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Pasteurgasse 6/7
A-1090 Wien, Austria

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Abhay Ashtekar
Jerzy Lewandowski

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Projective techniques and functional integration for gauge theories

Abhay Ashtekar[1,2] and Jerzy Lewandowski[3,4]

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[1]Center for Gravitational Physics and Geometry,
Physics Department, Penn State, University Park, PA 16802-6300.
[2]Isaac Newton Institute for Mathematical Sciences,
University of Cambridge, Cambridge CB3 OEH.
[3]Institute of Theoretical Physics,
Warsaw University, ul Hoza 69, 00-681 Warsaw.
[4]Erwin Schrödinger Institute for Mathematical Physics
Pasteurgasse 6/7, A-1090 Vienna.

Abstract

A general framework for integration over certain infinite dimensional spaces is first developed using projective limits of a projective family of compact Hausdorff spaces. The procedure is then applied to gauge theories to carry out integration over the non-linear, infinite dimensional spaces of connections modulo gauge transformations. This method of evaluating functional integrals can be used either in the Euclidean path integral approach or the Lorentzian canonical approach. A number of measures discussed are diffeomorphism invariant and therefore of interest to (the connection dynamics version of) quantum general relativity. The account is pedagogical; in particular prior knowledge of projective techniques is not assumed. [1]

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1 Introduction

Theories of connections are playing an increasingly important role in the current description of all fundamental interactions of Nature. They are also of interest from a purely mathematical viewpoint. In particular, many of the recent advances in the understanding of topology of low dimensional manifolds have come from these theories. To quantize such theories non-perturbatively, it is desirable to maintain manifest gauge invariance. In practice, this entails working directly on the quotient of the space of connections modulo (local) gauge transformations. In particular, one has to develop an integration theory on the quotient. Now, this is a *non-linear* space with a rather complicated topology, while the standard functional integration techniques [1] are geared to linear spaces. Therefore, a non-linear extension of these techniques is needed. This task has been carried out in a series of papers over the past two years [2-6]. (For earlier work with the same philosophy, see [7, 8].) The purpose of this article is to present a new, self-contained treatment of these results.

Let us begin with a chronological summary of these developments. We will also point out some of the limitations of the original methods which will be overcome in the present treatment.

Fix an n-dimensional manifold M and consider the space $\mathcal{A}$ of smooth connections on a given principal bundle $P(M, G)$ over M. Following the standard terminology, we will refer to G as the structure group and denote the space of smooth maps from M to G by $\mathcal{G}$. This $\mathcal{G}$ is the group of *local* gauge transformations. If M happens to be a Cauchy surface in a Lorentzian space-time, the quotient $\mathcal{A}/\mathcal{G}$ serves as the physical configuration space of the classical gauge theory. Following the familiar procedures from non-relativistic quantum mechanics, one might therefore expect that quantum states would be represented by square-integrable functions on this $\mathcal{A}/\mathcal{G}$. In the sum over histories approach, M is taken to be the Euclidean space-time and the quotient $\mathcal{A}/\mathcal{G}$ then represents the space of physically distinct classical histories. Therefore, one might expect that the problem of constructing Euclidean quantum field theory would reduce to that of finding suitable measures on this $\mathcal{A}/\mathcal{G}$.

While these expectations point us in the right direction, they are not quite accurate. For, it is well known that, due to the presence of an infinite number of degrees of freedom, in quantum field theory one must allow configurations and histories which are significantly more general than those that feature in the classical field theories. (See, e.g., [9].) For example, in scalar field theories in Minkowski space, quantum states arise as functionals on the space of *tempered distributions* on a $t =$ const surface in space-time. Similarly, in the Euclidean approach, measures are concentrated on *distributional* histories; in physically interesting theories, the space of smooth histories is typically a set of measure zero. One would expect the situation to be similar in gauge theories. Unfortunately, since the spaces $\mathcal{A}/\mathcal{G}$ are non-linear, with complicated topology, a canonical mathematical setting for discussing "generalized connections modulo gauge transformations" is not available. For example, the naive approach of substituting the smooth connections and gauge transformations in $\mathcal{A}/\mathcal{G}$ by distributional ones does not work because the space of distributional connections does not support the action of distributional local gauge transformations.

Recently, one such setting was introduced [2] using the basic representation theory of $C^\star$-algebras. The ideas underlying this approach can be summarized as follows. One first considers the space $\mathcal{HA}$ of functions on $\mathcal{A}/\mathcal{G}$ obtained by taking finite complex linear combinations of finite products of Wilson loop functions $W_\alpha(A)$ around closed loops α. (Recall that the Wilson loop functions are traces of holonomies of connections around closed loops; $W_\alpha(A) = \mathrm{Tr}\ \mathcal{P}\ \oint_\alpha Adl$. Since they are gauge invariant, they project down unambiguously to $\mathcal{A}/\mathcal{G}$.) By its very construction, $\mathcal{HA}$ has the structure of a $\star$-algebra, where the involution operation, $\star$, is just the complex-conjugation. Since G is compact, W_α are bounded. This enables one to introduce an obvious (i.e., the sup-) norm on $\mathcal{HA}$ and complete it to obtain a $C^\star$-algebra which we will denote by $\overline{\mathcal{HA}}$. In the canonical approach, $\overline{\mathcal{HA}}$ is the algebra of configuration observables. Hence, the first step in the construction of the Hilbert space of physical states is the selection of an appropriate representation of $\overline{\mathcal{HA}}$. It turns out that every cyclic representation of $\overline{\mathcal{HA}}$ by operators on a Hilbert space is of a specific type [2]. The Hilbert space is simply $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu)$ for some regular, Borel measure μ on a certain completion $\overline{\mathcal{A}/\mathcal{G}}$ of $\mathcal{A}/\mathcal{G}$ and, as one might expect of configuration operators, the Wilson loop operators act just by multiplication. Therefore, the space $\overline{\mathcal{A}/\mathcal{G}}$ is a candidate for the extension of the classical configuration space needed in the quantum theory.

$\overline{\mathcal{A}/\mathcal{G}}$ is called the Gel'fand spectrum of the $C^\star$-algebra $\overline{\mathcal{HA}}$ and can be constructed purely algebraically: Its elements are the maximal ideals of $\overline{\mathcal{HA}}$.

As we just saw above, if M is a Cauchy slice in a Lorentzian space-time, $\overline{\mathcal{A}/\mathcal{G}}$ serves as the domain space of quantum states. In the Euclidean approach, where $\mathcal{A}/\mathcal{G}$ is constructed from connections on the Euclidean space-time, $\overline{\mathcal{A}/\mathcal{G}}$ is the space of generalized histories, over which one must integrate to calculate the Schwinger functions of the theory. As in the linear theories, the space $\mathcal{A}/\mathcal{G}$ of classical configurations/histories is densely embedded in space $\overline{\mathcal{A}/\mathcal{G}}$ of quantum configurations/histories. A key difference from the linear case, however, is that, unlike the space of tempered distributions, which is linear, $\overline{\mathcal{A}/\mathcal{G}}$ is compact.

A central issue in this approach to quantum gauge theories is that of obtaining a convenient characterization of $\overline{\mathcal{A}/\mathcal{G}}$. For the cases when the structure group G is either $SU(n)$ or $U(n)$, this problem was solved in [3]. The resulting characterization is purely algebraic. Using piecewise analytic loops passing through an arbitrarily chosen but fixed base point, one first constructs a group, $\mathcal{HG}$, called the *hoop group*. One then considers the space $\mathrm{Hom}(\mathcal{HG}, G)$ from $\mathcal{HG}$ to the structure group G. Every smooth connection A defines such a homomorphism via the holonomy map, evaluated at the base point. However, $\mathrm{Hom}(\mathcal{HG}, G)$ has many other elements. The space $\overline{\mathcal{A}/\mathcal{G}}$ is then shown to be naturally isomorphic to the space $\mathrm{Hom}(\mathcal{HG}, G)/\mathrm{AdG}$, where the quotient by the adjoint action by G just serves to remove the gauge freedom at the base point. The proof of this characterization is however rather long and relies, in particular, on certain results due to Giles [10] which do not appear to admit obvious extensions to general compact gauge groups.

The space $\overline{\mathcal{A}/\mathcal{G}}$ is very large. In particular, every connection on *every* G-bundle over M defines a point in $\overline{\mathcal{A}/\mathcal{G}}$. (In particular, $\overline{\mathcal{A}/\mathcal{G}}$ is independent of the initial choice of the principal bundle $P(M, G)$ made in the construction of the holonomy algebra $\mathcal{HA}$.) Furthermore, there are points which do not correspond to *any* smooth connection; these are the generalized connections (defined on generalized principal G-bundles [11]) which are relevant to only the quantum theory. There is a precise sense in which this space provides a "universal home" for all interesting measures [12]. In specific theories, such as Yang-Mills, the support of the relevant measures is likely to be significantly smaller. For diffeomorphism invariant theories such as general relativity, on the other hand, the whole space appears to be relevant.

Even though $\overline{\mathcal{A}/\mathcal{G}}$ is so large, it is a compact, Hausdorff space. Therefore, we can use the standard results from measure theory directly. These imply that there is an interesting interplay between loops and generalized connections. More precisely, every measure μ on $\overline{\mathcal{A}/\mathcal{G}}$ defines a function Γ_μ on the space of multi-loops:

$$\Gamma_\mu(\alpha_1, ..., \alpha_n) := \int_{\overline{\mathcal{A}/\mathcal{G}}} W_{\alpha_1}(\bar{A})...W_{\alpha_n}(\bar{A})\ d\mu\ , \tag{1.1}$$

where $W_\alpha(\bar{A})$ is the natural extension of $W_\alpha(A)$ to $\overline{\mathcal{A}/\mathcal{G}}$ provided by the Gel'fand representation theory. Thus, in the terminology used in the physics literature, $\Gamma_\mu(\alpha_1, ..., \alpha_n)$ is just the "vacuum expectation value functional" for the Wilson loop operators. In the mathematics terminology, it is the Fourier transform of the measure μ. As in linear theories, the Fourier transform Γ_μ determines the measure μ completely. Furthermore, *every* multi-loop functional Γ which is consistent with all the identities in the algebra $\mathcal{HA}$ and is positive defines a measure with Γ as its Fourier transform. Finally, the measure μ is diffeomorphism invariant if Γ_μ is diffeomorphism invariant, i.e., is a functional of (generalized) knots and links (generalized, because the loops $\alpha_1, ...\alpha_n$ are allowed to have kinks, intersections and overlaps.)

Several interesting regular Borel measures have been constructed on $\overline{\mathcal{A}/\mathcal{G}}$. The basic idea [3] behind these constructions is to use appropriate families of cylindrical functions, introduce cylindrical measures and then show that they are in fact regular Borel measures. Not surprisingly, the first non-trivial measure μ_o' to be introduced on $\overline{\mathcal{A}/\mathcal{G}}$ is also the most natural one: it is constructed solely from the Haar measure on the structure group G. Furthermore, μ_o' has two attractive features; it is faithful and invariant under the induced action of the diffeomorphism group of M [3]. Marolf and Mourão have analyzed properties of this measure using projective techniques [4]. They showed that $\mathcal{A}/\mathcal{G}$ is contained in a set of zero μ_o'-measure; the measure is concentrated on genuinely generalized connections. Using the relation between the measures and knot invariants mentioned above, given a suitable knot invariant, one can use μ_o' to generate other diffeomorphism invariant measures [13]. The resulting family of measures is very large. The next family is obtained by examining the possible intersections and kinks in the loops and dividing corresponding vertices into diffeomorphism invariant classes of "vertex types." Then, by assigning to every n-valent vertex type a measure on G^n, G being the structure group, one can introduce a measure on $\overline{\mathcal{A}/\mathcal{G}}$. This large class of measures was provided by Baez [5, 6]. Another family arises from the use of heat kernel methods on Lie groups [14] and may be useful in field theories on space-times with a fixed metric. Finally, $\overline{\mathcal{A}/\mathcal{G}}$ is known to admit measures which are appropriate for the $SU(n)$ Euclidean Yang-Mills theory in 2-dimensions, corresponding to space-times M with topologies R^2 and $S^1 \times R$ [12]. For loops without self-intersections, the value of the generating functional Γ_μ is given just by the exponential of (a negative constant times) the area enlosed by the loop; confinement is thus manifest. These measures are invariant under the induced action of area-preserving diffeomorphisms of M.

Thus, large classes of diffeomorphism invariant measures have been constructed on $\overline{\mathcal{A}/\mathcal{G}}$. It was somewhat surprising at first that $\overline{\mathcal{A}/\mathcal{G}}$ admits *any* diffeomorphism invariant measures at all. Indeed, there was a general belief that, just as translation-invariant measures do not exist on infinite-dimensional topological vector spaces, diffeomorphism invaraint measures would not exist on spaces of connections modulo gauge transformations. That they exist is quite fortunate because there are strong indications they will play an important role in non-perturbative quantum general relativity [15]. Note finally that for each of these measures, the computation of the expectation values of the Wilson loop operators reduces to an integration over k-copies of the structure group, for some k, and can therefore be carried out explicitly.

This concludes the general summary of recent developments. In this paper, we will obtain these results from another perspective, that of projective limits. The general idea of using these limits is not new: it has been used already in [4] to analyze the support of the measure μ'_o mentioned above. However, whereas the discussion of [4] drew on earlier results [2, 3] rather heavily, here we will begin afresh. Thus, our treatment will serve three purposes. First, it will enable us to provide a concise treatment of the results discussed above without having to rely on external input, such as the results of Giles [10], whose proofs are somewhat involved. Second, it will enable us to generalize the previous results; we will be able to deal with all compact, connected gauge groups at once. Finally, this approach will enable us to show that functional integration over non-linear spaces such as $\overline{\mathcal{A}/\mathcal{G}}$ can be carried out using methods that are closely related to those used in the case of linear topological spaces [1, 16, 17], i.e., that there is an underlying coherence and unity to the subject.

In section 2, we present a general framework in which the standard projective techniques are applied to a projective family of compact Hausdorff spaces. This framework is then used in section 3 to obtain the space $\overline{\mathcal{A}/\mathcal{G}}$ and to develop integration theory on it.

2 Projective Techniques: The general framework

A general setting for functional integration over an infinite dimensional, locally convex, topological space V is provided by the notion of "projective families" [1, 16, 17] which are constructed from the quotients, $V/\tilde{S}$, of V by certain subspaces $\tilde{S}$ (see section 2.1). We wish to extend this framework gauge theories where the relevant space $\overline{\mathcal{A}/\mathcal{G}}$ is *non-linear*. One approach to this problem is to modify the projective family appropriately. In section 2.1, we will begin by presenting such a family; it will now be constructed using compact, Hausdorff topological spaces. In section 2.2, we will use some elementary results from the Gel'fand-Naimark representation theory to unravel the structure of the projective limit of this family. Section 2.3 provides a general characterization of measures on the limiting space in terms of measures on the projective family. Finally, in section 2.4, we consider the action of a compact group on the family to obtain a quotient projective family and show that the limit of the quotient family is the same as the quotient of the limit of the original family. In application to gauge theory, the group action will be given by the adjoint action of the structure group on holonomies, evaluated at a fixed base point.)

While the primary application of the framework –discussed in the next section– is to gauge theories, the results are quite general and may well be useful in other contexts.

2.1 Projective family and the associated $C^\star$ algebra of cylindrical functions

Let L be a partially ordered, directed set; i.e. a set equipped with a relation '$\geq$' such that, for all S, S' and S'' in L we have:

$$S \geq S\ ;\quad S \geq S' \text{ and } S' \geq S \Rightarrow S = S'\ ;\quad S \geq S' \text{ and } S' \geq S'' \Rightarrow S \geq S''\ ; \tag{2.1a}$$

and, given any $S', S'' \in L$, there exists $S \in L$ such that

$$S \geq S' \quad \text{and} \quad S \geq S''\ . \tag{2.1b}$$

L will serve as the label set. A *projective family* $(\mathcal{X}_S, p_{SS'})_{S,S' \in L}$ consists of sets $\mathcal{X}_S$ indexed by elements of L, together with a family of surjective *projections*,

$$p_{SS'} \ : \ \mathcal{X}_{S'} \to \mathcal{X}_S, \tag{2.2}$$

assigned uniquely to pairs (S', S) whenever $S' \geq S$ such that

$$p_{SS'} \circ p_{S'S''} = p_{SS''}. \tag{2.3}$$

A familiar example of a projective family is the following. Fix a locally convex, topological vector space V. Let the label set L consist of finite dimensional subspaces S of $V^\star$, the topological dual of V. This is obviously a partially ordered and directed set. Every S defines a unique sub-space $\tilde{S}$ of V via: $\tilde{v} \in \tilde{S}$ iff $< v, \tilde{v} >= 0 \ \ \forall v \in S$. The projective family can now be constructed by setting $\mathcal{X}_S = V/\tilde{S}$. Each $\mathcal{X}_S$ is a finite dimensional vector space and, for $S' \geq S$, $p_{SS'}$ are the obvious projections. As noted before, integration theory over infinite dimensional topological spaces can be developed starting from this projective family [1, 16, 17].

In this paper, we wish to consider another –and, in a sense, complementary– projective family which will be useful in certain kinematically non-linear contexts. We will assume that $\mathcal{X}_S$ are all topological, compact, Hausdorff spaces and that the projections $p_{SS'}$ are continuous. We will say that the resulting pairs $(\mathcal{X}_S, p_{SS'})_{S,S'\in L}$ constitute a *compact Hausdorff projective family*. In the application of this framework to gauge theories, carried out in the next section, the labels S can be thought of as general lattices (which are not necessarily rectangular) and the members $\mathcal{X}_S$ of the projective family, as the spaces of configurations/histories associated with these lattices. The continuum theory will be recovered in the limit as one considers lattices with increasing number of loops of arbitrary complexity.

Note that, in the projective family there will, in general, be no set $\overline{\mathcal{X}}$ which can be regarded as the largest, from which we can project to any of the $\mathcal{X}_S$. However, such a set does emerge in an appropriate limit, which we now define. The *projective limit* $\overline{\mathcal{X}}$ of a projective family $(\mathcal{X}_S, p_{SS'})_{SS'\in L}$ is the subset of the Cartesian product $\times_{S\in L}\mathcal{X}_S$ that satisfies certain consistency conditions:

$$\overline{\mathcal{X}} \ := \ \{(x_S)_{S\in L} \in \times_{S\in L}\mathcal{X}_S \ : \ S' \geq S \Rightarrow p_{SS'}x_{S'} = x_S\}. \tag{2.4}$$

(This is the limit that will give us the continuum gauge theory in the next section.) We provide $\overline{\mathcal{X}}$ with the product topology that descends from $\times_{S\in L}\mathcal{X}_S$.

This is the *Tychonov topology.* The topology of the product space is known to be compact and Hausdorff. Furthermore, as noted in [4], $\overline{\mathcal{X}}$ is closed in $\times_{S\in L}\mathcal{X}_S$, whence $\overline{\mathcal{X}}$ is also compact (and Hausdorff). We shall establish this property of $\overline{\mathcal{X}}$ independently in section 2.2. For now, we only note that the limit $\overline{\mathcal{X}}$ is naturally equipped with a family of projections:

$$p_S \ : \ \overline{\mathcal{X}} \to \mathcal{X}_S, \ \ p_S((x_{S'})_{S'\in L}) := x_S \ . \tag{2.5}$$

Next, we introduce certain function spaces. For each S consider the space $C^0(\mathcal{X}_S)$ of the complex valued, continuous functions on $\mathcal{X}_S$. In the union

$$\bigcup_{S\in L} C^0(\mathcal{X}_S)$$

let us define the following equivalence relation. Given $f_{S_i} \in C^0(\mathcal{X}_{S_i})$, $i = 1, 2$, we will say:

$$f_{S_1} \ \sim \ f_{S_2} \quad \text{if} \quad p^*_{S_1 S_3}\ f_{S_1} \ = \ p^*_{S_2 S_3}\ f_{S_2} \tag{2.6}$$

for every $S_3 \ \geq S_1, S_2$, where $p^*_{S_1, S_3}$ denotes the pull-back map from the space of functions on $\mathcal{X}_{S_1}$ to the space of functions on $\mathcal{X}_{S_3}$. Note that to be equivalent, it is sufficient if the equality (2.6) holds *just for one* $S_3 \ \geq S_1, S_2$. To see this, suppose that (2.6) holds for S_1, S_2 and S_3 and let $S_4 \geq S_1, S_2$. Take any $S_5 \geq S_i$, $i = 1, 2, 3, 4$. Then

$$p^*_{S_4 S_5} p^*_{S_1 S_4} f_{S_1} = p^*_{S_3 S_5} p^*_{S_1 S_3} f_{S_1} = p^*_{S_3 S_5} p^*_{S_2 S_3} f_{S_2} = p^*_{S_4 S_5} p^*_{S_2 S_4} f_{S_2}. \tag{2.7}$$

Since $p^*_{SS'} : C^0(\mathcal{X}_S) \to C^0(\mathcal{X}_{S'})$ is an embedding, (2.7) implies

$$p^*_{S_1 S_4} f_{S_1} = p^*_{S_2 S_4} f_{S_2}. \tag{2.8}$$

Using the equivalence relation we can now introduce the set of *cylindrical functions* associated with the projective family $(\mathcal{X}_S, p_{SS'})_{S,S'\in L}$,

$$\mathrm{Cyl}(\overline{\mathcal{X}}) \ := \ (\bigcup_{S\in L} C^0(\mathcal{X}_S) \ ) \ / \ \sim . \tag{2.9}$$

The quotient just gets rid of a redundancy: pull-backs of functions from a smaller set to a larger set are now identified with the functions on the smaller set. Note that in spite of the notation, as defined, an element of $\mathrm{Cyl}(\overline{\mathcal{X}})$ is *not* a function on $\overline{\mathcal{X}}$; it is simply an equivalence class of continuous functions on some of the members $\mathcal{X}_S$ of the projective family. The notation is, however, justified because, as we will show in the section 2.2, one *can* identify elements of $\mathrm{Cyl}(\overline{\mathcal{X}})$ with continuous functions on $\overline{\mathcal{X}}$.

Henceforth we shall denote the element of $\mathrm{Cyl}(\overline{\mathcal{X}})$ defined by $f_S \in C^0(\mathcal{X}_S)$ by $[f_S]_\sim$. We have:

Lemma 1 : *Given any $f, g \in \mathrm{Cyl}(\overline{\mathcal{X}})$, there exists $S \in L$ and $f_S, g_S \in C^0(\mathcal{X}_S)$ such that*

$$f = [f_S]_\sim, \quad g = [g_S]_\sim \tag{2.10}$$

Proof: Choose any two representatives $f_{S_1} \in f$ and $g_{S_1} \in g$; there exists $S \in L$ such that $S \geq S_i$, $i = 1, 2$. Take $f_S = p^*_{S_1 S} f_{S_1}$ and $g_S = p^*_{S_2 S} g_{S_2}$.

We will conclude this sub-section with a proposition that collects elementary properties of the cylindrical functions which makes it possible to construct a $C^\star$ algebra out of $\mathrm{Cyl}(\overline{\mathcal{X}})$.

Proposition 1 :
(i) Let $f, g \in \mathrm{Cyl}(\overline{\mathcal{X}})$; then the following operations are well defined:

$$f + g \;:=\; [f_S + g_S]_\sim, \quad fg \;:=\; [f_S g_S]_\sim, \tag{2.11a}$$

$$af \;:=\; [af_S], \quad f^\star \;:=[f^\star_S] \tag{2.11b}$$

where S is any element of L given by Lemma 1, $a \in \mathbf{C}$ and $\star$ within the bracket is the complex conjugation.
(ii) A constant function belongs to $\mathrm{Cyl}(\overline{\mathcal{X}})$.
(iii) Let $f_S \in C^0(\mathcal{X}_S)$, $f_{S'} \in C^0(\mathcal{X}_{S'})$ and $f_S \sim f_{S'}$; then

$$\sup_{x_S \in \mathcal{X}_S} |f_S(x_S)| \;=\; \sup_{x_{S'} \in \mathcal{X}_{S'}} |f_{S'}(x_{S'})|. \tag{2.11c}$$

The proofs are consequence of Lemma 1 and use an argument similar to that proving (2.8).

It follows from Proposition 1 that the set of cylindrical functions $\mathrm{Cyl}(\overline{\mathcal{X}})$ has the structure of a $\star$–algebra with respect to the operations defined in (i). The algebra contains the identity and we define on it a norm $\| \ \|$ to be

$$\|[f_S]_\sim\| \;:= \sup_{x_S \in \mathcal{X}_S} |f_S(x_S)|. \tag{2.12}$$

The norm is well defined according to part (iii) of Proposition 1. Consider the Banach algebra $\overline{\mathrm{Cyl}}(\overline{\mathcal{X}})$ obtained by taking the Cauchy completion of $\mathrm{Cyl}(\overline{\mathcal{X}})$. Since

$$\|f^\star\| \;=\; \|f\|, \tag{2.13}$$

$\overline{\mathrm{Cyl}}(\overline{\mathcal{X}})$ is a $C^\star$-algebra. By construction, it is Abelian and has an identity element. We shall refer to it as the *$C^\star$-algebra associated with the projective family* $(\mathcal{X}_S, p_{SS'})_{SS' \in \mathcal{G}}$.

2.2 The Gel'fand spectrum of $\overline{\mathrm{Cyl}}(\overline{\mathcal{X}})$.

A basic result in the Gel'fand-Naimark representation theory assures us that every Abelian $C^\star$-algebra $\overline{\mathcal{C}}$ with identity is realized as the $C^\star$-algebra of continuous functions on a compact Hausdorff space, called the *spectrum* of $\overline{\mathcal{C}}$. Furthermore, the spectrum can be constructed purely algebraically: Its elements are the maximal ideals of $\overline{\mathcal{C}}$, or, equivalently, the $\star$-preserving homomorphisms from $\overline{\mathcal{C}}$ to the $\star$-algebra of complex numbers. In this sub-section, we will use this result to show that the spectrum of $\overline{\mathrm{Cyl}}(\overline{\mathcal{X}})$ is precisely the projective limit $\overline{\mathcal{X}}$ of our projective family $(\mathcal{X}_S, p_{SS'})_{S,S' \in L}$. This will establish that, although the $C^\star$-algebra $\overline{\mathrm{Cyl}}(\overline{\mathcal{X}})$ was constructed using the projective family, its elements can be identified with continuous functions on the projective limit. In section 3, this result will enable us to obtain a characterization of the space $\overline{\mathcal{A}/\mathcal{G}}$ in a way that is significantly simpler and more direct than the original procedure [2, 3].

To show that $\overline{\mathcal{X}}$ is the spectrum of $\mathrm{Cyl}(\overline{\mathcal{X}})$, it is sufficient to establish the appropriate isomorphism from $\mathrm{Cyl}(\overline{\mathcal{X}})$ to the $C^\star$-algebra $C^0(\overline{\mathcal{X}})$ of continuous functions on $\overline{\mathcal{X}}$. There is an obvious candidate for this isomorphism. To see this, recall first, from Eq (2.5), that there is a natural projection map p_S from $\overline{\mathcal{X}}$ to $\mathcal{X}_S$. It is easy to check that

$$f_{S_1} \sim f_{S_2} \quad \Rightarrow \quad p^*_{S_1} f_{S_1} \;=\; p^*_{S_2} f_{S_2}\ . \tag{2.14}$$

Hence, we have a representation of the algebra of cylindrical functions $\mathrm{Cyl}(\overline{\mathcal{X}})$ by functions on the projective limit $\overline{\mathcal{X}}$

$$\mathrm{Cyl}(\overline{\mathcal{X}}) \ni f = [f_S]_\sim \;\mapsto F(f) \;:=\; p^*_S f_S \in \mathrm{Fun}(\overline{\mathcal{X}}), \tag{2.15}$$

Furthermore, this map satisfies $\sup_{x \in \overline{\mathcal{X}}} |F(f)(x)| \;\leq\; \|f\|$. Hence it extends to the completion $\overline{\mathrm{Cyl}}(\overline{\mathcal{X}})$. Thus the map F of Eq (2.15) is a candiate for the isomorphism we are seeking. Unfortunately, it is not obvious from the definition of the map that is it surjective; i.e. that *every* continuous function on $\overline{\mathcal{X}}$ arises from the pull-back of a function on a $\mathcal{X}_S$. We will therefore follow a different route towards our goal, using directly the algebraic definition of the spectrum. The final result will, in particular, establish that the map F has the desired properties.

As mentioned above, the Gel'fand spectrum of $\overline{\mathrm{Cyl}}(\overline{\mathcal{X}})$ is the set of homomorphisms $\mathrm{Hom}(\overline{\mathrm{Cyl}}(\overline{\mathcal{X}}), \mathbf{C})$ from the $\star$-algebra $\overline{\mathrm{Cyl}}(\overline{\mathcal{X}})$ into the $\star$-algebra of complex numbers. Now, since $\mathrm{Cyl}(\overline{\mathcal{X}})$ is dense in $\overline{\mathrm{Cyl}}(\overline{\mathcal{X}})$, every *continuous*, $\star$-operation preserving homomorphism from $\mathrm{Cyl}(\overline{\mathcal{X}})$ extends uniquely to an element of $\mathrm{Hom}(\overline{\mathrm{Cyl}}(\overline{\mathcal{X}}), \mathbf{C})$. Hence, to find the spectrum of $\overline{\mathrm{Cyl}}(\overline{\mathcal{X}})$ it suffices to find the space $\mathrm{Hom}_o(\mathrm{Cyl}(\overline{\mathcal{X}}), \mathbf{C})$ of the continuous homomorphisms from the $\star$-algebra $\mathrm{Cyl}(\overline{\mathcal{X}})$ to complexes. Our first observation is that there is a natural map ϕ from $\overline{\mathcal{X}}$ to $\mathrm{Hom}_o(\mathrm{Cyl}(\overline{\mathcal{X}}), \mathbf{C})$,

$$\phi \ : \ \overline{\mathcal{X}} \to \mathrm{Hom}_o(\mathrm{Cyl}(\overline{\mathcal{X}}), \mathbf{C}), \quad \phi(x)(f) \ := \ f_S(p_S(x)) \ , \tag{2.16}$$

where $x \in \overline{\mathcal{X}}$ and S is any element of L such that $f = [f_S]_\sim$ with some $f_S \in C^0(\mathcal{X}_S)$, since by definition of the projective limit, the right hand side of the Eq (2.16) is independent of a choice of S. We now show that ϕ is in fact a homeomorphism:

Theorem 1 : *Let* $\mathrm{Cyl}(\overline{\mathcal{X}})$ *be the* $\star$*–algebra of cylindrical functions of a Hausdorff, compact projective family* $(\mathcal{X}_S, p_{SS'})_{S,S'\in L}$*; then the map* ϕ *of Eq (2.16) is a homeomorphism from the projective limit* $\overline{\mathcal{X}}$ *equipped with the Tychonov topology onto the Gel'fand spectrum of* $\overline{\mathrm{Cyl}}(\overline{\mathcal{X}})$ *with its Gel'fand topology.*

Proof : The map ϕ is an injection because, for each $S \in L$, $C^0(\mathcal{X}_S)$ separates the points of $\mathcal{X}_S$. To see that ϕ is a surjection we construct the inverse mapping. Let us fix

$$h \in \mathrm{Hom}_o(\mathrm{Cyl}(\overline{\mathcal{X}}), \mathbf{C}) \ .$$

For each $\mathcal{X}_S$, h defines $h_S \in \mathrm{Hom}_o(C^0(\mathcal{X}_S), \mathbf{C})$ by

$$h_S(f_S) \ := \ h([f_S)]_\sim) \ .$$

Now, since $\mathcal{X}_S$ is compact and Hausdorff, given a h_S there exists $x^h_S \in \mathcal{X}_S$ such that

$$h_S(f_S) \ = \ f_S(x^h_S) \quad \text{for} \ \ \text{every} f_S \in C^0(\mathcal{X}_S).$$

In this way, for every $S \in L$ we have assigned to h a point x^h_S lying in $\mathcal{X}_S$. Pick any $S_1, S_2 \in L$ such that $S_2 \geq S_1$ and find the corresponding points $x^h_{S_i}$. Then, for each $f_{S_1} \in C^0(\mathcal{X}_{S_1})$, we have

$$f_{S_1}(p_{S_1S_2}(x^h_{S_2})) \ = \ p^*_{S_1S_2} f_{S_1}((x^h_{S_2})) \ = \ h_{S_2}(f_{S_2}) \ = \ h_{S_1}(f_{S_1}) \ = \ f_{S_1}(x^h_{S_1}), \tag{2.17}$$

from which we conclude that

$$p_{S_1S_2}(x^h_{S_2}) \ = \ x^h_{S_1}. \tag{2.18}$$

Hence, $(x_S)_{S\in L} \in \overline{\mathcal{X}}$ and

$$h([f_S]_\sim) \ = \ f_S(x_S). \tag{2.19}$$

Thus, we have proven that ϕ is a bijection.

The Gel'fand transform is a representation of $\overline{\mathrm{Cyl}}(\overline{\mathcal{X}})$,

$$\check{G} \ : \ \overline{\mathrm{Cyl}}(\overline{\mathcal{X}}) \ \rightarrow \ \mathrm{Fun}(\mathrm{Hom}_o(\mathrm{Cyl}(\overline{\mathcal{X}}), \mathbf{C}), \quad \check{G}(f)(h) \ := \ h(f) \ . \tag{2.20}$$

The Gel'fand topology on $\mathrm{Hom}_o(\mathrm{Cyl}(\overline{\mathcal{X}}), \mathbf{C})$ is the weakest topology with respect to which all the functions $\check{G}(\mathrm{Cyl}(\overline{\mathcal{X}}))$ are continuous. On the other hand, expressed in terms of functions, the Tychonov topology on $\overline{\mathcal{X}}$ is generated by the functions $\bigcup_{S\in L} p^*_S(C^0(\mathcal{X}_S))$. Because the map ϕ^* is a bijection of $\check{G}(\mathrm{Cyl}(\overline{\mathcal{X}}))$ onto $\bigcup_{S\in L} p^*_S(C^0(\mathcal{X}_S))$, ϕ^* carries the Gel'fand topology into that of Tychonov and ϕ^{-1*} maps the Tychonov topology into that of Gel'fand.

Theorem 1 has several interesting implications. We will conclude this sub-section by listing a few.

1. We can now conclude that the representation map F from $\mathrm{Cyl}(\overline{\mathcal{X}})$ to functions on $\overline{\mathcal{X}}$ of Eq (2.15) is faithful.

2. Second, the map F preserves the norms:

   $$\sup_{x\in\overline{\mathcal{X}}} |F(f)(x)| \ = \ \|f\| \ ;$$

   F can therefore be extended to an isomorphism between $\mathcal{C}(\overline{\mathcal{X}})$ and $C^0(\overline{\mathcal{X}})$.

3. Third, we can establish two properties of the projective limit $\overline{\mathcal{X}}$:
(a) The topology of the pointwise convergence in $\overline{\mathcal{X}}$ (i.e., $(x^n_S)_{S\in L} \rightarrow (x^0_S)_{S\in L}$ iff $x^n_S \rightarrow x^0_S$ for every $S \in L$) is Hausdorff and compact. This follows from the fact that the topology in question is just the Tychonov topology, and, from theorem 1, $\overline{\mathcal{X}}$ with Tychonov topology is homeomorphic with the Gel'fand spectrum which is compact, and Hausdorff.
(b)the map $\overline{\mathcal{X}} \ni (x_{S'})_{S'\in L} \mapsto x_S \in \mathcal{X}_S$ is surjective for every $S \in L$. This statement follows from the property 2 of the map F. Indeed, let $x^0_S \in \mathcal{X}_S$. There exists an everywhere positive $f_S \in C^0(\mathcal{X}_S)$ such that $f_S(x_S) - f_S(x^0_S) < 0$, for every $x_S \in L$. Hence $\sup_{x\in\overline{\mathcal{X}}} |F(f)(x)| = f(x^0_S)$. But since $F(f)$ is a continuous function on a compact space, there exists $x^1 \in \overline{\mathcal{X}}$ such that $F(f)(x^1) = \sup_{x\in\overline{\mathcal{X}}} |F(f)(x)|$. Then $f_S(x^1_S) = f_S(x^0_S)$ which implies that $x^0_S = p_S(x^1)$.

2.3 Regular Borel measures on the projective limit

The projective limit, $\overline{\mathcal{X}}$, is a compact, Hausdorff space in its natural –Tychonov– topology. Hence we can apply the standard results from measure theory to it. What we need is a procedure to construct interesting regular Borel measures on $\overline{\mathcal{X}}$. To simplify this task, in this sub-section we will obtain a convenient characterization of these measures.

Let us begin with a definition. Let us assign to each $S \in L$, a regular Borel, probability (i.e., normalized) measure, μ_S on $\mathcal{X}_S$. We will say that this is a *consistent family of measures* if

$$(p_{SS'})_* \, \mu_{S'} \; = \; \mu_S \; . \tag{2.21}$$

Using this notion, we can now characterize measures on $\overline{\mathcal{X}}$:

Theorem 2 : *Let $(\mathcal{X}_S, p_{SS'})_{SS'\in L}$ be a compact, Hausdorff projective family and $\overline{\mathcal{X}}$ be its projective limit.*

(a) Suppose μ is a regular Borel, probability measure on $\overline{\mathcal{X}}$; then μ defines a consistent family of regular, Borel, probability measures, given by:

$$\mu_S \; := \; p_{S*}\mu; \tag{2.22}$$

(b) Suppose $(\mu_S)_{S,S'\in L}$ is a consistent family of regular, Borel, probability measures. Then there is a unique regular, Borel, probability measure μ on $\overline{\mathcal{X}}$ such that $(p_S)_*\ \mu = \mu_S$;
(c) μ is faithful if $\mu_S\ :=\ (p_S)_*\ \mu$ is faithful for every $S \in L$.

Proof:
(a) Fix $\mathcal{X}_S$ and define the following functional on $C^0(\mathcal{X}_S)$:

$$C^0(\mathcal{X}_S)\ \ni\ f_S\ \mapsto\ \int_{\overline{\mathcal{X}}} d\mu[f_S]_\sim. \tag{2.23}$$

Being linear and positive, the functional defines a regular Borel measure μ_S on $\mathcal{X}_S$ such that

$$\int_{\mathcal{X}_S} d\mu_S f_S\ =\ \int_{\overline{\mathcal{X}}} d\mu[f_S]_\sim. \tag{2.24}$$

The unit function on $\mathcal{X}_S$, $I_S(x_S)\ =\ 1$ is lifted to $[I_S]_\sim \in \mathrm{Cyl}(\overline{\mathcal{X}})$, $[I_S]_\sim(x)\ =\ 1$. Hence, since μ is a probability measure, so is μ_S.
(b) Given a self consistent family of measures, we will define, now on $C^0(\overline{\mathcal{X}})$, a functional which is positive and linear. Using Theorem 1, let us identify the space of cylindrical functions $\mathrm{Cyl}(\overline{\mathcal{X}})$ with a dense subspace of $C^0(\overline{\mathcal{X}})$. Let $f \in \mathrm{Cyl}(\overline{\mathcal{X}}) \subset C^0(\overline{\mathcal{X}})$. Then $f = [f_S]_\sim$ for some $S \in \Gamma$ and $f_S \in C^0(\mathcal{X}_S)$. It follows from the definition of the equivalence relation $\sim$ and the consistency conditions (2.21), that the following functional is well defined:

$$\Gamma(f)\ :=\ \int_{\mathcal{X}_S} d\mu_S f_S\ . \tag{2.25}$$

Obviously, Γ is linear and positive on $\mathrm{Cyl}(\overline{\mathcal{X}})$, hence continuous. Because $\mathrm{Cyl}(\overline{\mathcal{X}})$ is dense in $C^0(\overline{\mathcal{X}})$ the functional extends to the positive linear functional defined on the entire $C^0(\overline{\mathcal{X}})$. Finally, there exists a regular Borel measure μ on $\overline{\mathcal{X}}$ such that

$$\int_{\overline{\mathcal{X}}} d\mu f\ =\ \Gamma(f), \quad f \in C^0(\overline{\mathcal{X}}).$$

This concludes the proof of (b).
(c) Suppose μ is faithful. If $f_S \in C^0(\mathcal{X}_S)$ then,

$$\int_{\mathcal{X}_S} d\mu_S f_S^\star f_S\ =\ \int_{\overline{\mathcal{X}}} d\mu ({p_S}^* f_S)^\star ({p_S}^* f_S).$$

Therefore the vanishing of the left hand side implies the vanishing of ${p_S}^* f_S$. But since we have proved in Theorem 1 that the map (2.15) is an injection, it follows that $f_S = 0$.

Suppose finally that the measures μ_S are all faithful. Then the linear functional Γ (2.25) is strictly positive on $\mathrm{Cyl}(\overline{\mathcal{X}}) \subset C^0(\overline{\mathcal{X}})$. Since $C^0(\overline{\mathcal{X}})$ is the closure of $\mathrm{Cyl}(\overline{\mathcal{X}})$, it follows from general results on C^* that Γ extends to a strictly positive linear functional on $C^0(\overline{\mathcal{X}})$. Alternatively, we can establish this result using measure theory. The product topology on $\overline{\mathcal{X}}$ is generated by the cylindrical open sets: pullbacks $p_S^* U_S$ of the open subsets U_S in $\mathcal{X}_S$, $S \in \Gamma$. It is easy to verify that the intersection of two open cylindrical sets is a cylindrical set. Hence, every non-empty open set in $\overline{\mathcal{X}}$ contains a non-empty cylinder set. Let U be a non-empty open set in $\overline{\mathcal{X}}$ and let $U' \subset U$ be cylindrical and non-empty. Then

$$\mu(U) \ \geq \ \mu(U') \ > \ 0$$

which concludes the proof of (c).

Remark: Note that, in establishing Theorem 2, we have used the results of section 2.1 in an essential way. First, for the generating function Γ of (2.25) to be well-defined, it is essential that the map (2.15) be one to one, which in turn is a consequence of Theorem 1. This property is also used in showing that the faithfulness of μ implies that of the projection $p_{S*}\mu$.

2.4 Quotient of a projective family.

Let now us suppose that a compact, Hausdorff topological group G acts on each component $\mathcal{X}_S$ of a projective family $(\mathcal{X}_S, p_{SS'})_{S,S' \in L}$,

$$\mathcal{X}_S \times G \ \ni \ (x_S, g_S) \ \mapsto \ x_S \, g_S \ \in \ \mathcal{X}_S. \tag{2.26}$$

Suppose further that the map (2.26) is continuous and the projections $p_{SS'}$ are G-equivariant:

$$p_{SS'}(x_{S'} \, g_{S'}) \ = \ x_S \, g_S. \tag{2.27}$$

Then, we will say that the G-action on the projective family is *consistent*. In this case, the initial projective family descends to a compact, Hausdorff projective family of the quotients, $(\mathcal{X}_S/G, p_{SS'})_{S,S' \in \Gamma}$.

While working simultanously with two projective families $(\mathcal{X}_S, p_{SS'})_{S,S'\in\Gamma}$ and $(\mathcal{X}_S/G, p_{SS'})_{S,S'\in\Gamma}$ we will mark the objects corresponding to the quotient family with the subscript $_G$. Thus for example, the two spaces of cylindrical functions will be denoted by Cyl and Cyl_G, respectively. However, for simplicity of notation, we will use the same symbol $p_{SS'}$ to denote the projection maps of both families; the intended projection should be clear from the context.

Now, the equivariance property (2.27) implies that the action (2.26) of G on the projective family $(\mathcal{X}_S, p_{SS'})_{S,S'\in L}$ induces a natural action of G on its projective limit $\overline{\mathcal{X}}$

$$\overline{\mathcal{X}} \ni (x_S)_{S\in\Gamma} \mapsto (x_S\ g_S)_{S\in\Gamma} \in \overline{\mathcal{X}}. \tag{2.28}$$

Taking the quotient, we obtain a compact Hausdorff space $\overline{\mathcal{X}}/G$. On the other hand, we can also take the projective limit of the quotient family $(\mathcal{X}_S/G, p_{SS'})_{S,S'\in\Gamma}$, to obtain a compact, Hausdorff space $\overline{\mathcal{X}}_G$. It is therefore natural to ask for the relation between the two. Not surprisingly, they turn out to be isomorphic. This is the main result of this sub-section.

We begin by noting that there is a natural sequence of maps:

$$\overline{\mathcal{X}}\ \rightarrow\ \overline{\mathcal{X}}/G \rightarrow \overline{\mathcal{X}}_G, \quad (x_S)_{S\in\Gamma} \mapsto [(x_S)_{S\in\Gamma}] \mapsto ([x_S])_{S\in\Gamma}, \tag{2.29}$$

where the square bracket denotes the orbits of G (in $\overline{\mathcal{X}}$ and in $\mathcal{X}_S$ respectively). Using this sequence, we can now state the theorem:

Theorem 3 : *Let G and $(\mathcal{X}_S, p_{SS'})_{S,S'\in\Gamma}$ be a Hausdorff compact group and a compact projective family respectively. Suppose G acts consistently on $(\mathcal{X}_S,\ p_{SS'})_{S,S'\in L}$; then, the map*

$$\phi: \quad \overline{\mathcal{X}}/G \ni [(x_S)_{S\in\Gamma}]\ \ \mapsto\ \ ([x_S])_{S\in\Gamma} \in \overline{\mathcal{X}}_G \tag{2.30}$$

is a homeomorphism.

Proof: This follows from the following two lemmas.

Lemma 2 : *The pullback map*

$$\phi^*\ \ : \overline{\mathrm{Cyl}}_G(\mathcal{X}_G)\ \rightarrow\ C^0(\overline{\mathcal{X}}/G) \tag{2.31}$$

is a bijection.

Proof: Let $f_G = [f_{GS}]_{\sim_G} \in \mathrm{Cyl}_G$ where $f_{GS} \in C^0(\mathcal{X}_S/G)$. Then, $\phi^* f_G$ coincides with the projection to $\overline{\mathcal{X}}/G$ of the function $[f_S]_\sim \in C^0(\overline{\mathcal{X}})$, f_S being the G invariant function on $\mathcal{X}_S$ corresponding to the function f_{GS}. Hence ϕ^* carries Cyl_G (thought of as the set of G invariant elements of Cyl) injectively into $C^0(\overline{\mathcal{X}}/G)$. Moreover, the image, $\phi^*\mathrm{Cyl}_G$ is dense in $C^0(\overline{\mathcal{X}}/G)$. Finally, ϕ^* preserves the norm: $\|f_G\|_G = \|[f_S]_\sim\| = \|\phi^* f_G\|$, where, the last term denotes the sup-norm in $C^0(\overline{\mathcal{X}}/G)$, and is therefore a continuous map. Hence the result. Next, we have:

Lemma 3 : *Let W and Z be two Hausdorff, compact spaces. Suppose there exists a map $\phi : W \to Z$ is such that $\phi^* : C^0(Z) \to C^0(W)$ is a bijection. Then ϕ is a homeomorphism.*

Proof: The map ϕ has to be injective because W is Hausdorff. Suppose that $z \in Z$ is not in the image of ϕ. z nonetheless defines an element of the spectrum of $C^0(Z)$, i.e., a ($\star$-preserving) homomorphism from $C^0(Z)$ to the space of complex numbers. Since ϕ^* is a bijection, here must exist $z' \in \phi(W)$ which represents the same homomorphism. But this contradicts the hypothesis that Z is Hausdorff, so $\phi(W) = Z$. Thus ϕ is also surjective. Finally, we establish the continuity of ϕ. We have just established that ϕ^{-1} exist. The hypothesis of the lemma implies that $\phi^{-1*} : C^0(W) \to C^0(Z)$ is also bijective. Since the topologies are given by the space of all continuous functions, it is clear that ϕ is a homeomorphism.

Combining the above lemmas we complete the proof of the theorem.

We will conclude with an observation. Define a *group projective family* to be a projective family $(\mathcal{G}_S, p_{SS'})_{S,S'\in L}$ such that every $\mathcal{G}_S$ is a topological group and the projections $p_{SS'}$ are homomorphisms. It is not hard to see that the projective limit $\mathcal{G}$ provided with the natural topology is a topological group. Furthermore, we have:

Proposition 2 : *Suppose $\overline{\mathcal{G}}$ is the projective limit of a group projective family $(\mathcal{G}_S, p_{SS'})_{S,S'\in L}$. If all the components $\mathcal{G}_S$ are compact, then $\overline{\mathcal{G}}$ is a compact topological group.*

Therefore, if to a projective family $(\mathcal{X}_S, p_{SS'})_{S,S'\in\Gamma}$ there is associated a group projective family $(\mathcal{G}_S, p_{SS'})_{S,S'\in L}$ of groups of motions of $\mathcal{X}_S$s which act consistently with the projections the projective limit of the quotients is again the quotient of the projective limits.

3 Application to gauge-theories

We will now apply the results of section 2 to gauge theories.

Section 3.1, introduces the relevant projective family. We begin with the notion of based loops and regard two as being equivalent if the holonomies of any smooth connection around them are the same. Each equivalence class is called a hoop. The space of equivalence classes has the structure of a group, which is called the hoop group and denoted by $\mathcal{HG}$. Subgroups S of $\mathcal{HG}$, generated by a finite number of hoops, will serve as the labels for our family. The set $\mathcal{X}_S$ will consist of all homomorphisms $\mathrm{Hom}(S, G)$ from S to the structure group G. If we regard S as forming a ("floating", i.e., irregular) lattice, $\mathcal{X}_S$ is in essence the space of configurations in the canonical approach and of histories in the Euclidean approach to the $\mathcal{X}_S$-lattice gauge theory. The projective limit $\overline{\mathcal{X}}$ then serves as the corresponding space for the continuum theory. In section 3.2, we consider the $C^\star$-algebra $\mathrm{Cyl}(\overline{\mathcal{X}})$ of cylindrical functions associated with this projective family. We show that this is naturally isomorphic with a certain $C^\star$-algebra of functions on $\mathcal{A}/\mathcal{G}$ which is known [3] to be naturally isomorphic to the holonomy algebra $\overline{\mathcal{HA}}$ of Wilson loop functions of the continuum gauge theory. Theorem 1 now implies that the projective limit $\overline{\mathcal{X}}$ of our projective family is naturally isomorphic with the Gel'fand spectrum $\overline{\mathcal{A}/\mathcal{G}}$ of $\overline{\mathcal{HA}}$. This provides a complete characterization of the spectrum without external inputs. Furthermore, since this characterization does not use results –such as those of Giles [10]– which are tied to $SU(n)$ or $U(n)$, we will now be able to treat all compact, connected structure groups G at once. Finally, this approach enables us to use directly the projective techniques of section 2.3, thereby streamlining the task of introducing measures on $\overline{\mathcal{A}/\mathcal{G}}$. This construction is carried out in section 3.3.

The specific projective family we use here was first introduced explicitly by Marolf and Mourão [4] using techniques developed in [3]. Our present treatment is a continuation of their work. However, to establish certain properties of the projective limit $\overline{\mathcal{X}}$ –such as the fact that $\overline{\mathcal{X}}$ is actually larger than $\mathcal{A}/\mathcal{G}$– they had to refer to the results of [2, 3] which in turn depend on the results due to Giles [10]. Using the general framework developed in section 2, we will now be able to establish these properties more directly.

Another projective family, based on graphs in M rather than subgroups of $\mathcal{HG}$, was introduced by Baez [5, 6]. It is better suited for constructing Baez measures and for introducing differential geometry on $\overline{\mathcal{A}/\mathcal{G}}$ [14].

3.1 The projective family

The set of labels.

Let us begin by recalling the notion of based loops. Let M be an analytic manifold and consider continuous, piecewise analytic (C^ω), parametrized *curves*, i.e., maps

$$p \;:\; [0,s_1]\cup\ldots\cup[s_{n-1},1]\to M$$

which are continuous on the whole domain and C^ω on the closed intervals $[s_k,s_{k+1}]$. Given two curves $p_1:\;[0,1]\to\; M$ and $p_2:\;[0,1]\to\; M$ such that $p_1(1)=p_2(0)$, we will denote by $p_2\circ p_1$ the natural composition:

$$p_2\circ p_1(s) = \begin{cases} p_1(2s), & \text{for } s\in[0,\frac{1}{2}] \\ p_2(2s-1), & \text{for } s\in[\frac{1}{2},1]. \end{cases}$$

The *inverse* of a curve $p:\;[0,1]\to M$ is a curve given by

$$p^{-1}(s) := p(1-s).$$

A curve which begins and ends at the same point is called a *loop.* Fix, once and for all, a point $x_0\in M$. Denote by $\mathcal{L}_{x_0}$ the set of (continuous, piecewise C^ω) loops which are based at x_0, i.e., which start and end at x_0.

Given a compact, connected Lie group G, two loops α_1,α_2 in M will be said to be G-holonomy equivalent if for every Lie algebra valued 1-form A on M, we have

$$\mathcal{P}\exp\int_{\alpha_1} Adl \;=\; \mathcal{P}\exp\int_{\alpha_2} Adl\ ,$$

where the holonomies are evaluated at the base point x_0. Each holonomically equivalent class of loops will be called a *hoop.* The space of all hoops has, naturally, the structure of a group, which we call the *G-hoop group* and denote by $\mathcal{HG}_G$.

It turns out, however, that the structure of $\mathcal{HG}_G$ is largely insensitive to the specific choice of the Lie group G [3]. More precisely, there are only two hoop groups: An Abelian one for the case when G is Abelian and a non-Abelian one for the case when G is non-Abelian. In this paper, we will assume that G is *non-Abelian* and denote the corresponding hoop group simply by $\mathcal{HG}$. (The Abelian case is discussed in some detail in Appendix A of [3].) $\mathcal{HG}$

can be described purely in terms of the geometry of the underlying manifold M: Two loops α and β in $\mathcal{L}_{x_0}$ define the same element in $\mathcal{HG}$ if they are related either by a reparametrization or by retracing of a line segment. More precisely, we have the following. An unparametrized loop is a class of loops any two elements of which differ by a piecewise analytic and orientation preserving diffeomorphism $[0,1] \rightarrow [0,1]$ and two unparametrized loops α and β define the same hoop in $\mathcal{HG}$, iff they can be written as

$$\alpha \; = \; p_2 \circ p_1, \quad \beta \; = \; p_2 \circ q^{-1} \circ q \circ p_1$$

for some curves p_i and q.

Next, we have the notion of *independent* hoops. A set of n loops $(\alpha_1, ..., \alpha_n)$ in $\mathcal{L}_{x_0}$ will be said to be independent if each α_i contains an open segment which traversed only once and which is shared by any other loop at most at a finite number of points. A set of n hoops is said to be independent if one can find a representative loop in each hoop such that the resulting collection of n loops is independent. A subgroup S of $\mathcal{HG}$ is said to be *tame* if it is generated by a finite number of independent hoops. It is straight forward to show that every subgroup of $\mathcal{HG}$ generated by a finite number of hoops (which are not necessarily independent) is in fact contained in a tame subgroup. [3].

Tame subgroups S of $\mathcal{HG}$ will serve as labels for our projective family. The partial ordering is given just by the inclusion relation. To show that this set is directed, i.e. satisfies Eq (2.b), we proceed as follows. Given any two tame subgroups S and S', denote by $\tilde{S}$ the subgroup of $\mathcal{HG}$ generated by the independent generators of S and S'. Since $\tilde{S}$ is finitely generated, it is contained in a tame subgroup S'' of $\mathcal{HG}$. Hence, in particular, there exists a tame group S'' such that $S'' \geq S$ and $S'' \geq S'$. Finally, note that since the labels S are all *finitely* generated, the hoop group $\mathcal{HG}$ does *not* belong to the label set L. In particular, therefore, L does not contain a largest element.

The projective family

To each $S \in L$, we assign the set $\mathcal{X}_S$ as follows:

$$\mathcal{X}_S \; := \; \mathrm{Hom}(S, G). \tag{3.1}$$

The projection maps are the obvious ones: if $S' \geq S$, we have the natural restriction map

$$p_{SS'} \; : \mathrm{Hom}(S', G) \rightarrow \mathrm{Hom}(S, G) \; . \tag{3.2}$$

Now, a key property of the tame subgroups of $\mathcal{HG}$ is that a given $S \in L$, every homomorphism from S to G is extendable to the entire $\mathcal{HG}$ [3]. Hence, in particular, it is extendable to every $S' \geq S$. Therefore, for any $S, S' \in L$, the projection $p_{SS'}$ is surjective as required in section 2.1.

Next, we will show that, for each S, the space $\mathcal{X}_S$ inherits from G the structure of a compact, analytic manifold. Choose a set $\beta_1, ..., \beta_n$ of independent generators of S and consider the map

$$\mathcal{X}_S \ni H \;\mapsto\; (H(\beta_1), ..., H(\beta_n)) \in G^n \; . \tag{3.3}$$

This map is bijective and provides $\mathcal{X}_S$ with the manifold structure of G^n. It is easy to verify that this structure is insensitive to the initial choice of the generators $(\beta_1, ..., \beta_n)$. Moreover, the projections $p_{S'S}$ are analytic with respect to the induced manifold structures on $\mathcal{X}_S$ and $\mathcal{X}_{S'}$ respectively.

Thus, $(\mathcal{X}_S, p_{SS'})_{S,S' \in L}$, where $\mathcal{X}_S$ is regarded as a compact, Hausdorff space, constitutes a compact, Hausdorff projective family in the sense of section 2.1. We will refer to it as the $\mathcal{HG}$-projective family.

Finally, note that the structure group G has a natural action on each component $\mathcal{X}_S$ of the projective family:

$$\mathcal{X}_S \ni H \;\mapsto\; \mathrm{Ad}(g) \circ H \in \mathcal{X}_S, \quad \forall g \in G \; , \tag{3.4}$$

where $(\mathrm{Ad}(g) \circ H)(\beta) = g^{-1} \cdot H(\beta) \cdot g, \; \forall \, \beta \in S$. It is easy to check that, in the terminology of section 2.4, G acts consistently on our projective family $(\mathcal{X}_S, p_{SS'})_{S,S' \in L}$. Therefore, it defines the quotient $\mathcal{HG}$-projective family. We will denote it by $(\mathcal{X}'_S, p_{SS'})_{S,S' \in L}$, where

$$\mathcal{X}'_S \; := \; \mathcal{X}_S / \mathrm{Ad}(G) \; , \tag{3.5}$$

and the projections descend from (3.2) in the obvious way. (As in section 2.4, we will use the same symbol for the two sets of projections; the context should suffice to indicate which projection is intended.)

The projective limit

We will first show that the projective limit $\overline{\mathcal{X}}$ of the $\mathcal{HG}$-projective family can be identified with the set of homomorphisms of the entire hoop-group $\mathcal{HG}$ into G. Pick any $H \in \mathrm{Hom}(\mathcal{HG}, G)$. For every $S \in L$, we can restrict H to S to obtain a homomorphism $H_{|_S} \in \mathcal{X}_S$ from S to G. It is obvious that:

$$p_{SS'} \, (H_{|_{S'}}) \; = \; H_{|_S}, \quad \text{whenever} \quad S' \geq S.$$

Hence, we have obtained a map

$$\mathrm{Hom}(\mathcal{HG}, G) \ \rightarrow\ \overline{\mathcal{X}}, \quad H \ \mapsto\ (H_{|_S})_{S\in L}\ . \tag{3.6}$$

This is the required identification:

Proposition 3 : *The map (3.6) is a bijection.*

Proof: That the map is injective as well as surjective follows easily from the fact that for every hoop $\alpha \in \mathcal{HG}$ there exists a tame subgroup $S \subset \mathcal{HG}$ which contains α [3].

We can now extend this result to the quotient $\mathcal{HG}$-projective family. Since the map (3.6) is G-equivariant with respect to the (adjoint) actions of G on $\overline{\mathcal{X}}$ and $\mathrm{Hom}(\mathcal{HG}, G)$, it projects to a bijective map of the quotients. Next, consider the map

$$\mathrm{Hom}(\mathcal{HG}, G)/\mathrm{Ad}(G) \ \rightarrow\ \overline{\mathcal{X}}/G \rightarrow\ \overline{\mathcal{X}'}, \tag{3.7a}$$

where $\overline{\mathcal{X}'}$ is the projective limit of the quotient family, given by

$$[H] \ \mapsto\ [(H_{|_S})_{S\in L}] \ \mapsto\ ([H_{|_S}])_{S\in L}. \tag{3.7b}$$

Theorem 3 now implies:

Proposition 4 : *The map (3.7) is a bijection between* $\mathrm{Hom}(\mathcal{HG}, G)/\mathrm{Ad}(G)$ *and the projective limit* $\overline{\mathcal{X}'}$ *of the quotient* $\mathcal{HG}$*-projective family.*

We conclude this subsection with a remark. A number of results used in this (as well as the next) subsection depend critically on the assumption that the loops are all (continuous and) *piecewise analytic*. (For example, our argument that the tame subgroups of $\mathcal{HG}$ constitute a directed set fails if the loops are allowed to be smooth.) It is not known how much of this analysis would go through if the loops were assumed to be only piecewise smooth. If the structure group is Abelian, piecewise smoothness does suffice (see Appendix A in [3]). However, the methods used there are tied to the special features of the Abelian case.

3.2 $C^\star$-algebras of cylindrical functions

In this sub-section, we will obtain a characterization of the Gel'fand spectrum of the holonomy $C^\star$-algebra $\overline{\mathcal{HA}}$ generated by the Wilson loop functions on the space $\mathcal{A}/\mathcal{G}$ of connections modulo gauge transformations.

Fix a principal bundle (P, M, π, G) over M with structure group G. Consider the fiber $\pi^{-1}(x_0) \subset P$ over the base point x_0 used in the hoop group, and fix a point $\tilde{x}_0$ in it. Then, given a hoop α, each smooth connection $A \in \mathcal{A}$ defines an element $H(A, \alpha)$ of the structure group G, via holonomy. This map is in fact a homomorphism of groups. Thus, we have the map

$$\mathcal{A} \ni A \ \mapsto \ H^A \in \mathrm{Hom}(\mathcal{HG}, G), \quad H^A(\alpha) \ = \ H(A, \alpha) \tag{3.8}$$

from the space $\mathcal{A}$ of connections to the space of homomorphisms from $\mathcal{HG}$ to G. Now, given a tame subgroup S of the hoop group, (3.8) induces a map from $\mathcal{A}/\mathcal{G}$ to the space $\mathrm{Hom}(S, G)/\mathrm{Ad}G$:

$$\tilde{p}_S \ : \ \mathcal{A}/\mathcal{G} \ \rightarrow \ \mathrm{Hom}(S, G)/\mathrm{Ad}(G) \quad \tilde{p}_S([A]) \ := \ [H^A{}_{|S}]. \tag{3.9}$$

This map is known to be surjective [3]. (It is, in effect, the projection from the space of classical configurations/histories of the continuum theory to the corresponding space for the lattice theory associated with S.) The pull-back $\tilde{p}_S^*$ of this map carries functions on $\mathcal{X}'_S$ $(= \mathrm{Hom}(S, G)/\mathrm{Ad}(G))$ to functions on $\mathcal{A}/\mathcal{G}$. We can use these pull-backs to introduce the notion of cylindrical functions on $\mathcal{A}/\mathcal{G}$. Thus, the space $\mathrm{Cyl}(\mathcal{A}/\mathcal{G})$ of *cylindrical functions* on $\mathcal{A}/\mathcal{G}$ is defined to be [3]

$$\mathrm{Cyl}(\mathcal{A}/\mathcal{G}) \ := \ \bigcup_{S \in L} \tilde{p}_S^* \ C^0(\mathcal{X}'_S). \tag{3.10}$$

Since each $\mathcal{X}'_S$ is in particular compact and Hausdorff, $\mathrm{Cyl}(\mathcal{A}/\mathcal{G})$ has the structure of a normed $\star$-algebra. Its completion, $\overline{\mathrm{Cyl}}(\mathcal{A}/\mathcal{G})$, will be called the *$C^\star$-algebra of cylindrical functions* on $\mathcal{A}/\mathcal{G}$.

A basic example of a cylindrical function on $\mathcal{A}/\mathcal{G}$ is the Wilson loop function. Fix a representation $\rho : G \rightarrow \mathrm{End}(V)$ of G where V is an n-dimensional vector space, and use it to define traces. Then, a Wilson loop function W_α on $\mathcal{A}/\mathcal{G}$, labelled by a hoop α, is defined by $W_\alpha([A]) := \frac{1}{n}\mathrm{Tr}\rho(H(\alpha, A))$. To see that this is a cylindrical function, choose any $S \in L$ which contains α. Then, $W_\alpha([A])$ is the pull-back of the function $f_{[A]}$ on $\mathcal{X}'_S$ defined by $f_{[A]}([H]) := \frac{1}{n}\mathrm{Tr} H$; it is a cylindrical function on S. Indeed, as one might intuitively expect, Wilson loop functions generate the entire algebra of cylindrical functions: $\overline{\mathrm{Cyl}}(\mathcal{A}/\mathcal{G})$ is naturally isomorphic to the holonomy $C^\star$-algebra $\overline{\mathcal{HA}}$ generated by the Wilson loop functions on $\mathcal{A}/\mathcal{G}$. (For the case when G is $SU(n)$ or $U(n)$, see [3]. For the general compact, connected gauge group, only a small modification required; one has to allow, all the fundamental representations.) The problem of characterizing the Gel'fand spectrum of the holonomy algebra therefore reduces to that of characterizing the spectrum of the $C^\star$-algebra $\overline{\mathrm{Cyl}}(\mathcal{A}/\mathcal{G})$. The task is further simplified by the following result:

Proposition 5 : *The $C^\star$-algebra $\overline{\mathrm{Cyl}}(\mathcal{A}/\mathcal{G})$ of cylindrical functions on $\mathcal{A}/\mathcal{G}$ is isometrically isomorphic with the $C^\star$-algebra $\mathrm{Cyl}(\overline{\mathcal{X}'})$ of cylindrical functions on the quotient $\mathcal{HG}$-projective family. The isomorphism is the continuous extension of the map*

$$\mathrm{Cyl}(\mathcal{A}/\mathcal{G}) \ni \tilde{p}_S^* \ f_S \ \mapsto \ [f_S]_\sim \in \mathcal{C}(\overline{\mathcal{X}'}) \ . \tag{3.11}$$

Proof: The result follows easily from the fact that the maps $\tilde{p}_S$ of (3.9) are surjections.

Using Theorem 1, we now have the desired characterization of the Gel'fand spectrum of $\overline{\mathcal{HA}}$:

Theorem 4 : *The Gel'fand spectrum of the $C^\star$-algebra $\overline{\mathrm{Cyl}}(\mathcal{A}/\mathcal{G})$ is naturally homeomorphic with the space $\overline{\mathcal{X}'} \ = \ \mathrm{Hom}(\mathcal{HG}, G)/Ad(G)$. The homeomorphism assigns to $[H] \in \overline{\mathcal{X}'}$ the following functional defined on $\overline{\mathrm{Cyl}}(\mathcal{A}/\mathcal{G})$:*

$$[H](f) \ := \ f_S([H]_{|S}) \tag{3.12}$$

for any $f_S \in C^0(\mathcal{X}'_S)$ such that $f \ = \ \tilde{p}_S^* \ f_S$.

Let us summarize. Proposition 5 implies that $\overline{\mathrm{Cyl}}(\mathcal{A}/\mathcal{G})$ –and hence the holonomy $C^\star$-algebra $\overline{\mathcal{HA}}$– is just a faithful representation of the $C^\star$-algebra $\overline{\mathrm{Cyl}}(\overline{\mathcal{X}'})$ of cylindrical functions associated with the quotient $\mathcal{HG}$-projective family. Theorem 1 ensures us that the Gel'fand spectrum $\overline{\mathcal{A}/\mathcal{G}}$ of $\overline{\mathcal{HA}}$ can be identified with the projective limit $\overline{\mathcal{X}'}$ of the quotient $\mathcal{HG}$-family. Now, since elements of $\overline{\mathcal{HA}}$ suffice to seperate points of $\mathcal{A}/\mathcal{G}$, and since $\overline{\mathcal{A}/\mathcal{G}}$ is the spectrum of $\overline{\mathcal{HA}}$, basic results of the Gel'fand theory imply that $\mathcal{A}/\mathcal{G}$

is densely embedded in $\overline{\mathcal{A}/\mathcal{G}}$. Hence, it now follows that $\mathcal{A}/\mathcal{G}$ is densely embedded in $\overline{\mathcal{X}'} \equiv \mathrm{Hom}(\mathcal{HG}, G)/\mathrm{Ad}(G)$. The projections $\tilde{p}_S : \mathcal{A}/\mathcal{G} \to \mathcal{X}'_S$ of (3.9) are just the restriction to $\mathcal{A}/\mathcal{G} \subset \overline{\mathcal{X}'}$ of the projections $p_S : \overline{\mathcal{X}'} \to \mathcal{X}'_S$. Hence the space of the cylindrical functions on $\mathcal{A}/\mathcal{G}$ may be viewed as the restriction to $\mathcal{A}/\mathcal{G} \subset \overline{\mathcal{X}'}$ of the cylindrical functions on $\overline{\mathcal{X}'}$, defined in terms of the projective family. Finally, we have an independent proof of the Marolf and Mourão [4] result that the Gel'fand topology on $\overline{\mathcal{A}/\mathcal{G}}$ coincides with the Tychonov topology on $\overline{\mathcal{X}'}$, a proof that now holds for general compact, connected gauge groups.

Recall that, given a tame subgroup S of $\mathcal{HG}$ generated freely by n independent hoops, the member $\mathcal{X}'_S$ of the projective family is isomorphic with $G^n/\mathrm{Ad}(G)$. Hence, we can think of it as the space of configurations (or histories) of the gauge theory associated with the lattice represented by S. The projection maps $\tilde{p}_S$ just serve to reduce the continuum theory to this lattice theory; they ignore what the connections do at points which are not on the lattice represented by S. As we enlarge the subgroup S, the lattice theory captures more and more information contained in the connection. The full information is recovered in the projective limit. Thus, this approach to the continuum theory is tied closely to the lattice approach. That, in particular, is the underlying reason why we can maintain manifest gauge invariance. However, the continuum limit is taken in a somewhat non-standard way: rather than refining the "mesh" of a rectangular lattice further and further, one allows bigger and bigger tame subgroups of the hoop group.

3.3 Measures on $\overline{\mathcal{A}/\mathcal{G}}$

We are now ready to introduce measures on $\overline{\mathcal{A}/\mathcal{G}}$. The basic idea is to apply results of section 2.3 to the $\mathcal{HG}$-projective family constructed above (using techniques from [5, 4]). Our aim in this subsection is to present only the key elements of the actual constructions of various measures and to provide the overall picture; a detailed account would make this article inordinately long.

Of the measures we will introduce, there are several families that are invariant under the induced action of the diffeomorphism group of M. It is therefore convenient to first spell out what this property entails. Consider an analytic diffeomorphism Φ:

$$\Phi \; : \; M \; \to \; M \; . \tag{3.13a}$$

Since Φ has a well-defined action on the space of continuous, piecewise analytic loops, and since the action obviously preserves the hoop equivalence relation, it induces an isomorphism on $\mathcal{HG}$ which in turn induces an isomorphism on the quotient $\mathcal{HG}$-projective family:

$$\Phi^S \ : \ \mathcal{X}'_{\Phi(S)} \ \rightarrow \ \ \mathcal{X}'_S \ . \qquad (3.13b)$$

Each Φ^S is a homeomorphism and this family of homeomorphisms induces a homeomorphism of the projective limit $\overline{\mathcal{X}'}$ on to itself. The question then is if a given measure μ on $\overline{\mathcal{X}'}$ is diffeomorphism invariant. It is easy to check that it is so if and only if the corresponding family μ_S of measures on $\mathcal{X}'_S$ satisfies [5]:

$$\Phi^S_* \ \mu_{\Phi(S)} \ = \ \mu_S \ . \qquad (3.13c)$$

The induced Haar measure

Let us begin with the $\mathcal{HG}$-projective family $(\mathcal{X}_S, p_{SS'})_{S,S' \in L}$. Recall from section 3.1 that, if the tame subgroup S of $\mathcal{HG}$ is generated by n independent hoops, then $\mathcal{X}_S$ is homeomorphic with G^n. Given a specific choice of n independent generators of S, the explicit homeomorphism is given by Eq (3.3). Denote by μ_H, the normalized Haar measure on G. The homeomorphism pushes the induced Haar measure on G^n forward and provides us with a measure on $\mathcal{X}_S$. Using properties of the Haar measure, it is easy to verify that this measure on $\mathcal{X}_S$ is insensitive to the initial choice of the independent generators. We will denote it by $\mu_H^{(n)}$. Thus, each member of our $\mathcal{HG}$-projective family is equipped with a regular, Borel, probability measure. It is easy to verify that this family $\mu_H^{(n)}$ of measures is consistent in the sense of Eq (2.21). Hence, by Theorem 2, it defines a regular, Borel, probability measure μ_o on the projective limit $\overline{\mathcal{X}}$. Since the Haar measure is faithful, so is the measure $\mu_H^{(n)}$ on $\mathcal{X}_S$ for any tame subgroups S. Hence, again by Theorem 2, it follows that μ_o is also faithful.

Next, consider the quotient $\mathcal{HG}$-family. Since its projective limit $\overline{\mathcal{X}'}$ can be expressed as $\overline{\mathcal{X}'} = \overline{\mathcal{X}}/G$, we can push forward μ_o on $\overline{\mathcal{X}}$ to obtain a measure μ'_o on $\overline{\mathcal{X}'} \equiv \overline{\mathcal{A}/\mathcal{G}}$. This is the required induced Haar measure. It is a faithful, regular, Borel, probability measure on $\overline{\mathcal{A}/\mathcal{G}}$. Finally, since its construction did not involve any structure –such as a metric, a fiducial connection, or a volume element on M– it follows that μ'_o is invariant under the induced action of Diff-M.

This measure was introduced in [3], where further details can be found.

Measures generated by knot invariants

One can use the induced Haar measure μ'_o to obtain a large class of diffeomorphism invariant measures, one corresponding to each knot invariant (of regular, embedded loops) of a suitable type. Here, we will present this construction for the simplest non-trivial case, that with the structure group $G = SU(2)$.

In this case, any regular, Borel, probability measure on $\overline{\mathcal{A}/\mathcal{G}}$ is completely determined by the generating function of Eq (1.1) involving only *single* loops [2]:

$$\Gamma_\mu(\alpha) := \int_{\overline{\mathcal{A}/\mathcal{G}}} W_\alpha(\bar{A})\ d\mu\ . \tag{3.14}$$

Let k_o be the characteristic function of an arbitrarily chosen knot class, say $\{\beta\}_o$ of smoothly embedded loops in M (with no self overlaps). Thus, $k_o(\beta) = 1$ if β belongs to $\{\beta\}_o$ of regular loops and zero otherwise. Consider the formal sum $\sum_\beta k_o(\beta)W_\beta(\bar{A})$, over all the loops in M, and set:

$$\Gamma_{\mu(k_o)}(\alpha) = \int_{\overline{\mathcal{A}/\mathcal{G}}} W_\alpha(\bar{A})\ [\sum_\beta k_o(\beta)W_\beta(\bar{A})]d\mu'_o\ . \tag{3.15}$$

Now, $SU(2)$ identities imply that $W_\alpha W_\beta = \frac{1}{2}(W_{\alpha\circ\beta} + W_{\alpha\circ\beta^{-1}})$, whence the integrand reduces to a formal sum of (k_o times) certain Wilson loop functions. Now, our measure μ'_o is such that the integral of all but a finite number of these terms is identically zero. Hence, the integral can actually be given a rigorous meaning. The resulting $\Gamma_{\mu(k_o)}$ can be shown to have the properties required to qualify as the Fourier transform of a signed (i.e. not necessarily positive definite) measure $\mu_o(k)$ on $\overline{\mathcal{A}/\mathcal{G}}$. Thus, while the sum $\sum_\beta k(\beta)W_\beta(\bar{A})$ is only formal and does not define a function on $\overline{\mathcal{A}/\mathcal{G}}$, there is a precise sense in which $\sum_\beta k_o(\beta)W_\beta(\bar{A})\ \times \mu'_o(\bar{A})$ can be regarded as a measure on $\overline{\mathcal{A}/\mathcal{G}}$.

Again, by construction, these measures are all invariant under the induced action of the Diff(M) group. Similar measures can be constructed if k_o is replaced by a more general knot invariant of an appropriate type. Details will appear in [13].

Baez vertex measures

We will now discuss another family of Diff(M) invariant measures introduced by Baez [5, 6], using a projective family labelled by graphs.

In terms of our projective families, Baez's construction can be summarized as follows. Given any tame subgroup S of the hoop group $\mathcal{HG}$, let us cut the loops in S to obtain a finite set of independent, analytic edges (with no overlaps) whose composition generates all the loops contained in S. (See [3] for details). These edges form a graph which is embedded in M and which contains all the loops in S. (Thus, S is contained in the fundamental group of the graph.) To an edge e, associate two G-valued random variables, h_{e-}, h_{e+}, each of which is assigned to an endpoint of the edge —i.e., to a vertex. For each vertex v in the graph, denote by n_v the number of edges incident at the point (any edge with both points at v being counted twice) and consider the associated n_v random variables. The space spanned by these variables is homeomorphic to G^{n_v}. The key idea is to introduce on G^{n_v} a measure μ_v which depends *only on the diffeomorphism invariant characteristics of the vertex* v. (Examples of such characteristics are: a cusp at v, a corner at v, a simple intersection of two smooth line segments, a multiple intersection, etc.) Now, given a function f on $\mathcal{X}_S$, we acquire, via Eq (3.3), a function $f(g_1, ..., g_n)$ on G^n. By expressing each independent generator of S as a composition of edges, we can reexpress f as a function $f(g_1(h), ..., g_n(h))$ on the space $G^{n_{v_1}+...+n_{v_V}}$ of random variables h associated with all the edges in the graph. Now, we can state the Baez ansätz:

$$\int_{\mathcal{X}_S} f_S \; d\mu_S^B \;:=\; \int_{G^{n_{v_1}} \times ... \times G^{n_{v_V}}} \; f(g_1(h), ..., g_n(h)) \; d\mu_{v_1}(h_{v_1}) ... d\mu_{v_V}(h_{v_V}) \;, \tag{3.16}$$

where V is the total number of vertices in the graph, and, as before, $h \in G^{n_{v_1}+...+n_{v_V}}$ and $h_v \in G^{n_v}$. This ansätz simplifies the task of solving the consistency conditions on the family μ_S of measures considerably. In particular, one can construct a solution of these conditions for *each* choice of a probability measure on the structure group G. Each solution in this large class defines a Diff(M) invariant measure on the projective limit $\overline{\mathcal{X}'} \equiv \overline{\mathcal{A}/\mathcal{G}}$.

By construction, the Baez maeasures are sensitive only to the presence of non-trivial vertices in graphs. Consequently, their support is contained within a proper subset of $(\overline{\mathcal{A}/\mathcal{G}})_B$ of $\overline{\mathcal{A}/\mathcal{G}}$:

$$(\overline{\mathcal{A}/\mathcal{G}})_B := \mathrm{Hom}_B(\mathcal{HG}, G) \;, \tag{3.17}$$

where $\mathrm{Hom}_B(\mathcal{HG}, G)$ is the set of all homomorphisms H such that $H(\alpha) = I_G$, whenever a hoop $\alpha \in \mathcal{HG}$ can be represented by an analytic embedding of a circle. Thus, the Baez measures are not faithful. One can of course obtain faithful measures by taking convex linear combination of any Baez measure with the induced Haar measure μ'_o.

The homotopy measure

We now turn to a measure [2] that is of interest to theories such as 3-dimensional gravity where only the flat connections are of physical interest. From the viewpoint of the general theory, however, this measure is rather trivial since it has support on a *finite* dimensional subspace of $\mathcal{A}/\mathcal{G}$.

Consider the space $\mathrm{Hom}(\pi_1(M), G)$ of homomorphisms from the first homotopy group of the manifold M into the structure group G and equip it with a topology such that for every tame subgroup of hoops S, the natural map

$$\mathrm{Hom}(\pi_1(M), G) \ \rightarrow \ \mathrm{Hom}(S, G) \tag{3.18}$$

is continuous. Then, for every measure μ_f defined on $\mathrm{Hom}(\pi_1(M), G)$, the push forward to $\mathrm{Hom}(\mathcal{HG}, G)$ defines a measure on $\overline{\mathcal{X}'}$, which we will denote by μ_F. The space $\mathrm{Hom}(\pi_1(M), G)/Ad(G)$ corresponds, of course, to the space of flat connections over M. Hence, the interpretation of the action of the measures μ_F on functions on $\overline{\mathcal{A}/\mathcal{G}}$ is as follows: given a function on $\overline{\mathcal{A}/\mathcal{G}}$, one first restricts it to a function on the space of flat connections and integrates this restriction using μ_f.

The measure μ_F is invariant with respect to group $\mathrm{Diff}_o(M)$ consisting of diffeomorphisms of M which are generated by analytic vector fields. Finally, we note that one can generalize this example by replacing the flat connections by those which are reducible (up to the conjugacy) to a given subgroup of G.

Heat-kernel measures.

The measures discussed above are all invariant under $\mathrm{Diff}(M)$ and therefore of interest primarily to diffeomorphism invariant theories such as general relativity. (As discussed below, however, μ'_o can also serve as a fiducial measure in the continuum Yang-Mills theory.) We will now discuss a class of measures [14] which do not share this invariance and may be useful in physical theories which, e.g., depend on a background space-time metric. On the mathematical side, these measures are, in a certain sense, the natural non-linear analogs of the Gaussian measures on linear spaces.

We begin by noting that since the structure group G is a compact, connected Lie group, it admits a family of heat kernel measures μ_t, with $t > 0$. These are obtained as follows. One first solves the heat equation on G:

$$\frac{d\rho_t}{dt} = \frac{1}{2} \ \triangle \rho_t \ , \quad \rho_{t=0}(g) = \delta(g, 1_G) \tag{3.19}$$

where $\triangle$ is the natural Laplacian on G and 1_G is the identity in G. It is known that ρ_t is strictly positive and smooth for all $t > 0$ [18]. (An explicit expression for ρ_t in terms of the characters of G is also available.) Hence, we can set $\mu_t = \rho_t \mu_H$. These are the heat-kernel measures on G.

The idea is to use these to obtain measures on $\overline{\mathcal{A}/\mathcal{G}}$. It turns out, however, that to satisfy the consistency conditions, it is necessary to introduce additional structure on the space of analytic curves, namely a "length functional". More precisely, consider a positive function l on the space of finite, unparametrized, anaytic curves on M, satisfying:

$$l(e) = l(e^{-1}), \quad l(e_1 \circ e_2) = l(e_1) + l(e_2), \tag{3.20}$$

for all curves e, e_1, e_2. If M is equipped with a positive definite metric, one can let $l(e)$ be just the length of the edge with respect to this metric. There are, however, many more solutions to (3.20).

Given a "length functional" we can introduce a measure on the members $\mathcal{X}_S$ of the $\mathcal{HG}$-projective family as follows. Fix a tame subgroup S of the hoop group. As in the case of the Baez measures, introduce a graph which contains the group S as a subgroup of its fundamental group. To each edge e_i in the graph assign a G-valued random variable h_i and, for every $t > 0$, a heat kernel measure $\mu_{h_i} = \rho_{s_i} \mu_H(h_i)$, with $s_i = l(e_i)t$, on G. Then, using the notation of Eq (3.16), we can define a measure μ_S on $\mathcal{X}'_S$ as follows:

$$\int_{\mathcal{X}'_S} f \; d\mu_S \; := \; \int_{G^E} \; f(g_1(h), ..., g_n(h)) \; \mu_H(h_1) .... \mu_H(h_E) \; , \tag{3.20}$$

where $f_S \in C^0(\mathcal{X'}_S)$, and where E denotes the number of edges in the graph. This family of measures is well-defined and consistent. Hence, it defines a regular, Borel, probability measure on $\overline{\mathcal{A}/\mathcal{G}}$. Note, however, that there is no non-trivial "length functional" which is Diff(M) invariant. Hence none of these heat-kernel measures on $\overline{\mathcal{A}/\mathcal{G}}$ are Diff(M) invariant.

Remarkably, however, with each of these heat-kernel measures, there is an associated Laplace operator defined on $\overline{\mathcal{A}/\mathcal{G}}$ which is essentially self adjoint on $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu'_0)$ [14]. Since such operators generally feature in the Hamiltonians of field theories, these measures may well play an important role in the canonical quantization of such theories. Finally, the heat-kernel measures can be used to construct the analog of the Segal-Bargmann transform of linear quantum field theories which provides a *holomorphic* representation of quantum states [19].

Measures for 2-dimensional Yang-Mills theories

Finally, let us discuss Yang-Mills theories. The $SU(n)$ Yang-Mills theories have been successfully quantized in two dimensions using measures on $\overline{\mathcal{A}/\mathcal{G}}$, in the cases when the topology is R^2 or $S^1 \times R$ [12] (see also the earlier work of Klimek & Kondracki [8]). Here we will sketch the construction of the Euclidean theory.

The idea, as in constructive quantum field theory, is to begin with a fiducial measure, but now on $\overline{\mathcal{A}/\mathcal{G}}$. This is chosen to be the natural measure μ'_o. Heuristically, the correct, physical measure may be written as $\mu_{\rm phy} = \exp\ -I(\bar{A})\mu'_o$. However, since the Yang-Mills action I is not well-defined on generalized connection, one must first introduce a regularized version thereof. The Wilson lattice action is an especially natural candidate for this since it is an integrable function on $\overline{\mathcal{A}/\mathcal{G}}$. One therefore introduces a lattice with lattice-separation a, writes down the associated Wilson action I_a, and computes the integrals

$$\Gamma_{(a)}(\alpha_1, ..., \alpha_n) = \int_{\overline{\mathcal{A}/\mathcal{G}}} W_{\alpha_1}(\bar{A})...W_{\alpha_n}(\bar{A})\ \exp(-I_a(\bar{A}))\ d\mu'_o\ . \tag{3.21}$$

This integral is well-defined; in fact the integrand is a cylindrical function on $\overline{\mathcal{A}/\mathcal{G}}$. One now takes the ultraviolet limit (lattice separation goes to zero) as well as the thermodynamic limit (lattice covers the entire Euclidean space-time) of $\Gamma_{(a)}$. The result is a well-defined function of multi-loops which satisfies all the conditions to qualify as the Fourier transform of a regular, Borel probability measure on $\overline{\mathcal{A}/\mathcal{G}}$. (In the case of the $U(1)$-theory, the Fourier transform can be written out in a closed form.) This is the physical measure of the theory.

This method interacts especially well with lattice approximations. In particular, it provides a natural framework for the continuum theory which one hopes will emerge from lattice methods in higher space-time dimensions. We should empahsize, however, that since higher dimensional Yang-Mills theories have not yet been analyzed in this framework, it is not yet clear if our general strategy will continue to be viable there.

4 Discussion

In this paper, we used projective techniques to construct a manifestly gauge invaraint framework for functional integration in gauge theories. Most of the final results reported in this paper have already appeared in the literature during the last two years. However, the approach adopted is new and it has enabled us to provide an essentially self-contained, concise and significantly simpler treatment of this material.

A key strength of these methods lies in the fact that they face the "kinematic non-linearities" of gauge theories squarely. More precisely, one recognizes early on that the space $\mathcal{A}/\mathcal{G}$ of *physically distinct* configurations or histories in gauge theories fails to admit a vector space structure and then takes these non-linearities seriously. This is to be contrasted with the more familiar approaches that attempt to fix a suitable gauge –ignoring Gribov ambiguities– to force a linear structure on $\mathcal{A}/\mathcal{G}$, and then look for measures which are perturbations of a (free) Gaussian measure on a linear space. The measures we discussed, by contrast, are all rather "far" from the ones obtained in such perturbative treatments. They are genuinely non-perturbative, geared to the kinematical non-linearities of gauge theories. As pointed out in section 3, the approach is closely related to the lattice approach; *the kinematics of the continuum theory is recovered from the lattice theories in a projective limit.*

Although these methods themselves are thus quite different from the ones used in the linear theories, as we saw in this paper, they do fall in the broad category of projective techniques. Thus, at a "primary" level, there *is* a structural similarity between our approach and the one based on promeasures [16, 17, 20], used in the linear case. In both cases, the basic objects are the projective families and one is ultimately interested in projective limits. The difference lies in the specific projective families used. The members of the family used in the linear cases are all finite dimensional vector spaces. In our case, they are compact, Hausdorff topological spaces. This difference does have important consequences; indeed even the sets of natural questions in the two cases are different. Nonetheless, when looked at from a sufficiently abstract perspective, one does find an underlying unity and coherence. Our non-linear generalization is thus not arbitrary; there is a precise sense in which it is a natural extension of the classical techniques [1] for functional integration on linear topological spaces.

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