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QUATERNIONIC TRANSFORMATIONS AND THE FIRST EIGENVALUES OF LAPLACIAN ON A QUATERNIONIC KÄHLER MANIFOLD

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Abstract. We consider quaternionic transformations of a quaternionic Kähler manifold (M, g, Q). General conditions for unicity of the quaternionic Kähler structure (g, Q) for given Q are applied to the case of two quaternionic Kähler metrics which are in correspondence through a quaternionic transformation. Characterization of compact quaternionic Kähler manifolds which admit a quaternionic transformation φ different from an isometry is given. A sharp estimate of first non zero eigenvalue for Laplacian on a compact quaternionic Kähler manifold with positive scalar curvature is also obtained, improving a previous result of C.M. Margerin. An analogous sharp estimate for Dirac operator was established by O. Hijazi and J.-L. Milhorat, [9].

Transformation quaternioniennes et prèmière valeur propre du Laplacien sur une variété Kähler-quaternionienne

Resumé. On considére les transormations quaternioniennes d'une variété Kähler-quaternionienne (M, g, Q). Des conditions génerales d'unicité de la structure Kähler-quaternionienne g, Q pour la structure quaternionienne Q donnée sont appliquées au cas de deux métriques Kähler-quaternioniennes qui se correspondent par une transormation quaternionienne. On donne une caractérisation des variétés Kähler-quaternioniennes compactes qui admettent une transormation quaternionienne φ differente d'une isometrie. On abtient aussi une estimation optimale de la première valeur propre positive du Laplacien sur une variété Kähler-quaternionienne compacte à courbure scalaire positive, en améliorant un résultat précedent de C.M. Margerin. Une estimation optimale analogue pour l'operateur de Dirac avait été établit par O. Hijazi et J.-L. Milhorat, [9].

Version française abrégée. Dans cette Note on donne une répose affirmative à une conjecture enoncée en [4], concernant la caractérisation des variétés Kähler-quaternioniennes admettant des transormations infinitésimales quaternioniennes qui ne sont pas des champs de Killing. On donne même des résultats plus complets

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soit en considérant le groupe des transformations quaternioniennes tout entier, et pas seulement sa composante connexe de l'identité, soit en établissant des liens précis entre des telles transformations et les fonctions propres du Laplacien sur la variété.

Soit g une métrique Kähler-quternionienne sur une varété M, c'est à dire une métrique Riemannienne avec groupe d'holonomie Hol $\subset Sp_1 \cdot Sp_n$ [5,17].

Soit Q la structure quaternionienne sur M associée a g: il s'agit d'un sous fibré 3-dimensionnel du fibré des endomorphismes, invariant par transport parallèle et engendré localement par une triple $H = (J_1, J_2, J_3)$ de structures presque-complexes qui anticommutent et telles que $J_3 = J_1 J_2$. On dit que le triple (M, g, Q) est une variété Kähler-quaternionienne. On dira aussi que la metrique g est subordonnée à la structure quaternionienne Q. On conviendra que $\dim M = 4n > 4$. Dans ce cas la métrique g est d'Einstein, c'est á dire $ric = (K/4n)g$, où ric et $K = const$ sont le tenseur de Ricci et la courbure scalaire. On définit la courbure scalaire réduite par $\nu = K/4n(n+2)$.

Definitions.

1) Une transformation φ de (M, g, Q) est dite quaternionienne si elle preserve la structure quaternionienne $Q : \varphi^* Q = Q$.

2) Un champ de vecteur $Z \in \chi(M)$ est dit quaternionien s'il engendre un group (local) á 1 paramètre de transformations quaternioniennes.

Nous prouvons les résultats suivant.

Théorème 1. *Soit (M, g, Q) une variété Kähler-quaternionienne telle que*

(1) la décomposition de de Rham de (M, g) n'as pas de facteur Euclidien.

Supposons qu'il existe une métrique Kähler-quaternionienne g' subordonnée à Q et qui n'est pas homothétique à la métrique g. Alors

1) la variété admet un champ de vecteur gradient quaternionien Z qui n'est pas un champ de Killing.

2) Si de plus la variété M est compacte, alors (M, g) est isométrique à l'espace projectif quaternionien $\mathbb{H}P^n$ doué de la métrique standard á courbure scalaire reduite $\nu, g_{can}(\nu)$.

Théorème 2. *Soit (M, g, Q) une variété Kähler-quaternionienne compacte et qui satisfait à (1). Si elle admet une transormation quaternionienne différente d'une isométrie alors (M, g, Q) est isometrique à $\mathbb{H}P^n$ doué de la métrique standard $g_{can}(\nu)$.*

Théorème 3. *Soit (M, g, Q) une variété Kähler-quaternionienne compacte, la courbure scalaire réduite est positive, égale à ν. Alors la première valeur propre positive λ_1 du Laplacien n'est past inférieure à $\nu_1 = 2(n+1)v$. De plus, si $\lambda_1 = \nu_1$, la variété est isométrique à $\mathbb{H}P^n$ doué de la métrique standard $g_{can}(\nu)$.*

Il faut remarquer que la minoration $\nu_1 \leq \lambda_1$ avait été déjà obtenue par C.M. Margerin et, aussi, que l'analogue du Théorème 3 pour l'operateur de Dirac a été établi par O. Hijazi et J.-L. Milhorat, [9].

1. Statement of the main results.

Let g be a quaternionic Kähler metric on a manifold M, that is a Riemannian metric with holonomy group Hol $\subset Sp_1 \cdot Sp_n$. We denote by Q the associated quaternionic structure on M that is a 3-dimensional subbundle of the bundle of endomorphisms, invariant under parallel transport, and locally generated by 3 anticommuting skew symmetric almost complex structures $J_\alpha, \alpha = 1, 2, 3$ with $J_3 = J_1 J_2$. The triple (M, g, Q) is called a quaternionic Kähler manifold. We will say also that the metric g is subordinated to the quaternionic structure Q. We shall assume that $\dim M = 4n > 4$. Then the metric g is Einstein, i.e. $ric = (K/4n)g$, where ric and $K = const$ are Ricci tensor and scalar curvature. We define the reduced scalar curvature as $\nu = K/4n(n+2)$.

Definitions.

1) A transformation φ of (M, g, Q) is called to be quaternionic if it preserves the quaternionic structure $Q : \varphi^* Q = Q$.

2) A vector field $Z \in \chi(M)$ is called to be quaternionic if it generates a (local) one-parameter group of quaternionic transformations.

We prove the following results.

Theorem 1. *Let (M, g, Q) be a quaternionic Kähler manifold such that*

(1) *the de Rham decomposition of (M, g) has no Euclidean factor.*

Assume that there exists a quaternionic Kähler metric g' subordinated to Q and is not homothetic to the metric g. Then

1) the manifold admits a gradient quaternionic vector field Z which is not a Killing field.

2) if, moreover, the manifold M is compact, then (M, g) is isometric to the quaternionic projective space $\mathbb{H}P^n$ with the standard metric $g_{can}(\nu)$ of reduced scalar curvature ν.

Theorem 2. *Let (M, g, Q) be a compact quaternionic Kähler manifold which satisfies* (1). *If it admits a quaternionic transformation different from isometry then (M, g, Q) is isometric to $\mathbb{H}P^n$ with the standard metric $g_{can}(\nu)$.*

Theorem 3. *Let (M, g, Q) be a compact quaternionic Kähler manifold with positive reduced scalar curvature ν. Then the first positive eigenvalue λ_1 for Laplacian is not less than $\nu_1 \equiv 2(n+1)\nu$. Moreover, if $\lambda_1 = \nu_1$, the manifold (M, g, Q) is isometric to $\mathbb{H}P^n$ with the standard metric $g_{can}(\nu)$.*

We must remark that the bound $\nu_1 \leq \lambda_1$ was previously obtained by C.M. Margerin and that the analogue of Theorem 3 for Dirac operator was established by O. Hijazi and J.-L. Milhorat, [9].

2. Properties of quaternionic vector fields.

Let (M, g, Q) be a quaternionic Kähler manifold with the reduced scalar curvature ν. For any vector field Z on M we denote by L_Z the field of endomorphisms $X \mapsto \nabla_X Z, X \in TM$, where ∇ is the Levi-Civita connection.

Lemma 1.

1) A vector field Z is quaternionic iff

$$[L_Z, Q] \subset Q \tag{2}$$

2) A gradient vector field $Z = grad\ f = g^{-1} \circ df$, $f \in C^\infty(M)$, is quaternionic iff

$$[L_Z, Q] = 0 \tag{3}$$

or in other terms, iff the form $\ell = g \circ L_Z = \nabla^2 f$ *is* Q*-Hermitian, i.e.*

$$\ell(JX, Y) + \ell(X, JY) = 0 \qquad \forall J \in Q, \quad X, Y \in TM.$$

Proof. 1) follows from the identity

$$Z \cdot J = \nabla_Z J + [J, L_Z],$$

where dot stands for the Lie derivative and J is a local section of Q. For gradient field Z the operator L_Z is symmetric and then 1) $\Rightarrow$ 2).

For any 1-form $\xi \in \Lambda^1 M$ we define a (1,2) tensor S^ξ on M by

$$S^\xi_X = \xi(X) Id + X \otimes \xi - \sum_\alpha \xi(J_\alpha X) J_\alpha - \sum_\alpha J_\alpha X \otimes (\xi \circ J_\alpha) \qquad X \in_\xi (M)$$

where $(J_1, J_2, J_3 = J_1 J_2)$ is a local base of Q.

Proposition 1. *Let* Z *be a quaternionic vector field. Then there is an associated 1-form* $\xi = \xi_Z$ *such that*

$$Z \cdot \nabla = S^\xi \tag{4}$$

Proof. Let $\{\varphi_t = \exp tZ\}$ be the (local) 1-parameter group of quaternionic transformations of M generated by Z. From [1, Note 11] (see Proposition and Corollary page 56), for any φ_t there exists a 1-form ξ_t such that

$$(\varphi_t^{-1})^* \nabla = \nabla + S^{\xi_t}$$

Then formula (4), with $\xi = (d/dt)\xi_t | t = 0$, follows immediately from definition of Lie derivative of ∇.

Now we define a parallel (1,3) tensor field P on M by

$$\begin{aligned} P(X,Y)Z = {} & 2g(X,Z)Y + g(Z,Y)X + g(X,Y)Z \\ & - \sum_\alpha g(Z, J_\alpha Y) J_\alpha X - \sum_\alpha g(X, J_\alpha Y) J_\alpha Z \end{aligned}$$

where $X, Y, Z \in TM$, $(J_1, J_2, J_3 = J_1 J_2)$ is a local base of Q, and denote by $\mathcal{P}$ the space of all gradient quaternionic vector fields.

Proposition 2. *Any vector field* $Z \in \mathcal{P}$ *satisfies the equation*

$$(\nabla^2 Z)_{X,Y} = -(\nu/4) P(X,Y) Z \qquad \forall X, Y \in TM \tag{5}$$

Corollary. *Assume that (M, g, Q) is locally irreducible manifold with zero reduced scalar curvature ν. Then for any $Z \in \mathcal{P}$ the operator $L_Z = fid$ for some $f \in C^\infty(M)$. In particular Z is a conformal field.*

$$[\ \textit{if}\ \nu = 0 : (4) \Rightarrow \nabla L_Z = 0 \Rightarrow [L_Z, Hol] = [L_Z, Sp_n] = 0 \Rightarrow L_Z = fid].$$

Proof of Proposition 2. Let Z be a quaternionic vector field. Then (4) holds for associated 1-form ξ and, by well known formula for Lie derivative $Z \cdot \nabla$ [10],

$$(\nabla^2 Z)_{X,Y} + R(Z, X)Y = S^\xi_X Y \qquad \forall X, Y \in \chi(M)$$

if, moreover, Z is a gradient field, then L_Z is symmetric with respect to g as well as its covariant derivative and it follows

$$2g(R(Z, X)Y, T) = g(S^\xi_X Y, T) - g(S^\xi_X T, Y) \qquad \forall X, Y, T \in_\xi (M)$$

By taking traces of both members with respect to g one finds $2ric \circ Z = (K/2n)g \circ Z = -4(n+2)\xi$, that is

$$\xi = -(\nu/2)g \circ Z$$

and hence, by a direct computation

$$R(Z, X)Y = (\nu/4)[S^{g \circ X}_Z Y - S^{g \circ Z}_X Y]$$

Then (5) follows by substitution in first identity.

Denote by $\mathcal{F}(\nu_1) \subset C^\infty(M)$ the space of eigenfunctions of the Laplacian with eigenvalue $\nu_1 = 2(n+1)\nu$. Now we characterize potentials for gradient quaternionic vector fields.

Proposition 3.

1) The divergence $divZ \equiv Tr\nabla Z$ of any field $Z \in \mathcal{P}$ belongs to $\mathcal{F}(\nu_1)$.

2) If (1) holds then the map

$$div : \mathcal{P} \to \mathcal{F}(\nu_1) \tag{6}$$

is injective.

3) If M is compact and $\nu \neq 0$, then (6) is an isomorphism and the inverse mapping is given by

$$\mathcal{F}(\nu_1) \ni f \mapsto (1/\nu_1)grad \quad f \tag{7}$$

To prove Proposition we define the projection Π of the space of bilinear forms onto the space of Q-Hermitian forms by

$$\Pi : \omega \to \Pi\omega = (1/4)[\omega + \sum_\alpha \omega(J_\alpha \cdot, J_\alpha \cdot)]$$

and for any 1-form ξ we put

$$\nabla^+ \xi = \Pi \cdot \nabla\xi, \qquad \nabla^- \xi = (1 - \Pi) \cdot \nabla\xi$$

such that $\nabla\xi = \nabla^+\xi + \nabla^-\xi$.

We denote by δ the codifferential (see [5]) and by $\Delta \equiv \delta d + d\delta$ the Laplacian. We remark that the restriction of δ to the space of Q-Hermitian (resp. skew-Q-Hermitian) bilinear forms is the (formal) adjoint of ∇^+ (resp. ∇^-).

Lemma 2. *(see [5], page 75). For any 1-form* ξ

$$\delta[\nabla^+\xi - (1/3)\nabla^-\xi] = \nu n\xi, \qquad \Delta\xi = 4\delta(\nabla^+\xi) - 2\nu(n-1)\xi$$

Corollary. *For any 1-form* ξ

1) $\Delta\xi = \nu_1\xi \Leftrightarrow \delta(\nabla^+\xi) = \nu n\xi \Leftrightarrow \delta(\nabla^-\xi) = 0$

2) If M *is compact, then* $\delta(\nabla^-\xi) = 0 \Leftrightarrow \nabla^-\xi = 0$.

Proof of Proposition 3.

1) For $Z \in \mathcal{P}$ the 1-form $\xi = g \circ Z$ is exact and $\nabla^-\xi = 0$. By Corollary this implies $\Delta\xi = \nu_1\xi$. Applying δ we obtain $\Delta f = \nu_1 f$ for $f = div\xi$ and $div\mathcal{P} \subset \mathcal{F}(v_1)$.

2) Assume that $divZ = 0$ for $Z \in \mathcal{P}$. Then Z preserves together with Q also the volume form vol. Hence Z preserves the canonical connection $\nabla^{Q,vol}$ associated with (Q, vol), see [1, 2]. This connection coincides with the Levi-Civita connection ∇. Hence, the field Z is affine and then condition (1) implies that $Z = 0$.

3) Let $f \in \mathcal{F}(\nu_1)$ and $Z = grad \quad f$. Then $divZ = \Delta f = \nu_1 f$ and hence, $\Delta\xi = \nu_1\xi$ for $\xi = df = g \circ Z$. The Corollary implies $\nabla^-\xi = 0$ that is $Z = g^{-1} \circ \xi \in \mathcal{P}$.

Now we assume that $\nu \neq 0$ and denote by $aut(Q)$, resp., $aut(g)$ the Lie algebra of quaternionic, resp., Killing vector fields on M.

Proposition 4. *There exists a reductive decomposition*

$$aut(Q) = aut(g) + \mathcal{P}, \qquad [aut(g), \mathcal{P}] \subset P.$$

The projection of $aut(Q)$ *on* $\mathcal{P}$ *is given by*

$$Z \mapsto Z' = Z + (2/\nu)g^{-1} \circ d(divZ)$$

3. Proof of the theorems.

Proof of Theorem 1.

1) Denote by ∇, ∇' (resp. ν, ν') the Levi-Civita connections (resp. reduced scalar curvatures) of the metrics g, g'. Then [2,7,15]

$$\nabla' = \nabla + S^\xi \tag{8}$$

$$(\nu'/4)g' = (\nu/4)g + [\xi \otimes \xi - \sum_\alpha (\xi \circ J_\alpha) \otimes (\xi \circ J_\alpha)] - \nabla\xi \tag{9}$$

where $\xi \in \Lambda^1 M$ is an 1-form. Equation (9) shows that the 1-form ξ is closed and the form $\nabla\xi - 2\xi \otimes \xi$ is symmetric and Q-Hermitian. Since M is simply connected, we can write $\xi = dh$ for some function h. We put $\eta := e^{-2h}\xi$. Then $\nabla_\eta = e^{2h}[\nabla\xi - 2\xi \otimes \xi]$ is symmetric Q-Hermitian form and $\eta = df$ for some function f. By Lemma 1, $Z = g^{-1} \circ \eta = gradf \in \mathcal{P}$. If $Z = 0$, then $\nabla' = \nabla$ and the metrics g, g' are homothetic [11]. This proves statement 1.

In the case $\nu > 0$, statement 2 follows from Proposition 2 and the following result (see [4]).

Theorem. *Let (M, g, Q) be a compact quaternionic Kähler manifold with positive reduced scalar curvature ν. If there exists a vector field $Z \neq 0$ which satisfies (5), then (M, g, Q) is isometric to the quaternionic projective space $\mathbb{H}P^n$.*

In the case $\nu \leq 0$, taking the trace of (5) and using the Weitzenböck formula we derive the identity

$$\Delta(||Z||^2) = 2(\nu n||Z||^2 - ||\nabla Z||^2).$$

Integrating it over M, we obtain $Z = 0$. This proves Theorem 1. Theorem 2 follows directly from Theorem 1. The second part of Theorem 3 follows from Theorem 1 and Proposition 2 (if $\nu > 0$). Now we prove the first statement of Theorem 3. Let f be an eigenfunction for the first positive eigenvalue λ_1 of the Laplacian, $\Delta f = \lambda_1 f$. Then $\xi = df$ is an eigenform for Laplacian $\Delta \equiv \Delta^1$ with eigenvalue λ_1, that is $\Delta\xi = \lambda_1\xi$. By taking the global scalar product $(\cdot,\cdot)_M$ on M by ξ of both members of two identities of Lemma 2, we get respectively $||\nabla^+\xi||^2_M = (1/3)||\nabla^-\xi||^2_M + \nu n||\xi||^2_M$, $(1/4)\lambda_1||\xi||^2_M = ||\nabla^+\xi||^2_M - (1/2)\nu(n-1)||\xi||^2_M$ Hence

$$(1/4)\lambda_1||\xi||^2_M \geq \nu n||\xi||^2_M - (1/2)\nu(n-1)||\xi||^2_M = [v(n+1)/2]||\xi||^2_M$$

that is

$$\lambda_1||\xi||^2_M \geq \nu_1||\xi||^2_M.$$

and statement follows immediately from the positivity of ν and $||\xi||^2_M$.

REFERENCES

1. D.V. Alekseevsky, S. Marchiafava, *Quaternionic-like structures on a manifold:Note 1. 1-integrability and integrability conditions, Note 2. Automorphism groups and their interrelations*, Rend.Mat.Acc.Lincei **s9,v4** (1993), 43-52,53-61.
2. D.V. Alekseevsky, S. Marchiafava, *Quaternionic structures on a manifold and subordinated structures, Dip.Mat.'G. Castelnuovo', Univ.Roma 'La Sapienza'*, Reprint 94/14 (giugno 1994).
3. D.V. Alekseevsky, S. Marchiafava, *Transformation groups of a quaternionic manifold*, in preparation.
4. D.V. Alekseevsky, S. Marchiafava, *Gradient quaternionic vector fields and a characterization of the quaternionic projective space*, Preprint ESI (1994).
5. A. Besse, *Einstein manifolds*, Springer Verlag, Berlin and New York, 1987.
6. E. Bonan, *Sur les G-structures de type quaternionien*, Cahiers de topologie et géomètrie différentielle **9** (1967), 389-461.
7. S. Fujimura, *Q-connections and their changes on almost quaternion manifolds*, Hokkaido Math. Jour. **5** (1976), 239-248.
8. J.F. Glazebrook, D. Sundararaman, *Deformations of foliated quaternionic structures and the twistor correspondence*, Preprint.
9. O. Hijazi, J.L. Milhorat, *Proprietés spectrales de l'operateur de Dirac sur les variétés Kähler-quaternioniennes. Rapport de recherche 92/06-3, Université de Nantes* (1993).
10. S. Kobayashi, *Transformation groups in differential geometry*, Springer Verlag, Berlin-Heidelberg-New York, 1972.
11. S. Kobayashi, K. Nomizu, *Foundations of differential geometry, vol.1 and vol.2*, Intersciences Publishers, New York-London, 1963.
12. A. Lichnerowicz, *Geometry of groups of transformations*, Noordhoff Int. Publishing Leyden, The Netherlands, 1977.
13. S. Marchiafava, *Some remarks on the geometry of quaternionic manifolds*, Proceedings of the '22nd Conference on Differential Geometry and Topology, Applications in Physics and Technics', Bucarest, September 9-13 (1991).

14. E. Musso, *On the transformation group of a quaternion manifold*, Bollettino U.M.I **(7) 6-B** (1992), 67-78.
15. V. Oproiu, *Integrability of almost quaternal structures*, An.st. Univ. 'Al.I. Cuza' Iazi **30** (1984), 75-84.
16. P. Piccinni, *On the infinitesimal automorphisms of quaternionic structures*, J. de Math. Pures et Appl. **72** (1993), 593-605.
17. S. Salamon, *Riemannian geometry and holonomy groups*, Ed. Longman Scientific & Technical, UK, 1989.
18. A. Swann, *Hyper Kähler and Quaternionic Kähler Geometry*, Ph.D. Thesis, Oxford (1990).

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