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A Hopf Bundle over a Quantum Four–Sphere from the Symplectic Group

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Abstract

We construct a quantum version of the $SU(2)$ Hopf bundle $S^7 \to S^4$. The quantum sphere S^7_q arises from the symplectic group $Sp_q(2)$ and a quantum 4-sphere S^4_q is obtained via a suitable self-adjoint idempotent p whose entries generate the algebra $A(S^4_q)$ of polynomial functions over it. This projection determines a deformation of an (anti-)instanton bundle over the classical sphere S^4. We compute the fundamental K-homology class of S^4_q and pair it with the class of p in the K-theory getting the value -1 for the topological charge. There is a right coaction of $SU_q(2)$ on S^7_q such that the algebra of coinvariants is the algebra $A(S^4_q)$. The algebra $A(S^7_q)$ turns out to be a $A(SU_q(2))$ faithfully flat Hopf-Galois extension over $A(S^4_q)$, a notion which extends to noncommutative geometry the one of a principal bundle in differential geometry; it is also not cleft, i.e. not trivial.

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1 Introduction

In this paper we study yet another example of how "quantization removes degeneracy" by constructing a new quantum version of the Hopf bundle $S^7 \to S^4$. This is the first outcome of our attempt to generalize to the quantum case the ADHM construction of $SU(2)$ instantons together with their moduli spaces. In analogy with the classical case [1], it is natural to start with the quantum version of the (compact) symplectic groups $A(Sp_q(n))$, i.e. the Hopf algebras generated by matrix elements T^j_i's with commutation rules (given in equation (4)) coming from the R matrix of the C-series [25]. These quantum groups have comodule-subalgebras $A(S_q^{4n-1})$ yielding deformations of the algebras of polynomials over the spheres S^{4n-1}, which apparently have been unnoticed before.

The relevant case for us is $n = 2$, i.e. the symplectic quantum 7-sphere $A(S^7_q)$, which is generated by the matrix elements of the first and the last column of T. Indeed, as we will see, $\overline{T}^4_i \propto T^1_{4-i}$; a similar conjugation occurs for the elements of the middle columns, but contrary to what happens at $q = 1$, they do not generate a subalgebra. The algebra $A(S^7_q)$ is the quantum version of the homogeneous space $Sp(2)/Sp(1)$ and the injection $A(S^7_q) \hookrightarrow A(Sp_q(2))$ is a faithfully flat Hopf-Galois extension with "structure Hopf algebra" $A(Sp_q(1))$ (see Sect. 6).

Most importantly, we show that S^7_q is the total space of a quantum $SU_q(2)$ principal bundle over a quantum 4-sphere S^4_q. The algebra $A(S^4_q)$ is constructed as the subalgebra of $A(S^7_q)$ generated by the matrix elements of a self-adjoint projection p which generalizes the anti-instanton of charge -1. This projection will be of the form vv^* with v a 4×2 matrix whose entries are made out of generators of $A(S^7_q)$. The naive generalization of the classical case produces a subalgebra with extra generators which vanish at $q = 1$. Luckily enough, there is just one alternative choice of v which gives the right number of generators of an algebra which deforms the algebra of polynomial functions of S^4. At $q = 1$ this gives a projection which is gauge equivalent to the standard one.

This good choice becomes even better because there is a natural coaction of $SU_q(2)$ on $A(S^7_q)$ with coinvariant algebra $A(S^4_q)$. In Sect. 6 we shall show that the injection $A(S^4_q) \hookrightarrow A(S^7_q)$ is a faithfully flat $A(SU_q(2))$-Hopf-Galois extension.

Finally, we set up the stage to compute the charge of our projection. We construct representations of the algebra $A(S^4_q)$ and the corresponding K-homology. The analogue of the fundamental class of S^4 is given by a non trivial Fredholm module μ. The natural coupling between μ and the projection p is computed via the pairing of the corresponding Chern characters $\mathrm{ch}^*(\mu) \in HC^*[A(S^4_q)]$ and $\mathrm{ch}_*(p) \in HC_*[A(S^4_q)]$ in cyclic cohomology and homology respectively [11]. As expected the result of this pairing, which is an integer by principle, is actually -1.

Clearly the example presented in this paper is very special and limited, since it is just a particular anti-instanton of charge -1. Indeed our construction is based on the requirement that the matrix v giving the projection is linear in the generators of $A(S^7_q)$ and such that $v^*v = 1$. This is false even classically at generic moduli and generic charge, except for the case considered here (and for a similar construction for the case of charge 1). A more elaborate strategy is needed to tackle the general case.

2 Odd spheres from quantum symplectic groups

We recall the construction of quantum spheres associated with the compact real form of the quantum symplectic groups $Sp_q(N,\mathbb{C})$ $(N=2n)$, the latter being given in [25]. Later we shall specialize to the case $N=4$ and the corresponding 7-sphere will provide the 'total space' of our quantum Hopf bundle.

2.1 The quantum groups $Sp_q(N,\mathbb{C})$ and $Sp_q(n)$

The algebra $A(Sp_q(N,\mathbb{C}))$ is the associative noncommutative algebra generated over the ring of Laurent polynomials $\mathbb{C}_q := \mathbb{C}[q,q^{-1}]$ by the entries $T_i{}^j$, $i,j=1,\dots,N$ of a matrix T which satisfies RTT equations:

$$R\,T_1T_2 = T_2T_1R\ , \qquad T_1 = T\otimes 1\ , \quad T_2 = 1\otimes T\ .$$

In components $(T\otimes 1)_{ij}{}^{kl} = T_i{}^k\delta_j{}^l$. Here the relevant $N^2\times N^2$ matrix R is the one for the C_N series and has the form [25],

$$\begin{aligned} R \ = \ & q\sum_{i=1}^{N} e_i{}^i\otimes e_i{}^i + \sum_{\substack{i,j=1\\ i\neq j,j'}}^{N} e_i{}^i\otimes e_j{}^j + q^{-1}\sum_{i=1}^{N} e_{i'}{}^{i'}\otimes e_i{}^i \\ & + (q-q^{-1})\sum_{\substack{i,j=1\\ i>j}}^{N} e_i{}^j\otimes e_j{}^i - (q-q^{-1})\sum_{\substack{i,j=1\\ i>j}}^{N} q^{\rho_i-\rho_j}\varepsilon_i\varepsilon_j\ e_i{}^j\otimes e_{i'}{}^{j'} \quad , \end{aligned} \tag{1}$$

where

$i' = N+1-i$;

$e_i{}^j\in M_n(\mathbb{C})$ are the elementary matrices, i.e. $(e_j{}^i)^k_l = \delta_{jl}\delta^{ik}$;

$\varepsilon_i = 1$, for $i=1,\dots,n$;

$\varepsilon_i = -1$, for $i=n+1,\dots,N$;

$(\rho_1,\dots,\rho_N) = (n,n-1,\dots,1,-1,\dots,-n)$.

The symplectic group structure comes from the matrix $C_i{}^j = q^{\rho_j}\varepsilon_i\delta_{ij'}$ by imposing the additional relations

$$TCT^tC^{-1} = CT^tC^{-1}T = 1 \quad .$$

The Hopf algebra co-structures (Δ,ε,S) of the quantum group $Sp_q(N,\mathbb{C})$ are given by

$$\Delta(T) = T\dot{\otimes}T\ , \quad \varepsilon(T) = I\ , \quad S(T) = CT^tC^{-1}\ .$$

In components the antipode explicitly reads

$$S(T)_i{}^j = -q^{\rho_{i'}+\rho_j}\varepsilon_i\varepsilon_{j'}T_{j'}{}^{i'}\ . \tag{2}$$

At $q=1$ the Hopf algebra $Sp_q(N,\mathbb{C})$ reduces to the algebra of polynomial functions over the symplectic group $Sp(N,\mathbb{C})$.

The compact real form $A(Sp_q(n))$ of the quantum group $A(Sp_q(N,\mathbb{C}))$ is given by taking $q\in\mathbb{R}$ and the anti-involution [25]

$$\overline{T} = S(T)^t = C^tT(C^{-1})^t\ . \tag{3}$$

2.2 The odd symplectic spheres

Let us denote

$$x_i = T_i{}^N \ , \quad v^j = S(T)_N{}^j \ , \quad i,j = 1,\ldots,N \ .$$

As we will show, these generators give subalgebras of $A(Sp_q(N,\mathbb{C}))$. With the natural involution (3), the algebra generated by the $\{x_i,\ v^j\}$ can be thought of as the algebra $A(S_q^{4n-1})$ of polynomial functions on a quantum sphere of 'dimension' $4n-1$.

From here on, whenever no confusion arises, the sum over repeated indexes is understood. In components the RTT equations are given by

$$R_{ij}{}^{kp}\ T_k{}^r\ T_p{}^s = T_j{}^p\ T_i{}^m\ R_{mp}{}^{rs} \ . \tag{4}$$

Hence

$$R_{ij}{}^{kl}\ T_k{}^r = T_j{}^p\ T_i{}^m\ R_{mp}{}^{rs} S(T)_s{}^l \ ,$$

and in turn

$$S(T)_p{}^j\ R_{ij}{}^{kl} = T_i{}^a\ R_{ap}{}^{rs}\ S(T)_s{}^l\ S(T)_r{}^k \ ,$$

so that

$$S(T)_a{}^i\ S(T)_p{}^j\ R_{ij}{}^{kl} = R_{ap}{}^{rs}\ S(T)_s{}^l\ S(T)_r{}^k \ . \tag{5}$$

Conversely, if we multiply $R_{ij}{}^{kp}\ T_k{}^r = T_j{}^l\ T_i{}^m\ R_{ml}{}^{rs}\ S(T)_s{}^p$ on the left by $S(T)$ we have

$$S(T)_l{}^j\ R_{ij}{}^{kp}\ T_k{}^r = T_i{}^m\ R_{ml}{}^{rs}\ S(T)_s{}^p \ . \tag{6}$$

We shall use equations (4), (5) and (6) to describe the algebra generated by the x_i's and by the v^i's.

The algebra $\mathbb{C}_q[x_i]$

From (4) with $r = s = N$ we have

$$R_{ij}{}^{kp}\ x_k x_p = T_j{}^p\ T_i{}^m\ R_{mp}{}^{NN} \ . \tag{7}$$

Since the only element $R_{mp}{}^{NN}\ \propto\ e_m{}^N \otimes e_p{}^N$ $(m,p \leq N)$ which is different from zero is $R_{NN}{}^{NN} = q$, it follows that

$$R_{ij}{}^{kp}\ x_k x_p = q\ x_j x_i \ , \tag{8}$$

and the elements x_i's give an algebra with commutation relations

$$\begin{aligned} x_i x_j &= q x_j x_i \ , \quad i<j, \quad i \neq j' \ , \\ x_{i'} x_i &= q^{-2} x_i x_{i'} + (q^{-2}-1) \sum_{k=1}^{i-1} q^{\rho_i - \rho_k} \varepsilon_i \varepsilon_k\ x_k x_{k'} \ , \quad i < i' \ . \end{aligned} \tag{9}$$

The algebra $\mathbb{C}_q[v^i]$

Putting $a = p = N$ in equation (5), we get

$$v^i v^j R_{ij}{}^{kl} = R_{NN}{}^{rs} S(T)_s{}^l S(T)_r{}^k \ .$$

The sum on the r.h.s. reduces to ${R_{NN}}^{NN} {S(T)_N}^l {S(T)_N}^k$ and the v^i's give an algebra with commutation relations

$$v^l v^k {R_{lk}}^{ji} = q v^i v^j . \tag{10}$$

Explicitly

$$\begin{aligned} v^i v^j &= q^{-1} v^j v^i \ , \quad i < j, \quad i \neq j' \ , \\ v^{i'} v^i &= q^2 v^i v^{i'} + (q^2 - 1) \sum_{k=i'+1}^{N} q^{\rho_k - \rho_{i'}} \ \varepsilon_k \varepsilon_{i'} v^k v^{k'} \ , \quad i < i' \ . \end{aligned} \tag{11}$$

The algebra $\mathbb{C}_q[x_i, v^j]$

Finally, for $l = r = N$ the equation (6) reads:

$$v^j {R_{ij}}^{kp} \ x_k = T_i^m {R_{mN}}^{Ns} {S(T)_s}^p .$$

Once more, the only term in R of the form ${e_m}^N \otimes {e_N}^s$ $(m \leq N)$ is ${e_N}^N \otimes {e_N}^N$ and therefore

$$v^j {R_{ij}}^{kp} \ x_k = q \ x_i v^p \ . \tag{12}$$

Explicitly the mixed commutation rules for the algebra $\mathbb{C}_q[x_i, v^j]$ read,

$$\begin{aligned} x_i v^i &= v^i x_i + (1 - q^{-2}) \sum_{k=1}^{i-1} v^k x_k + \underbrace{(1 - q^{-2}) q^{\rho_i - \rho_{i'}} v^{i'} x_{i'}}_{if \quad i > i'} \ , \\ x_i v^{i'} &= q^{-2} v^{i'} x_i \ , \\ x_i v^j &= q^{-1} v^j x^i \ , \quad i \neq j \quad \text{and} \quad i < j' \\ x_i v^j &= q^{-1} v^j x^i + (q^{-2} - 1) q^{\rho_i - \rho_{j'}} \ \varepsilon_i \varepsilon_{j'} v^{i'} x_{j'} \ , \quad i \neq j \quad \text{and} \quad i > j' \ . \end{aligned} \tag{13}$$

The quantum spheres S_q^{4n-1}

Let us observe that with the anti-involution (3) we have the identification $v^i = {S(T)_N}^i = \bar{x}^i$. The subalgebra $A(S_q^{4n-1})$ of $A(Sp_q(n))$ generated by $\{x_i, v^i = \bar{x}^i, \ i = 1, \ldots, 2n\}$ is the algebra of polynomial functions on a sphere. Indeed

$$S(T)T = I \ \Rightarrow \ \sum {S(T)_N}^i T_i^N = \delta_N^N = 1$$

i.e.

$$\sum_i \bar{x}^i x_i = 1 \ . \tag{14}$$

Furthermore, the restriction of the comultiplication is a natural left coaction

$$\Delta_L : A(S_q^{4n-1}) \longrightarrow A(Sp_q(n)) \otimes A(S_q^{4n-1}) \ .$$

The fact that Δ_L is an algebra map then implies that $A(S_q^{4n-1})$ is a comodule algebra over $A(Sp_q(n))$.

At $q = 1$ this algebra reduces to the algebra of polynomial functions over the spheres S^{4n-1} as homogeneous spaces of the symplectic group $Sp(n)$: $S^{4n-1} = Sp(n)/Sp(n-1)$.

3 The Hopf fibration $A(S^4_q) \hookrightarrow A(S^7_q)$

We now specialize the results of the previous section to the case $N = 4$. A central point will be the construction of a right coaction of $A(SU_q(2))$ over $A(S^7_q)$ such that the subalgebra of coinvariants is identified with the algebra of a 4-sphere $A(S^4_q)$. In Sect. 6, we shall also prove that $A(S^4_q) \subset A(S^7_q)$ is a faithfully flat Hopf-Galois extension, i.e. a 'good' quantum principal bundle.

3.1 The symplectic 7-sphere S^7_q

The algebra $A(S^7_q)$ is generated by the elements $x_i = T_i{}^4$ and $\bar{x}^i = S(T)_4{}^i = q^{2+\rho_i}\varepsilon_{i'}T_{i'}{}^1$, for $i = 1, \ldots, 4$. From $S(T)\,T = 1$ we have the sphere relation $\sum_{i=1}^4 \bar{x}^i x_i = 1$. Since we shall systematically use them in the following, we shall explicitly give the commutation relations among the generators.
From (9), the algebra of the x_i's is given by

$$\begin{array}{ll} x_1x_2 = qx_2x_1\ , & x_1x_3 = qx_3x_1\ , \\ x_2x_4 = qx_4x_2\ , & x_3x_4 = qx_4x_3\ , \\ x_4x_1 = q^{-2}x_1x_4\ , & x_3x_2 = q^{-2}x_2x_3 + q^{-2}(q^{-1}-q)x_1x_4\ , \end{array} \tag{15}$$

together with their conjugates (given in (11)).
We have also the commutation relations between the x_i and the $\bar{x}^j$ deduced from (12):

$$\begin{array}{l} x_1\bar{x}^1 = \bar{x}^1x_1\ , \qquad x_1\bar{x}^2 = q^{-1}\bar{x}^2x_1\ , \\ x_1\bar{x}^3 = q^{-1}\bar{x}^3x_1\ , \quad x_1\bar{x}^4 = q^{-2}\bar{x}^4x_1\ , \\ \\ x_2\bar{x}^2 = \bar{x}^2x_2 + (1-q^{-2})\bar{x}^1x_1\ , \\ x_2\bar{x}^3 = q^{-2}\bar{x}^3x_2\ , \\ x_2\bar{x}^4 = q^{-1}\bar{x}^4x_2 + q^{-1}(q^{-2}-1)\bar{x}^3x_1\ , \\ \\ x_3\bar{x}^3 = \bar{x}^3x_3 + (1-q^{-2})[\bar{x}^1x_1 + (1+q^{-2})\bar{x}^2x_2]\ , \\ x_3\bar{x}^4 = q^{-1}\bar{x}^4x_3 + (1-q^{-2})q^{-3}\bar{x}^2x_1\ , \\ \\ x_4\bar{x}^4 = \bar{x}^4x_4 + (1-q^{-2})[(1+q^{-4})\bar{x}^1x_1 + \bar{x}^2x_2 + \bar{x}^3x_3]\ , \end{array} \tag{16}$$

again with their conjugates.

Next we show that the algebra $A(S^7_q)$ can be realized as the subalgebra of $A(Sp_q(2))$ generated by the coinvariants under the right-coaction of $A(Sp_q(1))$, in complete analogy with the classical homogeneous space $Sp(2)/Sp(1) \simeq S^7$.

Lemma 1. *The two-sided *-ideal in $A(Sp_q(2))$ generated as*

$$I_q = \{T_1{}^1 - 1, T_4{}^4 - 1, T_1{}^2, T_1{}^3, T_1{}^4, T_2{}^1, T_2{}^4, T_3{}^1, T_3{}^4, T_4{}^1, T_4{}^2, T_4{}^3\}$$

with the involution (3) *is a Hopf ideal.*

Proof. Since $S(T)_i{}^j \propto T_{j'}{}^{i'}$, $S(I_q) \subseteq I_q$ which also proves that I_q is a *-ideal. One easily shows that $\varepsilon(I_q) = 0$ and $\Delta(I_q) \subseteq I_q \otimes A(Sp_q(2)) + A(Sp_q(2)) \otimes I_q$. □

Proposition 1. *The Hopf algebra* $B_q := A(Sp_q(2))/I_q$ *is isomorphic to the coordinate algebra* $A(SU_{q^2}(2)) \cong A(Sp_q(1))$.

Proof. Using $\overline{T} = S(T)^t$ and setting ${T_2}^2 = \alpha$, ${T_3}^2 = \gamma$, the algebra B_q can be described as the algebra generated by the entries of the matrix

$$T' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & -q^2\bar{\gamma} & 0 \\ 0 & \gamma & \bar{\alpha} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{17}$$

The commutation relations deduced from RTT equations (4) read:

$$\begin{array}{lll} \alpha\bar{\gamma} = q^2\bar{\gamma}\alpha \quad , & \alpha\gamma = q^2\gamma\alpha \ , & \gamma\bar{\gamma} = \bar{\gamma}\gamma \ , \\ \bar{\alpha}\alpha + \bar{\gamma}\gamma = 1 \quad ; & \alpha\bar{\alpha} + q^4\gamma\bar{\gamma} = 1 \ . & \end{array} \tag{18}$$

Hence, as an algebra B_q is isomorphic to the algebra $A(SU_{q^2}(2))$. Furthermore, the restriction of the coproduct of $A(Sp_q(2))$ to B_q endows the latter with a coalgebra structure, $\Delta(T') = T' \dot{\otimes} T'$, which is the same as the one of $A(SU_{q^2}(2))$. We can conclude that also as a Hopf algebra, B_q is isomorphic to the Hopf algebra $A(SU_{q^2}(2)) \cong A(Sp_q(1))$. ☐

The restriction of the coproduct of $A(Sp_q(2))$ to $A(Sp_q(1))$ does also yield a right coaction $\Delta_R : A(Sp_q(2)) \to A(Sp_q(2)) \dot{\otimes} A(Sp_q(1))$ by $\Delta_R(T) = T \dot{\otimes} T'$.
The following proposition holds

Proposition 2. *The algebra* $A(S^7_q) \subset A(Sp_q(2))$ *is the algebra of coinvariants with respect to the coaction* Δ_R.

Proof. It is straightforward to show that the generators of the algebra $A(S^7_q)$ are coinvariants:

$$\Delta_R(x_i) = \Delta_R(T^4_i) = x_i \otimes 1 \quad ; \quad \Delta_R(\bar{x}^i) = -q^{2+\rho_i}\varepsilon_i \, \Delta_R(T^1_i) = \bar{x}^i \otimes 1$$

thus the algebra $A(S^7_q)$ is made of coinvariants. We refer to Appendix B for a sketchy proof that these are the only coinvariants. ☐

In Sect. 6 we shall show that $A(S^7_q) \hookrightarrow A(Sp_q(2))$ is a faithfully flat $A(Sp_q(1))$-Hopf-Galois extension.

The next natural step is to construct a map from S^7_q into a deformation of the Stieffel variety of unitary frames of 2-planes in $\mathbb{C}^4$ to parallel the classical construction as recalled in the Appendix A. The naive choice we have is to take as generators the elements of two (conjugate) columns of the matrix T. We are actually forced to take the first and the last columns of the matrix T because the other choice (i.e. the second and the third columns) does not yield a subalgebra since commutation relations of their elements will involve elements from the other two columns. If we set

$$v = \begin{pmatrix} \bar{x}^4 & x_1 \\ q^{-1}\bar{x}^3 & x_2 \\ -q^{-3}\bar{x}^2 & x_3 \\ -q^{-4}\bar{x}^1 & x_4 \end{pmatrix} , \tag{19}$$

we have $v^*v = \mathbb{I}_2$ and the matrix $p = v\ v^*$ is a self-adjoint idempotent, i.e. $p = p^* = p^2$. At $q = 1$ the entries of p are invariant for the natural action of $SU(2)$ on S^7 and generate the algebra of polynomials on S^4. This fails to be the case at generic q due to the occurrence of extra generators e.g.

$$p_{14} = (1 - q^{-2}) x_1 \bar{x}^4, \quad p_{23} = (1 - q^{-2}) x_2 \bar{x}^3, \tag{20}$$

which vanish at $q = 1$.

3.2 The quantum sphere S_q^4

The facts at the end of the previous section indicate that the quantum analogue of the quaternionic projective line as a homogeneous space of $Sp_q(2)$ has not the right number of generators. Rather surprisingly, we shall anyhow be able to select another subalgebra of $A(S_q^7)$ which is a deformation of the algebra of polynomials on S^4 having the same number of generators. These generators come from a better choice of a projection.

On the free module $\mathcal{E} := \mathbb{C}^4 \otimes A(S_q^7)$ we consider the hermitean structure given by

$$h(|\xi_1\rangle, |\xi_2\rangle) = \sum_{j=1}^4 \bar{\xi}_1{}^j \xi_2^j .$$

To every element $|\xi\rangle \in \mathcal{E}$ one associates an element $\langle\xi|$ in the dual module $\mathcal{E}^*$ by the pairing

$$\langle\xi| \left(|\eta\rangle\right) := \langle\xi|\eta\rangle = h(|\xi\rangle, |\eta\rangle).$$

Guided by a variant of the classical construction which we review in Appendix A, we shall look for two elements $|\phi_1\rangle$, $|\phi_2\rangle$ in $\mathcal{E}$ with the property that

$$\langle\phi_1|\phi_1\rangle = 1 \ , \quad \langle\phi_2|\phi_2\rangle = 1 \ , \quad \langle\phi_1|\phi_2\rangle = 0 \ .$$

As a consequence, the matrix valued function defined by

$$p := |\phi_1\rangle \langle\phi_1| + |\phi_2\rangle \langle\phi_2| \ , \tag{21}$$

is a self-adjoint idempotent (a projection).
In principle, $p \in \mathrm{Mat}_4(A(S_q^7))$, but we can choose $|\phi_1\rangle$, $|\phi_2\rangle$ in such a way that the entries of p will generate a subalgebra $A(S_q^4)$ of $A(S_q^7)$ which is a deformation of the algebra of polynomial functions on the 4-sphere S^4. The two elements $|\phi_1\rangle$, $|\phi_2\rangle$ will be obtained in two steps as follows.

Firstly we write the relation $1 = \sum \bar{x}^i x_i$ in terms of the quadratic elements $\bar{x}^1 x_1$, $x_2 \bar{x}^2$, $\bar{x}^3 x_3$, $x_4 \bar{x}^4$ by using the commutation relations of Sect. 3.1. We have that

$$1 = q^{-6} \bar{x}^1 x_1 + q^{-2} x_2 \bar{x}^2 + q^{-2} \bar{x}^3 x_3 + x_4 \bar{x}^4 \ .$$

Then we take,

$$|\phi_1\rangle = (q^{-3} x_1, -q^{-1} \bar{x}^2, q^{-1} x_3, -\bar{x}^4)^t \ , \tag{22}$$

(t denoting transposition) which is such that $\langle\phi_1|\phi_1\rangle = 1$.

Next, we write $1=\sum \bar{x}^i x_i$ as a function of the quadratic elements $x_1\bar{x}^1, \bar{x}^2x_2, x_3\bar{x}^3, \bar{x}^4x_4$:

$$1 = q^{-2}x_1\bar{x}^1 + q^{-4}\bar{x}^2x_2 + x_3\bar{x}^3 + \bar{x}^4x_4 \ .$$

By taking,

$$|\phi_2\rangle = (\pm q^{-2}x_2, \pm q^{-1}\bar{x}^1, \pm x_4, \pm \bar{x}^3)^t$$

we get $\langle\phi_2|\phi_2\rangle = 1$. The signs will be chosen in order to have also the orthogonality $\langle\phi_1|\phi_2\rangle = 0$; for

$$|\phi_2\rangle = (q^{-2}x_2, q^{-1}\bar{x}^1, -x_4, -\bar{x}^3)^t \tag{23}$$

this is satisfied.

The matrix

$$v = (|\phi_1\rangle \,, |\phi_2\rangle) = \begin{pmatrix} q^{-3}x_1 & q^{-2}x_2 \\ -q^{-1}\bar{x}^2 & q^{-1}\bar{x}^1 \\ q^{-1}x_3 & -x_4 \\ -\bar{x}^4 & -\bar{x}^3 \end{pmatrix} \ . \tag{24}$$

is such that $v^*v = 1$ and hence $p = vv^*$ is a self-adjoint projection.

Proposition 3. *The entries of the projection $p = vv^*$, with v in (24), generate a subalgebra of $A(S^7_q)$ which is a deformation of the algebra of polynomial functions on the 4-sphere S^4.*

Proof. Let us compute explicitly the components of the projection p and their commutation relations.

1. The diagonal elements are given by

$$\begin{aligned} p_{11} &= q^{-6}x_1\bar{x}^1 + q^{-4}x_2\bar{x}^2 \ , & p_{22} &= q^{-2}\bar{x}^2x_2 + q^{-2}\bar{x}^1x_1 \ , \\ p_{33} &= q^{-2}x_3\bar{x}^3 + x_4\bar{x}^4 \ , & p_{44} &= \bar{x}^4x_4 + \bar{x}^3x_3 \ , \end{aligned}$$

   and satisfy the relation

$$q^{-2}p_{11} + q^2p_{22} + p_{33} + p_{44} = 2 \ . \tag{25}$$

   Only one of the p_{ii}'s is independent; indeed by using the commutation relations and the equation $\sum \bar{x}^i x_i = 1$, we can rewrite the p_{ii}'s in terms of

$$t := p_{22} \ , \tag{26}$$

   as

$$p_{11} = q^{-2}t \quad , \quad p_{22} = t \quad , \quad p_{33} = 1 - q^{-4}t \quad , \quad p_{44} = 1 - q^2t \ .$$

   Equation (25) is easily verified. Notice that t is self-adjoint: $\bar{t} = t$.

2. As in the classical case, the elements p_{12}, p_{34} (and their conjugates) vanish:

$$p_{12} = -q^{-4}x_1x_2 + q^{-3}x_2x_1 = 0 \ , \quad p_{34} = -q^{-1}x_3x_4 + x_4x_3 = 0 \ .$$

3. The remaining elements are given by

$$\begin{array}{ll} p_{13} = q^{-4} x_1 \bar{x}^3 - q^{-2} x_2 \bar{x}^4 \ , & p_{14} = -q^{-3} x_1 x_4 - q^{-2} x_2 x_3 \ , \\ p_{23} = -q^{-2} \bar{x}^2 \bar{x}^3 - q^{-1} \bar{x}^1 \bar{x}^4 \ , & p_{24} = q^{-1} \bar{x}^2 x_4 - q^{-1} \bar{x}^1 x_3 \ , \end{array}$$

   with $p_{ji} = \bar{p}_{ij}$ when $j > i$.
   By using the commutation relations of $A(S_q^7)$, one finds that only two of these are independent. We take them to be p_{13} and p_{14}; one finds $p_{23} = q^{-2} \bar{p}_{14}$ and $p_{24} = -q^2 \bar{p}_{13}$.

Finally, we also have the sphere relation,

$$(q^6 - q^8) p_{11}^2 + p_{22}^2 + p_{44}^2 + q^4 (p_{13} p_{31} + p_{14} p_{41}) + q^2 (p_{24} p_{42} + p_{23} p_{32}) = (\sum \bar{x}^i x_i)^2 = 1 \ . \quad (27)$$

Summing up, together with $t = p_{22}$, we set $a := p_{13}$ and $b := p_{14}$. Then the projection p takes the following form

$$p = \begin{pmatrix} q^{-2} t & 0 & a & b \\ 0 & t & q^{-2} \bar{b} & -q^2 \bar{a} \\ \bar{a} & q^{-2} b & 1 - q^{-4} t & 0 \\ \bar{b} & -q^2 a & 0 & 1 - q^2 t \end{pmatrix} . \quad (28)$$

By construction $p^* = p$ and this means that $\bar{t} = t$, as observed, and that $\bar{a}, \bar{b}$ are conjugate to a, b respectively. Also, by construction $p^2 = p$; this property gives the easiest way to compute the commutation relations between the generators. One finds,

$$\begin{array}{ll} ab = q^4 ba \ , & \bar{a} b = b \bar{a} \ , \\ ta = q^{-2} at \ , & tb = q^4 bt \ , \end{array} \quad (29)$$

together with their conjugates, and sphere relations

$$\begin{array}{ll} a\bar{a} + b\bar{b} = q^{-2} t (1 - q^{-2} t) \ , & q^4 \bar{a} a + q^{-4} \bar{b} b = t (1 - t) \ , \\ b\bar{b} - q^{-4} \bar{b} b = (1 - q^{-4}) t^2 \ . & \end{array} \quad (30)$$

It is straightforward to check also the relation (27). □

We define the algebra $A(S_q^4)$ to be the algebra generated by the elements $a, \bar{a}, b, \bar{b}, t$ with the commutation relations (29) and (30). For $q = 1$ it reduces to the algebra of polynomial functions on the sphere S^4. Otherwise, we can limit ourselves to $|q| < 1$, because the map

$$q \mapsto q^{-1}, \quad a \mapsto q^2 \bar{a}, \quad b \mapsto q^{-2} \bar{b}, \quad t \mapsto q^{-2} t$$

yields an isomorphic algebra.

At $q = 1$, the projection p in (28) is conjugate to the classical one given in Appendix A by the matrix $diag[1, -1, 1, 1]$ (up to renaming the generators).

Our sphere S^4_q seems to be different from the one constructed in [4]. Two of our generators commute and most importantly, it does not come from a deformation of a subgroup (let alone coisotropic) of $Sp(2)$. However, at the continuous level these two quantum spheres are the same since the C^*-algebra completion of both polynomial algebras is the minimal unitization $\mathcal{K} \oplus \mathbb{C}\mathbb{I}$ of the compact operators on an infinite dimensional separable Hilbert space, a property shared with Podleś standard sphere as well [24]. This fact will be derived in Sect. 4 when we study the representations of the algebra $A(S^4_q)$.

3.3 The $SU_q(2)$-coaction

We shall now construct a "fibration from S^7_q to S^4_q with structure group $SU_q(2)$". In Sect. 6 this will be proven to be a faithfully flat Hopf-Galois extension.

We start by constructing a coaction of the quantum group $SU_q(2)$ on the sphere S^7_q. Let us observe that the two pairs of generators $(x_1, x_2), (x_3, x_4)$ both yields a quantum plane,

$$\begin{aligned} x_1 x_2 &= q x_2 x_1 \ , \qquad & \bar{x}^1 \bar{x}^2 &= q^{-1} \bar{x}^2 \bar{x}^1 \ , \\ x_3 x_4 &= q x_4 x_3 \ , \qquad & \bar{x}^3 \bar{x}^4 &= q^{-1} \bar{x}^4 \bar{x}^3 \ . \end{aligned}$$

Then we shall look for a right-coaction of $SU_q(2)$ on the rows of the matrix v in (24). Other pairs of generators yields quantum planes but the only choice which yields a projection with the right number of generators is the one given above.

The defining matrix of the quantum group $SU_q(2)$ reads

$$\begin{pmatrix} \alpha & -q\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix} \tag{31}$$

with commutation relations [31],

$$\begin{aligned} &\alpha\gamma = q\gamma\alpha \ , &&\alpha\bar{\gamma} = q\bar{\gamma}\alpha \ , &&\gamma\bar{\gamma} = \bar{\gamma}\gamma \ , \\ &\alpha\bar{\alpha} + q^2 \bar{\gamma}\gamma = 1 \ , &&\bar{\alpha}\alpha + \bar{\gamma}\gamma = 1 \ . \end{aligned} \tag{32}$$

We define a coaction of $SU_q(2)$ on the matrix (24) by,

$$\delta(v) := \begin{pmatrix} q^{-3} x_1 & q^{-2} x_2 \\ -q^{-1} \bar{x}^2 & q^{-1} \bar{x}^1 \\ q^{-1} x_3 & -x_4 \\ -\bar{x}^4 & -\bar{x}^3 \end{pmatrix} \dot{\otimes} \begin{pmatrix} \alpha & -q\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix} \ . \tag{33}$$

We shall prove presently that this coaction comes from a coaction of $A(SU_q(2))$ on the sphere algebra $A(S^7_q)$. For the moment we remark that, by its form in (33) the entries of the projection $p = vv^*$ are automatically coinvariants, a fact that we shall also prove explicitly in the following.

On the generators, the coaction (33) is given explicitly by

$$\begin{aligned} \delta(x_1) &= x_1 \otimes \alpha + q\ x_2 \otimes \gamma \ , \quad & \delta(\bar{x}^1) &= q\bar{x}^2 \otimes \bar{\gamma} + \bar{x}^1 \otimes \bar{\alpha} = \overline{\delta(x_1)} \ , \\ \delta(x_2) &= -x_1 \otimes \bar{\gamma} + x_2 \otimes \bar{\alpha} \ , \quad & \delta(\bar{x}^2) &= \bar{x}^2 \otimes \alpha - \bar{x}^1 \otimes \gamma = \overline{\delta(x_2)} \ , \\ \delta(x_3) &= x_3 \otimes \alpha - q\ x_4 \otimes \gamma \ , \quad & \delta(\bar{x}^3) &= -q\bar{x}^4 \otimes \bar{\gamma} + \bar{x}^3 \otimes \bar{\alpha} = \overline{\delta(x_3)} \ , \\ \delta(x_4) &= x_3 \otimes \bar{\gamma} + x_4 \otimes \bar{\alpha} \ , \quad & \delta(\bar{x}^4) &= \bar{x}^4 \otimes \alpha + \bar{x}^3 \otimes \gamma = \overline{\delta(x_4)} \ , \end{aligned} \tag{34}$$

from which it is also clear its compatibility with the anti-involution, i.e. $\delta(\bar{x}^i) = \overline{\delta(x_i)}$. The map δ in (34) can be extended as an algebra homomorphism to the whole of $A(S^7_q)$. Then, as alluded to before, we have the following

Proposition 4. *The coaction* (34) *is a right coaction of the quantum group* $SU_q(2)$ *on the 7-sphere* S^7_q,

$$\delta : A(S^7_q) \longrightarrow A(S^7_q) \otimes A(SU_q(2)) \ . \tag{35}$$

Proof. By using the commutation relations of $A(SU_q(2))$ in (32), a lengthy but easy computation gives that the commutation relations of $A(S^7_q)$ are preserved. This fact also shows that extending δ as an algebra homomorphism yields a consistent coaction. □

Proposition 5. *The algebra* $A(S^4_q)$ *is the algebra of coinvariants under the coaction defined in* (36).

Proof. We have to show that $A(S^4_q) = \{x \in A(S^7_q) \mid \delta(x) = x \otimes 1\}$. By using the commutation relations of $A(S^7_q)$ and those of $A(SU_q(2))$, we first prove explicitly that the generators of $A(S^4_q)$ are coinvariants:

$$\begin{aligned}
\delta(a) &= q^{-4}\delta(x_1)\delta(\bar{x}^3) - q^{-2}\delta(x_2)\delta(\bar{x}^4) \\
&= q^{-4}x_1\bar{x}^3 \otimes (\alpha\bar{\alpha} + q^2\bar{\gamma}\gamma) - q^{-2}x_2\bar{x}^4 \otimes (\gamma\bar{\gamma} + \bar{\alpha}\alpha) \\
&= (q^{-4}x_1\bar{x}^3 - q^{-2}x_2\bar{x}^4) \otimes 1 = a \otimes 1
\end{aligned}$$

$$\begin{aligned}
\delta(b) &= -q^{-3}\delta(x_1)\delta(x_4) - q^{-2}\delta(x_2)\delta(x_3) \\
&= -q^{-3}x_1x_4 \otimes (\alpha\bar{\alpha} + q^2\bar{\gamma}\gamma) - q^{-2}x_2x_3 \otimes (\gamma\bar{\gamma} + \bar{\alpha}\alpha) \\
&= -(q^{-3}x_1x_4 + q^{-2}x_2x_3) \otimes 1 = b \otimes 1
\end{aligned}$$

$$\begin{aligned}
\delta(t) &= q^{-2}\delta(\bar{x}^2)\delta(x_2) + q^{-2}\delta(\bar{x}^1)\delta(x_1) \\
&= q^{-2}\bar{x}^2x_2 \otimes (\alpha\bar{\alpha} + q^2\bar{\gamma}\gamma) + q^{-2}\bar{x}^1x_1 \otimes (\gamma\bar{\gamma} + \bar{\alpha}\alpha) \\
&= (q^{-2}\bar{x}^2x_2 + q^{-2}\bar{x}^1x_1) \otimes 1 = t \otimes 1
\end{aligned}$$

By construction the coaction is compatible with the anti-involution so that

$$\delta(\bar{a}) = \overline{\delta(a)} = \bar{a} \otimes 1, \quad \delta(\bar{b}) = \overline{\delta(b)} = \bar{b} \otimes 1$$

In fact, this only shows that $A(S^4_q)$ is made of coinvariants but does not rule out the possibility of other coinvariants not in $A(S^4_q)$. However this does not happen; a proof will be given in Appendix B. □

The right coaction of $SU_q(2)$ on the 7-sphere S^7_q can be written as

$$\delta(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4)\ \dot{\otimes} \begin{pmatrix} \alpha & -\bar{\gamma} & 0 & 0 \\ q\gamma & \bar{\alpha} & 0 & 0 \\ 0 & 0 & \alpha & \bar{\gamma} \\ 0 & 0 & -q\gamma & \bar{\alpha} \end{pmatrix} , \tag{36}$$

together with $\delta(\bar{x}_i) = \overline{\delta(x_i)}$.
In the block-diagonal matrix which appears in (36) the second copy is given by $SU_q(2)$ while the first one is twisted as

$$\begin{pmatrix} \alpha & -\bar{\gamma} \\ q\gamma & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha & \bar{\gamma} \\ -q\gamma & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

A similar phenomenon occurs in [4].

Remark 1. *It is also interesting to observe that*

$$\delta(v^* v) = v^* v \otimes 1 = 1 \otimes 1 .$$

Indeed,

$$\begin{aligned}
\delta(\langle\phi_1|\phi_1\rangle) &= \delta(q^{-6}\bar{x}^1 x_1 + q^{-2} x_2 \bar{x}^2 + q^{-2}\bar{x}^3 x_3 + x_4 \bar{x}^4) \\
&= (-q^{-5}\bar{x}^2 x_1 + q^{-2} x_1 \bar{x}^2 + q^{-1}\bar{x}^4 x_3 - x_3 \bar{x}^4) \otimes \bar{\gamma}\alpha \\
&\quad +(q^{-4}\bar{x}^2 x_2 + q^{-2} x_1 \bar{x}^1 + \bar{x}^4 x_4 + x_3 \bar{x}^3) \otimes \bar{\gamma}\gamma \\
&\quad +(q^{-6}\bar{x}^1 x_1 + q^{-2} x_2 \bar{x}^2 + q^{-2}\bar{x}^3 x_3 + x_4 \bar{x}^4) \otimes \bar{\alpha}\alpha \\
&\quad +(-q^{-5}\bar{x}^1 x_2 + q^{-2} x_2 \bar{x}^1 + q^{-1}\bar{x}^3 x_4 - x_4 \bar{x}^3) \otimes \bar{\alpha}\gamma \\
&= \langle\phi_2|\phi_1\rangle \otimes \bar{\gamma}\alpha + \langle\phi_2|\phi_2\rangle \otimes \bar{\gamma}\gamma + \langle\phi_1|\phi_1\rangle \otimes \bar{\alpha}\alpha + \langle\phi_1|\phi_2\rangle \otimes \bar{\alpha}\gamma \\
&= 1 \otimes (\bar{\gamma}\gamma + \bar{\alpha}\alpha) = 1 \otimes 1 ,
\end{aligned}$$

$$\begin{aligned}
\delta(\langle\phi_2|\phi_2\rangle) &= \delta(q^{-2} x_1 \bar{x}^1 + q^{-4}\bar{x}^2 x_2 + x_3 \bar{x}^3 + \bar{x}^4 x_4) \\
&= (q^{-4}\bar{x}^2 x_1 - q^{-1} x_1 \bar{x}^2 - \bar{x}^4 x_3 + q x_3 \bar{x}^4) \otimes \alpha\bar{\gamma} \\
&\quad +(q^{-4}\bar{x}^2 x_2 + q^{-2} x_1 \bar{x}^1 + \bar{x}^4 x_4 + x_3 \bar{x}^3) \otimes \alpha\bar{\alpha} \\
&\quad +(q^{-4}\bar{x}^1 x_1 + x_2 \bar{x}^2 + \bar{x}^3 x_3 + q^2 x_4 \bar{x}^4) \otimes \gamma\bar{\gamma} \\
&\quad +(q^{-4}\bar{x}^1 x_2 - q^{-1} x_2 \bar{x}^1 - \bar{x}^3 x_4 + q x_4 \bar{x}^3) \otimes \gamma\bar{\alpha} \\
&= -q\,\langle\phi_2|\phi_1\rangle \otimes \alpha\bar{\gamma} + \langle\phi_2|\phi_2\rangle \otimes \alpha\bar{\alpha} + q^2\,\langle\phi_1|\phi_1\rangle \otimes \gamma\bar{\gamma} - q\,\langle\phi_1|\phi_2\rangle \otimes \gamma\bar{\alpha} \\
&= 1 \otimes (\alpha\bar{\alpha} + q^2\gamma\bar{\gamma}) = 1 \otimes 1 ,
\end{aligned}$$

$$\delta(\langle\phi_1|\phi_2\rangle) = q^{-5}\delta(\bar{x}^1)\delta(x_2) - q^{-2}\delta(x_2)\delta(\bar{x}^1) - q^{-1}\delta(\bar{x}^3)\delta(x_4) + \delta(x_4)\delta(\bar{x}^3) = 0$$

since δ *defines a coaction on* S^7_q *and so preserves its commutation relations.* □

Remark 2. *Let us define the determinant by*

$$det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} := a_{11}a_{22} - q\; a_{12}a_{21} . \tag{37}$$

(Note that a_{12}, a_{21} *do not commute and so in the previous formula the ordering between them is fixed.) Let* m_{ij} *be the minors of* (24) *obtained by considering the* i, j *rows. Then*

$$\begin{aligned}
&m_{12} = q^2 p_{11} = t , && m_{13} = p_{14} = b , \\
&m_{14} = -q\; p_{13} = -q\; a , && m_{23} = p_{24} = -q^2\bar{a} , \\
&m_{24} = -q\; p_{23} = -q^{-1}\bar{b} , && m_{34} = -q\; p_{33} = q^{-3}t - q .
\end{aligned} \tag{38}$$

This reflects the classical situation in which the minors of the matrix v *in* (24) *give Plücker coordinates [1].*

4 Representations of the algebra of S_q^4

Let us now construct irreducible $*$-representations of $A(S_q^4)$ as bounded operators on a separable Hilbert space $\mathcal{H}$. For the moment, we denote in the same way the elements of the algebra and their images as operators in the given representation. As mentioned before, since $q \mapsto q^{-1}$ gives an isomorphic algebra, we can restrict ourselves to $|q| < 1$. We will consider the representations which are t-finite [20], i.e. such that the eigenvectors of t span $\mathcal{H}$.

Since the self-adjoint operator t must be bounded due to the spherical relations, from the commutation relations $ta = q^{-2}at$, $t\bar{b} = q^{-4}\bar{b}t$, it follows that the spectrum should be of the form λq^{2k} and $a, \bar{b}$ (resp. $\bar{a}, b$) act as rising (resp. lowering) operators on the eigenvectors of t. Then boundedness implies the existence of an highest weight vector, i.e. there exists a vector $|0,0\rangle$ such that

$$t\,|0,0\rangle = t_{00}\,|0,0\rangle\,, \quad a\,|0,0\rangle = 0, \quad \bar{b}\,|0,0\rangle = 0\;. \tag{39}$$

By evaluating $q^4\bar{a}a + b\bar{b} = (1 - q^{-4}t)t$ on $|0,0\rangle$ we have

$$(1 - q^{-4}t_{00})t_{00} = 0$$

According to the values of the eigenvalue t_{00} we have two representations.

4.1 The representation β

The first representation, that we call β, is obtained for $t_{00} = 0$. Then, $t\,|0,0\rangle = 0$ implies $t = 0$. Moreover, using the commutation relations (29) and (30), it follows that this representation is the trivial one

$$t = 0, \quad a = 0, \quad b = 0\;, \tag{40}$$

the representation Hilbert space being just $\mathbb{C}$; of course, $\beta(1) = 1$.

4.2 The representation σ

The second representation, that we call σ, is obtained for $t_{00} = q^4$. This is infinite dimensional.

We take the set $|m,n\rangle = N_{mn}\bar{a}^m b^n\,|0,0\rangle$ with $n, m \in \mathbb{N}$, to be an orthonormal basis of the representation Hilbert space $\mathcal{H}$, with $N_{00} = 1$ and $N_{mn} \in \mathbb{R}$ the normalizations, to be computed below.
Then

$$\begin{aligned} t\,|m,n\rangle &= t_{mn}\,|m,n\rangle\;\;, \\ \bar{a}\,|m,n\rangle &= a_{mn}\,|m+1,n\rangle\;\;, \\ b\,|m,n\rangle &= b_{mn}\,|m,n+1\rangle\;\;. \end{aligned}$$

By requiring that we have a $*$-representation we have also that

$$a\,|m,n\rangle = a_{m-1,n}\,|m-1,n\rangle\;\;, \quad \bar{b}\,|m,n\rangle = b_{m,n-1}\,|m,n-1\rangle\;\;,$$

with the following recursion relations

$$a_{m,n\pm1} = q^{\pm2} a_{m,n} \ , \qquad b_{m\pm1,n} = q^{\pm2} b_{m,n} \ , \qquad b_{m,n} = q^2 a_{2n+1,m} \ .$$

By explicit computation, we find

$$\begin{aligned} t_{m,n} &= q^{2m+4n+4} \ , \\ a_{m,n} &= N_{mn} N^{-1}_{m+1,n} = (1-q^{2m+2})^{\frac{1}{2}} q^{m+2n+1} \ , \\ b_{m,n} &= N_{mn} N^{-1}_{m,n+1} = (1-q^{4n+4})^{\frac{1}{2}} q^{2(m+n+2)} \ . \end{aligned} \tag{41}$$

In conclusion we have the following action

$$\begin{aligned} t\,|m,n\rangle &= q^{2m+4n+4}\,|m,n\rangle \,, \\ \bar{a}\,|m,n\rangle &= (1-q^{2m+2})^{\frac{1}{2}} q^{m+2n+1}\,|m+1,n\rangle \,, \\ a\,|m,n\rangle &= (1-q^{2m})^{\frac{1}{2}} q^{m+2n}\,|m-1,n\rangle \,, \\ b\,|m,n\rangle &= (1-q^{4n+4})^{\frac{1}{2}} q^{2(m+n+2)}\,|m,n+1\rangle \,, \\ \bar{b}\,|m,n\rangle &= (1-q^{4n})^{\frac{1}{2}} q^{2(m+n+1)}\,|m,n-1\rangle \,. \end{aligned} \tag{42}$$

It is straightforward to check that all the defining relations (29) and (30) are satisfied.

In this representation the algebra generators are all trace class:

$$\begin{aligned} \mathrm{Tr}(t) &= q^4 \sum_m q^{2m} \sum_n q^{4n} = \frac{q^4}{(1-q^2)(1-q^4)} \ , \\ \mathrm{Tr}(|a|) &= q \sum_{m,n} (1-q^{2m+2})^{\frac{1}{2}} q^{m+2n} = \frac{q}{1-q^2} \sum_m (1-q^{2m+2})^{\frac{1}{2}} q^m \\ &\leq \frac{q}{1-q^2} \sum_m q^m = \frac{q}{(1-q)(1-q^2)} \ , \\ \mathrm{Tr}(|b|) &= q^4 \sum_{m,n} (1-q^{4n+4})^{\frac{1}{2}} q^{2(n+m)} = \frac{q^4}{1-q^2} \sum_n (1-q^{4n+4})^{\frac{1}{2}} q^{2n} \\ &\leq \frac{q^4}{1-q^2} \sum_n q^{2n} = \frac{q^4}{(1-q^2)^2} \ . \end{aligned} \tag{43}$$

From the sequence of Schatten ideals in the algebra of compact operators one knows [29] that the norm closure of trace class operators gives the ideal of compact operators $\mathcal{K}$. As a consequence, the closure of $A(S^4_q)$ is the C^*-algebra $\mathcal{C}(S^4_q) = \mathcal{K} \oplus \mathbb{C}\mathbb{I}$.

5 The index pairings

The 'defining' self-adjoint idempotent p in (28) determines a class in the K-theory of S^4_q, i.e. $[p] \in K_0[\mathcal{C}(S^4_q)]$. A way to prove its nontriviality is by pairing it with a nontrivial element in the dual K-homology, i.e. with (the class of) a nontrivial Fredholm module $[\mu] \in K^0[\mathcal{C}(S^4_q)]$. In fact, in order to compute the pairing of K-theory with K-homology, it is more convenient to first compute the corresponding Chern characters in the cyclic homology $\mathrm{ch}_*(p) \in HC_*[A(S^4_q)]$ and cyclic cohomology $\mathrm{ch}^*(\mu) \in HC^*[A(S^4_q)]$ respectively, and then use the pairing between cyclic homology and cohomology [11].

The Chern character of the projection p in (28) has a component in degree zero $\mathrm{ch}_0(p) \in HC_0[A(S^4_q)]$ simply given by the matrix trace,

$$\mathrm{ch}_0(p) := \mathrm{tr}(p) = 2 - q^{-4}(1-q^2)(1-q^4)\ t \ \in A(S^4_q). \tag{44}$$

The higher degree parts of $\mathrm{ch}_*(p)$ are obtained via the periodicity operator S; not needing them here we shall not dwell more upon this point and refer to [11] for the relevant details.

As mentioned, the K-homology of an involutive algebra $\mathcal{A}$ is given in terms of homotopy classes of Fredholm modules. In the present situation we are dealing with a 1-summable Fredholm module $[\mu] \in K^0[\mathcal{C}(S^4_q)]$. This is in contrast to the fact that the analogous element of $K_0(S^4)$ for the undeformed sphere is given by a 4-summable Fredholm module, being the fundamental class of S^4.

The module $\mu := (\mathcal{H}, \Psi, \gamma)$ is constructed as follows. The Hilbert space is $\mathcal{H} = \mathcal{H}_\sigma \oplus \mathcal{H}_\sigma$ and the representation is $\Psi = \sigma \oplus \beta$. Here σ is the representation of $A(S^4_q)$ introduced in (42) and β given in (40) is trivially extended to $\mathcal{H}_\sigma$. The grading operator is

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The corresponding Chern character $\mathrm{ch}^*(\mu)$ of the class of this Fredholm module has a component in degree 0, $\mathrm{ch}^0(\mu) \in HC^0[A(S^{2n}_q)]$. From the general construction [11], the element $\mathrm{ch}^0(\mu_{\mathrm{ev}})$ is the trace

$$\tau^1(x) := \mathrm{Tr}\left(\gamma \Psi(x)\right) = \mathrm{Tr}\left(\sigma(x) - \beta(x)\right). \tag{45}$$

The operator $\sigma(x) - \beta(x)$ is always trace class. Obviously $\tau^1(1) = 0$. The higher degree parts of $\mathrm{ch}^*(\mu_{\mathrm{ev}})$ can again be obtained via a periodicity operator.

We are ready to compute the pairing:

$$\begin{aligned} \langle [\mu], [p] \rangle := \left\langle \mathrm{ch}^0(\mu), \mathrm{ch}_0(p) \right\rangle &= -q^{-4}(1-q^2)(1-q^4)\ \tau^1(t) \\ &= -q^{-4}(1-q^2)(1-q^4)\,\mathrm{Tr}(t) = -q^{-4}(1-q^2)(1-q^4)q^4(1-q^2)^{-1}(1-q^4)^{-1} \\ &= -1\ . \end{aligned} \tag{46}$$

This result shows also that the right $A(S^4_q)$-module $p[A(S^4_q)^4]$ is not free. Indeed, any free module is represented in $K_0[\mathcal{C}(S^4_q)]$ by the idempotent 1, and since $\langle [\mu], [1] \rangle = 0$, the evaluation of $[\mu]$ on any free module always gives zero.

We can extract the 'trivial' element in the K-homology $K^0[\mathcal{C}(S^4_q)]$ of the quantum sphere S^4_q and use it to measure the 'rank' of the idempotent p. This generator corresponds to the trivial generator of the K-homology $K_0(S^4)$ of the classical sphere S^4. The latter (classical) generator is the image of the generator of the K-homology of a point by the functorial map $K_*(\iota) : K_0(*) \to K_0(S^{N-1})$, where $\iota : * \hookrightarrow S^{N-1}$ is the inclusion of a point into the sphere. Now, the quantum sphere S^4_q has just one 'classical point', i.e. the 1-dimensional representation β constructed in Sect. 4.1. The corresponding 1-summable Fredholm module $[\varepsilon] \in K^0[\mathcal{C}(S^4_q)]$ is easily described: the Hilbert space is $\mathbb{C}$ with representation β; the grading operator is $\gamma = 1$. Then the degree 0 component $\mathrm{ch}^0(\varepsilon) \in HC^0[A(S^{2n}_q)]$ of the corresponding Chern character is the trace given by the representation itself (since it is a homomorphism to a commutative algebra),

$$\tau^0(x) = \beta(x) \; , \tag{47}$$

and vanishes on all the generators whereas $\tau^0(1) = 1$.
Not surprisingly, the pairing with the class of the idempotent p is,

$$\langle [\varepsilon], [p] \rangle := \tau^0(\mathrm{ch}_0(p)) = \beta(2) = 2 \; . \tag{48}$$

6 The Hopf-Galois structure

We now prove that our fibrations are faithfully flat Hopf-Galois extensions, a notion which is an analogous in noncommutative geometry of the one of principal bundles in differential geometry. Let us first recall some relevant definitions, [21] (see also [22]). We remind that we work over the field $k = \mathbb{C}$.

Definition 1. *Let H be a Hopf algebra and P a right H-comodule algebra with multiplication $m : P \otimes P \to P$ and coaction $\Delta_R : P \to P \otimes H$. Let $B \subseteq P$ be the subalgebra of coinvariants, i.e. $B = \{p \in P \mid \Delta_R(p) = p \otimes 1\}$. The extension $B \subseteq P$ is called an H Hopf-Galois extension if the canonical map*

$$\begin{aligned} &\chi : P \otimes_B P \longrightarrow P \otimes H \; , \\ &\chi := (m \otimes id) \circ (id \otimes_B \Delta_R) \; , \quad p' \otimes_B p \mapsto \chi(p' \otimes_B p) = p' p_{(0)} \otimes p_{(1)} \; , \end{aligned} \tag{49}$$

is bijective.

We use the notation $\Delta_R p = p_{(0)} \otimes p_{(1)}$. The canonical map is left P-linear and right H-colinear and is a morphism (an isomorphism for Hopf-Galois extensions) of left P-modules and right H-comodules. It is also clear that P is both a left and a right B-module.

The injectivity of the canonical map dualizes the condition of a group action $X \times G \to X$ to be free: if α is the map $\alpha : X \times G \to X \times_M X$, $(x, g) \mapsto (x, x \cdot g)$ then $\alpha^* = \chi$ with P, H the algebras of functions on X, G respectively and the action is free if and only if α is injective. Here $M := X/G$ is the space of orbits with projection map $\pi : X \to M$, $\pi(x \cdot g) = \pi(x)$, for all $x \in X, g \in G$. Furthermore, α is surjective if and only if for all $x \in X$, the fibre $\pi^{-1}(\pi(x))$ of $\pi(x)$ is equal to the residue class $x \cdot G$, that is, if and only if G acts transitively on the fibres of π.

In the definition of a principal bundle in differential geometry there is much more that the requirement of injectivity of the canonical map. A better algebraic translation of the notion of a principal bundle is encoded in the requirement that the extension $B \subset P$, beside being Hopf-Galois, is also faithfully flat. We shall not elaborate more on this point and we refer to [27]. Here we shall give some basic definitions. Following [17] we have,

Definition 2. *A right module P over a ring R is faithfully flat if the functor $P \otimes_R \cdot$ is exact and faithful on the category ${}_R\mathcal{M}$ of left R-modules.*

Flatness means that the functor associates exact sequences of abelian groups to exact sequences of R-modules and the functor is faithful if it is injective on morphisms. Equivalently one could state that a right module P over a ring R is faithfully flat if a sequence $M' \to M \to M''$ in ${}_R\mathcal{M}$ is exact if and only if $P \otimes_R M' \to P \otimes_R M \to P \otimes_R M''$ is exact.

The crucial theorem by Schneider which characterizes faithfully flat Hopf-Galois extensions is the following ([28], Th. I).
Let ${}_B\mathcal{M}$ (resp. $\mathcal{M}_B$) be the category of left (resp. right) B-modules and $\mathcal{M}_P^H$ (resp. ${}_P\mathcal{M}^H$) the category of (P, H)-Hopf modules, that is right P-modules and right (resp. left) H-comodule with P-linear comodule structure.

Theorem 1. *Let H be a Hopf-algebra, let P be an algebra carrying a right H-comodule structure Δ_R with coinvariant algebra $B = \{p \in P \mid \Delta_R(p) = p \otimes 1\}$ and let $\chi : P \otimes_B P \to P \otimes H$ be the canonical map (see e.g. (49)). Assume that the antipode of H is bijective, then the following conditions are equivalent:*

1. *P is injective as right H-comodule and χ is surjective;*
2. *P is faithfully flat as left B-module and χ is an isomorphism;*
3. *P is faithfully flat as right B-module and χ is an isomorphism.*
4. *the map $\mathcal{M}_B \to \mathcal{M}_P^H, \quad M \mapsto M \otimes_B P$ is an equivalence;*
5. *the map ${}_B\mathcal{M} \to_P \mathcal{M}^H, \quad M \mapsto P \otimes_B M$ is an equivalence;*

In particular, if the Hopf algebra H is also cosemisimple (as it happens for $A(SU_q(2))$), then any right H-comodule is injective [22] and the above theorem states that the surjectivity of the canonical map is enough to ensure that it is bijective (so that we have an H-Galois structure) and (left and right) faithfully flat.

Finally, an additional useful result [26] is that the map χ is surjective whenever, for any generator h of H, the element $1 \otimes h$ is in its image. This follows from the left P-linearity and right H-colinearity of the map χ. Indeed, let $h,\ k$ be two elements of H and $\sum p'_i \otimes p_i,\ \sum q'_j \otimes q_j \in P \otimes P$ be such that $\chi(\sum p'_i \otimes_B p_i) = 1 \otimes h,\ \chi(\sum q'_j \otimes_B q_j) = 1 \otimes k$. Then $\chi(\sum p'_i q'_j \otimes_B q_j p_i) = 1 \otimes kh$, that is $1 \otimes kh$ is in the image of χ. But, since the map χ is left P-linear, this implies its surjectivity.

We need also the following,

Definition 3. *Let P be a bimodule over the ring B. Given any two elements $|\xi_1\rangle$ and $|\xi_2\rangle$ in the free module $\mathcal{E} = \mathbb{C}^m \otimes P$, we shall define $\left\langle \xi_1 \dot{\otimes}_B \xi_2 \right\rangle \in P \otimes_B P$ by*

$$\left\langle \xi_1 \dot{\otimes}_B \xi_2 \right\rangle := \sum_{j=1}^{m} \bar{\xi}_1^j \otimes_B \xi_2^j \,. \tag{50}$$

Analogously, one can define quantities $\left\langle \xi_1 \dot{\otimes} \xi_2 \right\rangle \in P \otimes P$ with the same formula as above and tensor products taken over the ground field $\mathbb{C}$.

Proposition 6. *The extension $A(S^7_q) \subset A(Sp_q(2))$ is a faithfully flat $A(Sp_q(1))$-Hopf-Galois extension.*

Proof. Now $P = A(Sp_q(2)$, $H = A(Sp_q(1))$ and $B = A(S^7_q)$ and the coaction Δ_R of H is given just before Prop. 2. Since $A(Sp_q(1)) \simeq A(SU_{q^2}(2))$ has a bijective antipode and is cosemisimple ([20], Chapter 11), from the general considerations given above in order to show the bijectivity of the canonical map

$$\chi : A(Sp_q(2)) \otimes_{A(S^7_q)} A(Sp_q(2)) \longrightarrow A(Sp_q(2)) \otimes A(Sp_q(1)) \ ,$$

it is enough to show that all generators $\alpha, \gamma, \bar{\alpha}, \bar{\gamma}$ of $A(Sp_q(1))$ in (17) are in its image. Let $|T^2\rangle \, , |T^3\rangle$ be the second and third columns of the defining matrix T of $Sp_q(2)$. We shall think of them as elements of the free module $\mathbb{C}^4 \otimes A(Sp_q(2))$. Obviously, $\langle T^i | T^j \rangle = \delta^{ij}$. Recalling that $A(Sp_q(2))$ is both a left and right $A(S^7_q)$-module and using Def. 3, we have that

$$\chi \begin{pmatrix} \left\langle T^2 \dot{\otimes}_{A(S^7_q)} T^2 \right\rangle & \left\langle T^2 \dot{\otimes}_{A(S^7_q)} T^3 \right\rangle \\ & \\ \left\langle T^3 \dot{\otimes}_{A(S^7_q)} T^2 \right\rangle & \left\langle T^3 \dot{\otimes}_{A(S^7_q)} T^3 \right\rangle \end{pmatrix} = 1 \dot{\otimes} \begin{pmatrix} \alpha & -q^2 \bar{\gamma} \\ & \\ \gamma & \bar{\alpha} \end{pmatrix} \ .$$

Indeed,

$$\begin{aligned} \chi(\left\langle T^2 \dot{\otimes}_{A(S^7_q)} T^2 \right\rangle) &= \overline{T}^2_i \Delta_R T^2_i = \left\langle T^2 | T^2 \right\rangle \otimes \alpha + \left\langle T^2 | T^3 \right\rangle \otimes \gamma = 1 \otimes \alpha \ , \\ \chi(\left\langle T^3 \dot{\otimes}_{A(S^7_q)} T^2 \right\rangle) &= \overline{T}^3_i \Delta_R T^2_i = \left\langle T^3 | T^2 \right\rangle \otimes \alpha + \left\langle T^3 | T^3 \right\rangle \otimes \gamma = 1 \otimes \gamma; \end{aligned}$$

a similar computation giving the other two generators. □

Proposition 7. *The extension $A(S^4_q) \subset A(S^7_q)$ is a faithfully flat $A(SU_q(2))$-Hopf-Galois extension.*

Proof. Now $P = A(S^7_q)$, $H = A(SU_q(2))$ and $B = A(S^4_q))$ and the coaction δ of H is given in Prop. 4. As already mentioned $A(SU_q(2))$ has a bijective antipode and is cosemisimple, then as before in order to show the bijectivity of the canonical map

$$\chi : A(S^7_q) \otimes_{A(S^4_q)} A(S^7_q) \longrightarrow A(S^7_q) \otimes A(SU_q(2)) \ ,$$

we have to show that all generators $\alpha, \gamma, \bar{\alpha}, \bar{\gamma}$ of $A(SU_q(2))$ in (31) are in its image. Recalling that $A(S^7_q)$ is both a left and right $A(S^4_q)$-module and using Def. 3, we have that

$$\chi \begin{pmatrix} \left\langle \phi_1 \dot{\otimes}_{A(S^4_q)} \phi_1 \right\rangle & \left\langle \phi_1 \dot{\otimes}_{A(S^4_q)} \phi_2 \right\rangle \\ & \\ \left\langle \phi_2 \dot{\otimes}_{A(S^4_q)} \phi_1 \right\rangle & \left\langle \phi_2 \dot{\otimes}_{A(S^4_q)} \phi_2 \right\rangle \end{pmatrix} = 1 \dot{\otimes} \begin{pmatrix} \alpha & -q \bar{\gamma} \\ & \\ \gamma & \bar{\alpha} \end{pmatrix} \ ,$$

where $|\phi_1\rangle \, , |\phi_2\rangle$ are the two vectors introduced in eqs. (22) and (23). Indeed

$$\begin{aligned} \chi(\left\langle \phi_1 \dot{\otimes}_{A(S^4_q)} \phi_1 \right\rangle) &= \chi \Big(q^{-6} \bar{x}^1 \otimes_{A(S^4_q)} x_1 + q^{-2} x_2 \otimes_{A(S^4_q)} \bar{x}^2 \\ &\qquad\qquad + q^{-2} \bar{x}^3 \otimes_{A(S^4_q)} x_3 + x_4 \otimes_{A(S^4_q)} \bar{x}^4 \Big) \\ &= q^{-6} \bar{x}^1 \delta(x_1) + q^{-2} x_2 \delta(\bar{x}^2) + q^{-2} \bar{x}^3 \delta(x_3) + x_4 \delta(\bar{x}^4) \\ &= q^{-6} \bar{x}^1 x_1 \otimes \alpha + q^{-5} \bar{x}^1 x_2 \otimes \gamma + q^{-2} x_2 \bar{x}^2 \otimes \alpha - q^{-2} x_2 \bar{x}^1 \otimes \gamma \\ &\quad + q^{-2} \bar{x}^3 x_3 \otimes \alpha - q^{-1} \bar{x}^3 x_4 \otimes \gamma + x_4 \bar{x}^4 \otimes \alpha + x_4 \bar{x}^3 \otimes \gamma \\ &= \langle \phi_1 | \phi_1 \rangle \otimes \alpha = 1 \otimes \alpha \ , \end{aligned}$$

$$\begin{aligned}\chi(\left\langle \phi_2 \dot{\otimes}_{A(S^4_q)} \phi_1 \right\rangle) &= q^{-5}\bar{x}^2\delta(x_1) - q^{-2}x_1\delta(\bar{x}^2) - q^{-1}\bar{x}^4\delta(x_3) + x_3\delta(\bar{x}^4) \\ &= q^{-5}\bar{x}^2 x_1 \otimes \alpha + q^{-4}\bar{x}^2 x_2\gamma - q^{-2}x_1\bar{x}^2 \otimes \alpha + q^{-2}x_1\bar{x}^1 \otimes \gamma \\ &\quad -q^{-1}\bar{x}^4 x_3 \otimes \alpha + \bar{x}^4 x_4 \otimes \gamma + x_3\bar{x}^4 \otimes \alpha + x_3\bar{x}^3 \otimes \gamma \\ &= \langle \phi_2|\phi_1\rangle \otimes \alpha + \langle \phi_2|\phi_1\rangle \otimes \gamma = 1 \otimes \gamma \,,\end{aligned}$$

with similar computations for the other generators. □

It was proven in [5] that the bundle constructed in [4] is a coalgebra Galois extension [10, 7]. The fact that our bundle $A(S^4_q) \subset A(S^7_q)$ is Hopf-Galois shows also that these two Hopf bundles are different.

On our extension $A(S^4_q) \subset A(S^7_q)$ there is a strong connection. Indeed an H-Hopf-Galois extension $B \subseteq P$ for which H is cosemisimple and has a bijective antipode is also equivariantly projective, that is there exists a left B-linear right H-colinear splitting $s : P \to B \otimes P$ of the multiplication map $m : B \otimes P \to P$, $m \circ s = id_P$ [27]. Such a map characterizes the so called strong connection [13, 14].

In particular, if H has an invertible antipode S, an equivalent description of a strong connection can be given in terms of a map $\ell : H \to P \otimes P$ satisfying a list of conditions [8] (see also [16, 6]). We denote by Δ the coproduct on H with Sweedler notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$, by $\delta : P \to P \otimes H$ the right-comodule structure on P with notation $\delta p = p_{(0)} \otimes p_{(1)}$, and $\delta_l : P \to H \otimes P$ is the induced left H-comodule structure of P defined by $\delta_l(p) = S^{-1}(p_{(1)}) \otimes p_{(0)}$. Then, for the map ℓ one requires that $\ell(1) = 1 \otimes 1$ and that for all $h \in H$,

$$\begin{aligned}&\chi(\ell(h)) = 1 \otimes h \,, \\ &\ell(h_{(1)}) \otimes h_{(2)} = (id \otimes \delta) \circ \ell(h) \,, \\ &h_{(1)} \otimes \ell(h_{(2)}) = (\delta_l \otimes id) \circ \ell(h)\end{aligned} \tag{51}$$

The splitting s of the multiplication map is then given by

$$s : P \to B \otimes P \,, \quad p \mapsto p_{(0)}\ell(p_{(1)}) \,.$$

Now, if $g, h \in H$ are such that $\ell(g) = g^1 \otimes g^2$ and $\ell(h) = h^1 \otimes h^2$ satisfy condition (51) so does $\ell(gh)$ defined by

$$\ell(gh) := h^1 g^1 \otimes g^2 h^2 \,. \tag{52}$$

If H has a PBW basis [19], this fact can be used to iteratively construct ℓ once one knows its value on the generators of H.

For $H = A(SU_q(2))$, with generators, $\alpha, \gamma, \bar{\alpha}$ and $\bar{\gamma}$, the PBW basis is given by $\alpha^k\gamma^l\bar{\gamma}^m$, with $k, l, m \in \{0, 1, 2, \dots\}$ and $\gamma^k\bar{\gamma}^l\bar{\alpha}^m$, with $k, l \in \{0, 1, 2, \dots\}$ and $m \in \{1, 2, \dots\}$ [31]. Then, for our extension $A(S^4_q) \subset A(S^7_q)$ the map ℓ can be constructed as follows. Firstly, we put $\ell(1) = 1 \otimes 1$. Then, on the generators we set

$$\begin{aligned}\ell(\alpha) &:= \left\langle \phi_1 \dot{\otimes} \phi_1 \right\rangle \,, \qquad \ell(\bar{\alpha}) := \left\langle \phi_2 \dot{\otimes} \phi_2 \right\rangle \,, \\ \ell(\gamma) &:= \left\langle \phi_2 \dot{\otimes} \phi_1 \right\rangle \,, \qquad \ell(\bar{\gamma}) := -q^{-1} \left\langle \phi_1 \dot{\otimes} \phi_2 \right\rangle \,.\end{aligned}$$

These expressions for ℓ satisfy all the properties (51):

Firstly, $\chi(\ell(\alpha)) = 1 \otimes \alpha$ follows from the proof of Prop. 7. Then,

$$\begin{aligned}
(id \otimes \delta) \circ \ \ell(\alpha) &= q^{-6} \bar{x}^1 \otimes \delta x_1 + q^{-2} x_2 \otimes \delta \bar{x}^2 + q^{-2} \bar{x}^3 \otimes \delta x_3 + x_4 \otimes \delta \bar{x}^4 \\
&= \left\langle \phi_1 \dot{\otimes} \phi_1 \right\rangle \otimes \alpha + \left\langle \phi_1 \dot{\otimes} \phi_2 \right\rangle \otimes \gamma \\
&= \ell(\alpha) \otimes \alpha - q\ell(\bar{\gamma}) \otimes \gamma = \ell(\alpha_{(1)}) \otimes \alpha_{(2)} \ .
\end{aligned}$$

Moreover

$$\begin{aligned}
(\delta_l \otimes id) \circ \ \ell(\alpha) &= q^{-6}(\alpha \otimes \bar{x}^1 - q^2 \bar{\gamma} \otimes \bar{x}^2) \otimes x_1 + q^{-2}(q\bar{\gamma} \otimes x_1 + \alpha \otimes x_2) \otimes \bar{x}^2 \\
&+ \ q^{-2}(q^2 \bar{\gamma} \otimes \bar{x}^4 + \alpha \otimes \bar{x}^3) \otimes x_3 + (-q\bar{\gamma} \otimes x_3 + \alpha \otimes x_4) \otimes \bar{x}^4 \\
&= \alpha \otimes \left\langle \phi_1 \dot{\otimes} \phi_1 \right\rangle - q\bar{\gamma} \otimes \left\langle \phi_2 \dot{\otimes} \phi_1 \right\rangle \\
&= \alpha \otimes \ell(\alpha) - q\bar{\gamma} \otimes \ell(\gamma) = \alpha_{(1)} \otimes \ell(\alpha_{(2)}) \ .
\end{aligned}$$

Similar computations can be carried for $\gamma, \bar{\alpha}$ and $\bar{\gamma}$.

That an iterative procedure constructed by using (52) on the PBW basis leads to a well defined ℓ on the whole of $H = A(SU_q(2))$ will be proven in the forthcoming paper [23] where other elaborations coming from the existence of a strong connection will be presented as well.

6.1 The associated bundle and the coequivariant maps

We now give some elements of the theory of associated quantum vector bundles [12] (see also [27]). Let $B \subset P$ be a H-Galois extension with Δ_R the coaction of H on P. Let $\rho : V \to V \otimes H$ be a corepresentation of H with V a finite dimensional vector space and set $\rho(v) = v_{(0)} \otimes v_{(1)}$, with $v \in V$. A corresponding coequivariant map is a linear map $\varphi : V \to P$, with the property that

$$\varphi(v_{(0)}) \otimes S(v_{(1)}) = \Delta_R \varphi(v) \, , \quad \forall \, v \in V \, ; \tag{53}$$

here S is the antipode of $A(SU_q(2))$. The collection $\Gamma_\rho(V, P)$ of coequivariant maps is a right B-module with this structure given by $(\varphi \cdot b)(v) := \varphi(v)b$.

The algebraic analogue of bundle nontriviality is translated in the fact that the Hopf-Galois extension $B \subset P$ is not cleft [27]. On the other hand, it is knows that for a cleft Hopf-Galois extension, the module of coequivariant maps $\Gamma_\rho(V, P)$ is isomorphic (as a right B-module) to the free module $\Gamma(V, B)$ of all linear maps from V to B [9, 15].

For our $A(SU_q(2))$-Hopf-Galois extension $A(S_q^4) \subset A(S_q^7)$, let $\rho : \mathbb{C}^2 \to \mathbb{C}^2 \otimes A(SU_q(2))$ be the fundamental corepresentation of $A(SU_q(2))$ with $\Gamma_\rho(\mathbb{C}^2, A(S_q^7))$ the right $A(S_q^4)$-module of corresponding coequivariant maps.

Now, the projection p in (28) determines a quantum vector bundle over S_q^4 whose module of section is $p[A(S_q^4)^4]$, which is clearly a right $A(S_q^4)$-module.

Proposition 8. *The modules $p[A(S_q^4)^4]$ and $\Gamma_\rho(\mathbb{C}^2, A(S_q^7))$ are isomorphic as right $A(S_q^4)$-modules.*

A proof of this Proposition will be given in a forthcoming paper [23]. There, we shall also extend our constructions to any irreducible corepresentation of $A(SU_q(2))$, and in particular we shall construct the projections giving the corresponding associated bundles.

Proposition 9. *The Hopf-Galois extension $A(S^4_q) \subset A(S^7_q)$ is not cleft.*

Proof. As mentioned, the cleftness of the extension does imply that all modules of co-equivariant maps are free. On the other hand, the nontriviality of the pairing (46) between the defining projection p in (28) and the Fredholm module μ constructed in Sect. 5 also shows that the module $p[A(S^4_q)^4] \simeq \Gamma_\rho(\mathbb{C}^2, A(S^7_q))$ is not free. $\square$

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A The classical Hopf fibration $S^7 \to S^4$

We shall review the classical construction of the basic anti-instanton bundle over the four dimensional sphere S^4 in a 'noncommutative parlance' following [18]. This has been useful in the main text for our construction of the quantum deformation of the Hopf bundle.

We write the generic element of the group $SU(2)$ as

$$w = \begin{pmatrix} w_1 & w_2 \\ -\bar{w}_2 & \bar{w}_1 \end{pmatrix} . \tag{54}$$

The $SU(2)$ principal fibration $SU(2) \to S^7 \to S^4$ over the sphere S^4 is explicitly realized as follows. The total space is

$$S^7 = \{z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \ , \ \sum_{i=1}^4 |z_i|^2 = 1\} , \tag{55}$$

with right diagonal action

$$S^7 \times SU(2) \ \to \ S^7 \ , \quad z \cdot w := (z_1, z_2, z_3, z_4) \begin{pmatrix} w_1 & w_2 & 0 & 0 \\ -\bar{w}_2 & \bar{w}_1 & 0 & 0 \\ 0 & 0 & w_1 & w_2 \\ 0 & 0 & -\bar{w}_2 & \bar{w}_1 \end{pmatrix} . \tag{56}$$

The bundle projection $\pi : S^7 \to S^4$ is just the Hopf projection and it can be explicitly given as $\pi(z_1, z_2, z_3, z_4) := (x, \alpha, \beta)$ with

$$\begin{aligned} x &= |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = -1 + 2(|z_1|^2 + |z_2|^2) = 1 - 2(|z_3|^2 + |z_4|^2) \ , \\ \alpha &= 2(z_1 \bar{z}_3 + z_2 \bar{z}_4) \ , \qquad \beta = 2(-z_1 z_4 + z_2 z_3) \ . \end{aligned} \tag{57}$$

One checks that $|\alpha|^2 + |\beta|^2 + x^2 = (\sum_{i=1}^4 |z_i|^2)^2 = 1$.

We need the rank 2 complex vector bundle E associated with the defining left representation ρ of $SU(2)$ on $\mathbb{C}^2$. The quickest way to get this is to identify S^7 with the unit sphere in the 2-dimensional quaternionic (right) $\mathbb{H}$-module $\mathbb{H}^2$ and S^4 with the projective line $\mathbb{P}^1(\mathbb{H})$, i.e. the set of equivalence classes $(w_1, w_2)^t \simeq (w_1, w_2)^t\lambda$ with $(w_1, w_2) \in S^7$ and $\lambda \in Sp(1) \simeq SU(2)$. Identifying $\mathbb{H} \simeq \mathbb{C}^2$, the vector $(w_1, w_2)^t \in S^7$ reads

$$v = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \\ z_3 & z_4 \\ -\bar{z}_4 & \bar{z}_3 \end{pmatrix} . \tag{58}$$

This is actually a map from S^7 to the Stieffel variety of frames for E. In particular, notice that the two vectors $|\psi_1\rangle, |\psi_2\rangle$ given by the columns of v are orthonormal, indeed $v^*v = \mathbb{I}_2$. As a consequence,

$$p := vv^* = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| \tag{59}$$

is a self-adjoint idempotent (a projector), $p^2 = p$, $p^* = p$. Of course p is $SU(2)$ invariant and hence its entries are functions on S^4 rather than S^7. An explicit computation yields

$$p = \frac{1}{2}\begin{pmatrix} 1+x & 0 & \alpha & \beta \\ 0 & 1+x & -\bar{\beta} & \bar{\alpha} \\ \bar{\alpha} & -\beta & 1-x & 0 \\ \bar{\beta} & \alpha & 0 & 1-x \end{pmatrix} , \tag{60}$$

where (x, α, β) are the coordinates (57) on S^4. Then $p \in \mathrm{Mat}_4(C^\infty(S^4, \mathbb{C}))$ is of rank 2 by construction.

The matrix v in (58) is a particular example of the matrices v given in [1], for $n = 1$, $k = 1$, $C_0 = 0$, $C_1 = 1$, $D_0 = 1$, $D_1 = 0$. This gives the (anti-)instanton of charge -1 centered at the origin and with unit scale. The only difference is that here we identify $\mathbb{C}^4$ with $\mathbb{H}^2$ as a right $\mathbb{H}$-module. This notwithstanding, the projections constructed in the two formalisms actually coincide. Finally recall that, as mentioned already, the classical limit of our quantum projection (28) is conjugate to (60).

The canonical connection associated with the projector,

$$\nabla := p \circ d \ : \ \Gamma^\infty(S^4, E) \ \to \ \Gamma^\infty(S^4, E) \otimes_{C^\infty(S^4,\mathbb{C})} \Omega^1(S^4, \mathbb{C}), \tag{61}$$

corresponds to a Lie-algebra valued $(su(2))$ 1-form A on S^7 whose matrix components are given by

$$A_{ij} = \langle\psi_i|d\psi_j\rangle \ , \qquad i, j = 1, 2 \ . \tag{62}$$

This connection can be used to compute the Chern character of the bundle. Out of the curvature of the connection $\nabla^2 = p(dp)^2$ one has the Chern 2-form and 4-form given respectively by

$$\begin{aligned} C_1(p) &:= -\frac{1}{2\pi i} \ \mathrm{tr}(p(dp)^2) \ , \\ C_2(p) &:= -\frac{1}{8\pi^2} \ [\mathrm{tr}(p(dp)^4) - C_1(p)C_1(p)] \ , \end{aligned} \tag{63}$$

with the trace tr just an ordinary matrix trace. It turns out that the 2-form $p(dp)^2$ has vanishing trace so that

$$C_1(p) = 0 \ . \tag{64}$$

As for the second Chern class, a straightforward calculation shows that,

$$\begin{aligned} C_2(p) &= -\frac{1}{32\pi^2}[(x_0dx_4 - x_4dx_0)(d\xi)^3 + 3dx_0dx_4\ \xi\ (d\xi)^2] \\ &= -\frac{3}{8\pi^2}[x_0dx_1dx_2dx_3dx_4 + x_1dx_2dx_3dx_4dx_0 \\ &\qquad +x_2dx_3dx_4dx_0dx_1 + x_3dx_4dx_0dx_1dx_2 + x_4dx_0dx_1dx_2dx_3] \\ &= -\frac{3}{8\pi^2}\ d(vol(S^4))\ . \end{aligned} \tag{65}$$

The second Chern number is then given by

$$c_2(p) = \int_{S^4} C_2(p) = -\frac{3}{8\pi^2}\int_{S^4} d(vol(S^4)) = -\frac{3}{8\pi^2}\frac{8}{3}\pi^2 = -1\ . \tag{66}$$

The connection A in (61) is (anti-)self-dual, i.e. its curvature $F_A := dA + A \wedge A$ satisfies (anti-)self-duality equations, $*_H F_A = -F_A$, with $*_H$ the Hodge map of the canonical (round) metric on the sphere S^4. It is indeed the basic Yang-Mills anti-instanton found in [2].

B Proof of Prop.s 5 and 2

We sketch how the proof of Prop. 5 on the fact that elements of $A(S^4_q)$ are the only coinvariants, can be completed. A similar argument will work also for Prop. 2 concerning the algebra of coinvariants $A(S^7_q)$.
Let us consider the grading $A(S^7_q) = \oplus A_d(S^7_q)$, where $A_d(S^7_q)$ denote the subspaces of homogeneous polynomials of degree d. First of all, notice that there cannot be coinvariants of odd degree d due to the fact that δ preserves the grading. For $d = 2m$, the basic idea is to notice that the coinvariant subspaces (of degree d) are the kernels of the linear map $\tilde{\delta}_d : A_d(S^7_q) \to A_d(S^7_q) \otimes A_d(SU_q(2))$ given by

$$\tilde{\delta}_d(x) = \delta(x) - x \otimes (\alpha\bar{\alpha} + q^2\bar{\gamma}\gamma)^m \tag{67}$$

One can easily convince oneself that it is possible to choose generators (not necessarily independent) in the source and in the target of $\tilde{\delta}_d$ in such a way that the matrix M_d representing $\tilde{\delta}_d$ has scalar entries which are independent of q. Choosing a basis among the generators amounts to delete some lines and columns of M_d yielding a matrix with constant entries as before. Accordingly the dimension of the kernel is independent of q and it is equal to the classical case ($q = 1$). It remains to show that we have as many coinvariants at each degree as classically. This has been already checked at $d = 2$. At higher degrees we use the diamond lemma [3], whose notation and terminology are adopted in the following. We use the reduction system

$$S = \left\{ \begin{array}{lll} (ba, q^{-4}ab)\ , & (b\bar{a}, \bar{a}b)\ , & (\bar{b}b, q^4t(1-t) - q^8\bar{a}a) \\ (at, q^2ta)\ , & (\bar{a}t, q^{-2}t\bar{a})\ , & (b\bar{b}, (1-q^{-4})t^2 + q^{-4}\bar{b}b) \\ (\bar{b}a, a\bar{b})\ , & (\bar{b}\bar{a}, q^4\bar{a}\bar{b})\ , & (b\bar{b}, q^{-2}t(1-q^{-2}t) - a\bar{a}) \\ (bt, q^{-4}tb)\ , & (\bar{b}t, q^4t\bar{b})\ , & (a\bar{a}, (1-q^{-2})t - q^4\bar{a}a) \end{array} \right\}\ ,$$

in such a way that the corresponding ideal of relations gives precisely the commutation relations (29), (30) and their conjugates. Furthermore we use the lexicographic ordering induced by the following ordering of the letters

$$t < \bar{a} < a < \bar{b} < b$$

which is compatible with S and obviously satisfies the descending chain condition. There are only overlap ambiguities in S, which can be all resolved. Then by Th. 1.2 [3], a vector space basis for the space $A_d(S^4_q)$ can be chosen to consist of the S-irreducible elements given by

$$\{t^i \bar{a}^j a^{j'} \bar{b}^k b^{k'} \text{ with } kk' = 0;\ i + j + j' + k + k' = d\}.$$

This result is in accordance with a similar one in [4]. These same elements form a basis for the classical coinvariants as well; thus the cardinality of the basis is the required one. $\square$

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