathrm{Z}_2 \to \mathrm{Pin}(\pm h) \xrightarrow{\mathrm{Ad}} \mathrm{O}(h)$$

are equivalent to the corresponding extensions

$$\mathrm{Z}_2 \to \mathrm{Pin}(\mp h) \xrightarrow{\rho} \mathrm{O}(h);$$

(ii) if $\mathrm{vol}(h)^2 = -1$, then the extensions

$$\mathrm{Z}_2 \to \mathrm{Pin}(\pm h) \xrightarrow{\mathrm{Ad}} \mathrm{O}(h)$$

are equivalent to the corresponding extensions

$$\mathrm{Z}_2 \to \mathrm{Pin}(\pm h) \xrightarrow{\rho} \mathrm{O}(h).$$

To summarize, we have

Proposition 1. *For every real quadratic space* (V, h), *there are two inequivalent central extensions of* $\mathrm{O}(h)$ *by* Z_2, *given by*

$$\mathrm{Z}_2 \to \mathrm{Pin}(h) \xrightarrow{\rho} \mathrm{O}(h) \quad \text{and} \quad \mathrm{Z}_2 \to \mathrm{Pin}(-h) \xrightarrow{\rho} \mathrm{O}(h), \tag{8}$$

where ρ *is as in* (4). *By restriction to* $\mathrm{Spin}(h)$ *each of these extensions reduces to the one given by* (7).

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[2] Recall that, by definition, two extensions, $K \xrightarrow{k} G \xrightarrow{l} H$ and $K \xrightarrow{k'} G' \xrightarrow{l'} H$, of the group H by the group K, are equivalent if there is an isomorphism of groups $f : G \to G'$ such that $f \circ k = k'$ and $l' \circ f = l$.

Note that for $k = l$ (neutral signature) the groups $\mathrm{Pin}(h)$ and $\mathrm{Pin}(-h)$ are isomorphic, but the extensions (8) are not. There are also extensions of $\mathrm{O}(h)$ by Z_2 that do not come from the Clifford construction [Da]. The (untwisted) adjoint representation seems to be the first to have attracted attention. It has been much used by physicists in the theory of the Dirac equation of the electron, see, e.g., [Pul] or any book on relativistic quantum mechanics. The twisted representation is implicit in É. Cartan's approach to spinors, cf. §58 and §97 in [C], and has been explicitly defined by Atiyah *et al.* in [ABS]. It follows from the preceding remarks that, for even-dimensional spaces, one can use either of the two representations, but in the case of odd dimensions, only ρ provides a cover of the full orthogonal group. For this reason and for uniformity, from now on, only ρ is used in the definition of pin structures.

3.2. The spinor representations. In this paper, by a *spinor representation* of a group $\mathrm{Pin}(h)$ or $\mathrm{Spin}(h)$ is understood a representation obtained by restriction, to the group, of a representation of the algebra $\mathrm{Cl}(h)$ in a finite-dimensional *complex* vector space S, the space of *spinors*. If $\gamma : \mathrm{Cl}(h) \to \mathrm{End}S$ is any representation of the algebra, then the group representation, obtained by restriction to $\mathrm{Pin}(h)$, is denoted by the same letter γ; similar abuses of notation and terminology are made throughout the paper. In particular, the derived representation of the Lie algebra $\mathrm{spin}(h) \subset \mathrm{Cl}^0(h)$ in S, is also denoted by the same letter. Given an orthonormal frame (e_μ) in V, one defines the Dirac 'matrices' (automorphisms of S) by $\gamma_\mu = \gamma(e_\mu)$.

(i) If m is *even*, $m = 2\nu$, $\nu \in \mathrm{N}$, then the algebra $\mathrm{Cl}(h)$ is central simple and, as such, has only one, up to equivalence, faithful and irreducible *Dirac representation* γ in a vector space S, which turns out to be of complex dimension 2^ν. The restriction of γ to $\mathrm{Cl}^0(h)$ decomposes into the direct sum $\gamma_+ \oplus \gamma_-$ of two complex-inequivalent *Weyl representations*. The carrier spaces T_+ and T_- of these representations are both of complex dimension $2^{\nu-1}$ and $S = T_+ \oplus T_-$. The elements of S and $T_\pm$ are called Dirac and Weyl spinors, respectively. In a notation adapted to the decomposition of S, the Dirac matrices are of the form

$$\gamma_\mu = \begin{pmatrix} 0 & \gamma_\mu^- \\ \gamma_\mu^+ & 0 \end{pmatrix}.$$

(ii) If m is *odd*, $m = 2\nu - 1$, then the algebra $\mathrm{Cl}^0(h)$ is central simple and has a faithful and irreducible *Pauli representation* in a space T of complex dimension $2^{\nu-1}$. This representation extends to two representations, σ and $\sigma \circ \alpha$ of the full algebra $\mathrm{Cl}(h)$ in T by putting $\sigma(\mathrm{vol}(h)) = i(h)\mathrm{id}_T$. These representation, also referred to as Pauli representations of $\mathrm{Cl}(h)$, are complex-inequivalent and irreducible, but faithful only when $i(h) = \mathrm{i}$. A faithful, but reducible, *Cartan representation* γ of $\mathrm{Cl}(h)$ in

$S = T \oplus T$ is defined as $\gamma = \sigma \oplus (\sigma \circ \alpha)$. Therefore, if $\sigma_\mu = \sigma(e_\mu)$, then

$$\gamma_\mu = \begin{pmatrix} \sigma_\mu & 0 \\ 0 & -\sigma_\mu \end{pmatrix}.$$

The elements of S and T are now called Cartan and Pauli spinors, respectively.

If γ is as in (i) or (ii), then the *helicity* automorphism of S is $\gamma(\mathrm{vol}(h))$ so that Weyl (resp., Pauli) spinors are its eigenvectors for m even (resp., odd). The foregoing remarks can be summarized in

Proposition 2. *Let $\nu \in \mathrm{N}$ and let (V, h) be a quadratic space of dimension $m = 2\nu$ (resp., $2\nu - 1$) with an orthonormal frame (e_μ), $\mu = 1, \dots, m$. There is a faithful representation γ of the Clifford algebra $\mathrm{Cl}(h)$ in a complex vector space S of dimension 2^ν such that the representations γ and $\gamma \circ \alpha$ are complex-equivalent. The representation is unique, up to complex equivalence, and irreducible (resp., decomposable into two irreducibles). By restriction to the even subalgebra $\mathrm{Cl}^0(h)$, the representation γ decomposes into the direct sum of two irreducible representations, each defined in a complex space of dimension $2^{\nu-1}$. The isomorphism γ_{m+1} intertwining the representations γ and $\gamma \circ \alpha$ can be taken to act on the Dirac (resp., Cartan) spinor $(\varphi, \psi) \in S$ so that $\gamma_{m+1}(\varphi, \psi) = (\mathrm{i}\varphi, -\mathrm{i}\psi)$ (resp., $\gamma_{m+1}(\varphi, \psi) = (\psi, -\varphi)$). Irrespective of the parity of m, one has*

$$\gamma_{m+1}^2 = -\mathrm{id}_S \quad \textit{and} \quad \gamma_{m+1}\gamma_\mu + \gamma_\mu\gamma_{m+1} = 0, \tag{9}$$

for $\mu = 1, \dots, m$.

By applying a spinor representation γ to both sides of (6), one obtains

$$\gamma_\nu \rho^\nu{}_\mu(a) = \gamma \circ \alpha(a)\gamma_\mu\gamma(a). \tag{10}$$

3.3. Extension of a spinor representation from dimension m to $m + 1$. The following statement is a direct consequence of (2) and Proposition 2.

Proposition 3. *For every pair (k, l) of non-negative integers, such that $m = k + l = 2\nu$ or $2\nu - 1$, there is an extension of the spinor representation γ of the group $\mathrm{Pin}_{k,l}$ to a spinor representation γ' of the group $\mathrm{Pin}_{k,l+1}$ in the same space of spinors,*

$$\mathrm{Pin}_{k,l} \xrightarrow{\iota} \mathrm{Spin}_{k,l+1} \xrightarrow{\mathrm{inj}} \mathrm{Pin}_{k,l+1} \xrightarrow{\gamma'} \mathrm{GL}(2^\nu, \mathrm{C}), \quad \gamma = \gamma' \circ \mathrm{inj} \circ \iota. \tag{11}$$

In particular, for $m = 2\nu$ (resp., $2\nu - 1$), the Dirac *(resp.,* Cartan*) representation of* $\mathrm{Pin}_{k,l}$ *extends to a* Pauli *(resp.,* Dirac*) representation of* $\mathrm{Pin}_{k,l+1}$. *Moreover, for $m = 2\nu - 1$, there are the extensions,*

$$\mathrm{Pin}_{k,l} \xrightarrow{\iota} \mathrm{Spin}_{k,l+1} \to \mathrm{GL}(2^{\nu-1}, \mathrm{C}), \tag{12}$$

of the Pauli *representations of* $\mathrm{Pin}_{k,l}$ *to the* Weyl *representations of* $\mathrm{Spin}_{k,l+1}$.

Remark. By iteration of the above, one can obtain, for *m odd*, two Pauli representations of $\mathrm{Pin}_{k,l+2}$ extending the Cartan representation of $\mathrm{Pin}_{k,l}$. Similarly, for *m even*, there are two Weyl representations of $\mathrm{Spin}_{k,l+2}$ extending the Dirac representation of $\mathrm{Pin}_{k,l}$. One cannot, however, go beyond that without changing the dimension of the space of spinors underlying the representations.

4. PIN STRUCTURES AND BUNDLES OF SPINORS

4.1. Definitions.

4.1.1. *Pin and spin structures.* Let M be a connected, m-dimensional Riemannian manifold with a metric tensor of the same signature (k,l) as that of the quadratic space (V,h); let $\pi : P \to M$ be the bundle of all orthonormal frames of M. A *pin(h)-structure* on M is a principal $\mathrm{Pin}(h)$-bundle $\varpi : Q \to M$, together with a morphism $\chi : Q \to P$ of principal bundles over M associated with the epimorphism $\rho : \mathrm{Pin}(h) \to \mathrm{O}(h)$. The morphism condition means that $\varpi = \pi \circ \chi$ and there is the commutative diagram

$$\begin{tikzcd} Q \times \mathrm{Pin}(h) \arrow[r] \arrow[d, "\chi\times\rho"'] & Q \arrow[d, "\chi"] \\ P \times \mathrm{O}(h) \arrow[r] & P \end{tikzcd} \tag{13}$$

where the horizontal arrows denote the action maps.

The expression *$\mathrm{pin}_{k,l}$-structure* is used when one wants the signature to appear explicitly. For brevity, we shall describe a pin(h)-structure by the sequence

$$\mathrm{Pin}(h) \to Q \xrightarrow{\chi} P \xrightarrow{\pi} M. \tag{14}$$

If M is orientable and admits a pin(h)-structure, then it has a spin structure. In an abbreviated style, similar to that of (14), it may be described by the sequence of maps

$$\mathrm{Spin}(h) \to SQ \to SP \to M, \tag{15}$$

where SP is now an $\mathrm{SO}(h)$-bundle, one of the two connected components of P.

4.1.2. *Bundles of spinors.* Let M be a Riemannian space with a pin(h)-structure (14) and let γ be a spinor representation of the group $\mathrm{Pin}(h)$ in S, as described in Prop. 2. The complex vector bundle $\pi_E : E \to M$, with typical fiber S, associated with Q by γ, is the *bundle of spinors of type* $.\gamma$. If the dimension m of M is even (resp., odd), then E is called a bundle of Dirac (resp., Cartan) spinors. For m odd, one can also take the representation $\sigma : \mathrm{Pin}(h) \to \mathrm{GL}(T)$ to define the bundle of Pauli spinors over M. Similarly, if m is even and M has a spin structure, then there are two bundles of Weyl spinors over M.

4.1.3. *Spinor fields.* Let M be a Riemannian manifold with a pin structure (14). A *spinor field*[3] of type γ on M is a section of π_E. The (vector) space of such sections is known to be in a natural and bijective correspondence with the set of all maps $\psi : Q \to S$ equivariant with respect to the action of $\mathrm{Pin}(h)$,

$$\psi \circ \delta(a) = \gamma(a^{-1})\psi, \tag{16}$$

for every $a \in \mathrm{Pin}(h)$. It is convenient to refer to ψ itself as a spinor field on M of type γ. Depending on whether E is a bundle of Dirac, Weyl, Cartan or Pauli spinors, one refers to its sections as Dirac, Weyl, Cartan or Pauli spinor fields, respectively.

4.2. Existence of pin structures. Let $TM = T^+M \oplus T^-M$ be the decomposition of the tangent bundle of M into the Whitney sum of two vector bundles such that the metric tensor restricted to T^+M (resp., T^-M) is positive- (resp., negative-) definite. Denoting by w_i^+ (resp., by w_i^-) the ith Stiefel-Whitney class [MS] of T^+M (resp., of T^-M), one can formulate

Theorem 1. (Karoubi) *A Riemannian space admits a pin*(h)*-structure* (14) *if, and only if,*

$$w_2^+ + w_2^- + w_1^-(w_1^+ + w_1^-) = 0. \tag{17}$$

A proof of the Theorem is in [Ka]. Introducing the Stiefel-Whitney classes w_i of TM, one can write (17) as

$$w_2 + (w_1^-)^2 = 0. \tag{18}$$

In particular, if M proper Riemannian, then the condition for M to have a $\mathrm{pin}_{m,0}$-structure is $w_2 = 0$, whereas the corresponding condition for a $\mathrm{pin}_{0,m}$-structure is $w_2 + w_1^2 = 0$. The conditions $w_1^\pm = 0$ and $w_1 = 0$ are equivalent to the orientability of $T^\pm M$ and TM, respectively.

4.3. Remarks on the triviality of associated bundles.

Proposition 4. *The vector bundle $E \to M$, associated with the principal G-bundle $Q \to M$ by a representation γ of G in S, is trivial if, and only if, there exists a group G', a homomorphism $\iota : G \to G'$, and an extension $\gamma' : G' \to \mathrm{GL}(S)$ of γ, such that the associated principal G'-bundle $Q \times_\iota G' \to M$ is trivial.*

Proof. If E is trivial as a vector bundle, then there is a trivializing map $E \to M \times S$, $[(q,\varphi)] \mapsto (\pi(q), g(q)\varphi)$, such that $g : Q \to \mathrm{GL}(S)$ and $g(qa) = g(q) \circ \gamma(a)$ for every $q \in Q$, $\varphi \in S$ and $a \in G$. Taking $G' = \mathrm{GL}(S)$ and $\iota = \gamma$ one sees that $\gamma' = \mathrm{id}$ extends γ. The principal bundle $Q \times_\iota G' \to M$ is trivial because it has a global section corresponding to the equivariant map $e : Q \to G'$, where $e(q) = \iota(q)^{-1}$ for every $q \in Q$. Conversely, given an extension γ' of γ and a homomorphism $\iota : G \to G'$ such

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[3] Some authors say: a 'pinor' field.

that $Q \times_\iota G' \to M$ is trivial, there is a map $e : Q \to G'$ such that $e(qa) = \iota(a^{-1})e(q)$ for every $q \in Q$ and $a \in G$. If $g : Q \to \mathrm{GL}(S)$ is given by $g(q) = \gamma'(e(q)^{-1})$, then the map $[(q, \varphi)] \mapsto (\pi(q), g(q)\varphi)$, which is well-defined because of $\gamma' \circ \iota = \gamma$, trivializes the vector bundle $E \to M$. □

If G is a subgroup of G', then there is the principal G-bundle $\pi : G' \to G'/G$. The action of G on G' given by the left translations defines the associated principal G'-bundle $G' \times_\gamma G' \to G'/G$ that is trivial: a trivializing map is given by $[(a, b)] \mapsto (\pi(a), ab)$, where $a, b \in G'$.

Corollary 1. *If there is a representation γ' of G' in S extending the representation $\gamma : G \to \mathrm{GL}(S)$, then the bundle $G' \times_\gamma S \to G'/G$, associated with $\pi : G' \to G'/G$ by γ, is trivial.*

Proof. Indeed, a trivializing isomorphism is given by $[(a, \varphi)] \mapsto (\pi(a), \gamma'(a)\varphi)$, where $a \in G'$ and $\varphi \in S$. □

4.4. Examples.

4.4.1. *The spheres.* For every $m > 1$, the unit sphere $\mathrm{S}_m \subset \mathrm{R}^{m+1}$ has a unique spin structure described by

$$\mathrm{Spin}_m \to \mathrm{Spin}_{m+1} \to \mathrm{SO}_{m+1} \to \mathrm{S}_m.$$

Its bundle of Dirac (m even) or Pauli (m odd) spinors is trivial [Gu] by virtue of Prop. 3 and Corollary 1. The projection $\varpi : \mathrm{Spin}_{m+1} \to \mathrm{S}_m$ is given by $\varpi(a) = ae_{m+1}a^{-1}$, where $(e_1, \dots, e_{m+1})$ is the canonical frame in R^{m+1}. Let γ be a Dirac or Pauli representation of Spin_m in $GL(S)$ and let γ' be one of its extensions to Spin_{m+1}. For every $\Psi : \mathrm{S}_m \to S$ the map $\psi : \mathrm{Spin}_{m+1} \to S$ given by $\psi(a) = \gamma'(a^{-1})\Psi(\varpi(a))$ is a spinor field on the sphere; every such field can be so obtained. This observation is implicit in the work of Schrödinger [S1, S2] on the Dirac equation on low-dimensional spheres; see also [Pu2].

4.4.2. *Real projective spaces.* The description of pin and spin structures given in this paragraph is based on [DaT1]. The real projective space P_m is orientable if, and only if, m is odd. If k is a positive integer, then

$$\mathrm{vol}_{4k}^2 = 1, \quad \text{but} \quad \mathrm{vol}_{4k+2}^2 = -1,$$

and

$$\mathrm{vol}_{4k+1,0}^2 = \mathrm{vol}_{0,4k+3}^2 = 1, \quad \text{but} \quad \mathrm{vol}_{4k+3,0}^2 = \mathrm{vol}_{0,4k+1}^2 = -1.$$

By an argument similar to the one used in [CaGuT] to determine the spin structures on real projective quadrics it follows that:

(i.) The space P_{4k+1} has no spin structure.

(ii) The space P_{4k+2} has two inequivalent $\mathrm{pin}_{0,4k+2}$-structures,

$$\mathrm{Pin}_{0,4k+2} \xrightarrow{i_\pm} \mathrm{Spin}_{4k+3} \to \mathrm{SO}_{4k+3} \to \mathrm{P}_{4k+2},$$

where the two monomorphisms of groups i_+ and i_- are given by

$$i_\pm(a) = \begin{cases} a & \text{for } a \in \mathrm{Spin}_{4k+2}, \\ \pm a \mathrm{vol}_{0,4k+3} & \text{for } a = -\alpha(a) \in \mathrm{Pin}_{0,4k+2}. \end{cases}$$

The bundle of Dirac spinors associated with each of the pin structures on P_{4k+2} is trivial: this follows from Corollary 1 and the observation that the Dirac representation of $\mathrm{Pin}_{0,4k+2}$ extends to the Pauli representation of Spin_{4k+3}.

(iii) The space P_{4k-1} has two inequivalent spin structures,

$$\mathrm{Spin}_{4k-1} \to \mathrm{Spin}_{4k}/\mathrm{Z}_2^\pm \to \mathrm{SO}_{4k}/\mathrm{Z}_2 \to \mathrm{P}_{4k-1},$$

where $\mathrm{Z}_2^\pm = \{1, \pm\mathrm{vol}_{4k}\}$. The bundle of Pauli spinors associated with each of these structures is trivial: the Pauli representation of Spin_{4k-1} extends to a representation γ' of the group $\mathrm{Spin}_{4k}/\mathrm{Z}_2^+$, descending from the Weyl representation γ_+ of Spin_{4k} such that $\gamma_+(\mathrm{vol}_{4k}) = \mathrm{id}$, namely $\gamma'([a]_+) = \gamma_+(a)$, where $[a]_+ = \{a, a\mathrm{vol}_{4k}\}$ is the class of $a \in \mathrm{Spin}_{4k}$ in $\mathrm{Spin}_{4k}/\mathrm{Z}_2^+$; similarly for the other structure.

(iv) The space P_{4k} has two inequivalent $\mathrm{pin}_{4k,0}$-structures

$$\mathrm{Pin}_{4k,0} \xrightarrow{i_\pm} \mathrm{Spin}_{4k+1} \to \mathrm{SO}_{4k+1} \to \mathrm{P}_{4k},$$

where the two monomorphisms of groups i_+ and i_- are given by

$$i_\pm(a) = \begin{cases} a & \text{for } a \in \mathrm{Spin}_{4k}, \\ \pm a \mathrm{vol}_{4k+1,0} & \text{for } a = -\alpha(a) \in \mathrm{Pin}_{4k,0}. \end{cases}$$

The bundle of Dirac spinors associated with each of these structures is trivial; the argument is similar to the one in case (ii). The inequivalence of the two structures described in (ii)-(iv) is proved in [DaT1].

5. The Dirac operator

5.1. Covariant differentiation of spinor fields. Let again (14) be a pin structure on an m-dimensional Riemannian space M. The Levi-Civita connection form on P lifts to a $\mathrm{spin}(h)$-valued spin connection 1-form ω on Q. For every $q \in Q$, there is the orthonormal frame $\chi(q) = (\chi_\mu(q)) \in P$, where $\chi_\mu(q) \in T_{\varpi(q)}M$ for $\mu = 1, \dots, m$. The spin connection defines on Q the collection (∇_μ) of m *basic* horizontal vector fields such that, for $\mu = 1, \dots, m$ and every $q \in Q$,

$$\nabla_\mu \lrcorner \omega = 0 \quad \text{and} \quad T_q\varpi(\nabla_\mu(q)) = \chi_\mu(q). \tag{19}$$

For every $a \in \mathrm{Pin}(h)$ they transform according to

$$\nabla_\mu(qa) = T_q\delta(a)\nabla_\nu(q)\rho^\nu{}_\mu(a). \tag{20}$$

Let (e_μ) be a frame in V, orthonormal with respect to h and let γ be a spinor representation of $\mathrm{Pin}(h)$ in S. Defining $\gamma^\mu = h^{\mu\nu}\gamma_\nu$ and using (5) and (6) one obtains

$$\gamma \circ \alpha(a)\gamma^\mu = \rho^\mu{}_\nu(a^{-1})\gamma^\nu\gamma(a). \tag{21}$$

Let $\psi : Q \to S$ be a spinor field of type γ. Its *covariant derivative* is a map $\nabla\psi : Q \to \mathrm{Hom}(V, S)$ such that, for every $v = v^\mu e_\mu \in V$, one has $\langle v, \nabla\psi\rangle = v^\mu\nabla_\mu\psi$, where

$$\nabla_\mu\psi = \nabla_\mu \lrcorner\, \mathrm{d}\psi.$$

5.2. The classical and the modified Dirac operators. In the notation of the preceding paragraph, the *classical Dirac operator* $\mathcal{D}$ is given by

$$\mathcal{D}\psi = \gamma^\mu\nabla_\mu\psi.$$

According to (20) and (21), the classical Dirac operator maps a spinor field of type γ into a spinor field of type $\gamma \circ \alpha$.

Let γ_{m+1} be the isomorphism intertwining the representations γ and $\gamma \circ \alpha$, as described in Prop. 2. The *modified Dirac operator*,

$$\mathcal{D}' = \gamma_{m+1}\mathcal{D},$$

preserves the type of the spinor field and the corresponding eigenvalue equation

$$\mathcal{D}'\psi = \lambda\psi$$

is meaningful on non-orientable pin manifolds.

Remark 1. If the dimension m of M is *even*, then one can use the vector representation Ad in the definition of the pin structure on M. The classical Dirac operator preserves then the type of spinor fields and there is no need for its modification.

Remark 2. If M is a spin manifold and ψ is a spinor field, then $\gamma_{m+1}\psi$ is a spinor field of the same type. From (9) one obtains $(1 + \gamma_{m+1})^{-1} = \frac{1}{2}(1 - \gamma_{m+1})$ and

$$\mathcal{D}' = (1 + \gamma_{m+1})\mathcal{D}(1 + \gamma_{m+1})^{-1}$$

so that if $\mathcal{D}\psi = \lambda\psi$ then $\mathcal{D}'\psi' = \lambda\psi'$, where $\psi' = (1 + \gamma_{m+1})\psi$.

Remark 3. Since γ_{m+1} anticommutes with $\mathcal{D}$, on a spin manifold the spectra of $\mathcal{D}$ and $\mathcal{D}'$ are both *symmetric*: if λ is an eigenvalue, then so is $-\lambda$.

Remark 4. If M is an *odd*-dimensional *spin* manifold, then the interesting object is the *Pauli operator*

$$\mathcal{P} = \sigma^\mu\nabla_\mu.$$

Its spectrum need not be symmetric. If φ is an eigenfunction of $\mathcal{P}$, i.e. a Pauli spinor field satisfying $\mathcal{P}\varphi = \lambda\varphi$ for some $\lambda \in \mathrm{C}$, then the Cartan spinor fields $\psi_+ = (\varphi, 0)$ and $\psi_- = (0, \varphi)$ satisfy $\mathcal{D}\psi_\pm = \pm\lambda\psi_\pm$.

6. Pin structures on hypersurfaces

6.1. Existence. Let M be a hypersurface in a proper Riemannian spin $(m+1)$-manifold N, defined by an isometric immersion $f : M \to N$. The hypersurface need not be orientable. The normal bundle T^-M is a line bundle and the Whitney sum $TM \oplus T^-M$ is isomorphic to the pullback of TN to M by Tf. Since N is spin, its first and second Stiefel-Whitney classes vanish and the Whitney theorem [MS] gives

$$w_1(TM) + w_1(T^-M) = 0 \quad \text{and} \quad w_2(TM) + w_1(TM)w_1(T^-M) = 0.$$

Therefore, according to the Karoubi theorem, the hypersurface M has a $\mathrm{pin}_{0,m}$-structure. For example, since P_n is a spin manifold for $n \equiv 3 \mod 4$, and the real projective quadric $Q_{k,l} = (\mathrm{S}_k \times \mathrm{S}_l)/\mathrm{Z}_2$ is orientable for $k + l$ even, the natural immersion $Q_{k,l} \to \mathrm{P}_{k+l+1}$ gives, for $k + l \equiv 2 \mod 4$, a spin structure on the quadric with a proper Riemannian metric, cf. [CaGuT].

6.2. Construction. Let

$$\mathrm{Spin}_{m+1} \to Q' \xrightarrow{\chi'} P' \xrightarrow{\pi'} N \tag{22}$$

be a spin structure on N and let $\pi : P \to M$ be the O_m-bundle of all orthonormal frames on the hypersurface M immersed by f isometrically in N. Define the map $f' : P \to P'$ so that if $p = (p_\mu) \in P$ and $x = \pi(p)$, then the frame $f'(p) = (f'_i(p))$ at $f(x) \in N$ $(i = 1, \ldots, m+1)$ is given by

$$f'_\mu(p) = T_x f(p_\mu) \quad \text{for} \quad \mu = 1, \ldots, m$$

and $f'_{m+1}(p)$ is a unit vector at $f(x)$, orthogonal to $T_x f(T_x M)$ and oriented in such way that $f'(p) \in P'$. It is clear that f' is an injection. Let $\iota : \mathrm{Pin}_{0,m} \to \mathrm{Spin}_{m+1}$ be as in (2) for $k = 0$ and $l = m$ and let $\kappa : \mathrm{O}_m \to \mathrm{SO}_{m+1}$ be the corresponding monomorphism of the orthogonal groups, $\kappa \circ \rho = \rho \circ \iota$. For every $A \in \mathrm{O}_m$ and $p \in P$, one has $f'(pA) = f'(p)\kappa(A)$. Therefore, the pair (f, f') is a morphism of principal bundles. The $\mathrm{pin}_{0,m}$-structure on M is now given as a bundle $\chi : Q \to P$ induced from the bundle $\chi' : Q' \to P'$ by the map f'. Explicitly,

$$Q = \{(p, q') \in P \times Q' : f'(p) = \chi'(q')\}, \quad \chi(p, q') = p.$$

The action of $\mathrm{Pin}_{0,m}$ on Q is given by

$$(p, q')a = (p\rho(a), q'\iota(a)) \tag{23}$$

so that $\chi((p, q')a) = \chi(p, q')\rho(a)$ for every $a \in \mathrm{Pin}_{0,m}$.

The pin-structure on a hypersurface M, constructed in this manner, is said to be *induced* by the immersion $f : M \to N$.

Let $\gamma' : \mathrm{Spin}_{m+1} \to \mathrm{GL}(S)$ be a spinor representation and let $\psi' : Q' \to S$ be a spinor field on N of type γ'. It follows from (23) that its restriction ψ to Q, $\psi(p, q') = \psi'(q')$, is a spinor field on M of type $\gamma = \gamma' \circ \iota$.

6.3. Spinors on hypersurfaces in Euclidean spaces. As an important special case, consider a hypersurface M immersed in R^{m+1}.

Proposition 5. *Let f be an isometric immersion of a hypersurface M in the Euclidean space R^{m+1}. The pin-structure on M, induced by the immersion,*

$$\mathrm{Pin}_{0,m} \to Q \xrightarrow{\chi} P \xrightarrow{\pi} M,$$

is such that the bundle of Dirac (m even) or Pauli (m odd) spinors on M is trivial.

Proof. Since the spin structure (22) reads now

$$\mathrm{Spin}_{m+1} \to \mathrm{R}^{m+1} \times \mathrm{Spin}_{m+1} \to \mathrm{R}^{m+1} \times \mathrm{SO}_{m+1} \to \mathrm{R}^{m+1},$$

the map $f' : P \to \mathrm{R}^{m+1} \times \mathrm{SO}_{m+1}$, defined in the preceding paragraph, can be written as $f' = (f \circ \pi, F)$, where

$$F : P \to \mathrm{SO}_{m+1} \tag{24}$$

satisfies $F(pA) = F(p)\kappa(A)$ for every $p \in P$ and $A \in \mathrm{O}_m$. Let $k : P \to M \times \mathrm{SO}_{m+1}$ be the map $k = (\pi, F)$. The definition of Q can be simplified to read

$$Q = \{(x, a) \in M \times \mathrm{Spin}_{m+1} : (x, \rho(a)) \in k(P)\}.$$

Since k is injective, the projection $\chi : Q \to P$ is well-defined. If $p = \chi(x, a)$, then the definition of Q implies

$$F(p) = \rho(a). \tag{25}$$

The action of $\mathrm{Pin}_{0,m}$ on Q is now given by $(x, a)b = (x, a\iota(b))$, where $(x, a) \in Q$ and $b \in \mathrm{Pin}_{0,m}$. The bundle $Q \times_\iota \mathrm{Spin}_{m+1} \to M$, obtained by extending the structure group $\mathrm{Pin}_{0,m}$ of Q to Spin_{m+1}, is isomorphic with the trivial bundle $M \times \mathrm{Spin}_{m+1} \to M$: an isomorphism is given by $[((x, a), a')] \mapsto (x, aa')$, where $(x, a) \in Q$ and $a' \in \mathrm{Spin}_{m+1}$. The spinor representation $\gamma' : \mathrm{Spin}_{m+1} \to \mathrm{GL}(S)$ extends $\gamma = \gamma' \circ \iota$. Applying Prop.4 to the present case, with $G = \mathrm{Pin}_{0,m}$ and $G' = \mathrm{Spin}_{m+1}$, one obtains that the bundle of spinors $Q \times_\gamma S \to M$ is trivial. □

Let $\psi : Q \to S$ be a spinor field of type γ on M, i.e. $\psi(x, a\iota(b)) = \gamma(b^{-1})\psi(x, a)$ for every $(x, a) \in Q$ and $b \in \mathrm{Pin}_{0,m}$. The map $Q \to S$, given by $(x, a) \mapsto \gamma'(a)\psi(x, a)$, is constant on the fibers of $Q \to M$. There thus exists a map

$$\varPsi : M \to S \tag{26}$$

such that

$$\varPsi(x) = \gamma'(a)\psi(x, a) \tag{27}$$

for every $(x,a) \in Q$. Conversely, for every map (26), the *Schrödinger transformation* (27) defines a spinor field ψ of type $\gamma = \gamma' \circ \iota$ on M. An equivalent way of defining the map Ψ associated with the spinor field ψ of type γ is to consider the latter's extension ψ' to the trivial bundle $M \times \mathrm{Spin}_{m+1} \to M$ such that $\psi'(x,a) = \psi(x,a)$ for every $(x,a) \in Q$ and $\psi'(x,ab) = \gamma'(b^{-1})\psi'(x,a)$ for every $x \in M$ and $a, b \in \mathrm{Spin}_{m+1}$. If $s : M \to M \times \mathrm{Spin}_{m+1}$ is the standard section $s(x) = (x,1)$, then

$$\Psi = \psi' \circ s. \tag{28}$$

7. A formula for the Dirac operator on orientable hypersurfaces

7.1. The general case. Assume, for simplicity, that the hypersurface M, immersed isometrically in R^{m+1}, is connected, orientable and has been oriented by distinguishing a connected component SP of its bundle P of all orthonormal frames. The pin structure on M, induced by the immersion f, can be now restricted to the group Spin_m by taking $SQ = \chi^{-1}(SP)$, cf. (14) and (15). The map (24) restricted to SP is the *Gauss map*: it defines a vector field $n : M \to \mathrm{S}_m \subset \mathrm{R}^{m+1}$ of unit normals to M. It is given by

$$n(\pi(p)) = F(p)e_{m+1}$$

and, by virtue of (25), for every $(x,a) \in Q$, one has

$$n(x) = ae_{m+1}a^{-1}, \tag{29}$$

where the product on the right is given by multiplication in $\mathrm{Pin}_{0,m+1}$. Let γ be a spinor representation of $\mathrm{Pin}_{0,m+1}$ in S; by restriction, it gives rise to representations of its subgroups,

$$\mathrm{Spin}_m \to \mathrm{Spin}_{m+1} \to \mathrm{Pin}_{0,m+1} \xrightarrow{\gamma} \mathrm{GL}(S).$$

Since the first two arrows are standard injections, there is now no need to introduce a separate notation for the restrictions and to distinguish γ and γ' as in Prop. 3. In particular, for every $i = 1, \dots, m+1$, one has the Dirac matrix $\gamma_i = \gamma(e_i)$ and

$$\text{if} \quad \sigma_{ij} = \gamma_i\gamma_j + \delta_{ij}, \quad \text{then} \quad \sigma_{ij} + \sigma_{ji} = 0 \tag{30}$$

for $i, j = 1, \dots m+1$. By applying γ to both sides of (29), one obtains, for every $(x,a) \in Q$,

$$\boldsymbol{n}(x) = \gamma(a)\gamma_{m+1}\gamma(a^{-1}), \quad \text{where} \quad \boldsymbol{n} = \gamma \circ n. \tag{31}$$

The modified Dirac operator $\mathcal{D}'$ maps a spinor field ψ of type γ into a spinor field of the same type. There thus exists a linear differential operator D, acting on maps from M to S, such that

$$\gamma(a)(\mathcal{D}'\psi)(x,a) = (D\Psi)(x), \tag{32}$$

where $(x,a) \in Q$ and Ψ is given by (27). Symbolically,

$$D = \gamma(a)\mathcal{D}'\gamma(a^{-1}).$$

Since $\mathcal{D}'$ anticommutes with γ_{m+1}, one obtains from (31)

$$D\boldsymbol{n} + \boldsymbol{n}D = 0. \tag{33}$$

Let ∂_i denote the (constant) vector field on R^{m+1} given by differentiation with respect to the ith Cartesian coordinate, i.e. $\partial_i\varphi = e_i \lrcorner \mathrm{d}\varphi$ for every function φ on R^{m+1}. The field of unit normals, $n(x) = \sum_i n_i(x)\partial_i$, defines a collection of $\frac{1}{2}m(m+1)$ vector fields $n_i\partial_j - n_j\partial_i$ $(1 \le i < j \le m+1)$ tangent to the hypersurface.

The 'Schrödinger transform' D of the modified Dirac operator can be computed to be [T1]

$$D = \sum_{i,j} \sigma_{ij} n_i \partial_j + \tfrac{1}{2}\mathrm{div}n, \tag{34}$$

where the 'intrinsic divergence' div is given by

$$\mathrm{div}n = \sum_{i,j} (\delta_{ij} - n_i n_j)\partial_i n_j.$$

The differential operator $\sum_{i,j} \sigma_{ij} n_i \partial_j$ has been studied by Delanghe and Sommen [DS] who give, in connection with it, a reference to an unpublished thesis by Lounesto; see also [DSSk].

7.2. The case of a foliation. Consider an open subset U of R^{m+1} foliated by a family of hypersurfaces. The Gauss map defines now a field n on U. Let $\partial/\partial r = \sum_i n_i \partial_i$ be the derivative along n. Define the classical Dirac operator in R^{m+1} as

$$\boldsymbol{\partial} = \sum_i \gamma_i \partial_i.$$

A simple computation, based on (30) gives

$$\boldsymbol{n}\boldsymbol{\partial} = D - (\tfrac{\partial}{\partial r} + \tfrac{1}{2}\mathrm{div}n). \tag{35}$$

7.3. The example of spheres. The set $U = \{x \in \mathrm{R}^{m+1} : x \neq 0\}$ is foliated by the spheres $r = \text{const.} > 0$, where $r = (x_1^2 + \cdots + x_{m+1}^2)^{1/2}$. Since now $n_i = x_i/r$, one has $\mathrm{div}n = m/r$.

Let $\varphi : \mathrm{R}^{m+1} \to S$ be a harmonic polynomial, homogeneous of degree $l+1$ $(l = 0, 1, \ldots)$. Since $\boldsymbol{\partial}^2 = -\sum_i \partial_i^2$, the polynomial $\boldsymbol{\partial}\varphi$ is annihilated by $\boldsymbol{\partial}$. Therefore, by virtue of (35), its restriction ψ^+ to the unit sphere $r = 1$, is an eigenfunction of D,

$$D\psi^+ = (l + \tfrac{1}{2}m)\psi^+ \quad \text{and} \quad \psi^+(-x) = (-1)^l \psi^+(x).$$

for $l = 0, 1, \ldots$.

Let φ' be an S-valued, harmonic homogeneous polynomial of degree l. The function $r^{-2l-m+1}\varphi'$ is also harmonic on U and the restriction ψ^- of

$$\boldsymbol{\partial} r^{-2l-m+1}\varphi'$$

to the unit sphere satisfies

$$D\psi^- = -(l + \tfrac{1}{2}m)\psi^- \quad \text{and} \quad \psi^-(-x) = (-1)^{l+1}\psi^-(x)$$

for $l = 0, 1, \ldots$.

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