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Viscously Damped Conservation Laws, with
Application to the 2–D Navier Stokes Equation**

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**OPTIMAL SMOOTHING AND DECAY ESTIMATES FOR
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APPLICATION TO THE 2-D NAVIER STOKES EQUATION**

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Dedicated to John Nash on his 60th birthday

Abstract

Optimal bounds on the $L^p(\mathbb{R}^n)$ smoothing and decay are established for certain viscously damped conservation laws, of which the vorticity formulation of the Navier–Stokes equation on \mathbb{R}^2 is a basic example. From the smoothing bounds, we obtain pointwise bounds that provide optimal control on the spatial decay of solutions. We apply this in a study of the physically important case in which the integral of the initial data (i.e., the total vorticity in the example) vanishes. We show in this case that as the time t increases, the $L^1(\mathbb{R}^n)$ norm of the solution decays to zero in two stages: for large initial data, there is a slow decay period during which the $L^1(\mathbb{R}^n)$ norm falls off with an inverse power of the logarithm of t . Then, once the norm has fallen below a critical value, it decays away to zero with $t^{-1/2}$. Again, this is optimal, and all of the constants in these estimates are explicitly computable in terms of the initial data.

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1. INTRODUCTION

A **viscously damped conservation law** is an equation of the form

$$\frac{\partial}{\partial t}u(t) = \Delta u(t) + \nabla \cdot (\mathbf{f}(x, u)u(x, t)) \quad (1.1)$$

where the vector field \mathbf{f} in (1.1) has some functional dependence on u such that whenever $\xi(x)$ is a smooth function of compact support,

$$\int_{\mathbb{R}^n} |\xi(x)|^q (\nabla \cdot \mathbf{f}(x, \xi)) dx = 0 \quad (1.2)$$

for all $q \geq 1$. In particular, if $u(\cdot, 0)$ is integrable, $\frac{d}{dt} \int u(x, t) dx = 0$, and $\int u(x, t) dx$ is a conserved quantity. The term involving the Laplacean represents the effects of some sort of “viscosity”.

Useful and familiar examples are Burger’s equation and the two dimensional Navier–Stokes equations in the vorticity formulation. In one spatial dimension with $\mathbf{f}(x, u(\cdot)) = u(x)/2$, (1.1) becomes Burger’s equation. (Note that (1.2) holds without \mathbf{f} being divergence free in this case.) In two spatial dimensions with

$$\mathbf{f}(x, \xi) = \int_{\mathbb{R}^2} S(x - y)\xi(y) d^2 y , \quad (1.3)$$

where

$$S(x) = \frac{1}{2\pi|x|^2}(-x_2, x_1) , \quad (1.4)$$

(1.1) becomes two dimensional Navier–Stokes vorticity equation. These are the basic concrete examples towards which the considerations in this paper are directed, though the main interest lies with the latter.

The main results are optimal smoothing and decay estimates for solutions of (1.1). The proofs involve ideas of many previous researchers, in particular Davies, Gross, Kato, Nash and Simon, as we shall explain. We shall first state the results, and then discuss in detail their relation to existing work. In the statements, and throughout the paper, $\|\cdot\|_p$ denotes the norm on $L^p(\mathbb{R}^n)$, and p' denotes the conjugate index to p , i.e., $p' = p/(p - 1)$.

Theorem 1: (Optimal Smoothing) *Let u be a solution of (1.1). Let p and q satisfy $1 \leq p \leq q \leq \infty$, and suppose that the vector field \mathbf{f} satisfies (1.2). Then for all $T > 0$ we have*

$$\|u(\cdot, T)\|_q \leq \left(\frac{K(q)}{K(p)}\right)^{n/2} \left(\frac{4\pi T}{1/p - 1/q}\right)^{-(n/2)((1/p)-(1/q))} \|u(\cdot, 0)\|_p \quad (1.5)$$

where $K(p) := (p')^{(1/p')}/p^{1/p}$. These bounds are saturated for appropriate Gaussian initial data and certain \mathbf{f} ; e.g., when $\mathbf{f} = 0$ or when \mathbf{f} is given by (1.3). Moreover when there is equality in (1.5) then the initial data is necessarily Gaussian.

As a particular case, suppose that u is a solution of (1) with integrable initial data. Then

$$\|u(\cdot, T)\|_{L^\infty(\mathbb{R}^n)} \leq (4\pi T)^{-n/2} \|u(\cdot, 0)\|_{L^1(\mathbb{R}^n)}. \quad (1.6)$$

That is, the $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ smoothing properties of (1) are *exactly* those of the standard heat equation.

This result is an improvement of a recent result of Kato's [K94]. Kato proved (1.5) for the vorticity equation with the optimal power $((1/q) - (1/p))$ of T , but with a larger multiplicative constant. We shall discuss the relation between Theorem 1 and Kato's work, which motivated the present paper, in more detail after all of our main results are stated.

It is remarkable that equality holds in (1.5) for certain solutions of the Navier–Stokes vorticity equation; i.e., in a genuinely nonlinear setting. This is because for radial initial data, the solutions of the heat equation also solve the Navier–Stokes vorticity equation and the heat equation takes Gaussians into Gaussians.

More specifically let ξ_α denote the Gaussian probability density on \mathbb{R}^n given by $\xi_\alpha(x) := (\alpha/\pi)^{n/2} e^{-\alpha|x|^2}$. Then with $T := (1/4p) - (1/4q)$, $e^{T\Delta}\xi_{p'} = \xi_{q'}$. Since $\|\xi_{p'}\|_p = \pi^{-(n/2p')} K(p)^{n/2}$ equality holds in (1.5). All other cases of equality are obtained from this by scaling.

Depending on the nature of the velocity functional \mathbf{f} , it may be possible to obtain pointwise velocity bounds as a consequence of Theorem 1. This is the case concerning our two main examples.

Theorem 2: (Pointwise Velocity Bounds) *Let u be a solution of Burger's equation. Then the velocity field \mathbf{f} satisfies*

$$\|\mathbf{f}(\cdot, u(\cdot, t))\|_\infty \leq (1/2)((4\pi t)^{-1/2}\|u(\cdot, 0)\|_1). \quad (1.7)$$

Let u be a solution of the Navier–Stokes vorticity equation. Then the velocity field \mathbf{f} satisfies

$$\|\mathbf{f}(\cdot, u(\cdot, t))\|_\infty \leq (4\pi t)^{-1/2}(2/\pi)^{1/2}\|u(\cdot, 0)\|_1. \quad (1.8)$$

That is, in both of these examples, $|\int_0^T \mathbf{f}(x, u(\cdot, t))dt| \leq BT^{1/2}$, and hence the speed at which the streaming induced by the vector field \mathbf{f} displaces the solution is comparable to the speed at which the viscosity diffuses it. Under such conditions, we have bounds on spatial decay of the same type as those that hold for the heat kernel.

Theorem 3: (Optimal Spatial Decay) *Suppose, in addition to the conditions of Theorem 1, that $\nabla \cdot \mathbf{f}(\cdot, u(\cdot, t)) = 0$, and that*

$$\|\mathbf{f}(\cdot, u(\cdot, t))\|_\infty \leq B(t) \quad (1.9)$$

for some integrable function $B(t)$. Then we have

$$|u(x, T)| \leq \int_{\mathbb{R}^n} G(T, x, y)|u(0, y)|d^n y \quad (1.10)$$

where

$$G(T, x, y) = (4\pi T)^{-n/2} \exp \left[-\frac{|x-y|^2}{4T} \left(1 - \frac{1}{|x-y|} \int_0^T B(\tau)d\tau \right)_+^2 \right] \quad (1.11)$$

and where $(\cdot)_+ = \max(\cdot, 0)$. In particular, if $B(\tau) \leq B\tau^{-1/2}$, then for any $0 < \beta < 1$,

$$G(T, x, y) \leq C(\beta, B)(\beta/4\pi T)^{n/2} e^{-\beta|x-y|^2/4T}, \quad (1.12)$$

where $C(\beta, B) := \beta^{-n/2} e^{B^2\beta/(1-\beta)}$.

Without imposing the condition $\nabla \cdot \mathbf{f}(\cdot, u(\cdot, t)) = 0$ the following integrated bounds hold:

$$\int |u(x, t)| \exp\left\{\frac{|x|^2}{4\gamma^4 t}\right\} dx \leq \exp\left\{\frac{2B^2}{\gamma^3(\gamma - 1)}\right\} \left(1 - \frac{1}{\gamma^2}\right)^{-n/2\gamma} \left(\int |u(x, 0)| e^{\epsilon|x|^2} dx\right),$$

for all $\gamma > 1$ and all $t > t_0 := \gamma/(2\epsilon(\gamma - 1)^2)$.

Note that when $B = 0$, the kernel $G(T, x, y)$ in (1.11) is *exactly* the heat kernel.

The next theorem concerns the rate of decay of solutions whose initial data $u(x, 0)$ satisfies $\int_{\mathbb{R}^n} u(x, 0) dx = 0$. In the case of the Navier–Stokes equation, the vorticity satisfies this condition for all finite energy initial data, as one sees by a simple argument using Stokes’ theorem.

The theorem below applies to the two dimensional Navier–Stokes equation since by (1.8), (1.14) holds with $F = 1/\pi$. In part of the theorem, we shall assume that the initial condition is well localized. This is not only for the convenience of obtaining computable constants but is entirely natural in the present setting since Theorem 3 controls the propagation of localization. Furthermore, it certainly covers the cases of physical interest when considering the Navier–Stokes equations.

However, the methods that we use to prove the following theorem can be easily adapted to estimate the rate of decay for general integrable initial data. The estimates depend only on the rate at which $\int_{|x| \geq R} |u(x, 0)| dx$ decreases as R tends to infinity, but in a complicated way. This is discussed in more detail in the proof of Theorem 7 in Section 4.

Previous results on decay of vorticity have been obtained for the case of small initial data [GK88].

Theorem 4: (Optimal Temporal Decay) *Suppose in addition to the hypotheses of Theorem 1 that $u(\cdot, 0)$ is integrable and satisfies*

$$\int_{\mathbb{R}^n} u(x, 0) dx = 0, \tag{1.13}$$

and \mathbf{f} is such that

$$|\mathbf{f}(x, u(\cdot, t))| \leq F \|u(\cdot, t/2)\|_1 t^{-1/2} \tag{1.14}$$

for some fixed constant F . Then $\lim_{t \rightarrow \infty} \|u(\cdot, t)\|_1 = 0$. Moreover, the rate of this decay can be explicitly bounded given certain additional information. In particular, suppose that

$$\int_{\mathbb{R}^n} u(x, 0) e^{\epsilon|x|^2} dx =: D < \infty \quad (1.15)$$

for some $\epsilon > 0$, and suppose also that \mathbf{f} is divergence free. Then for all $t > t_0$,

$$\|u(\cdot, t)\|_1 \leq \left(H e^{\{8\sqrt{2}HF e^{t_0/4} + t_0\}} \right) t^{-1/2}, \quad (1.16)$$

where t_0 is given by (4.37), and

$$H := (DC(3/4, B))(9/2)^{n/4}.$$

$C(\beta, B)$ is given in Theorem 3.

Remarks: (i) In the case of the two dimensional Navier–Stokes equations, these estimates on the decay of the vorticity also give an estimate on the decay of the energy when the initial vorticity $u(x, 0)$ is such that $\int_{\mathbb{R}^n} |x|^r |u(x, 0)|^r dx < \infty$ for some $r > 1$. This is certainly the case under the conditions of Theorem 4. Results for the decay of energy had been obtained earlier by Schönbek [S86] (see also [W86]). To our knowledge, the present bounds are the first such results for the vorticity.

(ii) It is easy to combine Theorems 1 and 4 to get improved rates of smoothing under the combined conditions of the two theorems: Simply use Theorem 4 to obtain an estimate on the time interval $[0, t/2]$, and then use Theorem 1 on the interval $[t/2, t]$. The result is that

$$\|u(\cdot, T)\|_\infty \leq CT^{-(n+1)/2} \|u(\cdot, 0)\|_{L^1}$$

in this case. That this is sharp for the class of equations is seen by considering scaling solutions of the heat equation.

We now briefly discuss the methods by which these results are proved, the relation between this work and related work of previous researchers.

First, the $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$ smoothing properties of the vorticity equation have been studied recently by Kato [K94]. He uses a method developed by Fabes and Stroock [FS86] for *linear* parabolic equations together with the sharp constant [CL93] in Nash’s inequality. This method leads to estimates that are only slightly weaker than the optimal estimates established here. For example, Kato obtains the two dimensional version of (1.6) with 4π replaced by $(j_{1,1}/2)^2\pi$, where $j_{1,1}$ is the first positive zero of the Bessel function J_1 . Since $(j_{1,1}/2)^2 \approx 3.670$, the constant in Kato’s estimate differs from the optimal constant by less than ten percent. All the same, we find it quite striking that the optimal constant is *exactly* that in the corresponding inequality for the linear heat flow problem.

We are very thankful to Professor Kato for a stimulating correspondence concerning his work [K94]. His interest in the sharp constant in Nash’s inequality, and his subsequent explanation of his application of it, have led us to take up the work presented here.

In proving Theorem 1, we use certain ideas of Davies [D87], and Davies and Simon [DS84] relating smoothing properties of parabolic equations with logarithmic Sobolev inequalities. The ideas of Davies and Simon in turn are closely related to ideas of Gross [G76], who showed that logarithmic Sobolev inequalities for Dirichlet forms are equivalent to hypercontractivity of the corresponding semigroups. Davies and Simon showed that an appropriate *family* of logarithmic Sobolev inequalities for a Dirichlet form are equivalent to “ultracontractivity” of the corresponding semigroup; i.e., L^1 to L^∞ bounds for the semigroup. To obtain sharp results we redevelop this approach, starting from Gross’s basic computation, and the following sharp form of the logarithmic Sobolev inequality on \mathbb{R}^n with reference to Lebesgue measure:

For all functions f on \mathbb{R}^n that, together with their distributional gradients ∇f are square integrable,

$$\int_{\mathbb{R}^n} |f|^2 \ln(|f|^2 / \|f\|_2^2) d^n x + \left(n + \frac{n}{2} \ln a \right) \int_{\mathbb{R}^n} |f|^2 d^n x \leq \frac{a}{\pi} \int_{\mathbb{R}^n} |\nabla f|^2 d^n x \quad (1.17)$$

for all $a > 0$. Moreover, there is equality in (1.17) if and only if f is a multiple and translate of $f_a(x) = a^{-n/4} e^{-\pi|x|^2/2a}$.

This inequality is equivalent to Gross’s logarithmic Sobolev inequality for \mathbb{R}^n equipped with

the Gaussian measure $dm(x) = e^{-\pi|x|^2} d^n x$; one simply inserts $f(x) := e^{\pi|x|^2/2} g(x)$ in Gross's inequality, and rewrites it as an inequality concerning g . The result is (1.17) for $a = 1$. The general case now follows by making a scale transformation. This relation has been used in [CL90] in a *proof* of Gross's inequality. The statement concerning cases of equality is established in [C91]. We stress the fact that it is natural to try to use an inequality saturated by Gaussian functions to prove an inequality that will be saturated by Gaussian functions: the heat flow transforms optimizers of (1.17) into optimizers of a rescaled version of (1.17). Note that this is not the case with the optimizers of Nash's inequality since they have compact support [CL93], a property that is not preserved by the heat flow. Finally, we mention that other sharp L^p inequalities (this time in an analytic function context) have been proved using (1.17) in [C91].

In proving Theorem 1, we shall rely on certain estimates for *linear* parabolic equations depending only on appropriate size estimates on the coefficients. This is an old idea, but it was one of the motivations for Nash's work [N58] on the regularity of uniformly parabolic equations. It was in this paper that Nash developed the methods that motivated Fabes and Stroock, and hence Kato and then ourselves. We are therefore especially glad to dedicate this paper to John Nash.

The rest of the paper is organized as follows: Section 2 contains the proofs of the linear estimates discussed above. In section 3, we adapt these estimate to our non-linear setting and prove the first three theorems. Section 4 takes up the issue of estimating the decay of L^1 norms and contains the proof of Theorem 4, as well as several results that amplify or render its content more precisely in various ways.

2. LINEAR ESTIMATES

Theorem 5: *Let u be a solution of*

$$\frac{\partial}{\partial t} u(x, t) = \Delta u(x, t) + \nabla \cdot (\mathbf{b}(x, t)u(x, t)) + c(x, t)u(x, t), \quad (2.1)$$

where $\mathbf{b}(x, t)$ and $c(x, t)$ are such that

$$\int_{\mathbb{R}^n} |u(x, t)|^q (\nabla \cdot \mathbf{b}(x, t)) dx = 0 \quad \text{for all } q \geq 1, \quad (2.2)$$

$$\|\mathbf{b}(\cdot, t)\|_\infty \leq B(t) \quad \text{and} \quad \|c(\cdot, t)\|_\infty \leq C(t) ,$$

where $B(t)$ and $C(t)$ are given functions continuous on $(0, \infty)$.

Then for all p and q satisfying $1 \leq p \leq q \leq \infty$ and all $T > 0$

$$\|u(T)\|_q \leq \left(\frac{K(q)}{K(p)} \right)^{n/2} \left(\frac{4\pi T}{1/p - 1/q} \right)^{-(n/2)((1/p)-(1/q))} \left(e^{\int_0^T C(\tau) d\tau} \right) \|u(0)\|_p . \quad (2.3)$$

Furthermore, when $B(t) = Bt^{-1/2}$ and $c = 0$, the time evolution preserves Gaussian decay in the sense that for any $\gamma > 1$

$$\begin{aligned} & \int |u(x, t)| \exp\left\{ \frac{|x|^2}{4\gamma^4 t} \right\} dx \\ & \leq \exp\left\{ \frac{2B^2}{\gamma^3(\gamma-1)} \right\} \left(1 - \frac{1}{\gamma^2}\right)^{-n/2\gamma} \left(\int |u(x, 0)| e^{\epsilon|x|^2} dx \right)^{1-1/\gamma} \left(\int |u(x, 0)| dx \right)^{1/\gamma} , \end{aligned} \quad (2.4)$$

for all $t > t_0 := \gamma/(2\epsilon(\gamma-1)^2)$.

Moreover when (2.2) is replaced by the stronger condition $\nabla \cdot \mathbf{b}(x, t) = 0$, the time evolution preserves Gaussian decay in the following pointwise sense: If $P(t, x, y)$ denotes the fundamental solution of (2.1) (i.e., $P(t, x, \cdot)$ satisfies (2.1) with the initial data $\lim_{t \rightarrow 0} P(t, x, \cdot) = \delta(x - \cdot)$), then

$$P(T, x, y) \leq (4\pi T)^{-n/2} \exp \left[-\frac{|x-y|^2}{4T} \left(1 - \frac{1}{|x-y|} \int_0^T B(\tau) d\tau \right)_+^2 + \int_0^T C(\tau) d\tau \right] \quad (2.5)$$

for all x and y . These results are optimal for both \mathbf{b} and c vanishing.

Proof: Let u be a solution to (2.1), which, without loss of generality, we may assume to be non-negative. Let $T > 0$ and $1 \leq p \leq q \leq \infty$ be given, and let $r : [0, T] \rightarrow [p, q]$ be a continuously differentiable increasing function to be specified later. By direct computation,

$$\begin{aligned}
& r(t)^2 \|u(t)\|_{r(t)}^{r(t)-1} \frac{d}{dt} \|u(t)\|_{r(t)} = \\
& \dot{r}(t) \int_{\mathbb{R}^n} u(t)^{r(t)} \ln(u(t)^{r(t)} / \|u(t)\|_{r(t)}^{r(t)}) d^n x \\
& \quad + r(t)^2 \int_{\mathbb{R}^n} u(t)^{r(t)-1} \frac{d}{dt} u(t) d^n x = \\
& \dot{r}(t) \int_{\mathbb{R}^n} u(t)^{r(t)} \ln(u(t)^{r(t)} / \|u(t)\|_{r(t)}^{r(t)}) d^n x \\
& \quad - r(t)^2 \int_{\mathbb{R}^n} \nabla(u(t)^{r(t)-1}) \cdot \nabla u(t) d^n x + r(t)^2 \int_{\mathbb{R}^n} u(t)^{r(t)} c(t) d^n x = \\
& \dot{r}(t) \int_{\mathbb{R}^n} u(t)^{r(t)} \ln(u(t)^{r(t)} / \|u(t)\|_{r(t)}^{r(t)}) d^n x \\
& \quad - 4(r(t) - 1) \int_{\mathbb{R}^n} |\nabla u(t)^{r(t)/2}|^2 d^n x + r(t)^2 \int_{\mathbb{R}^n} u(t)^{r(t)} c(t) d^n x,
\end{aligned}$$

where the second equality holds on account of (2.2).

We now apply the sharp logarithmic Sobolev inequality with $a = 4\pi(r(t) - 1)/\dot{r}(t)$, and obtain

$$\begin{aligned}
& r(t)^2 \|u(t)\|_{r(t)}^{r(t)-1} \frac{d}{dt} \|u(t)\|_{r(t)} \leq \\
& \quad - \dot{r}(t) \left(n + \frac{n}{2} \ln(4\pi(r(t) - 1)/\dot{r}(t)) + r(t)^2 C(t) \right) \|u(t)\|_{r(t)}^{r(t)}.
\end{aligned}$$

Thus, if we define $G(t) := \ln(\|u(t)\|_{r(t)})$, we have

$$\frac{dG(t)}{dt} \leq - \left(\frac{\dot{r}(t)}{r^2(t)} \right) \left(n + \frac{n}{2} \ln(4\pi(r(t) - 1)/\dot{r}(t)) \right) + C(t). \quad (2.6)$$

Introducing $s(t) := 1/r(t)$, we have

$$\begin{aligned}
\frac{dG(t)}{dt} & \leq \dot{s}(t) \left(n + \frac{n}{2} \ln(4\pi s(t)(1 - s(t))) \right) + \\
& \quad \frac{n}{2} (-\dot{s}(t)) \ln(-\dot{s}(t)) + C(t).
\end{aligned}$$

Integrating from 0 to T , we get:

$$G(T) - G(0) \leq \int_0^T \left(n + \frac{n}{2} \ln(4\pi s(t)(1-s(t))) \right) \dot{s}(t) dt + \frac{nT}{2} \left(\frac{1}{T} \int_0^T (-\dot{s}(t)) \ln(-\dot{s}(t)) dt \right) + \int_0^T C(t) dt . \quad (2.7)$$

The first integrand above is a total derivative of a function of $s(t)$, and therefore it depends only on $s(0) = 1/p$ and $s(T) = 1/q$. Evaluating this integral we obtain

$$\frac{n}{2} \left[\ln(4\pi)s + \ln(s^s(1-s)^{-(1-s)}) \right] \Big|_{1/p}^{1/q} . \quad (2.8)$$

Because $x \ln x$ is a convex function of x , we minimize the second integral by choosing $\dot{s}(t)$ to be constant; i.e.,

$$\dot{s}(t) := \frac{1}{T} \left(\frac{1}{q} - \frac{1}{p} \right) .$$

With this choice, we obtain

$$\frac{nT}{2} \left(\frac{1}{T} \int_0^T (-\dot{s}(t)) \ln(-\dot{s}(t)) dt \right) = (n/2) \left((1/q) - (1/p) \right) \ln \left(\frac{T}{(1/p) - (1/q)} \right) . \quad (2.9)$$

Now using (2.8) and (2.9) in (2.7) and exponentiating, we obtain (2.3).

To prove (2.4) we proceed as follows: It is a standard fact that $|u(x, t)| \leq v(x, t)$ for all x and t , where $v(x, t)$ is the solution of

$$\frac{\partial}{\partial t} v(x, t) = \Delta v(x, t) + \nabla \cdot (\mathbf{b}(x, t)v(x, t)) ,$$

with initial data $|u(x, 0)|$. Thus, it suffices to show (2.4) for $v(x, t)$.

Now $v(x, t) / \int |u(x, 0)| dx$ is the density corresponding to the solution X_t of the stochastic differential equation

$$dX_t = -\mathbf{b}(X_t, t) dt + \sqrt{2} dW_t$$

with initial condition X_0 which has the density $|u(x, 0)| / \int |u(x, 0)| dx$. From this it follows that

$$|X_t| \leq |X_0| + 2B\sqrt{t} + \sqrt{2}|W_t| .$$

By the arithmetic–geometric mean inequality

$$|X_t|^2 \leq (2 + \alpha)|X_0|^2 + 4(2 + \alpha)B^2t + 2(1 + 2/\alpha)|W_t|^2 ,$$

for $\alpha > 0$. Thus,

$$\frac{\int v(x, t)e^{\frac{|x|^2}{4\gamma^4 t}} dx}{\int |u(x, 0)| dx} = \mathbb{E} e^{\frac{|X_t|^2}{4\gamma^4 t}} \leq e^{\frac{2+\alpha}{\gamma^4} B^2} \left(\mathbb{E} \exp \left\{ \frac{p'(2 + \alpha)|X_0|^2}{4\gamma^4 t} \right\} \right)^{1/p'} \left(\mathbb{E} \exp \left\{ \frac{2p(1 + 2/\alpha)|W_t|^2}{4\gamma^4 t} \right\} \right)^{1/p} ,$$

for all $p > 1$ by Hölder's inequality. Noticing that $t^{-1/2}W_t$ is a unit Gaussian variable, the expectation involving it is easily computed to be $(1 - (p/\gamma^4)(2 + \alpha)/\alpha)^{-n/2}$. Choosing $p = (2 + \alpha)/\alpha = \gamma$ the remaining expectation can be controlled by $\int |u(x, 0)|e^{\epsilon|x|^2} dx$ for $t > t_0$, and the bound (2.4) is obtained.

Under the assumption $\nabla \cdot \mathbf{b} = 0$, an idea of Davies [D87] can be used to extract the bound (2.5) from (2.3) with $p = 1$ and $q = \infty$. Fix $\alpha \in \mathbb{R}^n$, and define $P^{(\alpha)}(T, x, y) := e^{-\alpha \cdot x} P(T, x, y) e^{\alpha \cdot y}$ where $P(T, x, y)$ is the fundamental solution of (2.1). Then $P^{(\alpha)}(T, x, y)$ is the fundamental solution of

$$\begin{aligned} \frac{\partial}{\partial t} w(t) &= \Delta w(t) + \nabla \cdot ((\mathbf{b}(x, t) + 2\alpha)w(x, t)) + \\ &\quad (c(x, t) + \alpha \cdot \mathbf{b}(x, t) + \alpha^2)w(x, t) . \end{aligned} \tag{2.10}$$

Since $\nabla \cdot (\mathbf{b}(x, t) + 2\alpha) = 0$, (2.2) will be satisfied for all solutions of (2.10). The $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ established above hold for solutions of (2.10) and imply that

$$P^{(\alpha)}(T, x, y) \leq (4\pi T)^{-n/2} \exp \left(|\alpha| \int_0^T B(\tau) d\tau + \alpha^2 T + \int_0^T C(\tau) d\tau \right) ,$$

for all x, y and α . This and the definition of $P^{(\alpha)}(T, x, y)$ yield

$$P^{(\alpha)}(T, x, y) \leq (4\pi T)^{-n/2} \exp \left(\alpha \cdot (x - y) + |\alpha| \int_0^T B(\tau) d\tau + \alpha^2 T + \int_0^T C(\tau) d\tau \right) ,$$

for all x, y and α . For fixed x and y choose

$$\alpha = -\frac{1}{4T} \frac{x - y}{|x - y|} \left(|x - y| - \int_0^T B(\tau) d\tau \right)_+ .$$

With this choice,

$$\left(\alpha \cdot (x - y) + |\alpha| \int_0^T B(\tau) d\tau + \alpha^2 T \right) = -\frac{1}{4T} \left(|x - y| - \int_0^T B(\tau) d\tau \right)_+^2,$$

and this proves the bound (2.5). It is easily seen to be sharp by direct comparison with the explicit form of the fundamental solution of (2.1) for $\mathbf{b} = 0$ and constant c . ■

Remark: As an aside, note that, contrary to one's expectation, the right hand side of (2.6) is not always positive along the optimizing $r(t)$. Indeed for $p = 1$ and $q = \infty$ the optimizer is $r(t) = T/(T - t)$ and for T sufficiently small, the right side of (2.6) will be negative for some $t < T$. This is in contrast to Gross' proof of the hypercontractive estimate where the corresponding term is always positive.

Remark: The bound (2.4) is of course only meaningful when $\int |u(x, 0)| e^{\epsilon|x|^2} dx$ is finite. But similar bounds, though less explicit and less strong, can be obtained for *general* integrable initial data. The point is that as long as the initial data is integrable, there is some function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $f(0) = 1$, and $f(r)$ increasing to infinity as r increases, such that $\int |u(x, 0)| f(|x|) dx = D < \infty$. The argument which led to (2.4) then leads to

$$\int_{\mathbb{R}^n} |u(x, t)| h\left(\frac{|x|}{\sqrt{t}}\right) dx \leq M \quad \text{for all } t \geq T \tag{2.11}$$

for some function h satisfying the same conditions as f , and some constants T and M .

3. PROOFS OF THEOREMS 1, 2 and 3.

Proof of Theorems 1 and 3: Let u be any solution of (1.1) with \mathbf{f} satisfying (1.2). As in the proof of Theorem 5, we compute $\frac{d}{dt} \|u(t)\|_{r(t)}$. Because of (1.2), all terms involving $\mathbf{f}(\cdot, u(\cdot, t))$, i.e., all terms involving the non-linearity, vanish. Thus, with $G(t) = \ln(\|u(t)\|_{r(t)})$, $G(t)$ satisfies the same differential inequality it satisfied in the proof of Theorem 5. Hence, the bound (1.5) holds for u .

Next, suppose that (1.9) is satisfied, and define $\mathbf{b}(x, t) = \mathbf{f}(x, u(x, t))$. Then $u(x, t)$ solves (2.1) with this choice of \mathbf{b} and $c = 0$. Hence (1.11) now follows from Theorem 5, and easy calculations lead to (1.12). \blacksquare

Proof of Theorem 2: The statement about Burger’s equation follows directly from Theorem 1. Next, consider the two dimensional vorticity equation. In this case

$$\begin{aligned} |\mathbf{f}(x, u(\cdot, t))| &\leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{1}{|x-y|} |u(y, t)| d^2y = \\ &\frac{1}{2\pi} \int_{|x-y| \leq R} \frac{1}{|x-y|} |u(y, t)| d^2y + \frac{1}{2\pi} \int_{|x-y| \geq R} \frac{1}{|x-y|} |u(y, t)| d^2y \leq \\ &R \|u(\cdot, t)\|_\infty + \frac{1}{2\pi R} \|u(\cdot, t)\|_1 . \end{aligned}$$

Optimizing with respect to R yields the bound $\sqrt{(2/\pi) \|u(\cdot, t)\|_\infty \|u(\cdot, t)\|_1}$. Applying Theorem 1 gives (1.8). \blacksquare

4. BOUNDS ON L^1 DECAY

Consider a solution of (1.1), and suppose, as in Theorem 2, that the velocity field \mathbf{f} is such that

$$\|\mathbf{f}(\cdot, u(\cdot, t))\|_\infty \leq Bt^{-1/2} . \tag{4.1}$$

Suppose also that the initial data is such that $\int u(x, 0) dx = 0$. Then $\|u(\cdot, t)\|_1$ will decrease, and our object in this section is to show that it actually decreases to zero, and that it does so at a computable rate depending on the initial data only. We now briefly describe the strategy.

Because of Theorem 3 we can rescale space and time in such a way that the solution to (1.1) “lives on a fixed scale”.

More precisely, given a solution of

$$\frac{\partial}{\partial t} u(x, t) = \Delta u(x, t) + \nabla \cdot (\mathbf{b}(x, t) u(x, t)), \tag{4.2}$$

we set

$$w(x, t) = (2\pi)^{n/2} (\sqrt{2}e^t)^n u(\sqrt{2}e^t x, e^{2t}) e^{-|x|^2/2} . \quad (4.3)$$

Then $w(x, t)$ satisfies

$$\frac{\partial}{\partial t} w(x, t) = (\Delta - x \cdot \nabla) w(x, t) + (\nabla - x) \cdot (\widehat{\mathbf{b}}(x, t) w(x, t)) , \quad (4.4)$$

where

$$\widehat{\mathbf{b}}(x, t) = \sqrt{2}e^t \mathbf{b}(\sqrt{2}e^t x, e^{2t}) . \quad (4.5)$$

Throughout this section, $d\mu$ denotes the Gaussian probability measure

$$d\mu = (2\pi)^{-n/2} e^{-|x|^2/2} dx . \quad (4.6)$$

Notice that

$$\int w(x, t) d\mu = \int u(x, e^{2t}) dx = 0 \quad (4.7)$$

and

$$\int |w(x, t)| d\mu = \int |u(x, e^{2t})| dx \quad (4.8)$$

for all t . Thus, it suffices to estimate the decay of $\int |w(x, t)| d\mu$.

The operator $(\Delta - x \cdot \nabla)$ in (4.4) is the generator of the Ornstein–Uhlenbeck semigroup Q_t given by

$$Q_t g(x) = \int_{\mathbb{R}^n} g(e^{-t}x + (1 - e^{-2t})^{1/2}y) d\mu . \quad (4.9)$$

If $\widehat{\mathbf{b}}$ were zero, then the equation (4.4) would be solved by applying Q_t to the initial data, and then it would be easy to see that $\int |w(x, t)| d\mu$ would tend to zero. In fact, fix any time t , and split the solution into its positive and negative parts; i.e., $w(x, t) = w_+(x, t) - w_-(x, t)$. Individually evolving these functions a further time step s produces an overlap in their supports no matter how far apart these initially are. The resulting cancellation when they are recombined is what decreases the L^1 norm. The strategy will be to compare the “overlap production” of the full equation (4.4) with that of the Ornstein–Uhlenbeck semigroup.

This overlap production depends crucially on the possible separation of the supports of $w_+(x, t)$ and $w_-(x, t)$. However, it follows easily from Theorem 3, (4.1) and (4.3) that this separation is bounded by a finite quantity depending only on the initial data. In the following estimates we handle this worst possible case, allowing the vector field $\widehat{\mathbf{b}}$ to be such that its full strength is applied to keeping the supports of $w_+(x, t)$ and $w_-(x, t)$ apart.

The first step in this program is to get pointwise lower bounds solutions of (4.4)

Theorem 6 *Consider a solution $w(x, \cdot)$ of the equation (4.4) on some interval $(T, T + S)$. Suppose that the vector field $\widehat{\mathbf{b}}(x, t)$ satisfies the bound*

$$\sup_{x,t} |\widehat{\mathbf{b}}(x, t)| \leq B < \infty , \quad (4.10)$$

and $w(x, T) \geq 0$ is integrable with respect to $d\mu$. Then for any bounded function ϕ , and all $p \geq 1$,

$$\frac{\int w(x, T + S)\phi(x)d\mu}{\int w(x, T)Q_S\phi(x)d\mu} \geq \exp[-(\|\phi\|_\infty/K)^{1/p}2Bn(\sqrt{pS} + BS)] , \quad (4.11)$$

where $K := \int w(x, T)Q_S\phi(x)d\mu / \int w(x, T)\phi(x)d\mu$, and Q_t is the Ornstein-Uhlenbeck semigroup.

Proof: It is convenient to use functional integrals to compare $\int w(x, T + S)\phi(x)d\mu$ and $\int w(x, T)Q_S\phi(x)d\mu$. To do this consider the diffusion process Y_t that solves the stochastic differential equation

$$dY_t = -Y_t dt - \widehat{\mathbf{b}}(Y_t, t)dt + \sqrt{2}dW_t \quad \text{for } t \geq T$$

such that the initial random variable Y_T has $w(\cdot, T) / \int w(x, T)\phi(x)d\mu$ as its density. A standard application of Ito's formula shows that $w(\cdot, t) / \int w(x, T)\phi(x)d\mu$ is the density of Y_t . (Here W_t denotes a standard Wiener process; see [SV79, Theorem 6.4] for further information.)

In the special case $\widehat{\mathbf{b}} = 0$, the corresponding stochastic differential equation

$$dX_t = -X_t dt + \sqrt{2}dW_t \quad \text{for } t \geq T$$

has an explicit solution called the Ornstein-Uhlenbeck process; namely

$$X_t = e^{T-t}X_T + \sqrt{2} \int_T^t e^{(s-t)} dW_s .$$

It is easy to see that

$$\mathbb{E}\phi(X_{T+S}) = \frac{\int w(x, T)Q_S\phi(x)d\mu}{\int w(x, T)\phi(x)d\mu}$$

where Q_t is given by (4.9). Then since

$$\frac{\int w(x, T+S)\phi(x)d\mu}{\int w(x, T)\phi(x)d\mu} = \mathbb{E}\phi(Y_{T+S}),$$

we are now prepared to estimate the effect of $\widehat{\mathbf{b}}$ because the measures on path-space induced by the two processes Y_t and X_t are equivalent, and the Girsanov formula (see [SV79]) gives the Radon–Nikodym derivative in explicit form:

$$\mathbb{E}\phi(Y_{T+S}) = \mathbb{E}\left(e^{-\int_T^{T+S}\widehat{\mathbf{b}}(X_s, s)\cdot dW_s - \int_T^{T+S}|\widehat{\mathbf{b}}(X_s, s)|^2 ds}\phi(X_{T+S})\right). \quad (4.12)$$

Here once again W_t denotes the standard Wiener process associated with X_t .

Then, with $K := \int w(x, T)Q_S\phi(x)d\mu / \int w(x, T)\phi(x)d\mu$, Jensen's inequality gives

$$\frac{\int w(x, T+S)\phi(x)d\mu}{K} \geq \exp\left\{\frac{\mathbb{E}\left(\left(-\int_T^{T+S}\widehat{\mathbf{b}}(X_s, s)\cdot dW_s - \int_T^{T+S}|\widehat{\mathbf{b}}(X_s, s)|^2 ds\right)\phi(X_{T+S})\right)}{K}\right\}. \quad (4.13)$$

The second ratio in the exponential, by direct use of (4.10), can be estimated below by $-B^2 t$ while the first ratio, by Hölder's inequality, is bounded below by

$$-\left(\mathbb{E}\left|\int_T^{T+S}\widehat{\mathbf{b}}(X_s, s)\cdot dW_s\right|^p\right)^{1/p} \left(\mathbb{E}(\phi(X_{T+S})^{p'})\right)^{1/p'}/K. \quad (4.14)$$

Now clearly,

$$\left(\mathbb{E}(\phi(X_{T+S})^{p'})\right)^{1/p'}/K \leq (\|\phi\|_\infty/\mathbb{E}(\phi(X_{T+S})))^{1/p} = \|\phi\|_\infty^{1/p} K^{-1/p}. \quad (4.15)$$

The first factor in (4.14) is controlled by Burkholder's inequality [B67] and (4.10):

$$\left(\mathbb{E}\left|\int_T^{T+S}\widehat{\mathbf{b}}(X_s, s)\cdot dW_s\right|^p\right)^{1/p} \leq 2n\sqrt{p}\left(\mathbb{E}\left(\int_T^{T+S}|\widehat{\mathbf{b}}(X_s, s)|^2 ds\right)^{p/2}\right)^{1/p} \leq 2n\sqrt{p}BS^{1/2}$$

for all $1 \leq p < \infty$. (The explicit constant $2n\sqrt{p}$ can be extracted [CK91] from a result of Davis [D76].) Combining these bounds yields the result. \blacksquare

Theorem 7 *Let $u(x, t)$ be a solution of (4.2), and suppose that $\mathbf{b}(x, t)$ satisfies the estimate*

$$\|\mathbf{b}(\cdot, t)\|_\infty \leq Bt^{-1/2} \quad (4.16)$$

for some constant B . Moreover assume that $u(\cdot, 0)$ is integrable and that

$$\int_{\mathbb{R}^n} u(x, 0) dx = 0 . \quad (4.17)$$

Then $\lim_{t \rightarrow \infty} \|u(\cdot, t)\|_1 = 0$. If we further suppose that for some $\epsilon > 0$,

$$\int |u(x, 0)| e^{\epsilon|x|^2} dx := D < \infty , \quad (4.18)$$

then for all $T \geq t_1$, where $e^{2t_1} = 5/\epsilon$, the following explicit estimate holds:

$$\|u(\cdot, e^{2(T+N)})\|_{L^1} \leq \frac{\|u(\cdot, e^{2T})\|_{L^1}}{(1 + 5(\frac{\epsilon+1}{\epsilon-1})G(B)N(\frac{\|u(\cdot, e^{2T})\|_{L^1}}{D})^{5(\frac{\epsilon+1}{\epsilon-1})})^{\frac{\epsilon-1}{5(\epsilon+1)}}} . \quad (4.19)$$

for all $N > 0$ where

$$G(B) = \frac{1}{4} \exp\{-5(3n+2)(\frac{\epsilon+1}{\epsilon-1})\} \exp\{-B^2 \left(10\frac{\epsilon+1}{\epsilon-1} + n(2\sqrt{\epsilon}+8)\right)\} .$$

Proof: First assume (4.18); we return to the general case at the end. Let $w(x, t)$ and $\widehat{\mathbf{b}}(x, t)$ be defined as in (4.3) and (4.5) in terms of $u(x, t)$. The precise version of our statement that the rescaled function w “lives on a fixed scale” is the bound

$$\int |w(x, t)| e^{|x|^2/8} d\mu \leq e^{2B^2} (8^n) D \quad (4.20)$$

for all $t \geq t_1$, which follows directly from (2.4) and (4.3).

Now define $a(t)$ by

$$a(t) := \frac{\int |w(x, t)| d\mu}{D} .$$

Our goal is to derive a lower bound for $a(t) - a(t + 1)$ in terms of $a(t)$ itself.

For $0 < a < 1$, define $R(a)$ by

$$e^{-R(a)^2/8} := \{e^{-2B^2} 2^{-(3n+2)}\} a .$$

It then follows from the bound (4.20) that

$$\frac{\int_{|x| > R(a)} w_{\pm}(x, t) d\mu}{D} \leq \frac{a}{4} ,$$

where $w_{\pm} = \max\{\pm w, 0\}$.

Having made these preparations, we proceed to the main part of the argument. Note, by the definition of $a(t)$,

$$\int_{|x| \leq R(a(t))} w_{\pm}(x, t) d\mu \geq \frac{Da(t)}{4} . \quad (4.21)$$

Indeed,

$$\begin{aligned} \int_{|x| \geq R(a(t))} w_+ d\mu + \int_{|x| \geq R(a(t))} w_- d\mu = \\ \int_{|x| \geq R(a(t))} |w| d\mu \leq \frac{Da(t)}{4} \end{aligned}$$

and hence $\int_{|x| \geq R(a(t))} w_{\pm} d\mu \leq Da(t)/4$. Then, since $\int_{\mathbb{R}^n} u(x, t) dx = 0$ for all t , it follows that $\int_{\mathbb{R}^n} w(x, t) d\mu = 0$ for all t , and thus that

$$\int_{\mathbb{R}^n} w_+(x, t) d\mu = \int_{\mathbb{R}^n} w_-(x, t) d\mu = \frac{Da(t)}{2}$$

from which our assertion follows.

Next fix any time $T > t_1$ and define $w^{\pm}(x, T + s)$ for $s \geq 0$ to be the solution of

$$\frac{\partial}{\partial s} w^{\pm}(x, T + s) = (\Delta - x \cdot \nabla) w^{\pm}(x, T + s) + (\nabla - x) \cdot (\widehat{\mathbf{b}}(x, T + s) w^{\pm}(x, T + s))$$

with initial condition $w^{\pm}(x, T) = w_{\pm}(x, T)$. Certainly, $w(x, T + s) = w^+(x, T + s) - w^-(x, T + s)$,

and $\int_{\mathbb{R}^n} w^{\pm}(x, T + s) d\mu = \int_{\mathbb{R}^n} w_{\pm}(x, T + s) d\mu = Da(T)/2$ for all s . Thus, choosing $s = 1$

$$\begin{aligned} Da(T + 1) = \\ \int_{\{w^+ \geq w^-\}} (w^+ - w^-) d\mu + \int_{\{w^- \geq w^+\}} (w^- - w^+) d\mu \leq \\ \int_{\mathbb{R}^n} w^+ d\mu - \int_{\{w^+ \geq w^-\} \cap \{|x| \leq R\}} w^- d\mu + \int_{\mathbb{R}^n} w^- d\mu - \int_{\{w^- \geq w^+\} \cap \{|x| \leq R\}} w^+ d\mu = \\ Da(t) - \left(\int_{\{w^+ \geq w^-\} \cap \{|x| \leq R\}} w^- d\mu + \int_{\{w^- \geq w^+\} \cap \{|x| \leq R\}} w^+ d\mu \right) , \end{aligned} \quad (4.22)$$

where we have increased the domain of integration in the positive terms, and decreased it in the negative terms. In other words

$$D(a(T+1) - a(T)) \leq - \int \phi^- w^- d\mu - \int \phi^+ w^+ d\mu, \quad (4.23)$$

where ϕ^+ and ϕ^- denote the characteristic functions of the sets $\{w^+ \geq w^-\} \cap \{|x| \leq R\}$ and $\{w^- \geq w^+\} \cap \{|x| \leq R\}$ respectively.

The proof now continues using Theorem 6 which yields

$$\int w^\pm(x, T+1) \phi^\pm(x) d\mu \geq \exp[-(1/K^\pm)^{1/p} 2Bn(\sqrt{p} + B)] \int w^\pm(x, T) Q_1 \phi^\pm(x) d\mu,$$

where $K^\pm := \int w^\pm(x, T) Q_1 \phi^\pm(x) d\mu / \int w^\pm(x, T) \phi^\pm(x) d\mu$. It follows easily from the definition (4.9) that

$$Q_1 \phi^\pm(x) \geq (2\pi)^{-n/2} \exp\left\{-\frac{R^2}{2} \frac{e+1}{e-1}\right\} \|\phi^\pm\|_1$$

for all $|x| \leq R$. Moreover since $\|\phi^+\|_1 + \|\phi^-\|_1 = \omega_n R^n$, at least one of the two terms is no less than $\omega_n R^n / 2$. Here $\omega_n = 2\pi^{n/2} / n\Gamma(n/2)$ is the volume of the unit ball. We retain only the corresponding term in (4.23). Hence one of the two bounds

$$K^\pm \geq \frac{R^n}{n2^{n/2}\Gamma(n/2)} \exp\left\{-\frac{R^2}{2} \frac{e+1}{e-1}\right\}$$

is valid. Using the definition of $R(a)$, one sees that $R(a)^2 \geq 8(3n+2)\ln 2$ which leads to

$$K^\pm \geq \exp\left\{-\frac{R^2}{2} \frac{e+1}{e-1}\right\}. \quad (4.24)$$

Choosing $p = R(a)^2(e+1)/(e-1)$ yields

$$\begin{aligned} \int w^\pm(x, T+1) \phi^\pm(x) d\mu &\geq \exp[-2Bn\sqrt{e}(R(\frac{e+1}{e-1})^{1/2} + B)] \int w^\pm(x, T) Q_1 \phi^\pm(x) d\mu \\ &\geq \exp[-(2n\sqrt{e} + 8\alpha en^2 \frac{e+1}{e-1})B^2 - \frac{R^2}{8\alpha}] \int w^\pm(x, T) Q_1 \phi^\pm(x) d\mu \\ &\geq \exp[-(2n\sqrt{e} + 8\alpha en^2 \frac{e+1}{e-1})B^2 - \frac{R^2}{8\alpha}] \exp\left\{-\frac{R^2}{2} \frac{e+1}{e-1}\right\} Da(T)/4. \end{aligned}$$

The second inequality is obtained by using the arithmetic–geometric mean inequality; the parameter $\alpha > 0$ is at our disposal. The third inequality follows from (4.24) and (4.21). Using the definition of $R(a)$ once more and returning to (4.23) we finally obtain

$$a(T+1) - a(T) \leq -G(B)a(T)^b, \quad (4.25)$$

where

$$G(B) = \frac{1}{4} \exp\{-5(3n+2)\left(\frac{e+1}{e-1}\right)\} \exp\left\{-B^2 \left(10\frac{e+1}{e-1} + n(2\sqrt{e}+8)\right)\right\} \quad (4.26)$$

and

$$b = 1 + 5\left(\frac{e+1}{e-1}\right).$$

Define $A_m := a(T+m)$, and note that this sequence satisfies a relation of the form

$$\frac{A_m - A_{m+1}}{A_m^b} \geq G \quad (4.27)$$

for $b > 1$ and $G > 0$. Since the sequence is monotone decreasing, simple estimations lead to

$$A_N^{1-b} - A_0^{1-b} \geq (b-1) \sum_{m=1}^N \frac{A_{m-1} - A_m}{A_{m-1}^b}.$$

Combining this with (4.27), we obtain $A_N^{1-b} - A_0^{1-b} \geq (b-1)NG$. Specializing to the values of b and G made relevant by (4.26), yields

$$a(T+N) \leq \frac{a(T)}{\left(1 + 5\left(\frac{e+1}{e-1}\right)G(B)N(a(T))^{5\left(\frac{e+1}{e-1}\right)\frac{e-1}{5(e+1)}}\right)},$$

which is (4.19).

Returning to the case of general initial data, we use the bound (2.11) in place of (2.4). With this bound, we establish, using the same line of argument, the existence of a function $R(a)$ so that the bounds (4.21) hold – though of course $R(a)$ is no longer given by the same explicit formula. Nonetheless, proceeding further along the same line of argument with this inexplicit $R(a)$, one still gets a bound of the form

$$a(t+1) - a(t) \leq -\psi(a(t))$$

for some function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\psi(r) > 0$ for all $r > 0$, and all t sufficiently large. This implies that $\lim_{t \rightarrow \infty} \|u(\cdot, t)\|_1 = 0$ ■

In the proof just completed, the decay is estimated in *discrete* time steps. This is not an artifice of our presentation; the constant $G(B)$ implicitly depends also on the size of the time step s , which we have fixed at $s = 1$. If s were taken toward zero, $G(B)$ would vanish rapidly. Thus the decay mechanism that is exploited in the proof of Theorem 7, depending as it does on the overlap of Gaussian tails, does not permit a proof based on a differential inequality.

However, when the constant B in Theorem 7 is sufficiently small that the displacement caused by streaming is comparable or less than that caused by diffusion, a faster decay mechanism takes over. In this circumstance, the decay can be estimated using a differential inequality, which, once again, comes from the logarithmic Sobolev inequality. We exploit this in Theorem 8 below, together with one more point: Because of Theorem 2, we know that in our examples, the constant B in the inequality $\|\mathbf{f}(\cdot, u(\cdot, t))\|_\infty \leq Bt^{-1/2}$ decreases as $\|u(\cdot, t)\|_1$ decreases. That is, this bound gets better and better as the decay proceeds. For example, it is easy to see from Theorem 2 that in the case of the two dimensional Navier–Stokes equation, we have

$$\|\mathbf{f}(\cdot, u(\cdot, t))\|_\infty \leq \pi^{-1} \|u(\cdot, t/2)\|_1 t^{-1/2} .$$

Thus, in application to this equation, the constant F in Theorem 8 below can be taken to be $1/\pi$.

Theorem 8: *Let u be a solution of (4.2) subject to the following conditions.*

$$\int_{\mathbb{R}^n} |u(x, 0)| e^{\epsilon|x|^2} dx := D < \infty \tag{4.28}$$

and

$$\int_{\mathbb{R}^n} u(x, 0) dx = 0 . \tag{4.29}$$

Suppose furthermore that $\mathbf{b}(x, t)$ is divergence free for all t and that for some fixed constant F ,

$$\|\mathbf{b}(\cdot, t)\|_\infty \leq F \|u(\cdot, t/2)\|_1 t^{-1/2} . \tag{4.30}$$

Then for all $t > t_0$,

$$\|u(\cdot, t)\|_1 \leq \left(H e^{\{8\sqrt{2}HF e^{t_0/4} + t_0\}} \right) t^{-1/2}, \quad (4.31)$$

where t_0 is given by (4.37), and

$$H := (DC(3/4, B))(9/2)^{n/4}.$$

$C(\beta, B)$ is given in Theorem 3.

Proof: Again, let $w(x, t)$ be given in terms of $u(x, t)$ by (4.3). The first step in the proof is to bound $\int w^2(x, t)d\mu$ for sufficiently large t . To do this we apply (1.12) of Theorem 3 and obtain

$$|u(x, t)| \leq C(\beta, B) \left(\frac{\beta}{4\pi t} \right)^{n/2} \exp\left\{-\frac{\beta(1-\alpha)|x|^2}{4t}\right\} \int \exp\left\{\frac{\beta(1/\alpha-1)|y|^2}{4t}\right\} |u(y, 0)| dy,$$

for all $\alpha > 1$ and all $0 < \beta < 1$. Choosing $\alpha = 1/4$ and $\beta = 3/4$ and using (4.28) yields

$$\int w^2(x, t)d\mu \leq (DC(3/4, B))^2 (9/2)^{n/2}, \quad (4.32)$$

for all $e^{-2t} \leq 2\epsilon$, where $C(\beta, B)$ is given in Theorem 3.

Our immediate goal is to derive a differential inequality for $\int w^2(x, t)d\mu$. Note that the equation (4.4) satisfied by $w(x, t)$ can be written as

$$\frac{\partial}{\partial t} w(x, t) = \nabla \cdot (\nabla w(x, t) e^{-|x|^2/2}) e^{|x|^2/2} + \nabla \cdot (\widehat{\mathbf{b}}(x, t) w(x, t) e^{-|x|^2/2}) e^{|x|^2/2}.$$

Integrating by parts, using the fact that $\nabla \cdot \widehat{\mathbf{b}}(x, t) = 0$, we obtain

$$\int_{\mathbb{R}^n} w(x, t) \nabla \cdot (\widehat{\mathbf{b}}(x, t) w(x, t) e^{-|x|^2/2}) e^{|x|^2/2} d\mu = -\frac{1}{2} \int_{\mathbb{R}^n} w^2(x, t) (x \cdot \widehat{\mathbf{b}}(x, t)) d\mu$$

and hence

$$\frac{d}{dt} \int_{\mathbb{R}^n} w^2(x, t) d\mu = -2 \int_{\mathbb{R}^n} |\nabla w(x, t)|^2 d\mu - \int_{\mathbb{R}^n} (x \cdot \widehat{\mathbf{b}}(x, t)) w^2(x, t) d\mu. \quad (4.33)$$

Now using Young's inequality in the form $yz \leq ae^{y-1} + z \ln(z/a)$ for all $a > 0$, and then the logarithmic Sobolev inequality [G76] for the Gaussian measure $d\mu$; i.e.,

$$\int_{\mathbb{R}^n} \phi^2(x) \ln \phi^2(x) d\mu - \left(\int_{\mathbb{R}^n} \phi^2(x) d\mu \right) \ln \left(\int_{\mathbb{R}^n} \phi^2(x) d\mu \right) \leq 2 \int_{\mathbb{R}^n} |\nabla \phi(x)|^2 d\mu,$$

we obtain

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} (x \cdot \widehat{\mathbf{b}}(x, t)) w^2(x, t) d\mu \right| \leq \\ & \|\widehat{\mathbf{b}}(\cdot, t)\|_\infty \left[\int_{\mathbb{R}^n} w^2(x, t) d\mu \int_{\mathbb{R}^n} e^{|x|-1} d\mu + 2 \int_{\mathbb{R}^n} |\nabla w(x, t)|^2 d\mu \right]. \end{aligned} \quad (4.34)$$

Next, define $k(t) := \int_{\mathbb{R}^n} w^2(x, t) d\mu$. Then from (4.33) and (4.34),

$$\dot{k}(t) \leq -2(1 - \|\widehat{\mathbf{b}}(\cdot, t)\|_\infty) \int_{\mathbb{R}^n} |\nabla w(x, t)|^2 d\mu + 2\|\widehat{\mathbf{b}}(\cdot, t)\|_\infty k(t). \quad (4.35)$$

By the definition (4.5) of $\widehat{\mathbf{b}}$ and (4.30)

$$\|\widehat{\mathbf{b}}(x, t)\|_\infty \leq \sqrt{2F} \|u(\cdot, e^t)\|_1. \quad (4.36)$$

Applying Theorem 7

$$\|u(\cdot, e^t)\|_1 \leq \frac{1}{4\sqrt{2F}},$$

for all $t \geq t_0$ where

$$t_0 := \ln(5/\epsilon) + 2N, \quad (4.37)$$

and N is the smallest integer satisfying

$$N \geq \frac{(4\sqrt{2FD})^{5(\epsilon+1)/(\epsilon-1)}}{G(F\|u(\cdot, 0)\|_1)5(\epsilon+1)/(\epsilon-1)}.$$

Thus, for $t \geq t_0$, $\|\widehat{\mathbf{b}}(x, t)\|_\infty \leq 1/4$ and the first term in (4.35) is negative. Applying (4.29) and the Poincaré inequality for the Gaussian measure $d\mu$; i.e.,

$$\int_{\mathbb{R}^n} |\nabla \phi(x)|^2 d\mu \geq \int_{\mathbb{R}^n} \phi^2(x) d\mu - \left(\int_{\mathbb{R}^n} \phi(x) d\mu \right)^2,$$

we obtain the differential inequality

$$\dot{k}(t) \leq -2k(t) + 4\|\widehat{\mathbf{b}}(\cdot, t)\|_\infty k(t). \quad (4.38)$$

In particular, since $\|\widehat{\mathbf{b}}(\cdot, t)\|_\infty \leq 1/4$ for all $t \geq t_0$, $k(t) \leq e^{-(t-t_0)} k(t_0)$. Note that t_0 is sufficiently large so that (4.32) bounds $k(t_0)$. Then by (4.8),

$$\begin{aligned} \|u(\cdot, e^{2t})\|_1 &= \int_{\mathbb{R}^n} |w(x, t)| d\mu \leq \\ & \left(\int_{\mathbb{R}^n} w^2(x, t) d\mu \right)^{\frac{1}{2}} \leq e^{-(t-t_0)/2} \left(\int_{\mathbb{R}^n} w^2(x, t_0) d\mu \right)^{\frac{1}{2}} \leq \\ & e^{-(t-t_0)/2} \sqrt{k(t_0)}. \end{aligned} \quad (4.39)$$

This estimate together with (4.36) yields

$$\|\widehat{\mathbf{b}}(\cdot, t)\|_\infty \leq \sqrt{2} F e^{t_0/2} \sqrt{k(t_0)} e^{-t/4} .$$

Inserting this into (4.38) and integrating the resulting differential inequality yields the estimate

$$k(t) \leq k(t_0) e^{\{16\sqrt{2k(t_0)} F e^{t_0/4} + 2t_0\} e^{-2t}} .$$

Estimating the L^1 norm of u in terms of the L^2 of w as above, we obtain the desired bound. ■

Proof of Theorem 4: Theorem 4 follows from Theorems 7 and 8 in the same fashion as Theorem 3 follows from Theorem 5. ■

References

- [B67] Burkholder, D.: Martingale transforms, *Ann. Math. Stat.* **37**, 1967, 1944-04
- [C91] Carlen, E.A.: Some integral identities and inequalities for entire functions and their application to the coherent state transform, *J. Funct. Anal.* **97**, 1991, 231-49
- [CK91] Carlen, E.A. and Kree, P.: L^p estimates on iterated stochastic integrals, *Anal. of Prob.*, **19**, 1991, 354-368
- [CL90] Carlen, E.A. and Loss, M.: Extremals of functionals with competing symmetries, *J. Funct. Anal.* **88**, 1990, 437-55
- [CL93] Carlen, E.A. and Loss, M.: Sharp constant in Nash's inequality, *Int. Math. Res. Not.*, No. 7 pp. 213-15 in *Duke Math. J.* **71**, 1993
- [D76] Davis, B.: On the L^p norms of stochastic integrals and other martingales, *Duke Math. J.*, **43**, 1976, 697-04
- [D87] Davies, E.B.: Explicit constants for Gaussian upper bounds on heat kernels, *Amer. J. Math.* **109**, 1987, 319-34
- [DS84] Davies, E.B. and Simon, B.: Ultracontractivity and the heat kernel for Schrödinger semigroups, *J. Funct. Anal.* **59**, 1984, 335-95
- [FS86] Fabes, E.B. and Stroock, D.W.: A new proof of Moser's parabolic Harnack inequality via the old ideas of Nash, *Arch. Rat. Mech. Anal.* **96**, 1986, 327-38

- [GK88] Giga, Y. and Kambe, T.: Large time behavior of the vorticity of two-dimensional viscous flow and its application to vortex formation, *Commun. Math. Phys.* **117**, 1988, 549-568.
- [G76] Gross, L.: Logarithmic Sobolev inequalities, *Amer. J. Math.* **97**, 1976, 1061-83
- [K84] Kato, T.: Strong L^p solutions of the Navier–Stokes equation in \mathbb{R}^m , with applications to weak solutions, *Math. Z.* **187**, 1984, 471-80
- [K92] Kato, T.: Strong solutions of the Navier–Stokes equation in Morrey spaces, *Bol. Soc. Bras. Mat.* **22**, 1992, 127-55
- [K94] Kato, T.: The Navier–Stokes equation for an incompressible fluid in \mathbb{R}^2 with a measure as the initial vorticity, Berkeley preprint, 1994
- [N58] Nash, J.: Continuity of solutions of parabolic and elliptic equations, *Amer. J. Math.* **80**, 1958, 931-54
- [S86] Schönbek, M.E.: Large time behavior of solutions to the Navier-Stokes equations, *Comm. Part. Diff. Eq.* **11**, 1986, 733-63
- [SV79] Stroock, D.W. and Varadhan, S.R.S.: *Multidimensional Diffusion Processes*, Grundlehren der mathematischen Wissenschaften 233, Springer–Verlag, Berlin, 1979
- [W86] Wiegner, M.: Decay results for weak solutions of the Navier–Stokes equations on \mathbb{R}^n , *J. London Math. Soc.* **35**, 1986, 303-13