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# The Inverse Problem for the Hill Operator , the Direct Approach

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**Abstract.** Let  $G_n = (A_n^-, A_n^+)$ ,  $n \geq 1$ , denote the set of gaps of the Hill operator  $T = -d^2/dx^2 + V(x)$  in  $L^2(\mathbf{R})$  where  $V$  is an even 1-periodic real potential from  $L^2(0, 1)$  and  $h_n$  be heights of the corresponding slits on the quasimomentum domain,  $M_n^\pm$  be effective masses associated with the edges of the gap  $G_n$ . Let  $g_n$ ,  $n \geq 1$ , denote the gaps of the operator  $T_0 = \sqrt{T - N_0} \geq 0$  where  $N_0$  is the beginning of the spectrum of  $T$ , and  $\mu_n^\pm$  be the reduced masses (analog of the effective masses) connected with the gap  $g_n$ . We study the inverse problem for the mappings  $V \rightarrow \{|g_n|\}$ ,  $V \rightarrow \{h_n\}$ ,  $V \rightarrow \{\mu_n^\pm\}$  and  $V \rightarrow \{M_n^\pm\}$  by a direct approach.

## 1 . Introduction

Consider the Hill operator  $T = -d^2/dx^2 + V(x)$  in  $L^2(\mathbf{R})$  where  $V$  is an even 1-periodic real potential from  $L^2(0, 1)$  and  $\int_0^1 V(x)dx = 0$ . It is well known that the spectrum of  $T$  is absolutely continuous and consists of intervals  $S_1, S_2, \dots$ , and where  $S_n = [A_{n-1}^+, A_n^-], \dots, A_n^- \leq A_n^+ < A_{n+1}^-$ ,  $n \geq 1$ . These intervals are separated by the gaps  $G_1, G_2, \dots$ , where  $G_n = (A_n^-, A_n^+)$ . If a gap degenerates i.e.  $G_n = \emptyset$  then the corresponding segments  $S_n, S_{n+1}$  merge. The spectrum of the Hill operator consists of closed non overlapping intervals which are called spectral bands. Let  $\varphi(x, E, V), \vartheta(x, E, V)$  be the solutions of the equation

$$-f'' + Vf = Ef, \quad E \in \mathbf{C}, \quad (1.1)$$

satisfying  $\varphi'(0, E, V) = \vartheta(0, E, V) = 1$ ,  $\varphi(0, E, V) = \vartheta'(0, E, V) = 0$ , and let the Lyapunov function  $F(E, V) = (\varphi'(1, E, V) + \vartheta(1, E, V))/2$ . Here and later on  $(\prime) = \partial/\partial x$ ,  $(\ddot{\phantom{x}}) = \partial/\partial E$ ,  $(\partial) = \partial/\partial V$ . The sequence  $A_0^+ < A_1^- \leq A_1^+ < \dots$  is the spectrum of the equation (1.1) with the periodic boundary conditions of period 2, i.e.  $f(x+2) = f(x)$ ,  $x \in \mathbf{R}$ . Here the equality means that  $A_n^- = A_n^+$  is the double eigenvalues. We note that  $F(A_n^\pm) = (-1)^n$ ,  $n \geq 1$ . The lowest eigenvalue  $A_0^+$  is simple,  $F(A_0^+) = 1$  and the corresponding eigenfunction has period 1. The eigenfunction corresponding to  $A_n^\pm$  have period 1 when  $n$  is even and they are antiperiodic,  $f(x+1) = -f(x)$ ,  $x \in \mathbf{R}$ , when  $n$  is odd. The derivative of Lyapunov function,  $\dot{F}$ , has a zero  $\lambda_n$  in "a closed gap"

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$[A_n^-, A_n^+]$ . Let  $D_n(V), n \geq 1$ , be the Dirichlet spectrum of  $V$ , i.e. the spectrum of (1.1) with the boundary condition  $f(0) = f(1) = 0$  and let  $N_n(V), n \geq 1$ , be the Neumann spectrum of  $V$ , i.e. the spectrum of (1.1) with the boundary condition  $f'(0) = f'(1) = 0$ . It is well known that  $D_n, N_n \in [A_n^-, A_n^+]$  and  $N_0 \leq 0$ . Moreover a potential  $V$  is even if and only if  $|G_n| = |D_n - N_n|, n > 0$ . We consider only the case of an even potential, i.e.  $V(x) = V(1-x), 0 < x < 1$ . Let us introduce a signed gap length  $L_n = D_n - N_n, n > 0$ , and a signed height  $h_n = |h_n| \text{sign} L_n$ , where  $|h_n| \geq 0$  is defined by the equation  $\cosh |h_n| = (-1)^n F(\lambda_n) \geq 1$ .

Let us consider the operator  $T_0 = \sqrt{T - N_0} \geq 0$ . It is clear that the spectrum of  $T_0$  is absolutely continuous. In the spectrum of  $T_0$  there are the gaps  $g_n = (a_n^-, a_n^+)$ , where the numbers  $a_n^\pm = \sqrt{A_n^\pm - N_0} > 0, n > 0$ . As above we introduce a signed gap length  $l_n = a_n - b_n$ , where the numbers  $a_n = \sqrt{D_n - N_0} > 0$ , and  $b_n = \sqrt{N_n - N_0} > 0, n > 0$ . Now we define a quasimomentum function (see[F], [MO])

$$k(z) = \arccos F(N_0 + z^2, V), \quad z \in Z = \mathbf{C} \setminus \cup g_n.$$

Here  $g_{-n} = -g_n, n \geq 1$ . The function  $k(z)$  is analytic and moreover  $k(z)$  is a conformal mapping from  $Z$  onto a quasimomentum domain  $K = \mathbf{C} \setminus \cup \Gamma_n$ , where  $\Gamma_n$  is an excised slit

$$\Gamma_n = \left\{ \text{Re} k = \pi n, \quad |\text{Im} k| \leq |h_{|n|}| \right\}, \quad n \in \mathbf{Z}, \quad h_0 = 0.$$

Any nondegenerate (degenerate) slit  $\Gamma_n$  is connected in some way with the non degenerate (degenerate) gap  $g_n$  and the energy gap  $G_n$ . With an edge of the energy gap  $G_n$ , we associate the effective mass  $M_0^- = 0, \quad M_0^+ = 1/E''(0)$ ,

$$M_n^\pm = 0, \quad \text{if } L_n = 0, \quad \text{and } M_n^\pm = 1/E''(k(a_n^\pm)), \quad \text{if } L_n \neq 0,$$

where  $E(k) = N_0 + z(k)^2$  and  $z(k)$  is the inverse function for  $k(z)$ . It is well known that if  $L_n \neq 0$ , then  $\pm M_n^\pm > 0$  and

$$E(k) = A_n^\pm + (k - \pi n)^2 (1/2M_n^\pm + o(1)), \quad \pm(k - \pi n) \downarrow 0.$$

We introduce "reduced masses" (some analog of the effective masses)

$$\mu_n^\pm = 1/z''(k(a_n^\pm)), \quad \text{if } l_n \neq 0 \quad \text{and} \quad \mu_n^\pm = 0, \quad \text{if } l_n = 0.$$

It is clear that  $\pm \mu_n^\pm > 0$  if  $l_n \neq 0$ . There is the equality  $\mu_n^\pm = 2a_n^\pm M_n^\pm$ . Let  $M_n$  be the effective masses corresponding to the Dirichlet eigenvalue  $D_n$  and  $\mu_n$  be the reduced mass for the point  $a_n$ . We have also  $\mu_n = 2a_n M_n$ .

Let us introduce the real Hilbert spaces

$$H = \left\{ f \in L_R^2(0, 1), \quad \int_0^1 V(x) dx = 0, \quad f(x) = f(1-x), 0 < x < 1 \right\},$$

$$\ell_m^2 = \{f = \{f_n\}_1^\infty, \|f\|_m < \infty\}, \quad \|f\|_m^2 = \sum_{n>0} n^{2m} f_n^2, \quad m \geq 0, \quad \ell_0^2 = \ell^2.$$

We define the maps

$$V \rightarrow h(V) = \{h_n\}, \quad V \rightarrow l(V) = \{l_n\}, \quad V \rightarrow \mu(V) = \{\mu_n\},$$

$$V \rightarrow M(V) = \{M_n\}, \quad V \rightarrow L(V) = \{L_n\}.$$

Remember that the derivative of a map  $f : H \rightarrow S$  between two Hilbert spaces  $H, S$  at a point  $x \in H$  is a bounded linear map from  $H$  into  $S$ , which we denote by  $d_x f$ . A map  $f : H \rightarrow S$  is compact on  $H$  if it maps a weakly convergent sequence in  $H$  into a strongly convergent sequence in  $S$ . A map  $f : H \rightarrow S$  is a real analytic isomorphism between  $H$  and  $S$  if  $f$  is one-to-one and onto and both  $f$  and  $f^{-1}$  are real analytic maps of the Hilbert space.

There are some methods to solve the inverse problems. Let us shortly describe "the direct approach" based on a theorem from nonlinear functional analysis. We illustrate it for the mapping  $h : H \rightarrow \ell_1^2$ . We assume:

*i) the map  $h : H \rightarrow \ell_1^2$  is real analytic and for any  $n > 0$  the map  $h_n : H \rightarrow \mathbf{R}$  is compact,*

*ii) the operator  $d_V h$  has inverse for any  $V \in H$ ,*

*iii) there is an estimate  $\|V\|$  in the terms of  $\|h\|_1$ .*

*Then the map  $h : H \rightarrow \ell_1^2$  is a real analytic isomorphism.*

Let us remark that we don't use the Gelfand-Levitan -Marchenco equation or trace formulae in this method.

Let us describe our goal. We solve the inverse problems for the maps  $l, h, \mu, M$  by the direct method.

Note that the operator  $T_0$  and hence  $l_n, h_n, \mu_n$  arise in the problem of the propagation of the acoustic waves in the periodic media. We have

$$u''_{tt} = -T_0^2 u, \quad u(0, x) = u_0(x), \quad u'_t(0, x) = u_1(x), \quad \text{in } L^2(\mathbf{R}).$$

The second author found the asymptotics of the Green function of this problem at large times [K1]. Under some conditions there are the reduced masses in the terms of these asymptotics. Then Theorem 2.1-3 (see Sect. 2) give us the solutions of the inverse problems for the propagation of acoustic waves in periodic media. In the case of the Schrödinger equation we have  $h_n, L_n, M_n$  and we use Theorem 2.2, 2.4 to solve the inverse problem ( for  $L_n$  see [GT2] and Theorem 5.3 ).

There are a lot of papers devoted to the inverse problem for the Hill operator. Marchenko-Ostrovski [MO] show that for any sequence  $p \in \ell_1^2$  there exists a periodic potential  $V \in L^2(0, 1)$  such that  $h(V) = p$ . Its-Matveev consider the case of finite band periodic potentials [IM]. Garnett-Trubowitz [GT1] prove that the maps  $h : H \rightarrow \ell_1^2$  and  $L : H \rightarrow \ell^2$  are real analytic isomorphisms. Considering the map  $V \rightarrow h(V)$  they study the composition  $V \rightarrow \{D_n(V)\}$  and  $\{D_n(V)\} \rightarrow h$  (the proof is not easy). In the next

paper [GT2] they reprove the result for the map  $L : H \rightarrow \ell^2$  by the direct method but they can't estimate  $\|V\|$  in the terms of  $\|L\|$ .

Let us describe the main results. We solve the inverse problem for the maps  $l(V), h(V), \mu(V), M(V)$  by the direct method. We estimate  $\|l\|_1, \|h\|_1, \|\mu\|_1, \|M\|_2, \|L\|$  in the term of  $\|V\|$  and  $\|V\|$  in the term of  $\|l\|_1, \|h\|_1, \|\mu\|_1$ . Note that the inverse problem for  $l, h$  is solved completely and for  $\mu, M$  under additional conditions. Let us shortly present a sketch our proof. We use the direct approach and we check all conditions i)-iii) for our mappings.

There are no "big problems" in checking of point i), except for the analyticity of  $h(V)$ . We use well known results on the analyticity of the fundamental solutions  $\varphi(x, E, V), \theta(x, E, V)$  and the Lyapunov function  $F(E, V)$  of  $E \in \mathbf{C}, V \in L_c^2(0, 1)$ . We also study the map  $\lambda_n(V)$  to get the analyticity of our maps. In the case of the map  $h(V)$  we have some problems with  $h_n(V)$  as a function of a complex potential  $V$ . But we solve this problem in the case  $n \gg 1$ . Moreover we solve this problem in the case of bounded  $n$  and where  $V = V_r + V_c$ . Here  $V_r$  is a real potential and  $V_c$  is a small complex potential. This result is essential and gives us the possibility to solve the inverse problem for the map  $V \rightarrow h(V)$ .

To check ii) we prove that the Frechet derivatives of our maps are Fredholm operators and that the corresponding kernels are zero ( to get the inverse operator.) We would like to say on the entire function  $U = \varphi(1, E, V)\psi_+(x, E, V)\psi_-(x, E, V)$  where  $\psi_{\pm}(x, E, V)$  is the Bloch function. We use  $U$  to prove that the Frechet derivative has inverse. It is necessary to note that using  $U$  simplifies the proof in comparison with the corresponding proof in [GT2]. Moreover it " helps to understand " the existence of the inverse operator for the Frechet derivative.

The check of iii) is based on the estimates in [KK]. The required estimates for  $l, h, \mu$  are proved in the Section 4.

## 2 . The main results

In this section we introduce the concepts and the facts needed to formulate the theorems, some results for the Hill operator.

At first we give more convenient formula for the effective mass. By  $\cos k = F(E)$  we get  $\sin k/E'(k) = \dot{F}(E)$  and by  $E'(\pi n) = 0$  we obtain  $\cos k(a_n^{\pm})/E''(k(a_n^{\pm})) = \dot{F}(A_n^{\pm})$  and  $M_n^{\pm} = -F(A_n^{\pm})\dot{F}(A_n^{\pm}), n \geq 1$ , and hence

$$M_n = -F(D_n)\dot{F}(D_n) = (-1)^{n+1}\dot{F}(D_n), \quad n \geq 1.$$

Let us introduce the Bloch functions  $\psi_{\pm} = \vartheta(1, E, V) + m_{\pm}(E, V)\varphi(x, E, V)$  where the Weyl function

$$m_{\pm}(E, V) = \frac{\rho(E) \pm i \sin k(z)}{\varphi(1, E)}, \quad E = N_0 + z^2,$$

and  $\rho = (\varphi'(1, E, V) - \vartheta(1, E, V))/2$ . It is well known that  $\exp(\mp k(z)x)\psi_{\pm}$  is 1-periodic function of  $x \in \mathbf{R}$  for any  $E \in \mathbf{C}$ . We introduce the function

$$U = \varphi(1, E, V)\psi_+(x, E, V)\psi_-(x, E, V),$$

and the coefficients

$$\hat{V}_n = \int_0^1 V(x) \cos 2\pi n x dx, \quad n \geq 0, \quad V \in L_c^2(0, 1).$$

Let  $I_m : \ell^2 \rightarrow \ell_m^2, m \geq 0$ , be the operator  $(I_m f)_n = n^{-m} f_n, n > 0$ , and  $\Phi$  be the Fourier transform  $(\Phi V)_0 = \hat{V}_0, (\Phi V)_n = \sqrt{2}\hat{V}_n, n > 0$ . We present the main result on  $h(V)$ .

**Theorem 2.1.** *The map  $h : H \rightarrow \ell_1^2$  is a real analytic isomorphism and for any  $n > 0$ , the map  $h_n : H \rightarrow \mathbf{R}$  is compact. Moreover for any  $V \in H$  the operator  $d_V h + (2\pi\sqrt{2})^{-1}I_1\Phi$  is compact,  $d_V h$  has inverse and there are estimates*

$$\|h\|_1 \leq 2\|V\|(1 + \|V\|)e^{2\|V\|}, \quad (2.1)$$

$$\|V\| \leq 16\|h\|_1(1 + \|h\|). \quad (2.2)$$

Note that the result on the analytical isomorphism of  $h(V)$  is proved in [GT1] but by the different method. Let us present the main result on  $l(V)$ .

**Theorem 2.2.** *The map  $l : H \rightarrow \ell_1^2$  is a real analytic isomorphism and for any  $n > 0$ , the map  $l_n : H \rightarrow \mathbf{R}$  is compact. Moreover for any  $V \in H$  the operator  $d_V l + (\pi\sqrt{2})^{-1}I_1\Phi$  is compact,  $d_V l$  has inverse and there are estimates*

$$\|l\|_1 \leq \|V\|(1 + \|V\|)e^{2\|V\|}, \quad (2.3)$$

$$\|V\| \leq 8\|l\|_1(1 + \|l\|)e^{\|l\|_1}. \quad (2.4)$$

We continue consider the inverse problem for the operator  $T_0$  and present the main result on the reduced masses  $\mu$ .

**Theorem 2.3.** *The map  $\mu : H \rightarrow \ell_1^2$  is real analytic and for any  $n > 0$ , the map  $\mu_n : H \rightarrow \mathbf{R}$  is compact. Moreover for any  $V \in H$  the operator  $d_V \mu + (2\pi\sqrt{2})^{-1}I_1\Phi$  is compact and there are estimates*

$$2\|\mu\|_1 \leq \|V\|(1 + \|V\|)e^{2\|V\|}, \quad (2.5)$$

$$\|V\| \leq 8\pi\|\mu\|_1(1 + \|\mu\|). \quad (2.6)$$

Suppose that the kernel  $\ker d_V \mu = \{0\}$  for any  $V \in H$ . Then the map  $\mu : H \rightarrow \ell_1^2$  is a real analytic isomorphism.

Let us introduce the ball  $B_r(f_0) = \{f \in H, \|f - f_0\| \leq r\} \subset H$ , with the central point  $f_0$  from the Hilbert space  $H$ . Now we consider the more difficult case, the effective masses. First we remark that the second author found the formula [K2]

$$V(x) = 4 \sum_{\pm, n \geq 0} A_n^{\pm} M_n^{\pm} \Psi(x, A_n^{\pm}),$$

where the function  $\Psi(x, E) = -U(x, E)/(2\dot{F}(E))$ . Then we see that a potential  $V$  is expressed by the effective masses and the function  $\Psi$ . On the contrary if we know the effective masses can we get a potential  $\Gamma$  We present the result on it.

**Theorem 2.4.** *a) The map  $M : H \rightarrow \ell_2^2$  is real analytic and for any  $n > 0$ , the map  $M_n : H \rightarrow \mathbf{R}$  is compact. Moreover for any  $V \in H$  the operator  $d_M\mu + (4\pi^2\sqrt{2})^{-1}I_2\Phi$  is compact and there is an estimate*

$$8\|M\|_2 \leq \|V\|(1 + \|V\|)e^{3\|V\|}. \quad (2.7)$$

*b) There exist  $r > 0, C_r > 0$  and a neighborhood of the zero  $W \subset H$  such that  $M$  is a real analytic isomorphism  $W$  onto the ball  $B_r(0) \subset \ell_2^2$ . If  $M(V) \in B_r$ , then  $V \in W$  and*

$$\|V\| \leq C_r\|M(V)\|_2. \quad (2.8)$$

*c) Suppose that for any  $V \in H$  the kernel  $\ker d_V M = \{0\}$  and there exists a nondecreasing function  $g : [0, \infty) \rightarrow [0, \infty), g(0) = 0$ , such that  $\|V\| \leq g(\|M(V)\|_2)$ . Then the map  $M : H \rightarrow \ell_2^2$  is a real analytic isomorphism.*

### 3 . The Analyticity

In this chapter useful results on the maps  $l, h, \mu, M$  will be presented. First we describe the properties of the fundamental solutions (see [PT]). Let  $H_c$  be the complexification of the real Hilbert space  $H$ . The functions  $\varphi(\cdot, E, V), \vartheta(\cdot, E, V)$  and their derivatives  $\varphi'(\cdot, E, V), \vartheta'(\cdot, E, V)$  are the entire functions of the spectral parameter  $E \in \mathbf{C}$  and a potential  $V \in L_c^2(0,1)$  as the mappings from  $\mathbf{C} \times L_c^2(0,1)$  into the space of continuous functions  $C([0, 1])$ . We have also

$$\partial\vartheta(E) = \vartheta(x, E)[\varphi(E)\vartheta(x, E) - \vartheta(E)\varphi(x, E)],$$

$$\partial\varphi'(E) = \varphi(x, E)[\varphi'(E)\vartheta(x, E) - \vartheta'(E)\varphi(x, E)],$$

here  $\varphi(E) = \varphi(1, E), \varphi'(E) = \varphi'(1, E)$ , and so on. There are also the similar equations for  $\vartheta', \varphi$ . Then we obtain

$$2\partial F(E, V) = U(x, E, V) = \varphi(E)\vartheta(x, E)^2 - \vartheta'(E)\varphi(x, E)^2 + 2\rho(E)\vartheta(x, E)\varphi(x, E). \quad (3.1)$$

Let  $f$  be one of the functions  $\varphi, \vartheta, \vartheta', \varphi'$  and  $V_n \rightarrow V$  weakly in  $L_c^2(0,1)$ . Then  $f(V_n) \rightarrow f(V)$  uniformly on compact sets of  $\mathbf{C}$ . We have the asymptotics

$$\varphi(x, E) = \frac{\sin x\sqrt{E}}{\sqrt{E}} + O\left(\frac{\exp|\operatorname{Im}x\sqrt{E}|}{|E|}\right), \quad (3.2)$$

$$\vartheta(x, E) = \cos x\sqrt{E} + O\left(\frac{\exp|\operatorname{Im}x\sqrt{E}|}{\sqrt{E}}\right),$$

as  $|E| \rightarrow \infty$ , uniformly on bounded subsets of  $[0, 1] \times L_c^2(0,1)$ . These asymptotics can be differentiated by  $x$  and any times by  $E$ .

The functions  $\varphi(x, D_n), \vartheta(x, N_n)$  are the Dirichlet and Neumann eigenfunctions. We define functions

$$\varphi_n(x, V) = \frac{\varphi(x, D_n)^2}{\varphi'(1, D_n)\dot{\varphi}(1, D_n)}, \quad \vartheta_n(x, V) = \frac{\vartheta(x, N_n)^2}{-\vartheta(1, N_n)\dot{\vartheta}'(1, N_n)}.$$

The functions  $\varphi_n, \vartheta_n$  are real analytic and we have

$$\int_0^1 \varphi_n(x) dx = 1, \quad n \geq 1 \quad \int_0^1 \vartheta_n(x) dx = 1, \quad n \geq 0. \quad (3.3)$$

We need the following results on the Lyapunov function.

**Lemma 3.1.** *The function  $F(E, \cdot)$  is compact entire on  $H$ . Its gradient is  $2\partial F = U$  and*

$$\int_0^1 U(x, E, V) dx = -2\dot{F}(E, V). \quad (3.4)$$

In particular if  $L_n \neq 0, V \in H$ , then

$$\Psi(x, D_n, V) = \frac{U(x, D_n, V)}{-2\dot{F}(D_n, V)} = \varphi_n(x, V), \quad n > 0, \quad (3.5)$$

$$\Psi(x, N_n, V) = \frac{U(x, N_n, V)}{-2\dot{F}(N_n, V)} = \vartheta_n(x, V), \quad n \geq 0,$$

but if  $L_n = 0$ , then

$$U(\cdot, D_n, V) = 0, \quad \dot{U}(\cdot, D_n, V) = -\ddot{F}(D_n, V)(\varphi_n(\cdot, V) + \theta_n(\cdot, V)), \quad n > 0. \quad (3.6)$$

Moreover there are asymptotics

$$F(E, V) = \cos \sqrt{E} + O\left(\frac{\exp |\operatorname{Im} \sqrt{E}|}{E}\right), \quad (3.7)$$

$$U(x, E, V) = \frac{\sin \sqrt{E}}{\sqrt{E}} \left(1 + O\left(\frac{1}{\sqrt{E}}\right)\right), \quad |\sqrt{E} - \pi n| \geq r > 0, \quad n \in \mathbf{Z}, \quad (3.8)$$

$$U(x, E, V) = \frac{\sin \sqrt{E}}{\sqrt{E}} + O\left(\frac{1}{E}\right), \quad |\operatorname{Im} \sqrt{E}| \leq 1,$$

as  $|E| \rightarrow \infty$ , uniformly on bounded subsets of  $[0, 1] \times L_c^2(0,1)$ . These asymptotics can be differentiated any times by  $E$ .

*Proof.* By the properties of  $\varphi', \theta$  we get that  $F$  is a compact analytic function on  $L_c^2(0,1)$ . The equality (3.4) is proved in [PT]. By  $\rho(D_n) = 0, \varphi(D_n) = 0$ , we have  $U(x, D_n, V) = -\theta'(D_n)\varphi(x, D_n)^2$ , and by (3.3) we obtain

$$-2\dot{F}(D_n, V) = \int_0^1 U(x, D_n, V) dx = \int_0^1 -\theta'(D_n)\varphi(x, D_n)^2 dx = -\theta'(D_n)\varphi'(D_n)\dot{\varphi}(D_n).$$

Hence  $2\dot{F}(D_n, V) = \theta'(D_n)\varphi'(D_n)\dot{\varphi}(D_n)$  and we get (3.5), the proof for the Neumann spectrum is the same.

We prove (3.6). Let  $L_n = 0$ , then  $\rho(D_n) = \varphi(D_n) = \theta'(D_n) = 0$  and we have  $U(\cdot, D_n, V) = 0$ . Suppose  $\dot{\rho}(D_n) = 0$ . Then by (3.3), (3.4) we obtain

$$\begin{aligned} -2\ddot{F}(D_n, V) &= \int_0^1 \dot{U}(x, D_n, V)dx = \int_0^1 \{\dot{\varphi}(D_n)\theta(x, D_n)^2 - \dot{\vartheta}'(D_n)\varphi(x, D_n)^2\}dx = \\ &= -\dot{\theta}'(D_n)\theta(D_n)\dot{\varphi}(D_n) - \dot{\vartheta}'(D_n)\varphi'(D_n)\dot{\varphi}(D_n) = -2\dot{\theta}'(D_n)\dot{\varphi}(D_n)F(D_n). \end{aligned}$$

Hence  $\ddot{F}(D_n, V) = \dot{\vartheta}'(D_n)\dot{\varphi}(D_n)F(D_n)$ , if  $L_n = 0$ , and by (3.3) we obtain

$$\dot{U}(x, D_n, V) = \dot{\varphi}(D_n)\vartheta(x, D_n)^2 - \dot{\vartheta}'(D_n)\varphi(x, D_n)^2 = -\dot{\varphi}(D_n)\dot{\vartheta}'(D_n)F(D_n)(\varphi_n + \theta_n)$$

and then we get (3.6).

We shall prove  $\dot{\rho}(D_n) = 0$ . Since  $\varphi(1-x, D_n) = -F(D_n)\varphi(x, D_n)$  and  $\theta(1-x, D_n) = F(D_n)\vartheta(x, D_n)$ ,  $0 < x < 1$ , we get

$$\int_0^1 \varphi(x, D_n)\vartheta(x, D_n)dx = 0.$$

We have (see [PT])

$$\dot{\varphi}'(E) = -\int_0^1 \varphi(x, E)[\varphi'(E)\theta(x, E) - \theta'(E)\varphi(x, E)]dx = 0,$$

then we get  $\dot{\varphi}'(D_n) = 0$  and by  $2\dot{F}(D_n) = \dot{\varphi}'(D_n) + \dot{\theta}(D_n) = 0$  we obtain  $\dot{\theta}(D_n) = 0$  and  $\dot{\rho}(D_n) = 0$ .

In [T] there is the asymptotics

$$\frac{U(x, E, V)}{-2\dot{F}(E)} = 1 + O\left(\frac{1}{\sqrt{E}}\right), \quad \text{if } |\sqrt{E} - \pi n| \geq r > 0, \quad n \in \mathbf{Z},$$

for some  $r > 0$  and by (3.7) we obtain the first asymptotics in (3.8). If we substitute (3.2) in the formula (3.1) we get the second estimate in (3.8).  $\square$

Later on we need the results on the Dirichlet and Neumann problems from [PT].

**Theorem 3.2.** *The functions  $D_n(\cdot), N_n(\cdot)$  are compact real analytic on  $L_c^2(0,1)$ . Its gradients are*

$$\partial D_n(\cdot) = \varphi_n, \quad N_n(\cdot) = \vartheta_n, \quad (3.9)$$

Moreover

$$D_n = \pi^2 n^2 + \hat{V}_0 - \hat{V}_n + O\left(\frac{1}{n}\right), \quad (3.10)$$

$$N_n = \pi^2 n^2 + \hat{V}_0 + \hat{V}_n + O\left(\frac{1}{n}\right),$$

and

$$\partial D_n(x) = \varphi_n(x) = 2 \sin^2 \pi n x + O\left(\frac{1}{n}\right), \quad (3.11)$$

$$\partial N_n(x) = \theta_n(x) = 2 \cos^2 \pi n x + O\left(\frac{1}{n}\right), \quad (3.12)$$

uniformly on bounded subsets of  $[0, 1] \times L_c^2(0.1)$ .  $\square$

Now we consider another maps. First we study the case  $\lambda_n(V)$ . The properties of  $\lambda_n(V)$  is important to study our maps  $h(V), \mu(V), M(V)$ .

Now we present the result about the behavior of  $\lambda_n(V)$ .

**Theorem 3.3.a)** *Let  $V \in L_c^2(0.1)$ ,  $n \geq \epsilon^{\|V\|}$ . Then*

$$|\sqrt{\lambda_n(V)} - \pi n| \leq \frac{\pi}{4}. \quad (3.13)$$

b) *The function  $\lambda_n(\cdot)$  is compact real analytic on  $L_c^2(0.1)$ . Its gradients are*

$$\partial \lambda_n = -\frac{\dot{U}(x, \lambda_n(V), V)}{2\ddot{F}(\lambda_n(V), V)}, \quad (3.14)$$

$$\partial \lambda_n = \frac{\varphi_n + \theta_n}{2}, \quad \text{if } L_n = 0.$$

Moreover

$$\lambda_n = \pi^2 n^2 + \hat{V}_0 + O\left(\frac{1}{n}\right), \quad (3.15)$$

$$\partial \lambda_n = 1 + O\left(\frac{1}{n}\right), \quad (3.16)$$

uniformly on bounded subsets of  $[0, 1] \times L_c^2(0.1)$ .

*Proof.* The proof of (3.13) repeat the case of Dirichlet eigenvalues in [PT].

To prove a real analyticity fix  $V_0 \in H$ . We have  $\dot{F}(\lambda_n(V_0), V_0) = 0$  and  $\ddot{F}(\lambda_n(V_0), V_0) \neq 0$ . Then the implicit function theorem applies and there exists a unique continuous function  $\tilde{\lambda}_n$ , defined on some small neighborhood  $W \subset L_c^2(0.1)$  of  $V_0$  such that

$$\dot{F}(\tilde{\lambda}_n(V), V) = 0, V \in W, \quad \text{and,} \quad \tilde{\lambda}_n(V_0) = \lambda_n(V_0).$$

Furthermore  $\tilde{\lambda}_n(V)$  is real analytic. On the other hand  $\lambda_n(V)$  is also a continuous function on  $W$  satisfying  $\dot{F}(\lambda_n(V), V) = 0$ . Therefore  $\tilde{\lambda}_n(V) = \lambda_n(V), V \in W$ , by the uniqueness and so  $\lambda_n(V), V \in W$  is real analytic.

To calculate the gradient note that  $\dot{F}(\lambda_n(V), V) = 0$ . Hence

$$0 = \frac{\partial}{\partial V} \{\dot{F}(\lambda_n(V), V)\} = \ddot{F}(\lambda_n(V), V) \partial \lambda_n + \partial \dot{F}(E, V) |_{E=\lambda_n} =$$

$$\ddot{F}(\lambda_n(V), V) \partial \lambda_n + \frac{1}{2} \dot{U}(x, \lambda_n(V), V),$$

and then we get (3.14).

By a) we have  $|\sqrt{\lambda_n(V)} - \pi n| \leq \frac{\pi}{4}$  for  $n \geq e^{\|V\|}$ . Hence by (3.7), (3.8) we obtain

$$\partial \lambda_n(V) = -\frac{\dot{U}}{2\dot{F}} = \frac{\cos \sqrt{E} + O(\frac{1}{\sqrt{E}})}{\cos \sqrt{E} + O(\frac{1}{\sqrt{E}})} = 1 + O(\frac{1}{\sqrt{E}}), \quad E = \lambda_n.$$

We can improve the asymptotics (3.13). By the analytical property we have

$$\lambda_n(V) - (\pi n)^2 = \lambda_n(V) - \lambda_n(0) = \int_0^1 \frac{d}{dt} \lambda_n(tV) dt = \int_0^1 (\partial \lambda_n(tV), V) dt$$

and by (3.16) we obtain (3.15).  $\square$

By this theorem we see that the map  $V \rightarrow \{\lambda_n(V) - (\pi n)^2\} \in \ell^2$  is real analytic. We have also the asymptotics of the gradient (3.16). These results will be used to study the maps  $\mu, M$  but particular  $h$ . Now we present results on the mapping  $l$ . Some estimates for  $l$  follow from the properties of the maps  $L_n, D_n, N_n$  (see Theorem 3.2).

**Theorem 3.4.** *The function  $l_n(\cdot)$  is compact real analytic on  $H$ . Its gradients is*

$$\partial l_n(\cdot) = \frac{\varphi_n - \vartheta_0}{2a_n} - \frac{\vartheta_n - \vartheta_0}{2b_n}, \quad (3.17)$$

and in particular, let  $U_0 = U(x, E, V) + 2\dot{F}(E)\vartheta_0(x)$ . Then

$$\partial l_n(\cdot) = \frac{U_0(\cdot, N_n, V)}{4b_n \dot{F}(N_n)} - \frac{U_0(\cdot, D_n, V)}{4a_n \dot{F}(D_n)}, \quad \text{if } l_n \neq 0, \quad (3.18)$$

$$\partial l_n = \frac{\varphi_n - \vartheta_n}{2a_n}, \quad \text{if } l_n = 0. \quad (3.19)$$

Moreover

$$l_n = -\frac{\hat{V}_n}{\pi n} + O(\frac{1}{n^2}), \quad (3.20)$$

$$\partial l_n(x) = -\frac{\cos 2\pi n x}{\pi n} + O(\frac{1}{n^2}), \quad (3.21)$$

uniformly on bounded subsets of  $[0, 1] \times H_c$ .

*Proof.* We have  $l_n = L_n/(a_n + b_n)$  and by the corresponding properties of  $L_n, D_n, N_n$  we get that the map  $l_n(\cdot)$  is compact real analytic on  $H$ .

By  $l_n = a_n - b_n$  and  $a_n^2 = D_n - N_0, b_n^2 = N_n - N_0$ , we find (3.17), (3.19) and by (3.5) we obtain (3.18).

By (3.10) we get (3.20) and by (3.17), (3.10-12) we obtain (3.21).  $\square$

Now we consider the case of the effective masses. Using the estimates on the  $M_n$  we obtain immediately results on the reduced masses by very simple formula  $\mu_n = 2a_n M_n$ . We present

**Theorem 3.5.** *The function  $M_n(\cdot)$  is compact real analytic on  $H$ . Its gradients is*

$$\partial M_n = (-1)^{n+1} \left\{ \frac{1}{2} \dot{U}(x, D_n) + \ddot{F}(D_n) \varphi_n(x) \right\} \quad (3.22)$$

and in particular

$$\partial M_n = -M_n \dot{\Psi}(x, D_n, V) = -M_n \left\{ \frac{U}{-2\dot{F}} \right\} \Big|_{E=D_n}, \quad \text{if } L_n \neq 0, \quad (3.23)$$

$$2\partial M_n = (-1)^{n+1} \ddot{F}(D_n) (\varphi_n - \vartheta_n), \quad \text{if } L_n = 0. \quad (3.24)$$

Moreover

$$M_n = -\frac{\hat{V}_n}{(2\pi n)^2} + O\left(\frac{1}{n^3}\right), \quad (3.25)$$

$$\partial M_n(x) = -\frac{\cos 2\pi n x}{(2\pi n)^2} + O\left(\frac{1}{n^3}\right), \quad (3.26)$$

uniformly on bounded subsets of  $[0, 1] \times H_c$ .

*Proof.* Remember that we have  $M_n = (-1)^{n+1} \dot{F}(D_n(V), V)$ . By Lemma 3.1 we get that  $M_n$  is a compact real analytic function on  $H$ . Its gradient is

$$\begin{aligned} (-1)^{n+1} \partial M_n &= \partial \{ \dot{F}(D_n(V), V) \} = \partial \dot{F}(E, V) \Big|_{E=D_n} + \ddot{F}(D_n(V), V) \partial D_n = \\ &= \frac{1}{2} \dot{U}(\cdot, D_n(V), V) + \ddot{F}(D_n(V), V) \varphi_n. \end{aligned}$$

Let  $L_n \neq 0$ , then by (3.5), (3.22) we obtain

$$\partial M_n = (-1)^{n+1} \left\{ \frac{1}{2} \dot{U} + \ddot{F} \frac{U}{-2\dot{F}} \right\} = (-1)^{n+2} \dot{F} \left[ \frac{\dot{U}}{-2\dot{F}} - \frac{\ddot{F}U}{-2\dot{F}^2} \right].$$

Then we get (3.23). Let  $L_n = 0$ , then by (3.6) we obtain

$$\partial M_n = (-1)^{n+1} \left[ -\frac{1}{2} \ddot{F}(D_n) (\varphi_n + \vartheta_n) + \ddot{F}(D_n) \varphi_n \right] = (-1)^{n+1} \frac{1}{2} \ddot{F}(D_n) (\varphi_n - \vartheta_n).$$

We have " the Taylor series " (let  $s_n = D_n - \lambda_n$ )

$$(-1)^{n+1} M_n = \dot{F}(D_n(V), V) = \dot{F}(\lambda_n) + \ddot{F}(\lambda_n) s_n + s_n^2 \int_0^1 (1-t) \dot{F}(\lambda_n + t s_n, V)''' dt$$

By (3.7), (3.10), (3.15) we get

$$\ddot{F}(\lambda_n)(D_n - \lambda_n) = \left( \frac{(-1)^{n+1}}{(2\pi n)^2} + O\left(\frac{1}{n^3}\right) \right) (-\hat{V}_n + O\left(\frac{1}{n^3}\right)) = \frac{(-1)^n \hat{V}_n}{(2\pi n)^2} + O\left(\frac{1}{n^3}\right),$$

and

$$|\int_0^1 (1-t)F(\lambda_n + ts_n, V)'''_E dt| \leq \sup_{t \in [0,1]} |F(\lambda_n + ts_n, V)'''_E| |s_n|^2 / 2 = O(\frac{1}{n^3}).$$

Then we obtain (3.25). By (3.22), (3.10), (3.11), (3.7), (3.8) we have

$$\begin{aligned} (-1)^{n+1} \partial M_n &= \frac{\cos \sqrt{D_n}}{4D_n} + O(\frac{1}{D_n^{3/2}}) + \frac{(-1)^{n+1}}{(2\pi n)^2} (1 + O(\frac{1}{n})) (2 \sin^2 \pi n x + O(\frac{1}{n})) = \\ &= \frac{(-1)^n \cos 2\pi n x}{(2\pi n)^2} + O(\frac{1}{n^3}) \end{aligned}$$

and hence we have (3.26).  $\square$

Now we consider the case of the reduced masses.

**Theorem 3.6.** *The function  $\mu_n(\cdot)$ ,  $n \geq 1$ , is compact real analytic on  $H$ . Its gradients is*

$$\partial \mu_n = \frac{\varphi_n - \vartheta_0}{a_n} M_n + 2a_n \partial M_n = \frac{\varphi_n - \vartheta_0}{2a_n^2} \mu_n - (-1)^n \left\{ \frac{\dot{U}(\cdot, D_n, V)}{2} + \ddot{F}(D_n) \right\} \varphi_n, \quad (3.27)$$

and in particular

$$\partial \mu_n = \mu_n \left\{ \frac{\varphi_n(x) - \vartheta_0(x)}{2a_n^2} - \dot{\Psi}(x, D_n) \right\}, \quad l_n \neq 0, \quad (3.28)$$

$$\partial \mu_n = 2a_n \partial M_n = (-1)^{n+1} \ddot{F}(D_n, V) (\varphi_n - \vartheta_n), \quad \text{if } l_n = 0. \quad (3.29)$$

Moreover

$$\mu_n = -\frac{\hat{V}_n}{2\pi n} + O(\frac{1}{n^2}), \quad (3.30)$$

$$\partial \mu_n(x) = -\frac{\cos 2\pi n x}{2\pi n} + O(\frac{1}{n^2}), \quad (3.31)$$

uniformly on bounded subsets of  $[0, 1] \times H_c$ .

*Proof.* Remember that we have  $\mu_n = 2a_n M_n$ . Then we get  $\partial \mu_n = a_n^{-1} \{\varphi_n - \vartheta_0\} M_n + 2a_n \partial M_n$  and by (3.22) we have (3.27). But if  $l_n \neq 0$  then by (3.23), (3.27) we obtain

$$\partial \mu_n = \frac{\varphi_n - \vartheta_0}{a_n} M_n - 2a_n M_n \dot{\Psi}(D_n) = \mu_n \left\{ \frac{\varphi_n - \vartheta_0}{2a_n^2} - \dot{\Psi}(D_n) \right\}.$$

But if  $l_n = 0$  then by (3.24), (3.27) we get (3.29).

Using (3.26), (3.10) and  $\mu_n = 2a_n M_n$  we get (3.30)

$$\mu_n = 2a_n M_n = 2\pi n (1 + O(\frac{1}{n^2})) \left( \frac{-\hat{V}_n}{(2\pi n)^2} + O(\frac{1}{n^3}) \right) = -\frac{\hat{V}_n}{2\pi n} + O(\frac{1}{n^2}),$$

and by (3.25-27), (3.10-11) we obtain (3.31).  $\square$

Now we consider the more difficult case in this section the map  $h$ . The main problem connected with small complex  $h_n$ . Introduce the function

$$y_n(V) = \frac{1}{2}(F(\lambda_n) + F(D_n))\{-\ddot{F}(\lambda_n) - s_n \int_0^1 (1-t)^2 F(\lambda_n + ts_n, V)''' dt\},$$

where  $s_n = D_n - \lambda_n$ . The functions  $F, \lambda_n, D_n$  are compact real analytic on  $H$ . Then  $y_n$  is also compact real analytic on  $H$ . Let  $V_0 \in H$  such that  $L_n(V_0) = 0$ . Then

$$y_n(V_0) = -F(D_n(V_0))\ddot{F}(\lambda_n(V_0), V_0) > 0 \quad (3.32)$$

and we can define the real analytic function  $\beta_n(V) = \sqrt{y_n(V)}$  at some ball  $B(V_0, \varepsilon_n), \varepsilon_n > 0$ , in  $H_c$ . By (3.7), (3.10), (3.15) we obtain

$$y_n(V) = \frac{1}{(2\pi n)^2} + O\left(\frac{1}{n^3}\right) \quad (3.33)$$

uniformly on bounded subsets of  $H_c$ . Then at large  $n \gg 1$  we can define the real analytic function  $\beta_n(V) = \sqrt{y_n(V)}$  uniformly on bounded subsets of  $H_c$ . Now we present the result on the map  $h(V)$ .

**Theorem 3.7.** *The function  $h_n(\cdot), n \geq 1$ , is compact real analytic on  $H$ . Its gradients is*

$$\partial h_n = (-1)^n \frac{U(\cdot, \lambda_n(V), V)}{2\sqrt{F^2(\lambda_n(V), V) - 1}}, \quad \text{if } L_n \neq 0. \quad (3.34)$$

*In particular, let  $L_n(V_0) = 0$  for some  $V_0 \in H, n \geq 1$ . Then*

$$\sinh h_n = (D_n - \lambda_n)\beta_n, \quad (3.35)$$

*at some ball  $B(V_0, \varepsilon_n), \varepsilon_n > 0$ , in  $H_c$  and*

$$(\partial h_n)(V_0) = \frac{\varphi_n - \vartheta_n}{2} \sqrt{(-1)^{n+1} \ddot{F}(\lambda_n(V_0), V_0)}. \quad (3.36)$$

*Moreover, let  $n \rightarrow \infty$ . Then*

$$\sinh h_n = (D_n - \lambda_n)\beta_n, \quad (3.37)$$

$$h_n = -\frac{\hat{V}_n}{2\pi n} + O\left(\frac{1}{n^2}\right), \quad (3.38)$$

$$\partial h_n(x) = -\frac{\cos 2\pi nx}{2\pi n} + O\left(\frac{1}{n^2}\right), \quad (3.39)$$

*uniformly on bounded subsets of  $[0, 1] \times H_c$ .*

*Proof.* Remember that we have  $\cosh h_n = (-1)^n F(\lambda_n)$  and  $F, \lambda_n$  are compact real analytic. Suppose that  $L_n(V) \neq 0$ . Then at some ball  $B(V_0, \varepsilon_n), \varepsilon_n > 0$ , in  $H_c$  the function  $h$  is also compact real analytic and differentiating  $\cosh h_n = (-1)^n F(\lambda_n(V), V)$  we get  $(-1)^n (\sinh h_n) \partial h_n = \partial \{F(\lambda_n(V), V)\}$  and hence

$$(-1)^n (\sinh h_n) \partial h_n = \dot{F}(\lambda_n(V), V) \partial \lambda_n + \partial F(E, V) |_{E=\lambda_n} = \frac{U(\cdot, \lambda_n(V), V)}{2}.$$

and we obtain (3.34).

Suppose that  $L_n(V_0) = 0$ . We have the Taylor series for  $F(E, V)$  at  $E = \lambda_n(V)$  for fixed  $V$  from some ball of  $V_0$ :

$$F(D_n(V), V) = F(\lambda_n) + \frac{1}{2} \ddot{F}(\lambda_n) s_n^2 + \frac{1}{2} s_n^3 \int_0^1 (1-t)^2 F(\lambda_n + t s_n, V)'''_E dt,$$

where  $s_n = D_n - \lambda_n$ , and

$$F(\lambda_n) = F(D_n) + \frac{s_n^2}{2} \{-\ddot{F}(\lambda_n) - s_n \int_0^1 (1-t)^2 F(\lambda_n + t s_n, V)'''_E dt\},$$

hence

$$\begin{aligned} F(\lambda_n)^2 - 1 &= (F(\lambda_n) + F(D_n))(F(\lambda_n) - F(D_n)) = \\ s_n^2 \frac{1}{2} (F(\lambda_n) + F(D_n)) \{-\ddot{F}(\lambda_n) - s_n \int_0^1 (1-t)^2 F(\lambda_n + t s_n, V)'''_E dt\} &= s_n^2 y_n. \end{aligned}$$

And by the property of  $y_n$  (see (3.32)) we get  $\sinh h_n = \sqrt{F^2(\lambda_n) - 1} = s_n \beta_n$  at some neighborhood of  $V_0$ . Differentiating (3.35) at the point  $V_0$  we obtain

$$\partial h_n = \frac{\partial(s_n \beta_n)}{\sqrt{1 + (s_n \beta_n)^2}} = \beta_n(V_0) (\partial s_n) = (\partial D_n - \partial \lambda_n) \sqrt{(-1)^{n+1} \dot{F}(\lambda_n(V_0), V_0)}$$

and by  $2\partial \lambda_n = \varphi_n + \vartheta_n$ , see (3.14), we get (3.36).

Let  $n \rightarrow \infty$ . Then by (3.33) we can define the function  $\beta_n$  on bounded subsets of  $H_c$  and hence we obtain (3.37) as in the case of a degenerate gap, (3.35). Differentiating (3.37) we get  $\sqrt{1 + (s_n \beta_n)^2} \partial h_n = \partial(s_n \beta_n)$ , and by (3.33), Theorem 3.1-3 we obtain (3.39). There is the equality

$$h_n(V) = h_n(V) - h_n(0) = \int_0^1 \frac{d}{dt} h_n(tV) dt = \int_0^1 (\partial h_n(tV), V) dt$$

and by (3.39) we get (3.38) :

$$h_n(V) = - \int_0^1 \int_0^1 \frac{\cos 2\pi n x}{2\pi n} V(x) dx dt + O\left(\frac{1}{n^2}\right) = -\frac{\hat{V}_n}{2\pi n} + O\left(\frac{1}{n^2}\right). \square$$

## 4 . The Estimates

We need some results on the function  $U$ . These ones are responsible for the local isomorphism of mappings  $l_n, h_n$ . First we write a general interpolation Lemma from [GT2].

**Lemma 4.1.** *Suppose a function  $f(E)$  is entire and*

$$\sup_{|E|=\pi^2(n+\frac{1}{2})^2} |f(E)/(\frac{\sin \sqrt{E}}{\sqrt{E}})| = o(1), \quad \text{as } n \rightarrow \infty,$$

and  $f(A_n) = 0$  for some sequence  $\{A_n\}_1^\infty$  of distinct complex numbers such that  $f(A_n) = \pi^2 n^2 + o(1)$  as  $A_n \rightarrow \infty$ . Then  $f \equiv 0$ .  $\square$

Now we present the lemma which is important for the existence of the inverse operators for the Frechet derivatives of  $l(V), h(V)$ .

**Lemma 4.2.** *Suppose that functions  $V, g \in H$  and  $(g, \varphi_n - \vartheta_n) = 0$  if  $L_n = 0$ . Let functions  $f(E) = \int_0^1 U(x, E, V)g(x)dx$  and  $f_0(E) = f(E) + 2\dot{F}(E)(\vartheta_0, g)$ ,  $E \in \mathbf{C}$ . Suppose that  $\{A_n\}_1^\infty$  is some sequence of distinct complex numbers such that  $f(A_n) = \pi^2 n^2 + o(1)$  as  $A_n \rightarrow \infty$  and one of the following condition is fulfilled*

a)  $f(A_n) = 0$  for any  $n \geq 1$ , or b)  $f_0(A_n) = 0$  for any  $n \geq 1$ . Then  $g \equiv 0$ .

*Proof.* First we consider the case a). By (3.8) we see that all conditions of Lemma 4.1 are fulfilled and hence  $f \equiv 0$ . Note that since  $V \in H$  the vectors  $\varphi_n - 1$  form a basis for  $H$  (see [PT]). There are two cases . First, let  $L_n \neq 0$ . Then by (3.5) we have  $f(D_n) = -2\dot{F}(D_n)(\varphi_n, g) = 0$  and hence  $(\varphi_n - 1, g) = 0$ . Second, let  $L_n = 0$ . Then by (3.6) we have  $f(D_n) = (\dot{U}(D_n), g) = -\dot{F}(D_n)(\varphi_n + \vartheta_n, g) = 0$  and by the condition  $(\varphi_n - \vartheta_n, g) = 0$  we obtain  $(\varphi_n - 1, g) = 0$  and hence  $g \equiv 0$ .

Let us consider the case b). We introduce the function  $f_1 = f_0(E)/(E - N_0)$ . The function  $f_1$  is entire since by (3.5)  $f_0(N_0) = 0$ . Then by (3.7), (3.8) we see that all conditions of Lemma 4.1 are fulfilled and hence  $f_1 \equiv 0, f_0 \equiv 0$ . By (3.7), (3.8) we get that  $U, \dot{F}$  have different asymptotics as  $|E| \rightarrow \infty$ . Hence  $(\vartheta_0, g) = 0$  and  $f = 0$ . Then by a) we get  $g \equiv 0$ .  $\square$

Letter on in this section we will obtain estimates on  $l, h, M, \mu$  in some Hilbert spaces and  $V \in H$ . First we bound the Neumann eigenvalue  $N_0$  in terms  $\|V\|$ . We do it because in [KK] we estimated  $l, h, M\mu$  and  $N_0^2 + \|V\|^2$ .

**Lemma 4.3.** *Suppose that  $V \in H$ . Then*

$$|N_0| \leq \|V\|(1 + \|V\|/3). \quad (4.1)$$

*Proof.* Let  $N_n(0) = (\pi n)^2$  be the Neumann eigenvalues for  $V = 0$  and  $e_0 = 1$ ,  $e_n = \sqrt{2} \cos 2\pi n x, n \geq 1$  be the corresponding eigenfunctions. Let  $f$  be the Neumann eigenfunction corresponding to  $N_0$ . Suppose that  $\|f\| = 1$  and  $f = f_0 + f_1$ , where  $f_0 = e_0(f, e_0)$ . By  $-f'' + Vf = N_0 f$  we obtain

$$f_1 = - \sum_{n>0} \frac{e_n(Vf, e_n)}{N_n(0) - N_0},$$

then

$$|f_1(x)| \leq 2\|V\| \sum_{n>0} \frac{1}{N_n(0) - N_0} \leq \frac{2\|V\|}{\pi^2} \sum_{n>0} \frac{1}{n^2} = \frac{2\|V\|}{\pi^2} \frac{\pi^2}{6} = \frac{\|V\|}{3},$$

where  $0 < x < 1$ , since  $N_0 \leq 0$  and  $\sum_{n>0} \frac{1}{n^2} = \frac{\pi^2}{6}$ . Hence we get

$$\begin{aligned} -N_0 &= -2(Vf_1, f_0) - (Vf_1, f_1) - \|f_1'\|^2 \leq \\ &2|f_0|\|V\|\|f_1\| + \|V\|\|f_1\| \frac{\|V\|}{3} \leq \|V\|(1 + \frac{\|V\|}{3}). \square \end{aligned}$$

First we estimate  $l, h, M, \mu, L$ , in terms of  $\|V\|$ . Introduce the constants  $h_+ = \sup_{n>0} |h_n|$ , and  $\omega = \cosh h_+$ . We have the theorem

**Theorem 4.4.** *Suppose that  $V \in H$ . Then*

$$\omega = \cosh h_+ \leq e^{\|V\|}, \quad (4.2)$$

$$\|L\|^2 \leq (\|V\|^2 + N_0^2) \leq 2\|V\|^2(1 + \|V\|)^2, \quad (4.3)$$

$$4\|\mu\|_1 \leq \sqrt{2}\omega^2\|V\|(1 + \|V\|), \quad (4.4)$$

$$4\|M\|_2 \leq \omega\|\mu\|_1, \quad (4.5)$$

$$|l_n| \leq 2|h_n| \leq 2\pi|\mu_n^\pm|, \quad |l_n| \leq 2|\mu_n^\pm|, \quad \text{if } n \geq 1. \quad (4.6)$$

*Proof.* In [PT] there is an estimate  $|F(E)| \leq \exp(\|V\|)$ ,  $E \in \mathbf{R}$ . Then by  $F(\lambda_n) = (-1)^n \cosh h_n$  we get (4.2). In [KK] there is the first estimate in (4.3) and by (4.1) we get the second one. In [KK] there are estimates (4.6) and  $2n \leq \omega a_n^\pm$ ,  $n \geq 1$ . Then by (4.15) we have  $\omega^2|L_n| = \omega^2|(a_n^+ + a_n^-)l_n| \geq 4n|\mu_n|$ . Hence  $\omega^2\|L\| \geq 4\|\mu\|_1$  and by (4.3) we obtain  $4\|\mu\|_1 \leq \omega^2\|L\| \leq \omega^2\sqrt{2}\|V\|(1 + \|V\|)$  then we get (4.4). Moreover by  $\mu_n = 2a_nM_n$  we get  $|\mu_n|\omega \geq 4n|M_n|$  and then we have (4.5).  $\square$

Now we are going to estimate  $(\|V\|^2 + N_0^2)$  in terms of  $\|l\|_1, \|h\|_1, \|\mu\|_1$ . This result is important in the checking of the point iii) of the Direct Method. Unfortunately, we can not obtain the same estimate for  $M, L$ .

**Theorem 4.5.** *Suppose that  $V \in H$ . Then*

$$\|V\|^2 + N_0^2 \leq \frac{16}{\pi} \sum_{n>0} |h_n l_n| (a_n^+)^2, \quad (4.7)$$

$$(a_n^+)^2 \leq 2((\pi n)^2 + n \sum_{m>0}^n l_m^2) \leq 2((\pi n)^2 + n\|l\|^2), \quad (4.8)$$

$$\|V\|^2 + N_0^2 \leq 64\pi \left\{ \|h\|_1^2 + \frac{4}{\pi^2} \|h\|^2 \|h\|_{\frac{1}{2}}^2 \right\}, \quad (4.9)$$

$$\|V\|^2 + N_0^2 \leq 64(\pi^2\|\mu\|_1^2 + 4\|\mu\|^2\|\mu\|_{\frac{1}{2}}^2), \quad (4.10)$$

$$\|V\|^2 + N_0^2 \leq \frac{16}{\pi}(\pi^2\|l\|_1^2 + \|l\|^2\|l\|_{\frac{1}{2}}^2)e^{\|l\|_1}. \quad (4.11)$$

*Proof.* In [KK] there are (4.7) and  $a_n^+ \leq \pi n + \sum_{m>0}^n |l_m|$ ,  $n > 0$ . Then we have (4.8) and by (4.6-8) we get

$$\|V\|^2 + N_0^2 \leq \frac{16}{\pi}2 \sum_{n>0} 2h_n^2((\pi n)^2 + 4n\|h\|^2),$$

and we have (4.9). Moreover by (4.6-8) we obtain (4.10):

$$\|V\|^2 + N_0^2 \leq \frac{16}{\pi}2\pi \sum_{n>0} 2\mu_n^2((\pi n)^2 + 4n\|\mu\|^2).$$

By (4.7), (4.8) and (4.14) we have

$$\|V\|^2 + N_0^2 \leq \frac{16}{\pi}\sqrt{\omega} \sum_{n>0} l_n^2((\pi n)^2 + n\|l\|^2),$$

and by (4.12) we obtain (4.11) .□

Estimating  $\|V\|$  in the terms of  $\|l\|_1$ , see (4.11), we used the following results.

**Lemma 4.6.** *Suppose that  $V \in L^1(0, 1)$ . Then*

$$h_+ \leq \sum_{n>0} |l_n| \leq \frac{\pi}{\sqrt{6}}\|l\|_1, \quad (4.12)$$

$$\cosh h_n \leq 1 + \frac{|l_n|}{2}\omega \min\left(\frac{|l_n|}{4}, 1\right), \quad (4.13)$$

$$h_n^2 \leq |l_n|\omega \min\left(\frac{|l_n|}{4}, 1\right), \quad (4.14)$$

$$\mu_n^+ - \mu_n^- \leq |l_n|\omega. \quad (4.15)$$

*Proof.* We need the following fact from [Fr],[L]. Let  $\mathcal{S}$  be a closed subset of a real axis such that for some values  $L < \infty$  and  $\delta > 0$  the Lebesgue measure of the intersection of  $\mathcal{S}$  and any interval of length  $L$  is not less than  $\delta$ . Then there exists the unique function  $v(z)$  which is harmonic in the domain  $\mathbf{C} \setminus \mathcal{S}$  and has the following properties:

- i) a.e. on  $\mathcal{S}$  the function  $v(z)$  has zero limit values,
- ii) for every  $z \in \mathbf{C}$ ,  $0 \leq v(z) - |\operatorname{Im}z| \leq \frac{2L}{\pi} \log \cot \frac{\delta\pi}{4L}$ .

Let  $v(z) = \operatorname{Im}k(z)$ ,  $z \in Z$ . We take  $L = al$ ,  $\delta = (a-1)l$ ,  $l = 2\sum_{n>0} |l_n|$ . Then we obtain

$$h_+ \leq \lim_{a \rightarrow \infty} \frac{2l}{\pi}a \log \cot \frac{\pi}{4}\left(1 - \frac{1}{a}\right) = \frac{l}{2}.$$

and hence

$$h_+ \leq \sum_{n>0} |l_n| \leq \left(\sum_{n>0} \frac{1}{n^2}\right)^{\frac{1}{2}}\|l\|_1,$$

and by  $\sum_{n>0} \frac{1}{n^2} = \frac{\pi^2}{6}$  we get (4.12).

Recall that  $F_0(z) = F(N_0 + z^2, V)$ . The function  $F_0$  is real entire and  $F_0(z) = \cos k(z)$ . We have  $k(z) = z + O(1/z)$ , as  $|z| \rightarrow \infty$  (see [KK]). The function  $v_1(z) = v(z) - y, z = x + iy$ , is positive harmonic in  $\mathbf{C}_+$  and has the maximum on the real line

$$0 \leq \sup_{z \in \mathbf{C}_+} (v(z) - y) = \sup_{x \in \mathbf{R}} v(x) = \sup_{n>0} |h_n| = h_+.$$

Hence

$$|F_0(z)| = |\cos k(z)| \leq \cosh v(z) \leq \cosh(y + h_+), z \in \mathbf{C}_+,$$

and  $F_0$  is of exponential type 1 and

$$\sup_{z \in \mathbf{R}} |F_0(z)| = \sup_{z \in \mathbf{R}} |\cos k(z)| = \sup_{z \in \mathbf{R}} \cosh v(z) = \sup_{n>0} \cosh(h_n) = \omega < \infty.$$

Let for simplicity  $n$  be even and  $2|a_n^- - \sqrt{\lambda_n}| \leq |l_n|$ , the proof for  $a_n^+$  is the same. Then  $F_0(a_n^-) = F_0(\sqrt{\lambda_n}) + F_0''(t_n)(a_n^- - \sqrt{\lambda_n})^2/2$ , for some  $t_n \in (a_n^-, \sqrt{\lambda_n})$ . Hence by Bernstein inequality we get

$$F_0(\sqrt{\lambda_n}) = \cosh h_n \leq 1 + \sup_{z \in \mathbf{R}} |F_0''(z)| l_n^2/8 \leq 1 + (\cosh h_+) l_n^2/8,$$

and by analogy

$$F(\lambda_n) = F_0(\sqrt{\lambda_n}) = \cosh h_n = 1 + F_0'(z_n)(a_n^- - \sqrt{\lambda_n}) \leq 1 + (\cosh h_+) |l_n|/2,$$

for some  $z_n \in g_n$ . Then by (4.13) we obtain (4.14).

By  $\mu_n^\pm = (-1)^{n+1} F_0(a_n^\pm)_z$  and again by Bernstein's inequality we get

$$\mu_n^+ - \mu_n^- = (-1)^{n+1} (F_0'(a_n^+) - F_0'(a_n^-)) \leq |l_n| \sup_{z \in g_n} |F_0''(z)| \leq |l_n| \omega. \square$$

Now we consider the estimates for the small effective masses.

**Lemma 4.7.** *There is an estimate*

$$\|\mu\|_1^2 \leq 8\pi^2 \{ \|M\|_2^2 + \frac{2}{3} \|\mu\|_1^2 \|M\|_1^2 \}. \quad (4.16)$$

Let  $4\pi \|M\|_1 \leq \sqrt{3}\varepsilon$  for some  $0 < \varepsilon < 1$ . Then

$$\|V\|^2 + N_0^2 \leq \frac{8^3 \pi^4 \|M\|_2^2}{1 - \varepsilon^2} \left\{ 1 + \frac{32 \|M\|_2^2}{1 - \varepsilon^2} \right\}. \quad (4.17)$$

*Proof.* In [KK] there is an estimate  $a_n^+ \leq \pi n + \sum_{m>0}^n |l_m|, n > 0$ . Then and by (4.6) we get

$$(a_n^+)^2 \leq 2((\pi n)^2 + 4(\sum_{n>0}^n |\mu_n|))^2 \leq 2\pi^2(n^2 + \frac{2}{3} \|\mu\|_1^2),$$

since  $\sum_{n>0} \frac{1}{n^2} = \frac{\pi^2}{6}$ . Hence by  $\mu_n = 2a_n M_n$  we obtain the first estimate

$$\|\mu\|_1^2 = 4 \sum_{n>0} n^2 a_n^2 M_n^2 \leq 8 \sum_{n>0} \pi^2 n^2 (n^2 + \frac{2}{3} \|\mu\|_1^2) M_n^2 = 8\pi^2 \{ \|M\|_2^2 + \frac{2}{3} \|\mu\|_1^2 \|M\|_1^2 \}.$$

Suppose we have  $4\pi \|M\|_1 \leq \sqrt{3}\varepsilon$ , then  $(1 - \varepsilon^2) \|\mu\|_1^2 \leq 8\pi^2 \|M\|_2^2$ , and by (4.10) we obtain

$$\|V\|^2 + N_0^2 \leq 64 \|\mu\|_1^2 (\pi^2 + 4 \|\mu\|_1^2) \leq \frac{8^3 \pi^2 \|M\|_2^2}{1 - \varepsilon^2} (\pi^2 + \frac{32\pi^2 \|M\|_2^2}{1 - \varepsilon^2}),$$

and then we get (4.17).  $\square$

## 5 . The Mappings

In this section we prove the main theorems. First we present "the basic estimate" from nonlinear functional analyses.

**Theorem 5.1.** *Let  $H, H_1$  be real separable Hilbert spaces with the norms  $\|\cdot\|, \|\cdot\|_1$ . Suppose that a map  $f : H \rightarrow H_1$  obeys the following conditions.*

- i) the map  $f$  is real analytic or from the class  $C^p, p \geq 1$ ,*
- ii) the operator  $d_V f$  has inverse for any  $V \in H$ ,*
- iii) there is a nondecreasing function  $g : [0, \infty) \rightarrow [0, \infty), g(0) = 0$ , such that  $\|V\| \leq g(\|f(V)\|_1)$  for any  $V \in H$ .*
- iv) there exists a basis  $\{e_n\}_1^\infty$  for  $H_1$  such that for any  $n > 0$ , the map  $(f(\cdot), e_n)_1 : H \rightarrow \mathbf{R}$  is compact.*

*Then the map  $f$  is a real analytic isomorphism between  $H, H_1$ .*

*Proof.* By i), ii) and by the Inverse Function Theorem the set  $f(H)$  is open. Let us prove that it is closed also.

Let  $h_n = f(V_n) \rightarrow h$  strongly as  $n \rightarrow \infty$ . Then by iii) we have  $\|V_n\| \leq g(\|h_n\|_1) \leq g(\sup_{n>0} \|h_n\|_1)$ . Hence there exists a subsequence  $\{V_{n_m}\}_{m=1}^\infty$  such that  $V_{n_m} \rightarrow V$  weakly as  $m \rightarrow \infty$ . Then by iv) we get  $(f(V_{n_m}), e_k)_1 \rightarrow (f(V), e_k)_1$  and  $(f(V), e_k)_1 = (h, e_k)_1$  at fixed  $k > 0$ . But  $\{e_n\}_1^\infty$  is the basis for  $H_1$ , hence  $h = f(V)$  and  $f(H) = H_1$  since  $H_1$  is connected.

Let us show that  $f$  is an injection. We define sets

$$K_m = \{h : (h, e_n)_1 = 0 \text{ as } n > m\} \subset H_1$$

and  $M_m = f^{-1}(K_m) \subset H$ . The map  $f$  is a smooth local isomorphism on  $H$ , then  $M_m$  is a real smooth submanifold of  $H$  with dimension  $m$ . Hence the topology on  $M_m$  generated by the norm  $\|\cdot\|$  is equal to the topology on  $M_m$  induced by the weak topology of  $H$ , i.e. from weak convergent we get strong convergent.

Denote by  $E_m$  the set of points in  $K_m$  having more than one preimage. Then  $E_m$  is open because  $f$  is a local isomorphism. But  $E_m$  is also closed. Indeed, let there exist

distinct points  $V_j$  and  $W_j$  in  $M_m$  such that  $f(V_j) = f(W_j) \rightarrow h$  as  $j \rightarrow \infty$ . Then by iii) there are two subsequences such that  $V_{j_q} \rightarrow V$ ,  $W_{j_q} \rightarrow W$ , as  $q \rightarrow \infty$ , and by iv) we have  $V, W \in E_m$ . If  $V = W$  then  $V_j = W_j$  for large  $j$ , because the map  $f$  is local homeomorphism. Hence  $V \neq W$  and  $E_m$  is closed. But  $0 \notin E_m$  and then  $E_m = \emptyset$ . Hence  $f : M_m \rightarrow K_m$  is isomorphism.

Suppose that  $f : H \rightarrow H_1$  is not an injection. Then some point  $h \in H_1$  has at least two preimages. Since  $f$  is a local homeomorphism the same is truth in some neighborhood of  $h$  and then for some  $h^0 \in K_m$  such that  $(h^0, e_k)_1 = (h, e_k)_1$ , if  $1 \leq k \leq m$  and  $(h^0, e_k)_1 = 0$  if  $k > m$ . Here  $m$  is sufficiently large. But this contradicts that  $f : M_m \rightarrow K_m$  is isomorphism.  $\square$

Now we check all conditions of this theorem for our maps. First we prove very simple result on analytical mappings.

**Lemma 5.2.** *Let  $f = \{f_n\}_1^\infty$  and a map  $f_n : H \rightarrow \mathbf{R}$  be compact real analytic. Suppose that there are asymptotics*

$$f_n = \hat{V}_n + O\left(\frac{1}{n}\right), \quad (5.1)$$

$$(\partial f_n)(x) = \cos 2\pi n x + O\left(\frac{1}{n}\right), \quad (5.2)$$

*uniformly on bounded subsets of  $[0, 1] \times H_c$ . Then the map  $f : H \rightarrow \ell^2$  is real analytic and  $d_V f - \Phi$  is compact operator for any  $V \in H$ .*

*Proof.* By (5.1) we get that the map  $f(V)$  is locally bounded and by the uniform boundness principle it is real analytic.

By (5.2) we have that  $d_V f$  is the sum of the operator  $\Phi$  and a compact operator.  $\square$

Now we shall prove main theorems.

**Proof of Theorem 2.1** We check all conditions of Theorem 5.1.

i) By Lemma 5.2 and Theorem 3.7 we get that the function  $h(V)$  is real analytic.

ii) By Lemma 5.2 and Theorem 3.7 we obtain that  $d_V h + (2\pi\sqrt{2})^{-1} I_1 \Phi$  is compact operator and hence  $d_V h$  is the Fredholm operator. Let  $g \in H$  be the solution of the equation

$$(d_V h)g = 0, \quad \text{or} \quad \{(\partial h_n, g) = 0, n > 0\}.$$

Introduce the function  $f(E) = \int_0^1 U(x, E, V)g(x)dx$ ,  $E \in \mathbf{C}$ . If  $L_n \neq 0$ , then  $h_n \neq 0$  and by (3.34) we obtain that  $f(\lambda_n) = 2(-1)^n (\sinh h_n)(\partial h_n, g) = 0$ . If  $L_n = 0$ , then by (3.6) we get that  $f(\lambda_n) = 0$ , and by (3.36) we obtain  $(\varphi_n - \vartheta_n, g) = 0$ . Hence  $f(\lambda_n) = 0$ , for any  $n > 0$  and by (3.15) we have  $\lambda_n = (\pi n)^2 + o(1)$  as  $n \rightarrow \infty$ . Then by Lemma 4.2 we get  $g = 0$ .

iii) The estimates (2.1), (2.2) are proved in Theorems 4.4 and 4.5.

iv) In Theorem 3.7 it was shown that the map  $h_n : H \rightarrow \mathbf{R}$  is compact for  $n > 0$ .  $\square$  We present the next proof on the small gaps.

**Proof of Theorem 2.2** We check all conditions of Theorem 5.1.

i) By Lemma 5.2 and Theorem 3.4 we get that the function  $l(V)$  is real analytic.

ii) By Lemma 5.2 and Theorem 3.4 we obtain that  $d_V l + (\pi\sqrt{2})^{-1}I_1\Phi$  is compact operator and hence  $d_V l$  is the Fredholm operator.

Let  $g \in H$  be the solution of the equation

$$(d_V l)g = 0, \quad \text{or,} \quad \{(\partial l_n, g) = 0, n > 0\}.$$

Introduce the functions  $f_0(e) = f(E) + 2\dot{F}(E)(\vartheta_0, g)$ ,  $f(E) = \int_0^1 U(x, E, V)g(x)dx$ ,  $E \in \mathbf{C}$ .

We have two cases. First let  $l_n \neq 0$ , then by (3.17), (3.5) we obtain that

$$0 = -\frac{f_0(D_n)}{4a_n \dot{F}(D_n)} + \frac{f_0(N_n)}{4b_n \dot{F}(N_n)},$$

but  $\dot{F}(D_n)\dot{F}(N_n) < 0$ . Hence  $f_0(D_n)f_0(N_n) \leq 0$  and there exists a point  $A_n \in [A_n^-, A_n^+]$  such that  $f(A_n) = 0$ .

Second, let  $l_n = 0$ , then by (3.6) and  $\dot{F}(D_n) = 0$  we get that  $f_0(A_n) = 0$ ,  $A_n = D_n$ , and by (3.19) we obtain  $(\varphi_n - \vartheta_n, g) = 0$ .

Hence we have the sequence  $A_n = (\pi n)^2 + o(1)$  as  $n \rightarrow \infty$ . Then by Lemma 4.2 we get  $g = 0$ .

iii) The estimates (2.3), (2.4) are proved in Theorems 4.4 and 4.5.

iv) In Theorem 3.4 it was shown that the map  $l_n : H \rightarrow \mathbf{R}$  is compact for  $n > 0$ .  $\square$

We gave the proofs of two theorems on  $h, l$ . We shown that all conditions of Theorem 5.1 are fulfilled. Unfortunately, we can not do it for  $\mu, M$ .

**Proof of Theorem 2.3** We check all conditions of Theorem 5.1.

i) By Lemma 5.2 and Theorem 3.6 we get that the function  $\mu(V)$  is real analytic.

ii) By Lemma 5.2 and Theorem 3.6 we obtain that  $d_V \mu + (2\pi\sqrt{2})^{-1}I_1\Phi$  is compact operator and hence  $d_V \mu$  is the Fredholm operator.

iii) The estimates (2.5), (2.6) are proved in Theorems 4.4 and 4.5.

iv) In Theorem 3.6 it was shown that the map  $\mu_n : H \rightarrow \mathbf{R}$  is compact for  $n > 0$ .

Suppose that  $\ker d_V \mu = \{0\}$  for any  $V \in H$  then the operator  $d_V \mu$  has inverse for any  $V \in H$ . Hence all conditions of Theorem 5.1 are fulfilled and  $\mu$  is a real analytic isomorphism.  $\square$

Now we consider the effective masses.

**Proof of Theorem 2.4** We check all conditions of Theorem 5.1.

i) By Lemma 5.2 and Theorem 3.5 we get that the function  $M(V)$  is real analytic.

ii) By Lemma 5.2 and Theorem 3.5 we obtain that  $d_V M + (4\pi^2\sqrt{2})^{-1}I_2\Phi$  is compact operator and hence  $d_V M$  is the Fredholm operator.

iii) The estimate (2.7) is proved in Theorems 4.4.

iv) In Theorem 3.5 it was shown that the map  $M_n : H \rightarrow \mathbf{R}$  is compact for  $n > 0$ .

Now we consider the case of the small effective masses. The estimate (2.8) is proved in Lemma 4.7. The derivative  $d_V M$  at the point  $V = 0$  has the inverse operator. Then by the inverse function theorem there exists local isomorphism  $M : W \rightarrow B(0, \varepsilon)$ , for some  $\varepsilon > 0$  and some neighborhood of zero.  $\square$

Now we reprove the result of Garnett-Trubowitz on the gaps.

**Theorem 5.3.** *The map  $L : H \rightarrow \ell^2$  is real analytic and for any  $n > 0$ , the map  $L_n : H \rightarrow \mathbf{R}$  is compact. Moreover for any  $V \in H$  the operator  $d_{L\mu} + \sqrt{2}\Phi$  is compact and there is an estimate*

$$\|L\|_0 \leq 2\|V\|(1 + \|V\|). \quad (5.3)$$

*Suppose that there exists a nondecreasing function  $g : [0, \infty) \rightarrow [0, \infty)$ ,  $g(0) = 0$ , such that  $\|V\| \leq g(\|L(V)\|_0)$ . Then the map  $L : H \rightarrow \ell^2$  is a real analytic isomorphism.*

*Proof.* We check all conditions of Theorem 5.1.

i) By Lemma 5.2 and Theorem 3.2 we get that the function  $L(V)$  is real analytic.

ii) By Lemma 5.2 and Theorem 3.2 we obtain that  $d_V L + \sqrt{2}\Phi$  is compact operator and hence  $d_V L$  is the Fredholm operator.

Let  $g \in H$  be the solution of the equation

$$(d_V L)g = 0, \quad \text{or,} \quad \{(\partial L_n, g) = 0, n > 0\}.$$

Introduce the function  $f(E) = \int_0^1 U(x, E, V)g(x)dx$ ,  $E \in \mathbf{C}$ . If  $L_n = 0$ , then by (3.6) we get that  $f(A_n) = 0$ , for  $A_n = \lambda_n$ . By (3.11) we obtain  $(\varphi_n - \vartheta_n, g) = 0$ .

Second, let  $L_n \neq 0$ , then by (3.5) we have that

$$0 = (\partial L_n, g) = -\frac{f(D_n)}{2\dot{F}(D_n)} + \frac{f(N_n)}{2\dot{F}(N_n)},$$

but  $\dot{F}(D_n)\dot{F}(N_n) < 0$ . Hence  $f(D_n)f(N_n) \leq 0$  and there exists a point  $A_n \in [A_n^-, A_n^+]$  such that  $f(A_n) = 0$ . Hence we obtain the sequence  $A_n = (\pi n)^2 + o(1)$  as  $n \rightarrow \infty$ . Then by Lemma 4.2 we get  $g = 0$  and we have proved the existence of the inverse operator.

iii) By (4.3) we get (5.3).

iv) By Theorem 3.2 we obtain that the map  $L_n = D_n - N_n$  is compact for  $n > 0$ .  $\square$

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