

**Twistor Theory, Complex Homogeneous Manifolds  
and G-Structures****Sergey A. Merkulov**

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# Twistor theory, complex homogeneous manifolds and $G$ -structures

Sergey A. Merkulov

School of Mathematics and Statistics, University of Plymouth  
Plymouth, Devon PL4 8AA, United Kingdom

**0. Introduction.** One of the most useful characteristics of an affine connection on a manifold  $M$  is its (restricted) holonomy group which is defined, up to a conjugation, as a subgroup of  $GL(T_tM)$  consisting of all automorphisms of the tangent space  $T_tM$  at a point  $t \in M$  induced by parallel translations along the  $t$ -based contractible loops in  $M$ . Which groups can occur as holonomies of affine connections? By Hano and Ozeki [H-O], any closed subgroup of a general linear group can be realised as a holonomy of some affine connection (which in general has a non-vanishing torsion tensor). The same question, if posed in the class of *torsion-free* (non-locally symmetric) affine connections only, is not yet answered. According to Berger [B], the list of all possible irreducibly acting holonomies of such connections is very restricted. How much is known about this list? In his seminal paper [B], Berger found a list of groups which embraces all possible holonomies of torsion-free *metric* connections, though his approach provides no method to distinguish which entries can indeed be realised as holonomies and which are superfluous. Later much work has been done to refine this list and to prove existence of Riemannian metrics with special holonomies [Al, Br1, Br2, S]. In the same paper Berger presented also a list of all but a finite number of possible candidates to irreducible holonomies of "non-metric" torsion-free affine connections. How many holonomies are missing from this second list is not known, but, as was recently shown by Bryant [Br3], the set of missing, or *exotic*, holonomies is non-empty. As usual in the representation theory, in order to get a deeper understanding of all irreducible real holonomies one should first try to address a complex version of the problem. The main result announced in this paper asserts that any torsion-free holomorphic affine connection with irreducibly acting holonomy group can be generated by twistor methods.

**1. Irreducible  $G$ -structures.** Let  $M$  be an  $m$ -dimensional complex manifold and  $\mathcal{L}^*M$  the holomorphic coframe bundle  $\pi : \mathcal{L}^*M \rightarrow M$  whose fibres  $\mathcal{L}_t^*M = \pi^{-1}(t)$  consist of all  $\mathbb{C}$ -linear isomorphisms  $e : \mathbb{C}^m \rightarrow \Omega_t^1M$ . The space  $\mathcal{L}^*M$  is a principle right  $GL(m, \mathbb{C})$ -bundle with the right action given by  $R_g(e) = e \circ g$ . If  $G$  is a closed subgroup of  $GL(m, \mathbb{C})$ , then a (holomorphic)  $G$ -structure on  $M$  is a principle subbundle  $\mathcal{G}$  of  $\mathcal{L}^*M$  with the group  $G$ . It is clear that there is a one-to-one correspondence between the set of  $G$ -structures on  $M$  and holomorphic sections  $\sigma$  of the quotient bundle  $\tilde{\pi} : \mathcal{L}^*M/G \rightarrow M$  whose typical fibre is isomorphic to  $GL(m, \mathbb{C})/G$ . A  $G$ -structure on  $M$  is called *locally flat* if  $\mathcal{L}^*M/G$  can be trivialised over a sufficiently small neighbourhood,  $U$ , of each point  $t \in M$  in such a way that the associated section  $\sigma$  of  $\mathcal{L}^*M/G$  is represented over  $U$  by a constant  $GL(m, \mathbb{C})/G$ -valued function. A  $G$ -structure is called *1-flat* if, for each  $t \in M$ , the first jet of the

associated section  $\sigma$  of  $\mathcal{L}^*M/G$  at  $t$  is isomorphic to the first jet of some locally flat section of  $\mathcal{L}^*M/G$ . It is easy to show that a  $G$ -structure admits a torsion-free affine connection if and only if it is 1-flat (cf. [Br2]). A  $G$ -structure on  $M$  is called *irreducible* if the action of  $G$  on  $C^m$  leaves no non zero invariant subspaces.

When studying a (torsion-free) affine connection  $\nabla$  on a connected simply connected complex manifold  $M$  with the irreducibly acting holonomy group  $G$ , one usually works with the associated irreducible (1-flat)  $G$ -structure  $\mathcal{G}_\nabla \subset \mathcal{L}^*M$  [Br1, S]. Define two points  $u$  and  $v$  of  $\mathcal{L}^*M$  to be equivalent,  $u \sim v$ , if there is a holomorphic path  $\gamma$  in  $M$  from  $\pi(u)$  to  $\pi(v)$  such that  $u = P_\gamma(v)$ , where  $P_\gamma : \Omega_{\pi(v)}^1 M \rightarrow \Omega_{\pi(u)}^1 M$  is the parallel transport along  $\gamma$ . Then  $\mathcal{G}_\nabla$  can be defined, up to an isomorphism, as  $\{u \in \mathcal{L}^*M \mid u \sim v\}$  for some coframe  $v$ . The  $G$ -structure  $\mathcal{G}_\nabla$  is the smallest subbundle of  $\mathcal{L}^*M$  which is invariant under  $\nabla$ -parallel translations.

**2. Complex contact structures.** Let  $Y$  be a complex  $(2n + 1)$ -dimensional manifold. A complex contact structure on  $Y$  is a rank  $2n$  holomorphic subbundle  $D \subset TY$  of the holomorphic tangent bundle to  $Y$  such that the Frobenius form

$$\begin{aligned} \Phi : D \times D &\longrightarrow TY/D \\ (v, w) &\longrightarrow [v, w] \text{ mod } D \end{aligned}$$

is non-degenerate. A complex  $n$ -dimensional submanifold  $X$  of the complex contact manifold  $Y$  is called a *Legendre submanifold* if  $TX \subset D$ . The normal bundle of a Legendre submanifold  $X \hookrightarrow Y$  is isomorphic to  $J^1 L_X$  [L2], where  $L_X = L|_X$  and  $L$  is the contact line bundle on  $Y$  defined by the exact sequence

$$0 \longrightarrow D \longrightarrow TY \longrightarrow L \longrightarrow 0.$$

Given a Legendre submanifold  $X \hookrightarrow Y$ , there is a naturally associated "flat" model,  $X \hookrightarrow J^1 L_X$ , consisting of the total space of the vector bundle  $J^1 L_X$  together with its canonical contact structure and the Legendre submanifold  $X$  realised as a zero section of  $J^1 L_X \rightarrow X$ . The Legendre submanifold  $X \hookrightarrow Y$  is called *k-flat* if the  $k$ th-order Legendre jet [L2] of  $X$  in  $Y$  is isomorphic to the  $k$ th-order Legendre jet of  $X$  in  $J^1 L_X$ . The obstruction for a complex Legendre submanifold to be 1-flat is a cohomology class in  $H^1(X, L_X \otimes S^2(J^1 L_X)^*)$ .

**3. Twistor theory and  $G$ -structures.** Recall that a generalised flag variety  $X$  is a compact simply connected homogeneous Kähler manifold [B-E]. Any such a manifold is of the form  $X = G/P$ , where  $G$  is a complex semisimple Lie group and  $P \subset G$  a fixed parabolic subgroup.

**Theorem 1** *Let  $X$  be a generalised flag variety embedded as a Legendre submanifold into a complex contact manifold  $Y$  with contact line bundle  $L$  such that  $L_X$  is very ample on  $X$ . Then*

- (i) *There exists a maximal family  $\{X_t \hookrightarrow Y \mid t \in M\}$  of compact complex Legendre submanifolds obtained by holomorphic deformations of  $X$  inside  $Y$ . Each submanifold  $X_t$  is isomorphic to  $X$ . The moduli space  $M$ , called a Legendre moduli space, is an  $m$ -dimensional complex manifold, where  $m = h^0(X, L_X)$ .*
- (ii) *The Legendre submanifold  $X \hookrightarrow Y$  is stable under holomorphic deformations of the contact structure on (the tubular neighbourhood of  $X$  in)  $Y$ .*

- (iii) For each  $t \in M$ , there is a canonical isomorphism  $s : T_t M \rightarrow H^0(X_t, L_{X_t})$  representing a tangent vector at  $t$  as a global holomorphic section of the line bundle  $L_{X_t} = L|_{X_t}$ .
- (iv) The Legendre moduli space  $M$  comes equipped with an induced irreducible  $G$ -structure,  $\mathcal{G}_{ind} \rightarrow M$ , with  $G$  isomorphic to the connected component of the identity of the group of all global biholomorphisms  $\phi : L_X \rightarrow L_X$  which commute with the projection  $\pi : L_X \rightarrow X$ .
- (v) The induced  $G$ -structure  $\mathcal{G}_{\setminus \Gamma}$  is 1-flat (i.e. torsion-free) if and only if the complete family  $\{X_t \hookrightarrow Y \mid t \in M\}$  consists of 1-flat Legendre submanifolds.
- (vi) If  $\mathcal{G}_{ind}$  is 1-flat, then the bundle of all torsion-free connections in  $\mathcal{G}_{ind}$  has as the typical fiber an affine space modelled on  $H^0(X, L_X \otimes S^2(J^1 L_X)^*)$ .
- (vii) If  $\mathcal{G}_{ind}$  is 1-flat, then the obstruction for  $\mathcal{G}_{ind}$  to be locally flat is given by a tensor field on  $M$  whose value at each  $t \in M$  is represented by a cohomology class  $\rho_t \in H^1(X_t, L_{X_t} \otimes S^3(J^1 L_{X_t})^*)$ .
- (viii) Let  $H \subset GL(k, \mathbb{C})$  be one of the following subgroups: (a)  $SO(2n+1, \mathbb{C})$  when  $k = 2n+2 \geq 8$ ; (b)  $Sp(2n+2, \mathbb{C})$  when  $k = 2n+2 \geq 4$ ; (c)  $G_2$  when  $k = 7$ . Suppose that  $G \subset GL(m, \mathbb{C})$  is a connected semisimple Lie subgroup whose decomposition into a locally direct product of simple groups contains  $H$ . If  $\mathcal{G}$  is any irreducible 1-flat  $G \times \mathbb{C}^*$ -structure on an  $m$ -dimensional manifold  $M$ , then there exists a complex contact manifold  $(Y, L)$  and a generalised flag variety  $X$  embedded into  $Y$  as a Legendre submanifold with  $L_X$  being very ample, such that, at least locally,  $M$  is canonically isomorphic to the associated Legendre moduli space and  $\mathcal{G} \subset \mathcal{G}_{\setminus \Gamma}$ . In particular, when  $G = H$  one has in the case (a)  $X = SO(2n+2, \mathbb{C})/U(n+1)$  and  $\mathcal{G}_{\setminus \Gamma}$  is a  $CO(2n+2, \mathbb{C})$ -structure; in the case (b)  $X = \mathbb{C}P^{2n+1}$  and  $\mathcal{G}_{\setminus \Gamma}$  is a  $GL(2n+2, \mathbb{C})$ -structure; and in the case (c)  $X = Q_5$  and  $\mathcal{G}_{\setminus \Gamma}$  is a  $CO(7, \mathbb{C})$ -structure.
- (ix) Let  $G \subset GL(m, \mathbb{C})$  be an arbitrary connected semisimple Lie subgroup whose decomposition into a locally direct product of simple groups does not contain any of the groups  $H$  considered in (viii). If  $\mathcal{G}$  is any irreducible 1-flat  $G \times \mathbb{C}^*$ -structure on an  $m$ -dimensional manifold  $M$ , then there exists a complex contact manifold  $(Y, L)$  and a Legendre submanifold  $X \hookrightarrow Y$  with  $X = G/P$  for some parabolic subgroup  $P \subset G$  and with  $L_X$  being very ample, such that, at least locally,  $M$  is canonically isomorphic to the associated Legendre moduli space and  $\mathcal{G} = \mathcal{G}_{\setminus \Gamma}$ .

The Lie algebra of the group  $G$  of all global biholomorphisms  $L_X \rightarrow L_X$  which commute with the projection  $\pi : L_X \rightarrow X$  is exactly the vector space  $H^0(X, L_X \otimes (J^1 L_X)^*)$  with its natural Lie algebra structure [Me1]. If  $X = G/P$ , then  $\mathcal{G}_{\setminus \Gamma}$  on the associated Legendre moduli space is often isomorphic to a  $G \times \mathbb{C}^*$ -structure, but there are exceptions [A] which are considered in Theorem 1(viii). In these exceptional cases the original  $G \times \mathbb{C}^*$ -structure may not be equal to the induced one, and one might try to identify some additional structures on the associated twistor spaces  $(Y, L)$  which ensure that  $\mathcal{G}_{\setminus \Gamma}$  admits a necessary reduction. However, the "exceptional"  $H$ -structures with  $H$  as in (a), (b) and (c) of Theorem 1(viii) are fairly well understood by now [B, Br1, Br2, S]. If there exists an exotic torsion-free  $G$ - or  $G \times \mathbb{C}^*$ -structure with simple  $G$  other than Bryant's  $H_3$  [Br3], it must be covered, up to a  $\mathbb{C}^*$  action, by the "generic" clause (ix) in Theorem 1.

Two particular examples of this general construction have been considered earlier [L1, Br3]. The first example is a pair  $X \hookrightarrow Y$  consisting of an  $n$ -quadric  $Q_n$  embedded into a  $(2n + 1)$ -dimensional contact manifold  $(Y, L)$  with  $L|_X \simeq i^* \mathcal{O}_{\mathbb{C}P^{n+1}}(1)$ ,  $i : Q_n \hookrightarrow \mathbb{C}P^{n+1}$  being a standard projective realisation of  $Q_n$ . It is easy to check that in this case  $H^0(X, L_X \otimes (J^1 L_X)^*)$  is precisely the conformal algebra implying that the associated  $(n + 2)$ -dimensional Legendre moduli space  $M$  comes equipped canonically with a conformal structure. This is in accord with LeBrun's paper [L1], where it has been shown how a conformal Weyl connection can be encoded into a complex contact structure on the space of complex null geodesics. Since  $H^1(X, L_X \otimes S^2(J^1 L_X)^*) = 0$ , the induced conformal structure must be torsion-free in agreement with the classical result of differential geometry. Easy calculations show that the vector space  $H^1(X, L_X \otimes S^3(J^1 L_X)^*)$  is exactly the subspace of  $TM \otimes \Omega^1 M \otimes \Omega^2 M$  consisting of tensors with Weyl curvature symmetries. Thus Theorem 1(vii) implies the well-known Schouten conformal flatness criterion. Since  $H^0(X, L_X \otimes S^2(J^1 L_X)^*)$  is isomorphic to the typical fibre of  $\Omega^1 M$ , the set of all torsion-free affine connections preserving the induced conformal structure is an affine space modelled on  $H^0(M, \Omega^1 M)$ , again in agreement with the classical result.

The second example, which also was among motivations behind the present work, is Bryant's [Br3] relative deformation problem  $X \hookrightarrow Y$  with  $X$  being a rational Legendre curve  $\mathbb{C}P^1$  in a complex contact 3-fold  $(Y, L)$  with  $L_X = \mathcal{O}(3)$ . Calculating  $H^0(X, L_X \otimes (J^1 L_X)^*)$ , one easily concludes that the induced  $G$ -structure,  $\mathcal{G}_{\lambda \setminus \Gamma}$ , on the associated 4-dimensional Legendre moduli space is exactly an exotic  $G_3$ -structure which has been studied by Bryant in his search for irreducibly acting holonomy groups of torsion-free affine connections which are missing in the Berger list [B]. Since  $H^1(X, L_X \otimes S^2(J^1 L_X)^*) = 0$ , Theorem 1(v) says the induced  $G_3$ -structure  $\mathcal{G}_{\lambda \setminus \Gamma}$  is torsion-free in accordance with [Br3]. Since  $H^0(X, L_X \otimes S^2(J^1 L_X)^*) = 0$ ,  $\mathcal{G}_{\lambda \setminus \Gamma}$  admits a unique torsion-free affine connection  $\nabla$ . The cohomology class  $\rho_t \in H^1(X, L_X \otimes S^3(J^1 L_X)^*)$  from Theorem 1(v) is exactly the curvature tensor of  $\nabla$ .

**4. Outline of the proof of Theorem 1.** Items (i)-(iii) of Theorem 1 follow from a more general theorem [Me1] which says that if  $X$  is a compact complex Legendre submanifold of a complex contact manifold  $(Y, L)$  such that  $H^1(X, L_X) = 0$ , then there exists a complete, maximal and stable analytic family  $\{X_t \hookrightarrow Y \mid t \in M\}$  of compact Legendre submanifolds containing  $X$  (completeness, e.g., means that the natural map  $T_t M \rightarrow H^0(X_t, L_{X_t})$  is isomorphic for each  $t \in M$ ). Indeed, if  $X$  is a generalised flag variety and  $L_X$  a very ample line bundle on  $X$ , then  $h^0(X, L_X) > 0$ , and hence  $H^1(X, L_X) = 0$  by Bott-Borel-Weil theorem and the fact that any holomorphic line bundle on  $X$  is homogeneous. Since  $X$  is rigid, each submanifold  $X_t$  of the family is isomorphic to  $X$ .

In view of the canonical isomorphism  $T_t M \rightarrow H^0(X_t, L_{X_t})$  the item (iv) is not a surprise. More precisely, defining  $F = \{(y, t) \in Y \times M \mid y \in X_t\}$  and using the fact that  $L_{X_t}$  is very ample on  $X_t$ , one can easily realise  $F$  as a subbundle of the projectivised conormal bundle  $\mathbb{P}_M(\Omega^1 M)$ . Fibrewise, this construction is the well-known projective realisation of a generalised flag variety  $X$  in  $\mathbb{C}P^{m-1} \simeq \mathbb{P}(H^0(X, L_X)^*)$  [B-E]. The subgroup  $G$  of  $GL(m, \mathbb{C})$  which leaves  $X \subset \mathbb{C}P^{m-1}$  invariant is exactly the one described in item (iv) of Theorem 1.

The next question we address is how to distinguish in terms of the holomorphic embedding data  $X \hookrightarrow Y$  the subclass of 1-flat induced  $G$ -structures. This leads us to explore the towers of infinitesimal neighbourhoods of two embeddings of analytic spaces,  $X_t \hookrightarrow Y$  and  $t \hookrightarrow M$ . At the first floors of these towers we have, by item (iii) of Theorem 1, an

isomorphism  $T_t M = H^0(X_t, L_{X_t})$  which is in the basis of the above conclusion about the induced  $G$ -structure on  $M$ . The second floors of these two towers are related to each other as follows. If  $J_t \subset \mathcal{O}_M$  is the ideal of holomorphic functions which vanish at  $t \in M$ , then the tangent space  $T_t M$  is isomorphic to  $(J_t/J_t^2)^*$ . Define a second order tangent bundle,  $T_t^{[2]} M$ , at the point  $t$  as  $(J_t/J_t^3)^*$ . This definition implies that  $T_t^{[2]} M$  fits into an exact sequence of complex vector spaces

$$0 \longrightarrow T_t M \longrightarrow T_t^{[2]} M \longrightarrow S^2(T_t M) \longrightarrow 0 \quad (1)$$

For each  $t \in M$  there exists a holomorphic line bundle,  $\Delta_{X_t}^{[2]}$ , on the associated Legendre submanifold  $X_t \hookrightarrow Y$  such that there are an exact sequence of locally free sheaves

$$0 \longrightarrow L_{X_t} \xrightarrow{\alpha} \Delta_{X_t}^{[2]} \longrightarrow S^2(N_t) \longrightarrow 0 \quad (2)$$

and a commutative diagram of vector spaces

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_t M & \longrightarrow & T_t^{[2]} M & \longrightarrow & S^2(T_t M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(X_t, L_{X_t}) & \longrightarrow & H^0(X_t, \Delta_{X_t}^{[2]}) & \longrightarrow & H^0(X_t, S^2(N_{X_t})) & \longrightarrow & 0 \end{array} \quad (3)$$

which extends the canonical isomorphism  $T_t M \rightarrow H^0(X_t, L_{X_t})$  to *second* order infinitesimal neighbourhoods of  $t \hookrightarrow M$  and  $X_t \hookrightarrow Y$ . All we need to know in this paper about  $\Delta_{X_t}^{[2]}$  is that this bundle exists and has the stated properties. For details of its definition we refer the interested reader to [Me1, Me2].

One can show that the Legendre submanifold  $X_t \hookrightarrow Y$  is 1-flat if and only if the obstruction,  $\delta_{X_t}^{[2]} \in H^1(X_t, L_{X_t} \otimes S^2(J^1 L_{X_t})^*)$ , for the global splitting of (2) is zero. If  $X_t$  is 1-flat, then any splitting of (2), i.e. a morphism  $\beta : \Delta_{X_t}^{[2]} \rightarrow L_{X_t}$  such that  $\beta \circ \alpha = id$ , induces via the above commutative diagram an associated splitting of the exact sequence (1) which is equivalent to a torsion-free affine connection at  $t \in M$ . This implies that if the family  $\{X_t \hookrightarrow Y \mid t \in M\}$  consists of 1-flat Legendre submanifolds, then the induced  $G$ -structure admits a torsion-free affine connection, i.e. is 1-flat. In reverse order, given a Legendre moduli space  $M$  with the induced  $G$ -structure  $\mathcal{G}_{\setminus \setminus \Gamma}$  as in Theorem 1(iv), one can use the commutative diagram (3) to show that that any torsion-free connection in  $\mathcal{G}_{\setminus \setminus \Gamma}$  induces canonically a splitting of the exact sequence (2) for each  $t \in M$ . The set of all splittings of (2) is an affine space modelled on  $H^0(X_t, L_{X_t} \otimes S^2(N_t^*)) \simeq H^0(X, L_X \otimes S^2(N^*))$ . These facts prove items (v) and (vi) of Theorem 1. The item (vii) can be proved by straightforward calculations in Darboux local coordinates on  $Y$ . For details of these calculations we refer to [Me1].

Consider now a complex  $m$ -dimensional manifold  $M$  and an irreducible  $G$ - or  $G \times \mathbb{C}^*$ -structure  $\mathcal{G} \subset \mathcal{L}^* \mathcal{M}$ , where  $G \subset GL(m, \mathbb{C})$  is a semisimple Lie subgroup (any irreducible  $H$ -structure with reductive  $H$  must be one of these). Since  $\mathcal{G}$  is irreducible, there is a naturally associated subbundle  $\tilde{F} \subset \Omega^1 M$  whose typical fiber is the cone in  $\mathbb{C}^m$  defined as the  $G$ -orbit of the line spanned by a highest weight vector. Denote  $\tilde{\mathcal{F}} = \tilde{F} \setminus 0_{\tilde{F}}$ , where  $0_{\tilde{F}}$  is the "zero" section of  $\tilde{p} : \tilde{F} \rightarrow M$  whose value at each  $t \in M$  is the vertex of the cone  $\tilde{p}^{-1}(t)$ . The quotient bundle  $\nu : \mathcal{F} \equiv \tilde{\mathcal{F}}/\mathbb{C}^* \rightarrow M$  is then a subbundle of the projectivized cotangent bundle  $\mathbb{P}_M(\Omega^1 M)$  whose fibres  $X_t$  are isomorphic to the generalised flag variety  $G/P$ , where  $P$  is the parabolic subgroup of  $G$  which preserves the highest weight vector in  $\mathbb{C}^m$  up to a

scale. Denote  $\dim G/P = n$ . The total space of the cotangent bundle  $\Omega^1 M$  has a canonical holomorphic symplectic 2-form  $\omega$ . Then the sheaf of holomorphic functions on  $\Omega^1 M$  is a sheaf of Lie algebras relative to the Poisson bracket  $\{f, g\} = \omega^{-1}(df, dg)$ . The pullback,  $i^*\omega$ , of the symplectic form  $\omega$  from  $\Omega^1 M \setminus 0_{\Omega^1 M}$  to its submanifold  $i : \tilde{\mathcal{F}} \rightarrow \Omega^1 M \setminus 0_{\Omega^1 M}$  defines a distribution  $\mathcal{D} \subset T\tilde{\mathcal{F}}$  as the kernel of the natural "lowering of indices" map  $T\tilde{\mathcal{F}} \xrightarrow{y\omega} \Omega^1 \tilde{\mathcal{F}}$ , i.e.  $\mathcal{D}_e = \{V \in T_e \tilde{\mathcal{F}} : Vy i^*\omega = 0\}$  at each point  $e \in \tilde{\mathcal{F}}$ . Using the fact that  $d(i^*\omega) = i^*d\omega = 0$ , one can show that this distribution is integrable and thus defines a foliation of  $\tilde{\mathcal{F}}$  by holomorphic leaves. We shall assume that the space of leaves,  $\tilde{Y}$ , is a complex manifold. This assumption imposes no restrictions on the local structure of  $M$ . The fact that the Lie derivative,  $\mathcal{L}_V i^*\omega = Vy i^*\omega + d(Vy i^*\omega) = 0$ , vanishes for any vector field  $V$  tangent to the leaves implies that  $i^*\omega$  is the pullback relative to the canonical projection  $\tilde{\mu} : \tilde{\mathcal{F}} \rightarrow \tilde{Y}$  of a closed 2-form  $\tilde{\omega}$  on  $\tilde{Y}$ . It is easy to check that  $\tilde{\omega}$  is non-degenerate. The quotient  $(\tilde{Y}, \tilde{\omega})$  is what is usually called a symplectic reduction of  $(\Omega^1 M \setminus 0_{\Omega^1 M}, \omega)$  along the submanifold  $\tilde{\mathcal{F}}$  (cf. [S]).

Let  $e$  be any point of  $\tilde{\mathcal{F}} \subset \Omega^1 M \setminus 0_{\Omega^1 M}$ . Restricting a "lowering of indices" map  $T_e(\Omega^1 M) \xrightarrow{y\omega} \Omega_e(\Omega^1 M)$  to the subspace  $\mathcal{D}_e$ , one obtains an injective map

$$0 \rightarrow \mathcal{D}_e \xrightarrow{y\omega} \mathcal{N}_e^*,$$

where  $\mathcal{N}_e^*$  is the fibre of the conormal bundle of  $\tilde{\mathcal{F}} \hookrightarrow \Omega^1 M \setminus 0_{\Omega^1 M}$ . Therefore, the rank of the distribution  $\mathcal{D}$  is equal at most to  $\text{rank } \mathcal{N}^* = m - n - 1$ . It is easy to show that  $\text{rank } \mathcal{D}$  is maximal possible if and only if the ideal sheaf,  $I_{\tilde{\mathcal{F}}}$ , of  $\tilde{\mathcal{F}}$  in  $\Omega^1 M \setminus 0_{\Omega^1 M}$  is a sheaf of Lie subalgebras, i.e.  $\{I_{\tilde{\mathcal{F}}}, I_{\tilde{\mathcal{F}}}\} \subset I_{\tilde{\mathcal{F}}}$ . An irreducible  $G$ -structure is called *Poisson* if  $I_{\tilde{\mathcal{F}}}$  is a Lie subalgebra (equivalently, if  $\text{rank } \mathcal{D} = \uparrow - \setminus - \infty$ ). It is easy to check that a locally flat  $G$ -structure is Poisson. Since  $\{, \}$  is a first order differential operation, this immediately implies that any irreducible 1-flat  $G$ -structure is also Poisson.

Next we show that if  $G$  is a reductive Lie group, then every complex  $m$ -manifold  $M$  with a Poisson  $G$ -structure is canonically isomorphic, at least locally, to a Legendre moduli space. Indeed, there is an integrable distribution  $\mathcal{D}$  of rank  $m - n - 1$  on the bundle  $\tilde{\mathcal{F}} \rightarrow M$ . Assuming that  $M$  is sufficiently "small", we obtain as the quotient a symplectic manifold,  $(\tilde{Y}, \tilde{\omega})$ , with  $\dim \tilde{Y} = (m + n + 1) - (m - n - 1) = 2n + 2$ . There is a double fibration

$$\tilde{Y} \xleftarrow{\tilde{\mu}} \tilde{\mathcal{F}} \xrightarrow{\tilde{\nu}} M$$

with fibres of  $\tilde{\mu}$  being leaves of the integrable distribution  $\mathcal{D}$  and fibres of  $\tilde{\nu}$  being  $(n + 1)$ -dimensional cones  $X_t \subset \Omega_t^1 M \setminus 0$  generated by  $G$ -orbits of highest weight vectors. It is clear that, for each  $t \in M$ , the submanifold  $\tilde{\mu}(X_t) \subset \tilde{Y}$  is a Lagrange submanifold relative to the induced symplectic form  $\tilde{\omega}$  on  $\tilde{Y}$ .

There is a natural action of  $C^*$  on  $\tilde{\mathcal{F}}$  which leaves  $\mathcal{D}$  invariant and thus induces an action of  $C^*$  on  $\tilde{Y}$ . The quotient  $Y = \tilde{Y}/C^*$  is a  $(2n + 1)$ -dimensional complex manifold which has a double fibration structure

$$Y \xleftarrow{\mu} \mathcal{F} = \tilde{\mathcal{F}}/C^* \xrightarrow{\nu} M$$

and thus contains a family of compact  $n$ -dimensional embedded manifolds  $\{X_t = \mu \circ \nu^{-1}(t) \hookrightarrow Y \mid t \in M\}$  with  $X_t = \tilde{X}_t/C^*$ . Next, inverting a well-known procedure of symplectivisation of a contact manifold, it is not hard to show that  $Y$  has a complex contact structure such that the family  $\{X_t \hookrightarrow Y \mid t \in M\}$  is a family of compact Legendre

submanifolds. The contact line bundle  $L$  on  $Y$  is just the quotient  $L = \tilde{\mathcal{F}} \times \mathbb{C}/\mathbb{C}^*$  relative to the natural multiplication map  $\tilde{\mathcal{F}} \times \mathbb{C} \rightarrow \tilde{\mathcal{F}} \times \mathbb{C}$ ,  $(p, c) \rightarrow (\lambda p, \lambda c)$ , where  $\lambda \in \mathbb{C}^*$ . Then  $L|_{X_t}$  is isomorphic to the restriction of the hyperplane section bundle on  $\mathbb{P}(\Omega_t^1 M)$  to its submanifold  $\nu^{-1}(t) \simeq X_t$  and hence is very ample on  $X_t$ . Therefore,  $h^0(X_t, L|_{X_t}) = m$  which implies that the Legendre family  $\{X_t \hookrightarrow Y \mid t \in M\}$  is complete.

Therefore we proved that if  $G \subset GL(m, \mathbb{C})$  is a semisimple Lie subgroup and  $\mathcal{G}$  any irreducible 1-flat  $G \times \mathbb{C}^*$ -structure on an  $m$ -dimensional manifold  $M$ , then there exists a complex contact manifold  $(Y, L)$  and a Legendre submanifold  $X \hookrightarrow Y$  with  $X = G/P$  for some parabolic subgroup  $P \subset G$  and with  $L_X$  being very ample, such that, at least locally,  $M$  is canonically isomorphic to the associated Legendre moduli space. To complete the proof of final items (viii) and (ix) of Theorem 1 one needs only the fact [A] that if  $X$  is a generalised flag variety of a connected complex semisimple Lie group  $G$ , then the connected component,  $\text{Aut}^0 X$ , of the group of global holomorphic automorphisms is simple and, as a rule, coincides with  $G$ . The only exceptions are listed below: (a)  $X = SO(2n+2, \mathbb{C})/U(n+1)$ ,  $n \geq 3$ ,  $G = SO(2n+1, \mathbb{C})$ ,  $\text{Aut}^0 X = SO(2n+2, \mathbb{C})$ ; (b)  $X = \mathbb{C}P^{2n+1}$ ,  $n \geq 1$ ,  $G = Sp(2n+2, \mathbb{C})$ ,  $\text{Aut}^0 X = SL(2n+2, \mathbb{C})$ ; (c)  $X = Q_5$ , the 5-dimensional compact quadric,  $G = G_2$ ,  $\text{Aut}^0 X = SO(7, \mathbb{C})$ . This fact completes the outline of the proof of Theorem 1.  $\square$

**5. On holonomy groups.** Let  $\{X_t \hookrightarrow Y \mid t \in M\}$  be a complete family of compact Legendre submanifolds. A torsion-free connection on  $M$  which arises at each  $t \in M$  from a splitting of the extension (2) is called an *induced* connection. In the previous subsection we proved also the following

**Theorem 2** *Let  $\nabla$  be a holomorphic torsion-free affine connection on a complex manifold  $M$  with irreducibly acting reductive holonomy group  $G$ . Then there exists a complex contact manifold  $(Y, L)$  and a Legendre submanifold  $X \hookrightarrow Y$  with  $X = G_s/P$  for some parabolic subgroup  $P$  of the semisimple factor  $G_s$  of  $G$  and with  $L_X$  being very ample, such that, at least locally,  $M$  is canonically isomorphic to the associated Legendre moduli space and  $\nabla$  is an induced torsion-free affine connection in  $\mathcal{G}_{\setminus \Gamma}$ .*

Theorem 2 and much of Theorem 1 are devoted to torsion-free affine connections and 1-flat  $G$ -structures. In fact, as follows from the outline of the proof of Theorem 1, the class of irreducible  $G$ -structures that can be interpreted as induced on Legendre moduli spaces is much larger than the class of 1-flat  $G$ -structures (but much smaller than the class of all possible irreducible  $G$ -structures). For motives explained in sect.4,  $G$ -structures in this class are called Poisson. The theorem of Hano and Ozeki [H-O] is no longer true in the category of affine connections in Poisson  $G$ -structures which implies that, in addition to the open problem of classifying all irreducibly acting holonomies of torsion-free affine connections, we get another seemingly non-trivial problem of classifying all irreducibly acting holonomies  $G$  of affine connections with *non-zero torsion* which are tangent to Poisson  $G$ -structures.

We conclude this paper with the remark that given a sufficiently "small" complex  $m$ -dimensional manifold  $M$  and an irreducible  $G$ -structure  $\mathcal{G} \subset \mathcal{L}^* \mathcal{M}$  with reductive  $G$ , one may proceed as in sect.4 to show that there is an associated complex contact  $(2q+1)$ -dimensional manifold  $(Y, D)$  and  $m$ -dimensional family of generalised flag varieties  $\{X_t \hookrightarrow Y \mid t \in M\}$  which are tangent to the contact distribution  $D$ , i.e.  $TX_t \subset D$  for each  $t \in M$ . If  $\mathcal{G}$  is 1-flat or, more generally, Poisson, then each contact submanifold  $X_t$  is of maximal possible dimension  $q$ , i.e. it is a Lagrange submanifold. In general,  $\dim X_t \leq q$ ,

and we can stratify all possible  $G$ -structures on  $M$  into classes parametrised by an integer  $l = q - \dim X_t$ . In this paper it is shown that in the case  $l = 0$  (much of) the original  $G$ -structure together with its basic geometric invariants can be reconstructed from the complex contact structure on  $Y$  by twistor methods. It seems unlikely that something similar can be done in the case  $l \geq 1$ .

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