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Quantum Energy Inequalities in two-dimensional conformal field theory

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Abstract. Quantum energy inequalities (QEIs) are state-independent lower bounds on weighted averages of the stress-energy tensor, and have been established for several free quantum field models. We present rigorous QEI bounds for a class of interacting quantum fields, namely the unitary, positive energy conformal field theories (with stress-energy tensor) on two-dimensional Minkowski space. The QEI bound depends on the weight used to average the stress-energy tensor and the central charge(s) of the theory, but not on the quantum state. We give bounds for various situations: averaging along timelike, null and spacelike curves, as well as over a spacetime volume. In addition, we consider boundary conformal field theories and more general ‘moving mirror’ models.

Our results hold for all theories obeying a minimal set of axioms which—as we show—are satisfied by all models built from unitary highest-weight representations of the Virasoro algebra. In particular, this includes all (unitary, positive energy) minimal models and rational conformal field theories. Our discussion of this issue collects together (and, in places, corrects) various results from the literature which do not appear to have been assembled in this form elsewhere.

1 Introduction

In classical theories of matter, the stress-energy tensor $T_{\mu\nu}$ is usually taken to satisfy “energy conditions”, encoding various physical assumptions. For example, the dominant energy condition (DEC) requires that $T^\mu{}_\nu v^\nu$ be a future-pointing causal (timelike or null) vector whenever v^ν is [reflecting the idea that energy-momentum should be propagated at or below the speed of light], while the weak energy condition (WEC) requires simply that the energy density seen by any observer is nonnegative. It is well-known that such conditions usually fail in quantum theoretical models of matter to the extent that, at a given spacetime point, the expectation value of the energy density can be made arbitrarily negative by a suitable choice of state. If such negative energy densities could in fact be sustained over a sufficiently large region of space and time, then all sorts of unexpected

physical phenomena ranging from exotic spacetimes to violations of the second law of thermodynamics could occur [35, 1, 16].

However, it has been shown that the duration and magnitude of negative energy density that can occur is constrained, at least in models of free fields, by so-called “quantum inequalities”. (We will use the more specific term “quantum energy inequalities” (QEIs).) Results are known for the free scalar [19, 39, 12, 6, 15], Dirac [52, 13, 9], Maxwell and Proca [19, 38, 10] and Rarita–Schwinger fields [54] in various levels of generality, including some quite general and rigorous results. These inequalities state that the weighted average of the expected energy density along a worldline is bounded from below by a negative constant depending only on the weighting function used in the averaging process, but not on the quantum state. Moreover, the bounds become more stringent if one increases the time interval over which the averaging is performed. These quantum energy inequalities arguably exclude, or at least severely constrain, the above-mentioned exotic physical phenomena (see, e.g., [18, 40, 45]).

Unfortunately, quantum inequalities of the above character are at present only known for free field theories, leaving open the possibility that physically interesting, interacting field theories might display a completely different behavior in this regard. Thus, one should also investigate quantum inequalities for interacting quantum field theories.

In the present paper, we take a first step in this direction, by deriving a sharp quantum energy inequality of the above character for arbitrary unitary, two-dimensional quantum field theories with conformal invariance and positive Hamiltonian¹. Our derivation is based on the realization that Flanagan’s bound [14] for a massless scalar field in two dimensions is in fact an argument in conformal field theory. Indeed, a close inspection shows that the essential part of his argument only relies upon the transformation law of the stress energy operator under diffeomorphisms, common to all two-dimensional unitary conformal field theories with positive Hamiltonian and a stress-energy tensor. As a result, our general bound differs from that for a massless scalar field in two dimensions only by a multiplicative factor of the central charge, c , of the conformal field theory under consideration (and the possibility that the left- and right-moving portions of the stress-energy tensor might have different central charges). We do not assume at any point that the theory is derived from a Lagrangian, nor do we invoke (but certainly do not exclude) at any point the existence of any fields other than the stress-energy tensor. The general arguments establishing the bound are sketched in Section 2, following Flanagan’s argument fairly closely.

Some non-trivial issues of mainly technical nature have to be dealt with in order to make the argument rigorous for the class of weighting functions that we want to consider, and to show that the bound is sharp. These issues mainly arise from the fact that the stress-energy tensor in two-dimensional conformal field theory has the familiar transformation property for those diffeomorphisms of the left- resp. right-moving light-ray that can be lifted to diffeomorphisms of the unit circle S^1 under the stereographic map. However, in order to prove our quantum inequality bound for weighting functions of compact support (and to show that it is sharp), one formally wants to consider diffeomorphisms outside this class. These difficulties were overcome in [14] by an appeal to general covari-

¹We are also assuming, of course, that the theory has a stress tensor. Not all theories with conformal invariance necessarily admit a stress-energy tensor [2, 28].

ance. In the setting explored in this paper, a different argument is needed, and this is elaborated in Section 4.

To make this argument, we need to have sufficient control over the unitary representations of the diffeomorphism group of S^1 which enter the transformation law of the stress-tensor in the given CFT model. We therefore begin in Sec. 3 by specifying — in an axiomatic fashion — the class of models to which our derivation applies. Our axioms are fairly minimal, and particular models will generally have extra structure. The main content of the axioms is that the theory should be covariant with respect to $\widetilde{\text{Diff}}_+(S^1)$, the universal covering group of the group of orientation preserving diffeomorphisms of the circle, and invariant under $\widetilde{\text{Möb}}$, the subgroup covering the Möbius transformations of the circle. Each independent component of the stress-energy tensor should correspond to an independent unitary multiplier representation of $\widetilde{\text{Diff}}_+(S^1)$ and the stress-energy tensor itself should be formed from the infinitesimal generators of these representations. As we will see (in Sect. 5.3), these axioms will be loose enough to embrace a wide range of theories: in particular, they encompass all unitary rational CFTs. Nonetheless, they are sufficient conditions for the theory to obey QEIs. We have also collected a number of facts about $\widetilde{\text{Diff}}_+(S^1)$ and its representations in Sect. 3; although much of this material is regarded as well-known, comprehensive references seem not to exist. Thus, our presentation may be of independent interest.

In Section 5, we verify that our axioms are satisfied by models constructed from unitary, highest-weight representations of the Virasoro algebra. Here, we draw on the results of Goodman and Wallach [25] and Toledano Laredo [49] which make precise the sense in which such representations may be ‘exponentiated’ to unitary multiplier representations of $\widetilde{\text{Diff}}_+(S^1)$. As particular models may be built as direct sums of tensor products of Virasoro representations, it is also necessary to maintain explicit control of the multiplier appearing in our representations, and we show that this may be defined in terms of the Bott cocycle. We have not found a full proof of this elsewhere in the literature.

We illustrate our main result by giving several applications in Sect.4.2. In particular, we derive QEIs valid along worldlines, or for averaging over spacetime volumes. A peculiarity of two-dimensional conformal field theory is that QEIs also exist for averages along spacelike or null lines, in contrast to the situation in four-dimensional theories [17, 11]. We also show that similar results hold for conformal field theories in the presence of moving boundaries (often called ‘moving mirrors’). Finally, we discuss the failure of QEIs for sharply cut-off averaging functions.

In conclusion, we mention that it is not clear that quantum energy inequalities involving averaging along worldlines will hold in generic non-conformally invariant theories in two dimensions, or in interacting quantum field theories in dimensions $d > 2$. Olum and Graham [37] have investigated a model with two nonlinearly coupled fields, one of which is in a domain wall configuration, and argued that a static negative energy density can be created in this fashion, which can be made large by tuning the parameters of the model. For these reasons, we suggest that spacetime-averaged QEIs might be a more profitable direction for future research (as mentioned, such QEIs hold in our present context). If one were required to scale the spatial support of the averaging with the temporal support, then averages of long duration would necessarily sense the large positive energy concen-

trated in the domain wall, preventing the overall average from becoming too negative. This may suggest an appropriate formulation for QEIs in more general circumstances.

2 Stress-energy densities of scale-invariant theories in two dimensions

Let us begin by considering a general scale-invariant theory in two-dimensional Minkowski space. The Lüscher–Mack theorem [32, 33, 21] asserts² that if such a theory possesses a symmetric and conserved stress-energy tensor field $T^{\mu\nu}$ obeying

$$\int T^{\mu 0}(x^0, x^1) dx^1 = P^\mu, \quad (2.1)$$

where P^μ are the energy-momentum operators generating spacetime translations, then $T^{\mu\nu}$ is traceless and the independent components T^{00} and T^{01} may be expressed in terms of left- and right-moving chiral components T_L and T_R which each depend on only one lightlike variable:

$$\begin{aligned} T^{00}(x^0, x^1) &= T_R(x^0 - x^1) + T_L(x^0 + x^1) \\ T^{01}(x^0, x^1) &= T_R(x^0 - x^1) - T_L(x^0 + x^1). \end{aligned} \quad (2.2)$$

These fields have scaling dimension two, i.e.,

$$U(\lambda)T_L(v)U(\lambda)^{-1} = \lambda^2 T_L(\lambda v) \quad (2.3)$$

(and an analogous relation for T_R) where $U(\lambda)$ is the unitary implementing the scaling $x^\mu \mapsto \lambda x^\mu$. Moreover, T_L and T_R commute with each other and satisfy relations of the form

$$[T_L(v_1), T_L(v_2)] = i \left(-T'_L(v_1)\delta(v_1 - v_2) + 2T_L(v_1)\delta'(v_1 - v_2) - \frac{c_L}{24\pi}\delta'''(v_1 - v_2)\mathbb{1} \right) \quad (2.4)$$

(and similarly for T_R) where the constants c_L , c_R are the central charges of the theory and are equal under the additional assumption of parity invariance. These commutation relations are closely related to those of the Virasoro algebra, a central extension of the (complexified) Lie algebra of $\text{Diff}_+(S^1)$, the group of orientation preserving diffeomorphisms of the circle.

One of the key properties of a QFT is the spectrum condition, which, in the present context, requires that $P^0 \pm P^1$ be positive operators. It is easy to see that

$$\begin{aligned} P_R &:= \frac{1}{2}(P^0 + P^1) = \int T_R(u) du \\ P_L &:= \frac{1}{2}(P^0 - P^1) = \int T_L(v) dv \end{aligned} \quad (2.5)$$

generate translations along null light-rays, so that P_R generates translations along a left-moving null ray and vice versa. Positivity of these operators does not, however, entail

²The theorem assumes that the theory obeys Wightman's axioms [48].

that the stress-energy densities themselves are everywhere nonnegative. On the contrary: for any v there is a sequence of states ψ_n (in the “Wightman domain” of the theory, and with the same norm) with

$$\langle T_L(v) \rangle_{\psi_n} \longrightarrow -\infty \quad \text{as } n \rightarrow \infty \quad (2.6)$$

(of course there is a similar statement for T_R). It is clearly enough to show this for $v = 0$. Let Ω be the vacuum state and write $T_L(f) = \int T_L(v)f(v)dv$, where f is a nonnegative test function. Now $\langle T_L(v) \rangle_{\Omega} = 0$ by translation- and scale-invariance of the vacuum, while $T_L(f)\Omega \neq 0$ by the Reeh–Schlieder theorem of Wightman theory (excluding the trivial possibility that $T_L(f) = 0$ for all f). Defining $\varphi_\lambda = \Omega - \lambda T_L(f)\Omega$ ($\lambda \in \mathbb{R}$), it is now evident that $\langle \varphi_\lambda | T_L(f)\varphi_\lambda \rangle = -2\lambda \|T_L(f)\Omega\|^2 + \lambda^2 \langle \Omega | T_L(f)^3 \Omega \rangle$ is negative for all sufficiently small positive λ . Hence $\langle T_L(v) \rangle_{\varphi_\lambda}$ must assume negative values for some point v , and we deduce the existence of a state ψ with $\langle T_L(0) \rangle_{\psi} < 0$. Defining $\psi_n = U(n)^{-1}\psi$ and using Eq. (2.3), we obtain Eq. (2.6).

Thus the stress-energy density at individual spacetime points is unbounded from below, as is the case in many other quantum field theories.³ In the following sections, we will formulate precise conditions under which averaged stress-energy densities such as $T_L(f)$ (for nonnegative f) obey state-independent lower bounds: Quantum Energy Inequalities. Our discussion is based on an argument given by Flanagan [14] for the particular case of the massless free scalar field (corresponding to the case $c_L = c_R = 1$). We now sketch the heart of the argument, proceeding rather formally and leaving details aside. This is based on the transformation property of a chiral stress-energy density T of a conformal field theory (representing T_L or T_R) under reparametrisations $v \mapsto V(v)$:

$$T(v) \longrightarrow V'(v)^2 T(V(v)) - \frac{c}{24\pi} \{V, v\} \mathbb{1} \quad (2.7)$$

where

$$\{V, v\} = \frac{V'''(v)}{V'(v)} - \frac{3}{2} \left(\frac{V''(v)}{V'(v)} \right)^2 = -2\sqrt{V'(v)} \frac{d^2}{dv^2} \frac{1}{\sqrt{V'(v)}} \quad (2.8)$$

is the Schwarz derivative of V . That is, to any state ψ there is a state ψ_V (of the same norm) such that

$$\langle T(v) \rangle_{\psi} = V'(v)^2 \langle T(V(v)) \rangle_{\psi_V} - \frac{c}{24\pi} \{V, v\}. \quad (2.9)$$

(The infinitesimal form of this transformation law is simply Eq. (2.4).)

Now suppose we are given a nonnegative test function H and choose a reparametrisation such that $V'(v) = H(v)^{-1}$. Then $\{V, v\} = -2H(v)^{-1/2} \frac{d^2}{dv^2} H(v)^{1/2}$ and

$$\begin{aligned} \int H(v) \langle T(v) \rangle_{\psi} dv &= \int V'(v) \langle T(V(v)) \rangle_{\psi_V} dv + \frac{c}{12\pi} \int \sqrt{H(v)} \frac{d^2}{dv^2} \sqrt{H(v)} dv \\ &= \int \langle T(V) \rangle_{\psi_V} dV - \frac{c}{12\pi} \int \left(\frac{d}{dv} \sqrt{H(v)} \right)^2 dv, \end{aligned} \quad (2.10)$$

³Arguments similar to those given here apply to any theory (in dimension $d \geq 2$) with a scaling limit of positive scaling dimension—see [7].

assuming that the integration by parts in the last term may be accomplished without producing any boundary terms. Since the first term on the right-hand side is $\langle P \rangle_{\psi_V}$, which is nonnegative, we conclude that

$$\int H(v) \langle T(v) \rangle_{\psi} dv \geq -\frac{c}{12\pi} \int \left(\frac{d}{dv} \sqrt{H(v)} \right)^2 dv \quad (2.11)$$

for all states ψ . Moreover, since $\langle P \rangle_{\Omega} = 0$, one expects the bound to be attained for a state ψ such that $\psi_V = \Omega$.

Although the above conveys the essential ideas underlying the QEI derivation (and differs from the scalar case only inasmuch as the central charge is not restricted to $c = 1$) one must exercise greater care to produce a satisfactory argument. There are various reasons for this. First, the reparametrisation rule (2.7) is expected to hold only for those reparametrisations of \mathbb{R} which correspond to a diffeomorphism of the compactified light-ray, and this will not generally be the case for the coordinate V invoked above. (Indeed the reparametrisation is not even defined for H vanishing outside a compact interval.) Second, it is clearly necessary to delineate the class of states for which the bound holds: for example, the left-hand side does not even exist for every state ψ ! Finally, one needs to ensure that the various formal manipulations relating to $\sqrt{H(v)}$ are valid—this technical point conceals some subtle nuances (for example, it is *not* the case that every smooth nonnegative function has a square root which is also smooth and positive [23]).

Flanagan addressed the first two points for the scalar field by an elegant appeal to general covariance in order to compare the theory on the full line with a theory restricted to the interior of the support of H . We have chosen not to make a parallel assumption for general conformal field theories and instead present an alternative resolution of the problem. The upshot is that the QEI (2.11) holds (on a specified domain of states) for any nonnegative H belonging to the Schwartz class⁴ $\mathcal{S}(\mathbb{R})$ and with the integrand on the right-hand side regarded as vanishing at any point where H vanishes. The formal statement and rigorous proof is given in Thm. 4.1.

3 Axiomatic framework

In this section, we delineate in a mathematically precise manner the class of models to which our rigorous QEI derivation in Sec. 4 applies. We will state the required properties of these models in an axiomatic fashion and demonstrate later in Sec. 5 (by drawing together various results in the literature) that there actually exists a wide class of models with those properties.

As we remarked in the previous section, independent components of the stress-energy are associated with independent representations of $\text{Diff}_+(S^1)$, the group of orientation-preserving diffeomorphisms of the circle. It is important for the validity of our arguments establishing the QEI's to have sufficient control over these representations, especially their continuity properties, as well as the spectral properties of certain generators. The essence of our axioms therefore consists in specifying the nature of the representations of $\text{Diff}_+(S^1)$

⁴That is, the class of functions which vanish, together with their derivatives, more rapidly than any inverse power at infinity.

that are allowed to occur in the given conformal field theory. In order to state these properties in a precise and efficient way, we will set the stage in the following subsections by recalling the salient facts about the group $\text{Diff}_+(S^1)$ and its unitary representations, especially the so-called “unitary multiplier representations”. With those facts at hand, we will then state our axioms for the conformal field theories considered in this paper in Subsec. 3.3.

Some of our later arguments in Sec. 5 establishing the existence of conformal field theories obeying our axioms will also require us to know certain properties of the phases that occur in the unitary multiplier representations. Our presentation will therefore include a discussion and analysis of those, even though this would not, strictly speaking, be necessary in order to present our axioms.

3.1 Preliminaries concerning $\text{Diff}_+(S^1)$

3.1.1 Group structure

Beginning with the circle itself, S^1 will be regarded as the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ in the complex plane. Under the Cayley transform $C : z \mapsto i(1 - z)/(1 + z)$, the circle (less -1) is mapped onto \mathbb{R} ; we will refer to this as the ‘light-ray picture’ in what follows. The real line will also enter as the universal covering group of S^1 , via the map $\theta \mapsto \tan \frac{1}{2}\theta$. We will call this copy of \mathbb{R} the ‘unrolled circle’ to distinguish it from the light-ray picture.

A function f on S^1 will be said to be differentiable if $\mathbb{R} \ni \theta \mapsto f(e^{i\theta})$ is, and the derivative f' will be given by

$$ie^{i\theta} f'(e^{i\theta}) = \frac{d}{d\theta} f(e^{i\theta}). \quad (3.1)$$

We may now define $\text{Diff}_+(S^1)$ to be the group (under composition) of all diffeomorphisms σ of the circle to itself which are orientation preserving, in the sense that $\sigma(z)$ winds once positively around the origin as z does. We will also be concerned with its universal covering group $\widetilde{\text{Diff}}_+(S^1)$, which may be identified with the group of diffeomorphisms ρ of \mathbb{R} obeying

$$\rho(\theta + 2\pi) = \rho(\theta) + 2\pi, \quad (3.2)$$

each such map determining a $\mathring{\rho} \in \text{Diff}_+(S^1)$ by

$$\mathring{\rho}(e^{i\theta}) = e^{i\rho(\theta)}. \quad (3.3)$$

As examples, let us note three particularly important one-parameter subgroups of $\widetilde{\text{Diff}}_+(S^1)$, which will appear in our discussion: namely R_ϕ ($\phi \in \mathbb{R}$) corresponding to rotations on the circle, and T_s ($s \in \mathbb{R}$) and D_λ ($\lambda > 0$) corresponding respectively to translations and dilations on the light-ray. On the unrolled circle, the rotations are defined by $R_\phi(\theta) = \theta + \phi$ [so that $\mathring{R}_\phi(z) = ze^{i\phi}$], while the translations and dilations are defined by

$$T_s(\theta) = 2 \tan^{-1} \left(s + \tan \frac{\theta}{2} \right) \quad \text{for } \theta \in (-\pi, \pi) \quad (3.4)$$

and

$$D_\lambda(\theta) = 2 \tan^{-1} \left(\lambda \tan \frac{\theta}{2} \right) \quad \text{for } \theta \in (-\pi, \pi) \quad (3.5)$$

and are extended to other values of θ by Eq. (3.2) and continuity. In each case the principal branch of arctangent should be understood.

The rotations and translations may be combined to obtain a further one-parameter subgroup of interest, namely the special conformal transformations $S_s = R_\pi T_s R_\pi^{-1}$ ($s \in \mathbb{R}$). We also observe that the elements $R_{2\pi k}$ ($k \in \mathbb{Z}$) constitute the centre of $\widetilde{\text{Diff}}_+(S^1)$ as a consequence of Eq. (3.2).

Taken together, the rotations, translations and dilations generate the universal cover $\widetilde{\text{Möb}}$ of Möb, the group of Möbius transformations of S^1 . This group will be the unbroken symmetry of conformal field theory; as we will see, these theories are only covariant (rather than invariant) with respect to the diffeomorphisms. Möbius transformations of the circle take the form

$$z \mapsto \frac{\alpha z + \beta}{\beta z + \alpha}, \quad (3.6)$$

where $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 - |\beta|^2 = 1$. Noting the invariance of Eq. (3.6) under simultaneous negation of α and β , we see that $\widetilde{\text{Möb}} \cong \text{PSU}(1, 1) = \text{SU}(1, 1)/\{\mathbb{1}, -\mathbb{1}\}$. In the light-ray picture, elements of Möb act according to

$$u \mapsto \frac{au + b}{cu + d}, \quad (3.7)$$

for real coefficients a, b, c, d with $ad - bc = 1$, and this provides a group isomorphism $\widetilde{\text{Möb}} \cong \text{PSL}(2, \mathbb{R})$.

3.1.2 Lie group structure

Let $C^\infty(\mathbb{R}; \mathbb{R})$ be the space of smooth, real-valued functions on \mathbb{R} equipped with the topology of uniform convergence of functions and their derivatives of all orders,⁵ which makes it into a Fréchet space. It is not hard to show that $\widetilde{\text{Diff}}_+(S^1)$ is an open subset of $C^\infty(\mathbb{R}; \mathbb{R})$ and we endow it with the relative topology. We may also equip $\widetilde{\text{Diff}}_+(S^1)$ with an atlas of charts modelled on $C_{2\pi}^\infty(\mathbb{R}; \mathbb{R})$, the (Fréchet) subspace of $C^\infty(\mathbb{R}; \mathbb{R})$ of (2π) -periodic functions, in such a way that the group multiplication and inversion of $\widetilde{\text{Diff}}_+(S^1)$ are smooth. With these definitions, $\widetilde{\text{Diff}}_+(S^1)$ becomes a Fréchet Lie group, and the same structure can be induced on $\text{Diff}_+(S^1)$ by the quotient map. (Cf., for example, Sec. 6 of [34] and example 4.2.6 in [26].)

The Lie algebra of these groups is therefore $C_{2\pi}^\infty(\mathbb{R}; \mathbb{R})$, which may be conveniently regarded as the space of real vector fields on the circle, $\text{Vect}_{\mathbb{R}}(S^1)$. Indeed, given any smooth one-parameter curve $t \mapsto \rho_t \in \widetilde{\text{Diff}}_+(S^1)$, we obtain a vector field X on S^1 by

$$(Xg)(z) = \left. \frac{d}{dt} g(\rho_t(z)) \right|_{t=0} \quad (g \in C^\infty(S^1)), \quad (3.8)$$

⁵That is, $f_k \rightarrow f$ iff $\sup_{x \in \mathbb{R}} |f_k^{(r)}(x) - f^{(r)}(x)| \rightarrow 0$ for all $r \geq 0$, where $f^{(r)}$ is the r 'th derivative of f .

which corresponds to the tangent vector to ρ_t at $t = 0$. This vector field is said to be real because it may be expressed in the form

$$(Xg)(e^{i\theta}) = f(e^{i\theta}) \frac{d}{d\theta} g(e^{i\theta}) \quad (g \in C^\infty(S^1)) \quad (3.9)$$

for some real-valued $f \in C^\infty(S^1)$. For our purposes, however, it will be more convenient to identify $\text{Vect}(S^1)$ and $C^\infty(S^1)$ so that $f \in C^\infty(S^1)$ corresponds to the vector field $\mathbf{f} \in \text{Vect}(S^1)$ with action

$$(\mathbf{f}g)(z) = f(z)g'(z). \quad (3.10)$$

With this identification, \mathbf{f} is real if and only if f is invariant under the antilinear conjugation $(\Gamma f)(z) = -z^2 \overline{f(z)}$. We will denote the space of $f \in C^\infty(S^1)$ obeying $\Gamma f = f$ by $C_{\Gamma}^\infty(S^1)$. As examples, it is straightforward to check that the tangent vector to the curve $\phi \mapsto R_\phi$ at $\phi = 0$ corresponds to the function $z \mapsto iz$, while those of $s \mapsto T_s$ and $s \mapsto S_s$ at $s = 0$ correspond to $z \mapsto \frac{i}{2}(1+z)^2$ and $z \mapsto -\frac{i}{2}(1-z)^2$ respectively. All three functions are invariant under Γ , as $\bar{z} = z^{-1}$ on the circle.

3.1.3 The Bott cocycle

As already remarked, the Virasoro algebras underlying CFT are central extensions of the complexified Lie algebra of $\text{Diff}_+(S^1)$. At the level of groups, these extensions are described by the Bott cocycle $B : \text{Diff}_+(S^1) \times \text{Diff}_+(S^1) \rightarrow \mathbb{R}$ given by⁶

$$B(\sigma_1, \sigma_2) = -\frac{1}{48\pi} \text{Re} \int_{S^1} \log((\sigma_1 \circ \sigma_2)'(z)) \frac{d}{dz} \log(\sigma_2'(z)) dz, \quad (3.11)$$

which lifts to a cocycle $\tilde{B}(\rho_1, \rho_2) = B(\overset{\circ}{\rho}_1, \overset{\circ}{\rho}_2)$ on $\widetilde{\text{Diff}_+(S^1)}$. Note that the logarithms do not introduce any ambiguity into this formula, because $\sigma'(z)$ has winding number zero about the origin for $\sigma \in \text{Diff}_+(S^1)$.

Let us now collect some properties of B and \tilde{B} . First, it is immediate from the definition that

$$B(\text{id}, \sigma) = B(\sigma, \text{id}) = 0, \quad B(\sigma, \sigma^{-1}) = 0 \quad (\sigma \in \text{Diff}_+(S^1)) \quad (3.12)$$

and that the cocycle property

$$B(\sigma_1, \sigma_2) + B(\sigma_1 \sigma_2, \sigma_3) = B(\sigma_2, \sigma_3) + B(\sigma_1, \sigma_2 \sigma_3) \quad (3.13)$$

holds for all $\sigma_1, \sigma_2, \sigma_3 \in \text{Diff}_+(S^1)$ (analogous results hold also for \tilde{B}).

Second, B vanishes on $\text{Möb} \times \text{Möb}$ by the Cauchy integral formula because the integrand is holomorphic in the unit disk in that case [47]. Similarly, \tilde{B} vanishes on $\widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$. Third, the following first derivatives are easily computed:

$$D_1 \tilde{B}|_{(\text{id}, \rho)}(\mathbf{f}) = -\frac{1}{48\pi} \text{Re} \int_{S^1} f'(\overset{\circ}{\rho}(z)) \frac{\overset{\circ}{\rho}''(z)}{\overset{\circ}{\rho}'(z)} dz \quad (3.14)$$

⁶This differs slightly from the form usually given, to which it is cohomologous, but which corresponds to the Gel'fand–Fuks (rather than Virasoro) cocycle at the level of Lie algebras. The form given here is drawn from [47] with some typographical errors corrected.

and

$$D_2 \widetilde{B}|_{(\rho, \text{id})}(\mathbf{f}) = -\frac{1}{48\pi} \text{Re} \int_{S^1} \left\{ \frac{\dot{\rho}'''(z)}{\dot{\rho}'(z)} - \left(\frac{\dot{\rho}''(z)}{\dot{\rho}'(z)} \right)^2 \right\} f(z) dz, \quad (3.15)$$

from which the second derivative

$$D_{12} \widetilde{B}|_{(\text{id}, \text{id})}(\mathbf{f}, \mathbf{g}) = -\frac{1}{48\pi} \text{Re} \int_{S^1} f'(z) g''(z) dz = \frac{1}{2} \omega(f, g) \quad (3.16)$$

follows easily, where

$$\omega(f, g) = \frac{1}{48\pi} \int_{S^1} (f(z) g'''(z) - f'''(z) g(z)) dz \quad (3.17)$$

is the Virasoro cocycle, i.e., the Lie algebra cocycle corresponding to \widetilde{B} . Note that the integral in Eq. (3.17) is automatically real for $f, g \in C_1^\infty(S^1)$.

3.2 Unitary multiplier representations of $\widetilde{\text{Diff}}_+(S^1)$

Let \mathcal{H} be a Hilbert space, and suppose that each $\rho \in \widetilde{\text{Diff}}_+(S^1)$ is assigned a unitary operator $U(\rho)$ on \mathcal{H} so that

$$U(\rho)U(\rho') = e^{ic\widetilde{B}(\rho, \rho')} U(\rho\rho') \quad (3.18)$$

holds for all $\rho, \rho' \in \widetilde{\text{Diff}}_+(S^1)$, where \widetilde{B} is the Bott cocycle introduced above. Then the map $\rho \mapsto U(\rho)$ will be called a *unitary multiplier representation* of $\widetilde{\text{Diff}}_+(S^1)$ with cocycle \widetilde{B} and central charge c . Representations of this type will form the main component of our axioms for CFT and we now collect some of their properties.

We begin by noting that U restricts to $\widetilde{\text{Möb}}$ as a bona fide unitary representation because \widetilde{B} vanishes on $\widetilde{\text{Möb}} \times \widetilde{\text{Möb}}$. It therefore obeys $U(\text{id}) = \mathbb{1}$, and, because we also have $\widetilde{B}(\rho, \rho^{-1}) = 0$ for all $\rho \in \widetilde{\text{Diff}}_+(S^1)$, we easily obtain $U(\rho^{-1}) = U(\rho)^{-1}$ from Eq. (3.18).

Now assume, in addition, that the map $\rho \mapsto U(\rho)\psi$ is continuous for each fixed $\psi \in \mathcal{H}$, i.e., the representation is *strongly continuous*. This assumption permits us to obtain the infinitesimal generators of the representation, which are interpreted as smeared stress-energy densities. In more detail: for each $f \in C_1^\infty(S^1)$, let $\mathbf{f} \in \text{Vect}_{\mathbb{R}}(S^1)$ be the corresponding real vector field and define a self-adjoint operator $\Theta(f)$ by

$$\Theta(f)\psi = \frac{1}{i} \frac{d}{ds} U(\exp(sf))\psi \Big|_{s=0} \quad (3.19)$$

on the dense domain of ψ for which the derivative exists.⁷ We then define $\Theta(f)$ for arbitrary $f \in C^\infty(S^1)$ by $\Theta(f) = \Theta(\frac{1}{2}(f + \Gamma f)) + i\Theta(\frac{1}{2i}(f - \Gamma f))$ on the appropriate

⁷The additive group of real numbers does not admit nontrivial smooth cocycles (see, e.g., Thm. 10.38 in [50]). Thus, because $s \mapsto U(\exp(sf))$ is a strongly continuous unitary multiplier representation of $(\mathbb{R}, +)$ with a smooth multiplier, we may write $U(\exp(sf)) = e^{i\alpha(s)}V(s)$ where $V(s)$ is a strongly continuous one-parameter group of unitaries and α is a smooth and real-valued. Stone's theorem and the Leibniz rule then guarantee that Eq. (3.19) does indeed define a self-adjoint operator with domain equal to the set of ψ for which the derivative exists.

intersection of domains, so that

$$\Theta(f)^* = \Theta(\Gamma f) \quad (3.20)$$

holds on $D(\Theta(f))$. We will also use the informal notation

$$\Theta(f) = \int_{S^1} f(z)\Theta(z) dz \quad (3.21)$$

as a convenient book-keeping device, but $\Theta(z)$ should not be interpreted as an operator in its own right. For example, let H be the generator of the 1-parameter subgroup R_ϕ of $\widetilde{\text{Möb}}$. Then

$$H\psi = \frac{1}{i} \frac{d}{d\phi} U(R_\phi)\psi \Big|_{\phi=0} = \frac{1}{i} \frac{d}{d\phi} U(\exp(\phi\mathbf{f}))\psi \Big|_{\phi=0} \quad (3.22)$$

for any $\psi \in D(H)$, where \mathbf{f} is the tangent vector to $\phi \mapsto R_\phi$ at $\phi = 0$. As shown above, this corresponds to the function $f(z) = iz$, so we write

$$H = \int_{S^1} iz\Theta(z) dz. \quad (3.23)$$

Similarly, the generators P and K of the 1-parameter subgroups $s \mapsto T_s$ and $s \mapsto S_s$ may be written as

$$P = \frac{i}{2} \int_{S^1} (1+z)^2 \Theta(z) dz \quad (3.24)$$

$$K = -\frac{i}{2} \int_{S^1} (1-z)^2 \Theta(z) dz, \quad (3.25)$$

so that

$$H = \frac{1}{2}(P + K), \quad (3.26)$$

on the intersection $D(H) \cap D(P) \cap D(K)$.

One of the key properties we will require is the transformation law of the smeared stress-energy densities under diffeomorphisms, provided by the following result.

Proposition 3.1 *Assume that \mathcal{H} carries a strongly continuous unitary multiplier representation of $\widetilde{\text{Diff}}_+(S^1)$ obeying Eq. (3.18). Suppose \mathcal{H} contains a dense domain \mathcal{D} which is invariant under each $U(\rho)$ ($\rho \in \widetilde{\text{Diff}}_+(S^1)$) and contained in each $D(\Theta(f))$ ($f \in C^\infty(S^1)$). Then \mathcal{D} is a core for each $\Theta(f)$ with $f = \Gamma f$. Moreover, the $\Theta(f)$ transform according to*

$$U(\rho)\Theta(f)U(\rho)^{-1} = \Theta(f_\rho) - \frac{c}{24\pi} \int_{S^1} \{\dot{\rho}, z\} f(z) dz \mathbb{1}, \quad (3.27)$$

where $f_\rho(z) = \dot{\rho}'(\dot{\rho}^{-1}(z))f(\dot{\rho}^{-1}(z))$ corresponds to the vector field $\mathbf{f}_\rho = \text{Ad}(\rho)(\mathbf{f})$, and obey the commutation relations

$$i[\Theta(g), \Theta(f)] = \Theta(g'f - f'g) + c\omega(g, f)\mathbb{1}, \quad (3.28)$$

where ω is the Virasoro cocycle. Here, both Eqs. (3.27) and (3.28) are valid as identities when applied to vectors in \mathcal{D} , and f, g are arbitrary elements of $C^\infty(S^1)$.

Remark: Eq. (3.27) may also be written in the ‘unsmeared form’

$$U(\rho)\Theta(z)U(\rho)^{-1} = \dot{\rho}'(z)^2\Theta(\dot{\rho}(z)) - \frac{c}{24\pi}\{\dot{\rho}, z\}\mathbb{1}. \quad (3.29)$$

Proof: That \mathcal{D} is a core follows from Theorem VIII.11 in [43]. To obtain the stated transformation property, choose $f \in C^\infty_\Gamma(S^1)$ and let \mathbf{f} be the corresponding vector field. Then for any $\psi \in \mathcal{D}$ and $\rho \in \widetilde{\text{Diff}}_+(S^1)$,

$$\begin{aligned} U(\rho)\Theta(f)U(\rho)^{-1}\psi &= \left. \frac{1}{i} \frac{d}{ds} U(\rho)U(\exp(s\mathbf{f}))U(\rho^{-1})\psi \right|_{s=0} \\ &= \left. \frac{1}{i} \frac{d}{ds} e^{ic\varphi(s)} U(\rho \exp(s\mathbf{f})\rho^{-1})\psi \right|_{s=0} \\ &= \left. \frac{1}{i} \frac{d}{ds} e^{ic\varphi(s)} U(\exp(s\mathbf{f}_\rho))\psi \right|_{s=0} \\ &= \Theta(f_\rho)\psi - c\varphi'(0)\psi, \end{aligned} \quad (3.30)$$

where $\varphi(s) = \tilde{B}(\rho, \exp(s\mathbf{f})) + \tilde{B}(\rho \exp(s\mathbf{f}), \rho^{-1})$. Using the fact that $\rho \exp(s\mathbf{f}) = \exp(s\mathbf{f}_\rho)\rho$, the cocycle relation Eq. (3.13), and the elementary properties Eq. (3.12), φ may be rewritten in the form

$$\varphi(s) = \tilde{B}(\rho, \exp(s\mathbf{f})) - \tilde{B}(\exp(s\mathbf{f}_\rho), \rho). \quad (3.31)$$

It is now a straightforward exercise, using the first derivatives of \tilde{B} given in the previous subsection and the definition (2.8) of the Schwarz derivative, to show that

$$\varphi'(0) = -\frac{1}{24\pi} \int_{S^1} \{\dot{\rho}, z\} f(z) dz \quad (3.32)$$

(the integral is real because $\dot{\rho} \in \text{Diff}_+(S^1)$ and $f \in C^\infty_\Gamma(S^1)$). Substituting this in Eq. (3.30), we have obtained Eq. (3.27) (applied to ψ). By now putting $\rho = \exp s\mathbf{g}$, where the vector field \mathbf{g} corresponds to some $g \in C^\infty_\Gamma(S^1)$, and then differentiating with respect to s at $s = 0$, we obtain the commutation relations Eq. (3.28) as an identity on \mathcal{D} . The relations Eq. (3.28) then extend by linearity to general $f, g \in C^\infty(S^1)$. ■

The following are simple applications of the above result:

$$U(D_\lambda)PU(D_\lambda)^{-1} = \lambda P; \quad U(D_\lambda)KU(D_\lambda)^{-1} = \lambda^{-1}K; \quad K = U(R_\pi)PU(R_\pi)^{-1} \quad (3.33)$$

(note that the Schwarz derivative of a Möbius transformation vanishes). In particular, we observe that K and P must have the same spectrum, which (as it is nonempty, closed and dilation-invariant) must be one of the four possibilities $\{0\}$, $[0, \infty)$, $(-\infty, 0]$ or \mathbb{R} . Restricting attention to the first two cases, in which $P \geq 0$, we find that $H \geq 0$ by Eq. (3.26), because H is thereby positive on \mathcal{D} , on which it is essentially self-adjoint. Conversely, if $H \geq 0$, we use the identity⁸

$$U(D_\lambda)HU(D_\lambda)^{-1} = \frac{1}{2}(\lambda P + \lambda^{-1}K) \quad (3.34)$$

⁸The fact that $P \geq 0$ iff $H \geq 0$ is well-known, but is usually obtained from a detailed knowledge of the unitary representations of Möb. Combine, for example, the proof of Prop. 9.2.6 in [41] with the representation theory given in [31, 29, 42]. The approach given here is adapted from Prop. 1 of [27] (note that the conventions differ slightly).

on \mathcal{D} to deduce that $P \geq 0$ because

$$\langle \psi | P\psi \rangle = \lim_{\lambda \rightarrow \infty} \lambda^{-1} \langle U(D_\lambda)^{-1} \psi | HU(D_\lambda)^{-1} \psi \rangle \geq 0 \quad (3.35)$$

for all $\psi \in \mathcal{D}$, which is again a core for P . Clearly, $P = 0$ if and only if $H = 0$, so $\text{spec}(P) = [0, \infty)$ if and only if H is a non-zero positive operator.

3.3 Axioms

We now come to the statement of the axioms we shall adopt for conformal field theory. These are to be regarded as minimal requirements: specific models will have more structure and possibly an enlarged symmetry group. Nonetheless, the following axioms are already sufficient to establish the QEIs, and are satisfied in models built from Virasoro representations (see Sect. 5). Note that, as they include the assumptions of Sect. 3.2, all the conclusions of that subsection apply to such theories, particularly Prop. 3.1.

For simplicity, we state our axioms for a conformal field theory with a single component of stress-energy; at the end of this section we describe the (straightforward) extension to two independent components.

A. Hilbert space, diffeomorphism group and energy positivity

1. The Hilbert space \mathcal{H} of the theory carries a strongly continuous unitary multiplier representation $\rho \mapsto U(\rho)$ of $\widehat{\text{Diff}}_+(S^1)$ obeying Eq. (3.18), with central charge $c > 0$.
2. Up to phase there is a unique unit vector $\Omega \in \mathcal{H}$ which is invariant under the restriction of U to Möb , and which will be called the vacuum vector.
3. The generator P of the one-parameter translation subgroup $s \mapsto U(T_s)$ is assumed to be a positive self-adjoint operator. (An equivalent requirement is that the generator H of the rotation subgroup $\phi \mapsto U(R_\phi)$ be positive, by the remarks above.)

B. Stress-energy density

The (smeared) stress-energy density $\Theta(f)$ is defined as the generator of $U(\rho)$, as described in the previous subsection, see Eq. (3.19). We assume that \mathcal{H} contains a dense subspace $\mathcal{D} \subset \mathcal{H}$ such that:

1. \mathcal{D} is invariant under each $U(\rho)$, contains Ω and is contained in each $D(\Theta(f))$ for all $f \in C^\infty(S^1)$.
2. For each $\psi \in \mathcal{D}$, the map $f \mapsto \Theta(f)\psi$ is a vector-valued distribution on $C^\infty(S^1)$ (equipped with its usual topology of uniform convergence of functions and all their derivatives).
3. For each $\psi \in \mathcal{D}$, $\langle \Theta(z) \rangle_\psi$ is smooth on S^1 .

Given a theory of the above type living on a circle, we may define a stress-energy density $T(v)$ living on a light-ray by the ‘unsmeared’ formula

$$T(v) = \left(\frac{dz}{dv} \right)^2 \Theta(z(v)) = -\frac{4}{(1-iv)^4} \Theta(z(v)), \quad (3.36)$$

where

$$z(v) = C^{-1}(v) = \frac{1 + iv}{1 - iv} \quad (3.37)$$

maps \mathbb{R} to S^1 (less -1 , which represents the ‘point at infinity’). The class of allowed smearing functions in this picture consists of all $F \in C^\infty(\mathbb{R})$ for which $z \mapsto \frac{i}{2}(1+z)^2 F(C(z))$ is smooth on S^1 [with an appropriate limiting definition at $z = -1$]. As before, we use an integral notation to denote such smearings, thus, for example, the relationship Eq. (3.24) now reads

$$P = \int T(v) dv. \quad (3.38)$$

We may also deduce from axiom B.3 and Eq. (3.36) that $\langle T(v) \rangle_\psi$ decays as $O(v^{-4})$ as $|v| \rightarrow \infty$ for $\psi \in \mathcal{D}$.

Finally, suppose $\rho \in \widetilde{\text{Diff}}_+(S^1)$ fixes the point at infinity, i.e., $\rho^\circ(-1) = -1$, and define a reparametrisation $v \mapsto V(v)$ of \mathbb{R} implicitly by $z(V(v)) = \rho^\circ(z(v))$. Then the transformation law Eq. (3.29) becomes

$$U(\rho)T(v)U(\rho)^{-1} = V'(v)^2 T(V(v)) - \frac{c}{24\pi} \{V, v\} \mathbb{1}. \quad (3.39)$$

Here, we have used the chain rule for Schwarz derivatives

$$\{z, x\} = \{z, y\} \left(\frac{dy}{dx} \right)^2 + \{y, x\}, \quad (3.40)$$

where $z = z(y)$, $y = y(x)$, and the fact that the Schwarz derivative of a Möbius transformation vanishes identically, so $\{z(v), v\} = 0$.

The above structure is already enough to encompass an interesting class of theories in Minkowski space: namely, boundary conformal field theories (see, e.g., [55], or [30] for a recent treatment in terms of algebraic quantum field theory). In these theories, there is a single underlying representation U of $\widetilde{\text{Diff}}_+(S^1)$ with corresponding stress-energy density T , and the theory lives on the right-hand half $x^1 > 0$ of Minkowski space with stress-energy tensor given by Eq. (2.2) where $T_L = T_R = T$. In particular, T^{01} vanishes on the timelike line $x^1 = 0$, reflecting the boundary condition that no energy-momentum should flow out of the half-space $x^1 > 0$.

A more general class of theories corresponds to the ‘moving mirror’ models studied in [20] (for particular case of the massless scalar field). Instead of an inertial boundary $x^1 = 0$, we consider a moving boundary with trajectory $v = p(u)$, where $u = x^0 - x^1$ and $v = x^0 + x^1$ are null coordinates on Minkowski space. The theory is defined on the portion of Minkowski space to the right of this curve, i.e., $v > p(u)$. Restricting, for simplicity, to the case in which $u \mapsto p(u)$ lifts to an element $\rho \in \widetilde{\text{Diff}}_+(S^1)$, the stress-energy tensor is again defined by Eq. (2.2), where we now put

$$T_L(v) = T(v), \quad T_R(u) = U(\rho)T(u)U(\rho)^{-1}. \quad (3.41)$$

(Boundary CFT corresponds, of course, to the case $p(u) = u$ and hence $U(\rho) = \mathbb{1}$.) It follows Eq. (3.39) and $\langle T(v) \rangle_\Omega$ that the energy density in the vacuum state Ω is then

$$\langle T_{00}(x^0, x^1) \rangle_\Omega = -\frac{c}{24\pi} \{p, u\} = \frac{c}{12\pi} \sqrt{p'(u)} \frac{d^2}{du^2} \frac{1}{\sqrt{p'(u)}}, \quad (3.42)$$

which reduces to the result of [20] in the case $c = 1$. In fact the moving mirror spacetime is conformally related to the boundary spacetime considered above (under the transformation $(u, v) \mapsto (p(u), v)$) and this dictates the form of Eq. (3.41), together with the boundary condition that Ω should be the ‘in’ vacuum at past null infinity. It is intended to discuss this more fully elsewhere.

Conformal field theories on the whole of Minkowski space must have two independent components of stress-energy, by the Lüscher–Mack theorem (see Sec. 2). We now briefly explain the required modifications to our axioms to permit the description of this situation. There are now two commuting projective unitary representations U_L and U_R of $\widetilde{\text{Diff}}_+(S^1)$ each restricting to $\widetilde{\text{Möb}}$ as a unitary representation. We assume the existence of a unique vacuum vector Ω invariant under both copies of $\widetilde{\text{Möb}}$ and assume that the two translation generators P_L, P_R are positive. The domain \mathcal{D} is assumed to be invariant under both U_L and U_R , and each representation is generated (in the sense of Eq. (3.19)) by a corresponding stress-energy density Θ_L, Θ_R , each of which obeys the regularity assumptions of axiom B. Each stress-energy density transforms according to the Eq. (3.29) (with central charge c_L or c_R as appropriate) under the corresponding representation of $\widetilde{\text{Diff}}_+(S^1)$ but is invariant under the adjoint action of the other copy. We also define light-ray fields T_L and T_R in the same way as above, and then define the stress-energy tensor by Eq. (2.2). In particular, one may construct such a theory as a tensor product of two conformal field theories with a single component of stress-energy, but this is by no means the only possibility.

Clearly, we could envisage theories with any number of independent components of stress-energy in a similar fashion, but the interpretation as a theory in Minkowski space is no longer clear.

4 Quantum Energy Inequalities in CFT

4.1 Main result

We are now in a position to state our main result. The notation is as in the previous section.

Theorem 4.1 *Consider a conformal field theory with a single component T of stress-energy. For any nonnegative $G \in \mathcal{S}(\mathbb{R})$, the quantum energy inequality*

$$\int G(v) \langle T(v) \rangle_\psi dv \geq -\frac{c}{12\pi} \int \left(\frac{d}{dv} \sqrt{G(v)} \right)^2 dv \quad (4.1)$$

holds for all $\psi \in \mathcal{D}$, where the derivative $d/dv \sqrt{G}$ is defined to be zero for points at which G vanishes:

$$\frac{d}{dv} \sqrt{G(v)} = \begin{cases} G'(v)/(2\sqrt{G(v)}) & G(v) \neq 0 \\ 0 & G(v) = 0. \end{cases} \quad (4.2)$$

Moreover, this bound is sharp: the right-hand side is the infimum of the left-hand side as ψ varies in \mathcal{D} .

In a conformal field theory with two independent components of stress-energy, both T_L and T_R obey bounds of the above type (with weight functions $G_L, G_R \in \mathcal{S}(\mathbb{R})$) which are simultaneously sharp in the sense that there is a sequence of states $\psi_n \in \mathcal{D}$ with

$$\begin{aligned} \int G_L(v) \langle T_L(v) \rangle_{\psi_n} dv &\longrightarrow -\frac{c_L}{12\pi} \int \left(\frac{d}{dv} \sqrt{G_L(v)} \right)^2 du \\ \int G_R(u) \langle T_R(u) \rangle_{\psi_n} du &\longrightarrow -\frac{c_R}{12\pi} \int \left(\frac{d}{du} \sqrt{G_R(u)} \right)^2 dv \end{aligned} \quad (4.3)$$

as $n \rightarrow \infty$.

Remarks: 1) It is proved in Corollary A.2 in the Appendix that the square root \sqrt{G} of a non-negative Schwartz function is in fact a distribution in the Sobolev space $W^1(\mathbb{R})$ (i.e., has square-integrable first derivative) and that the above rule (4.2) for defining its derivative coincides with the usual notion of the distributional (or “weak”) derivative of such a distribution. In particular, this formally establishes that the integral representing our QEI bound on the right side of Eq. (4.1) is actually finite even for smearing functions G that are not *strictly* positive.

2) As \mathcal{D} is a core for any smeared energy density the QEIs can be stated as operator inequalities, e.g.,

$$\int G(v) T(v) dv \geq -\frac{c}{12\pi} \int \left(\frac{d}{dv} \sqrt{G(v)} \right)^2 dv \mathbb{1} \quad (4.4)$$

by standard quadratic form arguments (see, e.g., Theorem X.23 in [44]). The fact that QEIs for T_L and T_R are simultaneously sharp is simply the statement that the sequence ψ_n in Eq. (4.3) belongs to the joint spectrum of the two operators concerned.

3) The above results can of course be transformed to give QEIs on the field Θ on the circle; one can also follow the general strategy given below to derive QEIs based on positivity of H (rather than P), which would be more natural in that setting. In addition, the results can be extended to any number of independent stress-energy operators. We will not pursue these directions here.

Proof: The proof is broken down into various stages. We start with the case in which the nonnegative function G is smooth and compactly supported, and then extend to the Schwartz class. As mentioned above, the obstruction to a straightforward use of the argument summarised in Sec. 2 is that the equation $V'(v) = 1/G(v)$ does not define a diffeomorphism which can be lifted to the circle. To circumvent this problem, we modify G to a function $H_{\epsilon,n}$ depending upon regulators ϵ and n . The function $H_{\epsilon,n}$ is constructed in such a way that the formal argument given Sec. 2 holds rigorously, and so that the desired bound is obtained as the regulators are removed.

The two regulators have the following effect. First, we add the constant ϵ to $G(v)$, thus obtaining a reparametrisation of the whole line by $V'(v) = 1/(G(v) + \epsilon)$. Although this reparametrisation fixes the point at infinity, it does not lift to a diffeomorphism of the circle as it has a discontinuous second derivative at $z = -1$ (unless G is identically zero). The remedy is to subtract from $G(v) + \epsilon$ a small compactly supported correction, which is translated to the right (and slightly rescaled) as n increases. We can then exploit

the decay of $\langle T(v) \rangle_\psi$ as $v \rightarrow \infty$ in order to control the limit $n \rightarrow \infty$. Other approaches to this issue are probably possible.⁹

The construction and properties of $H_{\epsilon,n}$ are summarised by the following lemma, whose proof is deferred to the end of this section.

Lemma 4.2 *Given a nonnegative $G \in C_0^\infty(\mathbb{R})$, let*

$$\lambda_\epsilon = \frac{1}{|\text{supp } G|} \int \frac{G(v)}{G(v) + \epsilon} dv, \quad (4.5)$$

where $|\text{supp } G|$ denotes the Lebesgue measure of the support of G . Then λ_ϵ increases as $\epsilon \rightarrow 0^+$, with $\lim_{\epsilon \rightarrow 0^+} \lambda_\epsilon = 1$. Let $\eta \in C_0^\infty(\mathbb{R})$ obey $0 \leq \eta(v) \leq 1/2$ for all v and

$$\int \frac{\eta(v)}{1 - \eta(v)} dv = |\text{supp } G|, \quad (4.6)$$

and set

$$\eta_{n,\epsilon}(v) = \eta\left(\frac{v-n}{\lambda_\epsilon}\right). \quad (4.7)$$

Then there exists an n_0 such that, for all $n \geq n_0$ and $\epsilon > 0$,

1. the support of $\eta_{n,\epsilon}$ lies to the right of $\text{supp } G$
2. there is a diffeomorphism $\rho_{n,\epsilon} \in \widetilde{\text{Diff}}_+(S^1)$ corresponding to a reparametrisation $v \mapsto V_{n,\epsilon}(v)$ of the light-ray with

$$V'_{n,\epsilon}(v) = \frac{1}{H_{n,\epsilon}(v)}, \quad (4.8)$$

where $H_{n,\epsilon}(v) = G(v) + \epsilon(1 - \eta_{n,\epsilon}(v))$.

Now let $\psi \in \mathcal{D}$ be arbitrary. Then the formal calculation of Sec. 2 holds rigorously if H is replaced by the function $H_{n,\epsilon}$ given in item (2) of the above lemma, and if ψ_V is replaced by $U(\rho_{n,\epsilon})\psi$. This yields

$$\int H_{n,\epsilon}(v) \langle T(v) \rangle_\psi \geq -\frac{c}{12\pi} \int \left(\frac{d}{dv} \sqrt{H_{n,\epsilon}(v)} \right)^2 dv, \quad (4.9)$$

the required integration by parts being valid because $H_{n,\epsilon}$ is constant outside a compact interval. For $n \geq n_0$, the supports of G and $\eta_{n,\epsilon}$ are disjoint by item (1) of the lemma, so the integral on the right-hand side falls into two pieces

$$\begin{aligned} 4 \int \left(\frac{d}{dv} \sqrt{H_{n,\epsilon}(v)} \right)^2 dv &= \int \frac{G'(v)^2}{G(v) + \epsilon} dv + \epsilon \int \frac{\eta'_{n,\epsilon}(v)^2}{1 - \eta_{n,\epsilon}(v)} dv \\ &= \int \frac{G'(v)^2}{G(v) + \epsilon} dv + \frac{\epsilon}{\lambda_\epsilon} \int \frac{\eta'(v)^2}{1 - \eta(v)} dv. \end{aligned} \quad (4.10)$$

⁹As we were completing this paper, Carpi and Weiner released a preprint [3] in which they point out that certain nonsmooth smearings of the stress-energy density also yield self-adjoint operators. It is likely that one could use this to find a unitary implementation of the reparametrisation of the line defined by $V'(v) = 1/(G(v) + \epsilon)$, removing the need for the second stage of regulation.

On the other hand, we have

$$\int H_{n,\epsilon}(v)\langle T(v)\rangle_\psi = \int G(v)\langle T(v)\rangle_\psi dv + \epsilon\langle P\rangle_\psi - \epsilon \int \eta_{n,\epsilon}(v)\langle T(v)\rangle_\psi. \quad (4.11)$$

As $n \rightarrow \infty$, $\eta_{n,\epsilon}$ is translated off to infinity, so the last term drops out in the limit owing to the decay of $\langle T(v)\rangle_\psi$. We therefore have

$$\int G(v)\langle T(v)\rangle_\psi dv \geq -\frac{c}{48\pi} \int \frac{G'(v)^2}{G(v) + \epsilon} dv - \frac{\epsilon c}{48\pi\lambda_\epsilon} \int \frac{\eta'(v)^2}{1 - \eta(v)} dv - \epsilon\langle P\rangle_\psi, \quad (4.12)$$

and the limit $\epsilon \rightarrow 0^+$ yields the QEI (4.1), owing to Corollary A.2 in the Appendix and the fact that ψ was an arbitrary element of \mathcal{D} .

We now turn to the case in which G is a nonnegative function of Schwartz class. According to Corollary A.2, \sqrt{G} belongs to the Sobolev space $W^1(\mathbb{R})$. It follows that we may find nonnegative $h_k \in C_0^\infty(\mathbb{R})$ with $h_k \rightarrow \sqrt{G}$ and $h'_k \rightarrow d/dv \sqrt{G}$ in $L^2(\mathbb{R})$ as $k \rightarrow \infty$ (the derivative $d/dv \sqrt{G}$ being understood in the sense of distributions). Thus for each $\psi \in \mathcal{D}$ and k , we have

$$\int \langle T(v)\rangle_\psi h_k(v)^2 dv \geq -\frac{c}{12\pi} \int h'_k(v)^2 dv. \quad (4.13)$$

In the limit $k \rightarrow \infty$ the right-hand side clearly converges to $-c/(12\pi) \int (d/dv \sqrt{G})^2 dv$, while the left-hand side converges to $\int \langle T(v)\rangle_\psi G(v) dv$ because $\langle T(v)\rangle_\psi$ is bounded in v . The QEI (4.1) therefore holds for all nonnegative $G \in \mathcal{S}(\mathbb{R})$.

To show that the bound is sharp, we employ another lemma:

Lemma 4.3 *If $F \in \mathcal{S}(\mathbb{R})$ and $G \in C_0^\infty(\mathbb{R})$ are nonnegative, then*

$$\inf_{\psi \in \mathcal{D}} \int F(v)\langle T(v)\rangle_\psi dv \leq -\frac{c}{12\pi} \int \left(\frac{d}{dv} \frac{F(v)}{\sqrt{G(v) + \epsilon}} \right) \left(\frac{d}{dv} \sqrt{G(v) + \epsilon} \right) dv. \quad (4.14)$$

Proof: Using the notation of Lemma 4.2, let $n > n_0$ and $\epsilon > 0$, and define $\psi_{n,\epsilon} = U(\rho_{n,\epsilon})^{-1}\Omega$ in terms of G . Since $\langle T(V_{n,\epsilon}(v))\rangle_\Omega$ vanishes identically, the transformation law Eq. (3.39) gives

$$\begin{aligned} \langle T(v)\rangle_{\psi_{n,\epsilon}} &= -\frac{c}{24\pi} \{V_{n,\epsilon}, v\} = \frac{c}{12\pi} \frac{1}{\sqrt{H_{n,\epsilon}(v)}} \frac{d^2 \sqrt{H_{n,\epsilon}(v)}}{dv^2} \\ &= \frac{c}{12\pi} \left(\frac{1}{\sqrt{G(v) + \epsilon}} \frac{d^2 \sqrt{G(v) + \epsilon}}{dv^2} + \frac{1}{\sqrt{1 - \eta_{n,\epsilon}(v)}} \frac{d^2 \sqrt{1 - \eta_{n,\epsilon}(v)}}{dv^2} \right) \end{aligned} \quad (4.15)$$

because G and $\eta_{n,\epsilon}$ have disjoint supports. Note that the effect of increasing n is merely to translate the final term to the right. This term therefore vanishes in the limit $n \rightarrow \infty$ when we integrate against F , because it is pushed off into the tail of F . Thus we have

$$\lim_{n \rightarrow \infty} \int F(v)\langle T(v)\rangle_{\psi_{n,\epsilon}} dv = \frac{c}{12\pi} \int \frac{F(v)}{\sqrt{G(v) + \epsilon}} \frac{d^2 \sqrt{G(v) + \epsilon}}{dv^2} dv \quad (4.16)$$

and Eq. (4.14) is obtained after integration by parts. ■

Now suppose that G is a nonnegative Schwartz-class function and set $G_n(v) = \chi(v/n)G(v)$, where $\chi \in C_0^\infty(\mathbb{R})$, $0 \leq \chi(x) \leq 1$ and $\chi(x) = 1$ for $|x| \leq 1$. One may verify that

$$\lim_{m \rightarrow \infty} \frac{d}{dv} \frac{G(v)}{\sqrt{G_m(v) + \epsilon}} = \frac{d}{dv} \sqrt{G(v) + \epsilon} = \lim_{m \rightarrow \infty} \frac{d}{dv} \sqrt{G_m(v) + \epsilon} \quad (4.17)$$

in $L^2(\mathbb{R})$. Applying Lemma 4.3 with F and G replaced by G and G_m respectively, these limits and the continuity of the right-hand side of Eq. (4.14) in both factors [it is effectively an L^2 -inner product] yield

$$\inf_{\psi \in \mathcal{D}} \int G(v) \langle T(v) \rangle_\psi dv \leq -\frac{c}{12\pi} \int \left(\frac{d}{dv} \sqrt{G(v) + \epsilon} \right)^2 dv. \quad (4.18)$$

On taking $\epsilon \rightarrow 0^+$, we conclude that the bound Eq. (4.1) is sharp.

Turning to conformal field theories with two independent components of stress-energy, it is immediate from the above that both T_L and T_R satisfy QEIs of the form required. That the bounds are simultaneously sharp follows from the fact that each stress-energy density transforms under its corresponding copy of $\widetilde{\text{Diff}}_+(S^1)$ but is invariant under the other copy. Thus the construction used to establish sharpness of the QEI (4.1) may be adapted in a straightforward fashion to prove Eq. (4.3). This concludes the proof of our main theorem 4.1. ■

It remains to establish the lemma used above.

Proof of Lemma 4.2: It is clear (e.g., by monotone convergence) that λ_ϵ increases to unity as $\epsilon \rightarrow 0^+$. Thus the support of $\eta_{n,\epsilon}$ will lie to the right of $\text{supp } G$ for all n greater than some n_0 and all $\epsilon > 0$. We define

$$V_{n,\epsilon}(v) = \int_0^v \frac{1}{H_{n,\epsilon}(v')} dv', \quad (4.19)$$

which evidently satisfies Eq. (4.8) and, as it is smooth and strictly increasing with $\lim_{v \rightarrow \pm\infty} V_{n,\epsilon}(v) = \pm\infty$ gives a diffeomorphism of \mathbb{R} . We wish to see that this diffeomorphism can be extended to the circle. Suppose the support of G is contained within $[-R, R]$ for some $R > 0$ and that $n > n_0$. Then, for $v < -R$ we have

$$V_{n,\epsilon}(v) = \frac{v}{\epsilon} + \alpha, \quad (4.20)$$

where

$$\alpha = \frac{R}{\epsilon} + \int_0^{-R} \frac{1}{G(v) + \epsilon} dv. \quad (4.21)$$

Now choose S to the right of $\text{supp } \eta_{n,\epsilon}$, so $\text{supp } \eta_{n,\epsilon} \subset (R, S)$. Then, for $v > S$ we have

$$\begin{aligned} V_{n,\epsilon}(v) &= \frac{v}{\epsilon} - \frac{S}{\epsilon} + \int_0^S \frac{1}{G(v) + \epsilon(1 - \eta_{n,\epsilon}(v))} dv \\ &= \frac{v}{\epsilon} + \alpha, \end{aligned} \quad (4.22)$$

which follows after a small amount of calculation using the definitions of η and λ_ϵ .

Thus $v \mapsto V_{n,\epsilon}(v)$ differs from the Möbius transformation $v \mapsto v/\epsilon + \alpha$ only on a compact set and may therefore be lifted to $\rho_{n,\epsilon} \in \widetilde{\text{Diff}_+(S^1)}$ defined by $\rho_{n,\epsilon}(\theta) = 2 \tan^{-1}(V_{n,\epsilon}(\tan \frac{1}{2}\theta))$ for $\theta \in (-\pi, \pi)$ and extended to other values by continuity and Eq. (3.2). ■

4.2 Applications

We now use Theorem 4.1 to give various useful QEI bounds for conformal field theories (on two-dimensional Minkowski space).

4.2.1 Worldline bounds

Consider a smooth curve $\lambda \rightarrow \gamma^\mu(\lambda)$ in Minkowski space, and set $u = \gamma^0 - \gamma^1$, $v = \gamma^0 + \gamma^1$. It is straightforward to show that

$$\rho_\gamma(\lambda) := T_{\mu\nu}(\gamma(\lambda))\dot{\gamma}^\mu(\lambda)\dot{\gamma}^\nu(\lambda) = T_R(u(\lambda))\dot{u}(\lambda)^2 + T_L(v(\lambda))\dot{v}(\lambda)^2. \quad (4.23)$$

To avoid technicalities, let us assume that our curve γ is either timelike or spacelike, with no endpoints. The curve can then be parametrized by proper time (resp. proper distance) λ ranging from $-\infty$ to $+\infty$, and we assume this has been done. We assume furthermore that both $\dot{u}(\lambda)$ and $\dot{v}(\lambda)$ are bounded away from zero on the parameter range of the curve (i.e., greater or equal to some fixed $\epsilon > 0$), meaning that the curve does not become null asymptotically. We also restrict consideration to curves that do not “wobble” too rapidly by assuming moreover that all derivatives of $\dot{u}(\lambda)$ and $\dot{v}(\lambda)$ vanish faster than polynomially. Our assumptions imply that the functions $u(\lambda)$ and $v(\lambda)$ can therefore be inverted with smooth inverses $\lambda(u)$ resp. $\lambda(v)$, the derivatives of which are Schwartz functions.

Let G be a smooth, non-negative Schwartz function. Our assumptions then ensure that the smearing functions $G_R(u) = G(\lambda(u))$ and $G_L(v) = G(\lambda(v))$ and consequently $G_R(u)|d\lambda(u)/du|^{-1}$ and $G_L(v)|d\lambda(v)/dv|^{-1}$ are in the Schwartz class. Thus, using the simultaneously sharp QEIs for both left- and right-moving stress-energy densities, we obtain the worldline QEI

$$\begin{aligned} & \inf_{\psi \in \mathcal{D}} \int \langle \rho_\gamma(\lambda) \rangle_\psi G(\lambda) d\lambda \\ &= -\frac{c_R}{12\pi} \int \left(\frac{d}{du} \sqrt{\frac{G_R(u)}{|d\lambda(u)/du|}} \right)^2 du - \frac{c_L}{12\pi} \int \left(\frac{d}{dv} \sqrt{\frac{G_L(v)}{|d\lambda(v)/dv|}} \right)^2 dv, \quad (4.24) \end{aligned}$$

where the integrands on the right side are set to zero for points such that G_L resp. G_R vanish. This bound can be generalized to smooth parametrized curves γ^μ satisfying less stringent conditions, but we will not go into this here. We only remark that we may also obtain a bound for the affinely parametrized left-moving null ray $u = \lambda, v = \text{const.}$ for any non-negative $G(\lambda)$ in the Schwartz class. In that case, $\rho_\gamma = T_R$ and the worldline bound is given by the QEI bound for the right-moving stress tensor (with $G_R = G$)

given in our theorem. A similar statement holds of course also for the right moving light ray. In general, therefore, averages of the null-contracted stress-energy density $T_{\mu\nu}k^\mu k^\nu$ are bounded below along an affinely parametrised null line with tangent k^μ . As noted in [15], no other component of the stress tensor can be bounded below along such a curve because all other components involve T_R or T_L evaluated at a single point and therefore not averaged.

For the case of a static worldline parametrized by proper time, $\gamma^0 = x^0, \gamma^1 = x^1 = \text{const.}$, we find

$$\inf_{\psi \in \mathcal{D}} \int \langle T_{00}(x^0, x^1) \rangle_\psi G(x^0) dx^0 = -\frac{c_L + c_R}{12\pi} \int \left(\partial_0 \sqrt{G(x^0)} \right)^2 dx^0 \quad (4.25)$$

which reduces to Flanagan's bound [14] for the massless scalar field ($c_L = c_R = 1$) and Vollick's bound [52] for the massless (complex) Dirac field, which also has $c_L = c_R = 1$. [The Majorana field has $c_L = c_R = 1/2$ and a correspondingly tighter bound.]

It is worth noting a feature of conformal quantum field theories in two dimensions: namely that one can obtain a (nontrivial) worldline quantum energy inequality even along spacelike or null curves. This can be traced back to the fact that one is free to interchange the role of space and time in two-dimensional conformal field theories (by "turning Minkowski space on its side") as far as the stress-tensor is concerned. Neither is possible in any other dimension [17, 11] (even for free scalar fields), nor for non-conformally invariant field theories in two dimensions. In those cases, we expect however that there still hold bounds for *spacetime* averages of the stress tensor, to which we now turn.

4.2.2 Worldvolume bounds

Let $f^{\mu\nu}$ be a smooth tensor field whose components (with respect to global inertial coordinates) are Schwartz class. Then

$$\int T_{\mu\nu} f^{\mu\nu}(x^0, x^1) dx^0 dx^1 = \int T_R(u) F_R(u) du + \int T_L(v) F_L(v) dv \quad (4.26)$$

where the *null averages* F_L and F_R are given by

$$F_R(u) = \int f^{uu}(u, v) dv, \quad F_L(u) = \int f^{vv}(u, v) du \quad (4.27)$$

with f^{uu}, f^{vv} appropriate components in (u, v) -coordinates, related to the components in (x^0, x^1) coordinates by

$$\begin{aligned} f^{uu} &= f^{00} + f^{11} - f^{01} - f^{10} \\ f^{vv} &= f^{00} + f^{11} + f^{01} + f^{10}. \end{aligned} \quad (4.28)$$

If $f^{\mu\nu}$ has nonnegative null averages¹⁰, then we have the worldvolume QEI

$$\begin{aligned} \inf_{\psi \in \mathcal{D}} \int \langle T_{\mu\nu} f^{\mu\nu}(x^0, x^1) \rangle_\psi dx^0 dx^1 \\ = -\frac{c_L}{12\pi} \int \left(\frac{d}{dv} \sqrt{F_L(v)} \right)^2 dv - \frac{c_R}{12\pi} \int \left(\frac{d}{du} \sqrt{F_R(u)} \right)^2 du, \end{aligned} \quad (4.29)$$

¹⁰This follows of course in particular if $f^{\mu\nu}$ satisfies the conditions $f^{uu}, f^{vv} \geq 0$ pointwise.

where the integrands on the right side are as usual defined to be zero for points u (resp., v) where $F_L(u)$ (resp., $F_R(v)$) vanishes. In particular, if s^μ and t^ν are Schwartz-class timelike vector fields, $f^{\mu\nu} = s^\mu t^\nu$ obeys the above condition and so we obtain a quantum dominated energy inequality (QDEI).

4.2.3 Moving mirrors and boundary CFT

As a variation on the foregoing results, let us consider a moving mirror model, with central charge c , living in the portion $v > p(u)$ of Minkowski space, where $u \mapsto p(u)$ lifts to some $\rho \in \text{Diff}_+(S^1)$. As described in Sec. 3.3, the left- and right-moving components of the stress-energy density are given in terms of a single field T by the relations $T_L(v) = T(v)$, $T_R(u) = U(\rho)T(u)U(\rho)^{-1}$. If $f^{\mu\nu}$ is a smooth tensor field compactly supported in $v > p(u)$, then Eq. (4.26) and the transformation law (3.39) entail

$$\int T_{\mu\nu} f^{\mu\nu}(x^0, x^1) dx^0 dx^1 = \int T(v)G(v) dv - \frac{c}{24\pi} \int \{p, u\} F_R(u) du, \quad (4.30)$$

where

$$G(v) = F_L(v) + p'(p^{-1}(v))F_R(p^{-1}(v)), \quad (4.31)$$

and an obvious change of variables has also been employed. Thus we have the modified worldvolume QEI

$$\begin{aligned} \inf_{\psi \in \mathcal{D}} \int \langle T_{\mu\nu} f^{\mu\nu}(x^0, x^1) \rangle_\psi dx^0 dx^1 \\ = -\frac{c}{12\pi} \int \left(\frac{d}{dv} \sqrt{G(v)} \right)^2 dv - \frac{c}{24\pi} \int \{p, u\} F_R(u) du, \end{aligned} \quad (4.32)$$

in which the last term relates to the stress-energy density created by the motion of the mirror.

If the support of $f^{\mu\nu}$ is such that the supports of F_L and $F_R \circ p^{-1}$ (i.e., the two ‘null projections’ of $f^{\mu\nu}$ onto the mirror trajectory) are disjoint, the first term in the above bound splits into terms involving F_L and F_R separately. The term in F_R may be recombined with the final term in Eq. (4.32), leading to the same overall result as in Eq. (4.29). This is to be expected on grounds of locality, as measurements in (a diamond neighbourhood of) the support of $f^{\mu\nu}$ should be unaware of the presence of the boundary. (See also [30] for a detailed discussion of boundary CFT in which these ideas also appear.)

4.2.4 Unweighted averages

Finally, we discuss unweighted averages of the stress-energy tensor along portions of a worldline γ . First, let us note that, if γ is an infinite straight line (with \dot{u} and \dot{v} constant) then

$$\int \langle \rho_\gamma(\lambda) \rangle_\psi d\lambda \geq 0 \quad (4.33)$$

for all $\psi \in \mathcal{D}$, because the left-hand side is simply a weighted sum of P_L and P_R with positive coefficients. Accordingly, conformal field theories in Minkowski space obey the averaged weak energy condition, and the averaged null energy condition.

However, unweighted averaging along a bounded, or even semi-infinite, portion of such a worldline leads to very different results. For simplicity, we consider a theory with only one independent component of stress-energy, and averaging over $(-\infty, 0)$, but it is easy to extend these arguments. We begin by constructing a particular family of states as follows. Let $f \in C_0^\infty((-1, 1))$ obey $f(v) \geq -1$, $\int f(v) dv = 0$, and suppose f is not identically zero on $(-1, 0)$. Then the map $v \mapsto V(v)$ defined by

$$V(v) = v + \int_{-1}^v f(v') dv' \quad (4.34)$$

is a diffeomorphism of the line which lifts to some element $\rho \in \widetilde{\text{Diff}}_+(S^1)$ (as it agrees with the identity outside a compact interval). If f obeys, additionally,

$$-1 \leq \frac{d^2}{dv^2} \frac{1}{\sqrt{1+f(v)}} \leq 0 \quad (4.35)$$

for $v \in (-1, 0)$, then $\{V, v\} \geq 0$ on this interval, and no conflict need arise with our previous assumptions because the left-hand inequality ensures that $\int_{-1}^0 f(v) dv < 1$. Moreover $\{V, v\}$ must be strictly positive on some open subset of $(-1, 0)$, since f is not identically zero there. Owing to the identity

$$\int \frac{\{V, v\}}{\sqrt{V'(v)}} dv = -2 \int \frac{d^2}{dv^2} \frac{1}{\sqrt{V'(v)}} dv = 0, \quad (4.36)$$

it follows that $\{V, v\}$ is strictly negative on some open subset of $(0, 1)$ (note that $\{V, v\}$ is supported in $(-1, 1)$).

With the above assumptions in force, we may use the resulting diffeomorphism to create a state $\psi = U(\rho)^{-1}\Omega$ by acting on the vacuum. The corresponding energy density,

$$\langle T(v) \rangle_\psi = -\frac{c}{24\pi} \{V, v\}, \quad (4.37)$$

is smooth and compactly supported in $(-1, 1)$, nonpositive for $v \leq 0$, and strictly negative (resp., positive) on some open subset of $(-1, 0)$ (resp., $(0, 1)$). In particular,

$$\int_{-\infty}^0 \langle T(v) \rangle_\psi dv = -\frac{c}{24\pi} \int_{-\infty}^0 \{V, v\} dv < 0. \quad (4.38)$$

We now consider the family of states obtained by scaling ψ , namely $\psi_\lambda = U(D_\lambda)^{-1}\psi$, for which

$$\int_{-\infty}^0 \langle T(v) \rangle_{\psi_\lambda} dv = -\frac{c\lambda}{24\pi} \int_{-\infty}^0 \{V, v\} dv \rightarrow -\infty \quad (4.39)$$

as $\lambda \rightarrow \infty$. The reason for this is that the negative energy density becomes more and more sharply peaked near zero under the dilations, with magnitude growing like λ^2 and support shrinking as λ^{-1} . Thus we have shown explicitly that sharp averages of the stress-energy density are not subject to QEI restrictions. A related result holds for general quantum fields with mass-gap in two dimensions, as shown by Verch (Prop. 3.1 of [51]).

However, there is no contradiction between this observation and the QEIs proved above. An average taken against a weight function $G \in C_0^\infty(-\infty, 0)$ in the states ψ_λ would in fact tend to zero as $\lambda \rightarrow \infty$ because the negative peak eventually leaves the support of G . If one used a weight function which did not vanish at the origin, its support would spill over into the right-hand half line and sense the energy density there. However, the family of states ψ_λ also has an increasingly sharply peaked *positive* energy density within the interval $(0, \lambda^{-1})$, which must at least compensate for the negative contribution (because $\int \langle T(v) \rangle_{\psi_\lambda} dv$ is nonnegative). It is the competition between these two differently weighted contributions which permits the QEI to hold.

To emphasise the point, let us consider averages over half the light-ray, but with a smoothed-off end. Let G be a nonnegative, smooth and compactly supported function, which equals unity in a neighbourhood of the origin. Define a sequence of smooth functions G_n by

$$G_n(v) = \vartheta(-v)G(v/n) + \vartheta(v)G(v), \quad (4.40)$$

where ϑ is the Heaviside function (and we take $\vartheta(0) = 1/2$). As $n \rightarrow \infty$, these functions approach $H(v) = \vartheta(-v) + \vartheta(v)G(v)$. Now for any state $\psi \in \mathcal{D}$ we have

$$\int \langle T(v) \rangle_\psi G_n(v) dv \geq -\frac{c}{12\pi} \int \left(\frac{d}{dv} \sqrt{G_n(v)} \right)^2 dv \quad (4.41)$$

$$= -\frac{c}{12\pi} \int \left[\frac{\vartheta(-v)}{n} + \vartheta(v) \right] \left(\frac{d}{dv} \sqrt{G(v)} \right)^2 dv \quad (4.42)$$

for each n . Taking $n \rightarrow \infty$ and using the fact that $\langle T(v) \rangle_\psi$ decays as $O(v^{-4})$, we obtain

$$\int \langle T(v) \rangle_\psi H(v) dv \geq -\frac{c}{12\pi} \int_0^\infty \left(\frac{d}{dv} \sqrt{G(v)} \right)^2 dv \quad (4.43)$$

for arbitrary $\psi \in \mathcal{D}$. As expected, the bound depends only on the way the averaging is rounded-off.

5 Highest-weight Virasoro representations

In this section, we describe how CFT models satisfying our axioms may be constructed by taking direct sums of unitary, highest-weight representations of the Virasoro algebra. In particular, this demonstrates that our QEI applies to so-called minimal models and to rational conformal field theories. As part of our discussion we will need to consider the unitary multiplier representations of $\widetilde{\text{Diff}}_+(S^1)$ carried by any such Virasoro representation; in particular, we need to show that the representation can be normalised so that the multiplier is of the Bott form assumed in Axiom A.1. We have not found this elsewhere in the literature.

5.1 Highest-weight representations of the Virasoro algebra

We recall that the Virasoro algebra is generated by elements L_n ($n \in \mathbb{Z}$) and a central element κ , obeying the relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}m(m^2 - 1)\delta_{m+n,0}\kappa \quad (m, n \in \mathbb{Z}) \quad (5.1)$$

and $[\kappa, L_m] = 0$ for all $m \in \mathbb{Z}$. A unitary highest-weight representation amounts to the specification of a pair (c, h) of real constants, a Hilbert space $\mathcal{H}_{(c,h)}$, a dense domain $\mathcal{D}_0 \subset \mathcal{H}_{(c,h)}$, a vector $|h\rangle \in \mathcal{D}_0$, and operators L_n ($n \in \mathbb{Z}$) defined on \mathcal{D}_0 such that

1. $L_0|h\rangle = h|h\rangle$ and $L_n|h\rangle = 0$ for $n > 0$.
2. \mathcal{D}_0 coincides with the set of vectors obtained from $|h\rangle$ by acting with polynomials in the L_n with $n < 0$ (including the trivial polynomial $\mathbb{1}$).
3. $L_n^* = L_{-n}$ on \mathcal{D}_0 and Eq. (5.1) holds as an identity on \mathcal{D}_0 with $\kappa = c\mathbb{1}$.

Such representations are irreducible; moreover, the ‘highest weight’ (c, h) is restricted to particular values first classified in [22, 24]. (See, e.g., Theorems 6.17(3) and 6.13 in [46].) However, we will not need the precise details of this classification beyond the fact that both c and h are nonnegative, which follows immediately from the observation that $0 \leq \|L_{-n}|h\rangle\|^2 = 2nh + n(n^2 - 1)c/12$ for all $n \geq 1$ as a consequence of Eq. (5.1).

In the course of our analysis, we will need more detailed information on the domain of definition of the L_n and various other operators. Our first observation is that, by virtue of the Virasoro relations, \mathcal{D}_0 contains an orthonormal basis of L_0 -eigenvectors. Indeed, this follows by the Gram–Schmidt process applied to vectors of the form $L_{-n_1}L_{-n_2}\cdots L_{-n_k}|h\rangle$ (for $n_1, \dots, n_k > 0$), which are L_0 -eigenvectors with eigenvalue $h + n_1 + n_2 + \cdots + n_k$. Thus L_0 is essentially self-adjoint on \mathcal{D}_0 and we will use L_0 from now on to denote the unique self-adjoint extension of this operator, writing $D(L_0)$ for its domain. The above remarks also show that L_0 is a positive operator, with spectrum contained in $h + \mathbb{N}_0$ and finite-dimensional eigenspaces. Secondly, estimates obtained by Goodman and Wallach [25]¹¹ entail that

$$\|L_n\psi\| \leq C(1 + |n|)^{3/2}\|L_0\psi\| \quad (5.2)$$

for all $\psi \in \mathcal{D}_0$ and $n \in \mathbb{Z}$, where the constant C is determined by the central charge and is independent of both ψ and n . Accordingly, the L_n may be extended uniquely to $D(L_0)$, and we now use L_n to denote these extensions. The relation $L_n = L_{-n}^*$ continues to hold, and the Virasoro relations hold as identities on $D(L_0^2)$. A further consequence is that the formula

$$\Theta(z) = -\frac{1}{2\pi} \sum_{n \in \mathbb{Z}} z^{-n-2} L_n, \quad (5.3)$$

defines $\Theta(\cdot)\psi$ as a vector-valued distribution on $C^\infty(S^1)$ for each $\psi \in D(L_0)$. Furthermore,

$$\Theta(f)^* = \Theta(\Gamma f) \quad (5.4)$$

¹¹See [2] for related bounds.

on $D(L_0)$ for $f \in C^\infty(S^1)$. In particular, if $\Gamma f = f$ (i.e., $f \in C_\Gamma^\infty(S^1)$) then $\Theta(f)$ is symmetric on $D(L_0)$ and an application of Nelson's commutator theorem (Theorem X.37 in [44]¹²) shows that $\Theta(f)$ is essentially self-adjoint on any core of L_0 . Henceforth we will use $\Theta(f)$ to denote the unique self-adjoint extension. It is easy to verify that the $\Theta(f)$'s defined in this way obey the commutation relations Eq. (3.28) on $D(L_0^2)$.

Finally, let us define the space \mathcal{H}^∞ to be the intersection $\mathcal{H}^\infty = \bigcap_{n \in \mathbb{N}_0} D(L_0^n)$, equipped with the Fréchet topology induced by the seminorms $\psi \mapsto \|L_0^n \psi\|$ ($n \in \mathbb{N}_0$). As $\mathcal{D}_0 \subset D(L_0^n)$ for each n , it follows that \mathcal{H}^∞ is dense in \mathcal{H} and is a core for L_0 .

5.2 Integration to a unitary representation of $\widehat{\text{Diff}}_+(S^1)$

We now need to demonstrate that Θ generates a unitary multiplier representation of $G = \widehat{\text{Diff}}_+(S^1)$ as in Axiom A.1 and Eq. (3.19). The relevant results are all present in the literature, but do not appear to have been assembled in this form before. Explicit control of the multiplier is necessary when we come to assemble Virasoro representations to form more general CFT models below: the direct sum of two projective representations is not generally a projective representation!

Let $\mathcal{U}_{(c,h)}$ be the group of unitary operators on $\mathcal{H}_{(c,h)}$ and let $P\mathcal{U}_{(c,h)}$ be the projective unitary group (i.e., unitaries modulo phases) $P\mathcal{U}_{(c,h)} = \mathcal{U}_{(c,h)}/\mathbb{T}$. In the following we distinguish unitary multiplier representations (which take values in $\mathcal{U}_{(c,h)}$) from projective unitary representations (which take values in $P\mathcal{U}_{(c,h)}$). As shown by Goodman and Wallach [25]¹³ and Toledano Laredo [49], $\mathcal{H}_{(c,h)}$ carries a projective unitary representation \mathbf{U} of G , so the remaining problem is to assign phases in such a way that Axiom A.1 and Eqs. (3.18) and (3.19) are satisfied.

It is helpful (and standard) to rephrase this problem in a geometric fashion. Let \widehat{G} be the subgroup of $G \times \mathcal{U}_{(c,h)}$ defined by

$$\widehat{G} = \{(g, V) \in G \times \mathcal{U}_{(c,h)} : \mathbf{U}(g) = p(V)\} \quad (5.5)$$

where $p : \mathcal{U}_{(c,h)} \rightarrow P\mathcal{U}_{(c,h)}$ is the quotient map. As shown in Proposition 5.3.1 of [49] \widehat{G} is a central extension of G by \mathbb{T} which may be given the structure of a Lie group. In particular, it is a smooth principal \mathbb{T} -bundle over G (with projection $\pi(g, V) = g$): the problem of assigning local (respectively, global) phases to \mathbf{U} is then equivalent to selecting a local (resp., global) section of \widehat{G} .

The local problem was addressed by Toledano Laredo in the course of proving the result just mentioned. He showed that phases can be assigned to \mathbf{U} in a neighbourhood N of id to provide a local unitary multiplier representation U_{loc} of G so that (i) the map $(g, \psi) \mapsto U_{\text{loc}}(g)\psi$ is smooth from $N \times \mathcal{H}^\infty$ to \mathcal{H}^∞ and (ii) for each $f \in C_\Gamma^\infty(S^1)$ and each $\psi \in \mathcal{H}^\infty$,

$$\left. \frac{d}{ds} U_{\text{loc}}(e_f(s))\psi \right|_{s=0} = i\Theta(f)\psi \quad (5.6)$$

where $s \mapsto e_f(s)$ is a smooth curve in G with $e_f(0) = \text{id}$ and $\dot{e}_f(0) = \mathbf{f}$, the corresponding vector field to f . [These curves, and U_{loc} , are determined by a choice of coordinates

¹²In the notation of [44], set $A = \Theta(f)$, $N = L_0 + \mathbb{1}$ and $D = \mathcal{D}_0$, for example.

¹³In fact [25] addresses $\text{Diff}_+(S^1)$ rather than its universal cover.

near id.] By (i) we may replace $e_f(s)$ by $\exp sf$ in Eq. (5.6), so U_{loc} obeys Eq. (3.19) and provides a local solution to our problem. A further consequence of (i) is that U_{loc} is strongly continuous on \mathcal{H} , because \mathcal{H}^∞ is dense in \mathcal{H} and the $U_{\text{loc}}(g)$ have unit operator norms. Toledano Laredo also uses U_{loc} to show that the Lie algebra cocycle of \widehat{G} is cohomologous to $c\omega$, where ω is the Virasoro cocycle of Eq. (3.17).

The global assignment of phases is achieved by the following result.

Proposition 5.1 *There is a global smooth section $g \mapsto (g, U_{(c,h)}(g))$ of \widehat{G} such that $g \mapsto U_{(c,h)}(g)$ is a strongly continuous unitary multiplier representation of G obeying*

$$U_{(c,h)}(g)U_{(c,h)}(g') = e^{ic\tilde{B}(g,g')}U_{(c,h)}(gg') \quad (g, g' \in G). \quad (5.7)$$

Moreover, if $\mathbf{f} \in \text{Vect}_{\mathbb{R}}(S^1)$ is the vector field corresponding to $f \in C_{\Gamma}^\infty(S^1)$ then $D(\Theta(f))$ consists precisely of those $\psi \in \mathcal{H}$ for which $s \mapsto U_{(c,h)}(\exp sf)\psi$ is differentiable, and we have

$$\left. \frac{d}{ds} U_{(c,h)}(\exp sf)\psi \right|_{s=0} = i\Theta(f)\psi \quad (5.8)$$

for such ψ .

Remark: \widehat{G} admits smooth global sections because G is contractible.

Proof: By Proposition 4.2 in [36] \widehat{G} may be described in terms of a group 2-cocycle mapping $G \times G$ to \mathbb{T} which is smooth near (id, id) . Because G is simply connected, the equivalence class of group cocycles describing \widehat{G} is fixed by the infinitesimal class of $c\omega$ (see, e.g., the long exact sequence of Theorem 7.12 in [36]) and therefore includes the Bott cocycle $\Omega_c(g, g') = e^{ic\tilde{B}(g,g')}$ for central charge c . Let $g \mapsto (g, V(g))$ be any smooth global section of \widehat{G} and define the corresponding (everywhere smooth) cocycle $m : G \times G \rightarrow \mathbb{T}$ by $V(g)V(g') = m(g, g')V(gg')$. Since m and Ω_c are cohomologous there exists $\mu : G \rightarrow \mathbb{T}$, smooth near id , such that

$$m(g, g') = \Omega_c(g, g') \frac{\mu(gg')}{\mu(g)\mu(g')}. \quad (5.9)$$

As both m and Ω_c are smooth it follows that μ is everywhere smooth; the required global section is given by $U_{(c,h)}(g) = \mu(g)V(g)$.

Near the identity, we must have $U_{(c,h)}(g) = e^{i\nu(g)}U_{\text{loc}}(g)$ for some smooth $\nu : N \rightarrow \mathbb{R}$. It follows that $U_{(c,h)}$ is strongly continuous on \mathcal{H} and has well-defined generators $\Xi(f)$ given on \mathcal{H}^∞ by

$$i\Xi(f)\psi = \left. \frac{d}{ds} U_{(c,h)}(\exp sf)\psi \right|_{s=0}, \quad (5.10)$$

and obeying $\Xi(f) = \Theta(f) + \alpha(f)\mathbb{1}$ (on \mathcal{H}^∞) where $\alpha(f) = \nu'_{\text{id}}(\mathbf{f})$ is continuous and linear in $f \in C_{\Gamma}^\infty(S^1)$ because ν is smooth. By Prop. 3.1, applied to $U_{(c,h)}$ and \mathcal{H}^∞ , the generators Ξ obey the same algebraic relations on \mathcal{H}^∞ as the Θ 's on \mathcal{H}^∞ . In particular they obey Eq. (3.28), from which it follows that $\alpha(fg' - f'g) = 0$ for all $f, g \in C_{\Gamma}^\infty(S^1)$. It is now straightforward to show that α vanishes on a basis for $C_{\Gamma}^\infty(S^1)$ and hence identically. Accordingly, Eq. (5.8) holds for $\psi \in \mathcal{H}^\infty$ and, in particular, on \mathcal{D}_0 . Now, the argument of footnote 7 above guarantees that the left-hand side of Eq. (5.8) defines a self-adjoint

operator whose domain consists precisely of those ψ for which the derivative exists. As this operator agrees with $\Theta(f)$ on a core, it must in fact be $\Theta(f)$. ■

We have thus established that the stress-energy density in a unitary highest-weight Virasoro representation is the infinitesimal generator of a unitary multiplier representation of $\widetilde{\text{Diff}}_+(S^1)$ with the Bott cocycle. Thus $\mathcal{H}_{(c,h)}$ and $U_{(c,h)}$ satisfy axiom A.1 of Sect. 3.3. Moreover the algebraic relations Eqs. (3.27) and (3.28) hold when applied to vectors in \mathcal{H}^∞ . Let us observe that it is *not* the case that

$$U_{(c,h)}(\exp sf) = e^{is\Theta(f)} \quad (\text{FALSE}) \quad (5.11)$$

for all $s \in \mathbb{R}$ and $f \in C_1^\infty(S^1)$ because the Bott cocycle does not vanish along all one-parameter subgroups (although it is of course a coboundary). In passing we mention that Goodman and Wallach [25] appear to claim that their unitary multiplier representation of $\text{Diff}_+(S^1)$ can be normalised in such a way that Eq. (5.11) holds. However, this cannot be true, as it is not possible for exponentiations of $\mathfrak{sl}(2, \mathbb{R})$ representations with noninteger highest weight.

Turning to axiom A.2, we note that representations with $h \neq 0$ do not contain a vacuum vector invariant under $U_{(c,h)}|_{\widetilde{\text{Möb}}}$, because this representation of $\widetilde{\text{Möb}}$ is generated by L_0 and linear combinations of $L_{\pm 1}$, and we know that $\text{spec}(L_0) \subset h + \mathbb{N}_0$. If $h = 0$ the highest-weight vector $|0\rangle$ is indeed the unique invariant vector, as required by axiom A.2.¹⁴ We will return to this when constructing more general CFT models.

Continuing with general highest-weight Virasoro representations, Axiom A.3 clearly holds, because the generator of rotations $H = L_0$ is positive. To check the remaining axioms, we construct a new $U_{(c,h)}$ -invariant domain by

$$\mathcal{D}_{(c,h)} = \bigcup_{g \in G} U_{(c,h)}(g)\mathcal{D}_0, \quad (5.12)$$

which lies within \mathcal{H}^∞ (and obviously contains the vacuum vector $|0\rangle$ if $h = 0$). The argument of Eq. (3.30) establishes that $D(\Theta(f)) = U(g^{-1})D(\Theta(f_g))$; as $\mathcal{D}_0 \subset D(\Theta(f_g))$, it follows that $U(g^{-1})\mathcal{D}_0 \subset D(\Theta(f))$ for any $g \in G$. Thus $\mathcal{D}_{(c,h)} \subset D(\Theta(f))$ for any $f \in C^\infty(S^1)$, verifying axiom B.1 (apart from the statement regarding the vacuum). In particular, $\mathcal{D}_{(c,h)} \subset D(L_0)$, so $\Theta(\cdot)\psi$ is a vector valued distribution on $C^\infty(S^1)$ for each $\psi \in \mathcal{D}_{(c,h)}$ by the comment after Eq. (5.3). Accordingly, $\mathcal{H}_{(c,h)}$, $U_{(c,h)}$ and $\mathcal{D}_{(c,h)}$ satisfy axiom B.2.

We also wish to see that expectation values of $\langle \Theta(z) \rangle_\psi$ for $\psi \in \mathcal{D}_{(c,h)}$ are smooth. This can be verified directly for $\psi \in \mathcal{D}_0$, in which case the expectation values are polynomial in z and z^{-1} ; the extension to $\mathcal{D}_{(c,h)}$ then follows from the transformation law Eq. (3.27) (which holds on \mathcal{H}^∞ and hence on $\mathcal{D}_{(c,h)}$). Thus axiom B.3 holds.

To summarise: we have established that $\mathcal{H}_{(c,h)}$, $\mathcal{D}_{(c,h)}$ and $U_{(c,h)}$ obey all the axioms for a CFT on S^1 except those relating to the vacuum state; all the axioms are obeyed if $h = 0$.

¹⁴For arbitrary highest-weight h , the highest-weight vector $|h\rangle$ obeys $L_0|h\rangle = h|h\rangle$, $L_1|h\rangle = 0$, $\|L_{-1}|h\rangle\|^2 = 2h$. The assertion follows on taking $h = 0$.

5.3 CFT models obeying the axioms

It is now easy to construct a large class of theories obeying our axioms, simply by taking direct sums of Virasoro representations. Starting with CFTs with a single component of stress-energy, we may take, for example,

$$\mathcal{H} = \bigoplus_{k=0}^K \mathcal{H}_{(c,h_k)}, \quad U = \bigoplus_{k=0}^K U_{(c,h_k)}, \quad (5.13)$$

where $0 \leq K \leq \infty$ and $0 = h_0 < h_1 \leq h_2 \leq h_3 \cdots$ with each (c, h_k) an allowed highest weight for a unitary representation of the Virasoro algebra. Here we take \mathcal{D} to be the space of vectors in \mathcal{H} with only finitely many nonzero components, each belonging to the appropriate $\mathcal{D}_{(c,h_k)}$, and set $\Omega = (|0\rangle, 0, 0, \dots)$. Since we argued in the last subsection that the multipliers in each $U_{(c,h_k)}$ are all equal, their direct sum is also a unitary multiplier representation with the same multiplier. In addition, by insisting on a unique summand with $h = 0$, we have guaranteed the existence of a unique vacuum vector.

In a similar fashion, CFTs with two independent components of stress-energy may be constructed as direct sums of tensor products of the form

$$\mathcal{H} = \bigoplus_{k=0}^K \mathcal{H}_{(c_L, h_{L,k})} \otimes \mathcal{H}_{(c_R, h_{R,k})}, \quad (5.14)$$

in which $0 \leq K \leq \infty$ as before, and we require that $(h_{L,k}, h_{R,k}) = (0, 0)$ if and only if $k = 0$. The vacuum is $\Omega = (|0\rangle \otimes |0\rangle, 0, 0, \dots)$ (and is again unique) and the space \mathcal{D} is constructed as before. Thus our axioms embrace (and are more general than) minimal models—in which case K is finite—and rational conformal field theories—in which case K may be infinite but the theory is minimal for an extended algebra, e.g., in minimal superconformal models [22] or WZW models [53].

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A Square roots of Schwartz class functions

In the body of the paper, we use various properties of square roots of functions in the Schwartz class. The following results are quite probably known, but are included for completeness. Related results, also based on the use of Taylor's theorem, may be found in e.g. Lemma 1 of [23] and p. 86 of [4].

Lemma A.1 *Let $G \in \mathcal{S}(\mathbb{R})$ be nonnegative. Then there exists $M > 0$ such that*

$$G'(v)^2 \leq \frac{4MG(v)}{1+v^2} \quad (\text{A.1})$$

for all v . In particular, $|d/dv \sqrt{G(v)}|^2 \leq M/(1+v^2)$ where $G(v) \neq 0$.

Proof: For $k = 0, 2$, let $M_k = \sup_{v \in \mathbb{R}} |(1+|v|^k)G''(v)|$. If $\epsilon > 0$, Taylor's theorem entails that

$$0 \leq G(v - \epsilon G'(v)) = G(v) - \epsilon G'(v)^2 + \frac{1}{2} \epsilon^2 G'(v)^2 G''(\eta) \quad (\text{A.2})$$

for some η lying between v and $v - \epsilon G'(v)$. We apply this in two ways. First, for any v , we use $G''(\eta) < M_0$ and put $\epsilon = M_0^{-1}$ to find

$$0 \leq G(v) - \epsilon G'(v)^2 + \frac{1}{2} \epsilon^2 G'(v)^2 M_0 = G(v) - \frac{G'(v)^2}{2M_0} \quad (\text{A.3})$$

so $G'(v)^2 \leq 2M_0 G(v)$ for all v . Second, we observe that

$$2 \left| \frac{G'(v)}{v} \right| \leq \frac{M_2}{1+(v/2)^2} \quad (\text{A.4})$$

holds for all sufficiently large $|v|$, so setting $\epsilon = (1+(v/2)^2)/M_2$ the η in Eq. (A.2) obeys $|\eta| \geq |v|/2$ and we find

$$0 \leq G(v) - \epsilon G'(v)^2 + \frac{\epsilon^2 G'(v)^2 M_2}{2(1+(v/2)^2)} = G(v) - \frac{1+(v/2)^2}{2M_2} G'(v)^2 \quad (\text{A.5})$$

for all sufficiently large $|v|$. Thus Eq. (A.1) holds with $M = \max\{\frac{1}{2}M_0, 4M_2\}$. ■

Corollary A.2 *Given $0 \leq G \in \mathcal{S}(\mathbb{R})$ define*

$$\varphi(v) = \begin{cases} G'(v)/(2\sqrt{G(v)}) & G(v) \neq 0 \\ 0 & G(v) = 0. \end{cases} \quad (\text{A.6})$$

Then $\varphi \in L^2(\mathbb{R})$ and $\varphi = d/dv \sqrt{G}$, where d/dv denotes the derivative in the sense of distributions. Thus \sqrt{G} belongs to the Sobolev space $W^1(\mathbb{R})$. Furthermore,

$$\int \varphi(v)^2 dv = \lim_{\epsilon \rightarrow 0^+} \int \frac{G'(v)^2}{4(G(v) + \epsilon)} dv. \quad (\text{A.7})$$

Proof: For $\epsilon > 0$ define $G_\epsilon(v) = (\sqrt{G(v) + \epsilon} - \sqrt{\epsilon})^2$. Then $\sqrt{G_\epsilon} \rightarrow \sqrt{G}$ in $L^2(\mathbb{R})$ as $\epsilon \rightarrow 0^+$. Moreover

$$\left| \frac{d}{dv} \sqrt{G_\epsilon(v)} - \varphi(v) \right| = \left| \frac{G'(v)}{2\sqrt{G(v) + \epsilon}} - \varphi(v) \right| \leq 2 \left(\frac{M}{1+v^2} \right)^{1/2}, \quad (\text{A.8})$$

where M is the constant furnished by Lemma A.1. (In the case $G(v) \neq 0$, this follows from the triangle inequality; the case $G(v) = 0$ is trivial as we must also have $G'(v) = 0$ by Eq. (A.1), so the left-hand side vanishes.) Since $d/dv \sqrt{G_\epsilon(v)} \rightarrow \varphi(v)$ pointwise as $\epsilon \rightarrow 0^+$, we deduce that the convergence occurs in $L^2(\mathbb{R})$ by the dominated convergence theorem. Thus $\varphi = D\sqrt{G} \in L^2(\mathbb{R})$. The expression for $\|\varphi\|^2$ is also proved by dominated convergence. ■

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