

The Second Order Estimates for the Hill Operator

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THE SECOND ORDER ESTIMATES FOR THE HILL OPERATOR

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Abstract. Let H be the Hill operator and let $G_n = (A_n^-, A_n^+)$ and $M_n^\pm, n \geq 1$, be the corresponding gaps and the effective masses for the Hill operator. Denote by $F(E)$ the Lyapunov function for the Hill operator. Let $\lambda_n \in [A_n^-, A_n^+]$ be such that $F'(\lambda_n) = 0$ and $h_n \geq 0$ be the solution of the equation $\cosh h_n = (-1)^n F(\lambda_n)$. We prove some identities for the effective masses M_n^\pm . Then we find "second order estimates" with respect to $|G_n|$ for λ_n, h_n, M_n^\pm (for example $|\lambda_n - \frac{1}{2}(A_n^- + A_n^+)| \leq C|G_n|^2/n^2$), and we get ones for more general cases (the Dirac operator with periodic coefficients etc.).

1 Introduction

Let us consider the Hill operator $H = -d^2/dx^2 + V(x)$ in $L^2(\mathbf{R})$ where V is a 1-periodic real potential in $L^1(0, 1)$. It is well known that the spectrum of H is absolutely continuous and consists of the intervals S_1, S_2, \dots . Let $S_n = [A_{n-1}^+, A_n^-], \dots, A_n^- \leq A_n^+ < A_{n+1}^-$, $n \geq 1$, and $A_0^+ = 0 < A_1^-$. These intervals are separated by the gaps G_1, G_2, \dots , where $G_n = (A_n^-, A_n^+)$. If a gap degenerates i.e. $G_n = \emptyset$ then the corresponding segments S_n, S_{n+1} merge. The spectrum of the Hill operator consists of closed non overlapping intervals which are called spectral bands. We introduce a parameter $w, w^2 = E$, and numbers $a_n^\pm = \sqrt{A_n^\pm} \geq 0$ and gaps

$$g_n = (a_n^-, a_n^+), \quad g_{-n} = -g_n, \quad n \geq 1, \quad g_0 = \emptyset.$$

Later on g_n will be called a gap and G_n an energy gap. Now we can define a quasimomentum function (see [F1], [MO]) $k(w) = \arccos F(w^2)$, $w \in W = \mathbf{C} \setminus \bar{g}$, $g = \cup g_n$, where F is the Lyapunov function of the Hill operator (see Section 2). The function $k(w)$ is analytic and moreover k is a conformal mapping from W onto a quasimomentum domain $K = \mathbf{C} \setminus \cup \Gamma_n$, where Γ_n is an excised slit

$$\Gamma_n = \{\operatorname{Re} k = \pi n, \quad |\operatorname{Im} k| \leq h_n\}, \quad h_n = h_{-n} \geq 0, \quad n \in \mathbf{Z}, \quad h_0 = 0.$$

Any non degenerate (degenerate) slit Γ_n is connected in some way with the non degenerate (degenerate) gap g_n and the energy gap G_n . With an edge of the energy gap G_n , having the length L_n , we associate the effective mass

$$M_n^\pm = \begin{cases} 0, & \text{if } L_n = 0, \\ 1/E''(k(a_n^\pm)), & \text{if } L_n \neq 0, \end{cases}$$

and $M_0^- = 0$, $M_0^+ = 1/E''(0)$, where $E(k) = w(k)^2$ and $w(k)$ is the inverse function for $k(w)$. It is well known that if $L_n \neq 0$ then

$$E(k) = A_n^\pm + \frac{(k - \pi n)^2}{2M_n^\pm}(1 + o(1)), \quad \text{as } E \rightarrow A_n^\pm.$$

Let us describe the main results of the present paper.

- a) *We prove the equality $\int_0^1 V(t)dt = \sum_{n \geq 1} (A_n^+ M_n^+ + A_n^- M_n^-)$ for $V \in L^1(0, 1)$.*
- b) *"Second order estimates" for M_n^\pm, h_n, λ_n with respect to L_n^2 are obtained.*
- c) *We find "second order" asymptotics of M_n^\pm, h_n, λ_n as $L_n \rightarrow 0$.*
- d) *We extend the results b)-c) for more general cases: the Dirac operator with periodic coefficients, the Schrödinger operator with limit periodic potentials .*

The identities, the estimates and asymptotics for the effective masses M_n^\pm , the gap lengths L_n , heights h_n and so on were studied in many articles (see lit. in [KK1] and [K2]). Firsova [F2] proved that the sum of all effective masses is equal to the physical mass. First uniform "first order" estimates for the effective masses, the heights, the gap lengths and so on were obtained in the papers [KK1], [K1], [K2]. The estimates from [KK1] helped to solve few inverse problems for the Hill operator [KK2]. Moreover in [KK2] were obtained the estimates of the important parameter $s = \text{mindust}(g_n, g \setminus g_n)$ in terms of one of $V, \{h_n\}, \{l_n\}$ (see (2.12)). Note that some estimates in terms of $\max h_n$ were obtained in [M]. Firsova found the relation between M_n^\pm, L_n, h_n and the Fourier coefficients of a potential V at large integers n . In [K1], [K2] the propagation of acoustic waves in periodic media is studied. It was shown that any spectral band (with number n) "creates" the wave with the velocity $c_n < 1$. The velocity c_n is equal to the maximum of the function $w'(k(w))$ when w^2 belongs to the energy band with the number n . In [K1], [K2] the "first order" estimates for c_n in terms of M_n^\pm, h_n are obtained.

To prove b)-d) we consider the quasimomentum of the Hill operator as a conformal mapping. That makes possible to reformulate the problem for the differential operator as a problem of the conformal mapping theory (see [MO] and [KK1]). Then we use the Poisson integral for the imaginary part of the quasimomentum and for the domain $\mathbf{C}_+ \cup \mathbf{C}_- \cup g_n$, for some n , and then some estimates of this integral from [KK1]. We also use the estimates of the spectral band length from [KK2]. Using the results of b)-c) we obtain a).

2 The main results

In this section we introduce the concepts and the facts needed to formulate the theorems, some results for the Hill operator, the Dirac operator with periodic coefficients and some results from the conformal mapping theory.

Let us consider the case of " a general quasimomentum " (see [KK1]). At first we give some definitions and facts from the theory of conformal mappings. We call the set

$K_+ = \mathbf{C}_+ \setminus \cup \Gamma_n$ the "comb" where

$$\Gamma_n = \{ \operatorname{Re} k = p_n, |\operatorname{Im} k| \leq h_n \}, \quad h_n \geq 0, \quad n \in \mathbf{Z}, \quad h_0 = 0,$$

while p_n is a strongly increasing sequence of real numbers such that $p_n \rightarrow \pm\infty$ as $n \rightarrow \pm\infty$. We call a conformal mapping $k(w)$ from the upper half plane \mathbf{C}_+ onto some comb K_+ a general quasimomentum (GQ) if 1) $k(0) = 0$, 2) $k(iv) = iv(1 + o(1))$ as $v \rightarrow \infty$. It is well known that a GQ $k(w)$ is a continuous function in $w \in \bar{\mathbf{C}}_+$. In this case we introduce the sets

$$s_n = [a_{n-1}^+, a_n^-] = k^{-1}([p_{n-1}, p_n]), \quad n \in \mathbf{Z}.$$

We call $\sigma = \cup s_n$ the spectrum of the corresponding general quasimomentum $k(w)$. We also denote by $g_n = (a_n^-, a_n^+)$, a gap in the spectrum of GQ and we let $g = \cup g_n$. It is well known that the set σ can not be the spectrum of two different GQ [L]. Note that the function $k(w)$ may be continued onto the domain $W = \mathbf{C} \setminus \bar{g}$ by the formula $k(\bar{w}) = \bar{k}(w)$, $w \in W$. If a gap g_n is empty then the components s_n, s_{n+1} merge. We denote the length of the gap g_n by l_n . For GQ we introduce "reduced masses" (some analog of the effective masses for the Dirac operator)

$$\mu_n^\pm = \begin{cases} 0, & \text{if } l_n = 0, \\ 1/w''(k(a_n^\pm)), & \text{if } l_n \neq 0, \end{cases}$$

It is clear that $\pm\mu_n^\pm > 0$ if $l_n \neq 0$. We emphasize that a symmetric (i.e. $g = -g$, and $k(-w) = -k(w)$, $w \in W$) GQ with $p_n = \pi n$, $n \in \mathbf{Z}$, corresponds to the quasimomentum for the Hill operator, GQ with $p_n = \pi n$, $n \in \mathbf{Z}$ corresponds to the quasimomentum for the Dirac operator with periodic coefficients. Furthermore a GQ is an integrated density of states (or the rotation number) for the Schrödinger operator with some limit periodic potential (see [JM]).

Later on n, m are the integers. We introduce the real functions p, q by the formula $k(w) = p(w) + iq(w)$ where $w = u + iv$, and the numbers $Q_m = \frac{1}{\pi} \int u^m q(u) du$, $m \geq -1$. Here and below an integral with no limits indicated denotes integration over \mathbf{R}^d , $d \geq 1$.

Let us describe the connection between GQ and the Hill operator. Let $\varphi(x, E), \vartheta(x, E)$ be the solutions of the equation $-f'' + Vf = Ef$, $E \in \mathbf{C}$, satisfying $\varphi'(0, E) = \vartheta(0, E) = 1$ and $\varphi(0, E) = \vartheta'(0, E) = 0$. Let us introduce the Lyapunov function (the discriminant) $F(E) = \frac{1}{2}(\varphi'(1, E) + \vartheta(1, E))$. The quasimomentum k for the Hill operator is defined by $k(w) = \arccos F(w)$, $w \in W$ (see [F1], [MO]). Recall that the spectrum of H consists of the segments S_n , $n \in \mathbf{N}$, with the energy gaps G_n . In the case of the Hill operator the numbers a_n^\pm satisfy $a_n^\pm = \sqrt{A_n^\pm} \geq 0$, $a_{-n}^\pm = -a_n^\mp$, $n \geq 0$, and the gaps g_n satisfy $g_n = (a_n^-, a_n^+)$, $g_{-n} = -g_n$, $n \in \mathbf{Z}$, $g_0 = \emptyset$. For an energy gap G_n and a gap g_n we have the equality $L_n = A_n^+ - A_n^- = l_n(a_n^+ + a_n^-)$, $n \geq 1$. Moreover, for the Hill operator we have (see [M])

$$2Q_0 = \int_0^1 V(x) dx, \quad Q_1 = 0, \quad 8Q_2 = \int_0^1 V(x)^2 dx, \dots$$

and (see [KK1])

$$\frac{1}{\pi} \int q(t)dt = \frac{1}{\pi} \int \int |k'(w) - 1|^2 dudv = \int_0^1 V(x)dx, \dots \quad (2.1)$$

$$M_0^+ = k'(0)^2/2 = 1/2w'(0)^2, \quad \mu_n^\pm = 2a_n^\pm M_n^\pm, \quad n \geq 1. \quad (2.2)$$

Let us formulate the main theorems.

Theorem 2.1. *Suppose $V \in L^1(0,1)$. Then*

$$\frac{1}{\pi} \int q(t)dt = \frac{1}{\pi} \int \int |k'(w) - 1|^2 dudv = \int_0^1 V(x)dx = 2 \sum_{n \geq 1} (A_n^+ M_n^+ + A_n^- M_n^-). \quad (2.3)$$

Furthermore, let $V \in L^2(0,1)$ and $m = 1$ then

$$\begin{aligned} \frac{1}{4\pi} \int \int |(w(k(w) - w))'|^2 dudv + \frac{1}{\pi} \int q(u)p(u)udu &= \frac{1}{\pi} \int u^2 q(u)du = \\ \frac{1}{8} \int_0^1 V^2(x)dx &= \frac{1}{3} \sum_{n \geq 1} [(A_n^+)^2 M_n^+ + (A_n^-)^2 M_n^-] - \frac{1}{6} Q_0^2 \end{aligned}$$

and etc. for V belonging Sobolev space $W_2^{m-1}(\mathbf{R}/\mathbf{Z})$ and $m = 2, 3, \dots$. All series converge absolutely.

Now we present the main estimates obtained in this paper. First we introduce the number $s = \min \text{dust}(g_n, g \setminus g_n)$, $n \geq 1$. This is the more small distance between two gaps non degenerate gaps. Let us introduce the functions $r_n(x) = |(x - a_n^-)(x - a_n^+)|^{1/2} = |(l_n/2)^2 - (t - a_n^0)|^{1/2}$, $u \in \mathbf{R}$, and

$$J_n(u) = \frac{1}{\pi} \int_{g \setminus g_n} \frac{q(t)dt}{r_n(t)|t - u|}, \quad u \in g_n. \quad (2.4)$$

and the numbers $J_n^0 = \max J_n(u)$, $u \in g_n$. The maximum J_n^0 obeys the estimates (see [KK1])

$$J_n^0 \leq \min\left\{\frac{Q_0}{s^2}, \frac{T}{n^2}\right\}, \quad n \geq 1, \quad T = \frac{\pi^2(1 + Q_0 s^{-2}) \max L_n^2}{48s^4}.$$

The function F'_E has a zero λ_n in a "closed gap" $[A_n^-, A_n^+]$. If $G_n = \emptyset$, then $\lambda_n = A_n^- = A_n^+$. In the case of non degenerate gap $A_n^- < \lambda_n < A_n^+$. The function $q(u)$, $u \in g_n$ is strongly convex and has a unique maximum at some point $u_n^0 \in g_n$. We present the following result.

Theorem 2.2. *Let $V \in L^1(0,1)$, $l_n \neq 0$, and $h_n^\pm = \sqrt{l_n \mu_n^\pm}/2$. Then*

$$\left|u_n^0 - \frac{a_n^+ + a_n^-}{2}\right| \leq \frac{l_n^2}{8s} \min\{1, J_n^0\} \leq \frac{L_n^2}{32s^3 n^2} \min(1, J_n^0), \quad (2.5)$$

$$\left|\lambda_n - \frac{A_n^+ + A_n^-}{2} - \frac{l_n^2}{4}\right| \leq \frac{L_n^2}{8s a_n} \min\{1, J_n^0\}, \quad (2.6)$$

$$|h_n - h_n^\pm| \leq \frac{l_n^2 J_n^0}{2s} \left(1 + \frac{l_n \min(1, J_n^0)}{8s}\right) \leq \frac{L_n^2 J_n^0}{8s^3 n^2} \left(1 + \frac{L_n \min(1, J_n^0)}{16s^2 n}\right). \quad (2.7)$$

In the case $L_n \neq 0$ we introduce the numbers $\mu_n^0 = 2h_n^2/l_n, M_n^0 = 2h_n^2/L_n$. Now we consider the case of the masses.

Theorem 2.3. *Suppose that $V \in L^1(0, 1)$ and $L_n \neq 0$. Then*

$$|\mu_n^+ + \mu_n^-| \leq \frac{l_n^2 J_n^0 (1 + J_n^0)}{s} \leq \frac{L_n^2 J_n^0 (1 + J_n^0)}{4s^3 n^2}, \quad (2.8)$$

$$|\mu_n^\pm \mp \mu_n^0| \leq \frac{l_n^2 J_n^0 (1 + J_n^0)}{s} \left(1 + \frac{l_n}{4s}\right). \quad (2.9)$$

$$|M_n^+ + M_n^-| \leq L_n^2 \frac{(1 + J_n^0)}{8a_n^+ a_n^2} \left\{ \frac{1 + J_n^0}{2a_n^-} + \frac{J_n^0}{s} \right\}, \quad (2.10)$$

$$|M_n^+ \mp M_n^0| \leq L_n^2 \frac{(1 + J_n^0)}{8a_n^3} \left\{ \frac{1 + J_n^0}{a_n^\pm} + \frac{J_n^0 (2s + l_n)}{s^2} \right\}. \quad (2.11)$$

Remark 1 . In [KK2] there are the estimates of s . Let $\omega = \cosh h_+$, where $h_+ = \sup h_n$. Then

$$2 \leq s\omega, \quad \omega \leq e^{\|V\|}, \quad h_+ \leq \sum_{n \geq 1} l_n. \quad (2.12)$$

Remark 2 . By Theorem 2.2-3 we also have the asymptotics in terms of both $L_n \rightarrow 0$ and $n \rightarrow \infty$. It is important that we have the estimates of s in terms of $\{h_n\}, \|V\|, \{l_n\}$.

Remark 3 . It is possible to improve (2.5-11). In some estimates instead of s we can write s_n , where s_n is the distance between two sets g_n and $g \setminus g_n$.

Some analogs of Theorems 2.2, 2.3 for the Dirac operator with periodic coefficients will be considered in Theorems 2.4 .

Let us consider the case of GQ. Suppose that GQ has a gap $g^0 = (a^-, a^+), l = |g_0|$, in his spectrum. Denote by μ^\pm the corresponding reduced masses and by h the height of the corresponding slit. We introduce the constant $h^\pm = \sqrt{l\mu^\pm}/2, \mu^0 = 2h^2/l$. The following statements hold true.

Theorem 2.4. *Let $k(z)$ be a general quasimomentum with a gap $g_0 = (a^-, a^+)$ and a point $a^0 = (a^- + a^+)/2$, and $\mu^0 = 2h^2/l, h^\pm = \sqrt{l\mu^\pm}/2$. Then*

$$|u^0 - a^0| \leq \frac{l}{8s} \min\{l, 2h - l\} \leq \frac{l^2}{8s} \min\{1, J(u^0)\}, \quad (2.13)$$

$$|h - h^\pm| \leq \frac{l^2 J^0}{2s} \left(1 + \frac{l}{8s} \min\{1, J^0\}\right), \quad (2.14)$$

$$|\mu^+ + \mu^-| \leq \frac{l}{2s} \left(\sqrt{2l\mu^+} + \sqrt{-2l\mu^-}\right) \min J_\pm \leq \frac{l^2}{s} (1 + J^0) J^0 \quad (2.15)$$

$$|\mu^\pm \mp \mu^0| \leq \frac{l J^0 (h + h_\pm)}{s} \left(1 + \frac{l}{8s} \min\{1, J^0\}\right) \leq \frac{l^2 J^0}{s} \left(1 + \left(1 + \frac{l}{4s}\right) J^0\right). \quad (2.16)$$

3 The estimates for the general quasimomentum

In this section we consider the case of GQ. We assume that GQ k has a gap $g_0 = (a_-, a_+)$ in the spectrum and denote by μ^\pm the corresponding reduced masses. It is well-known that for any GQ $k = p + iq$ the function $q(w) > 0, w = u + iv \in \mathbf{C}_+$ (see [L]). Let us introduce the gap length $l = |g_0|$ and the number $a^0 = \frac{1}{2}(a^+ + a^-)$. We define the function $J(u), u \in g_0$, by the formula

$$J(u) = \frac{1}{\pi} \int_{g \setminus g_0} \frac{q(t)dt}{r(t)|t - u|}, \quad u \in g_0,$$

$$r(t) = |(t - a^-)(t - a^+)|^{1/2} = |(l/2)^2 - (t - a^0)^2|^{1/2}, \quad t \in \mathbf{R},$$

and let $J^0 = \max J(u), u \in g_0$. Later on we need the following formulae from [KK1]

$$q(u) = r(u)(1 + J(u)), \quad u \in g_0, \quad (3.1)$$

$$\pm 2\mu^\pm = l(1 + J^\pm)^2, \quad J^\pm = J(a^\pm). \quad (3.2)$$

Let us prove some simple properties of the function $J(u), u \in g_0$. Note that by the definition of J we see that J is the real analytic function of $u \in g_0$. By the equality

$$J'(u) = \frac{1}{\pi} \int_{g \setminus g_0} \frac{q(t)\text{sign}(t - a^0)dt}{r(t)|t - u|^2}, \quad u \in g_0,$$

we obtain

$$|J'(u)| \leq \frac{J(u)}{s} \leq \frac{J^0}{s}, \quad u \in g_0. \quad (3.3)$$

where s is the distance between the gap g_0 and the set $g \setminus g_0$, and

$$|J(u) - J(u_1)| \leq \frac{l}{s} \max_{u \in g_0} J(u) = \frac{lJ^0}{s}. \quad (3.4)$$

The function $q(u), u \in g_0$, has a unique maximum h on the segment g_0 and the function $k'(u), u \in g_0$ has a zero $u^0 \in g_0$. Let $\delta = u^0 - a^0$ and $h = q(u^0)$. Now we find some formulae. We differentiate (3.1) and get $q' = r'(1 + J) + rJ' = 0$ at $u = u^0$. Then by $r'r = -\delta$ we have

$$\delta(1 + J(u^0)) = r(u^0)^2 J'(u^0). \quad (3.5)$$

The function $q(u), u \in g_0$, has a unique maximum at the point u_0 . Hence by (3.1), (3.7) we get

$$h = r(u^0)(1 + J(u^0)). \quad (3.6)$$

Later on we shall use a simple estimate

$$(1 + y) \min\{1, y\} \leq 2y, \quad y > 0. \quad (3.7)$$

Now we consider some estimates for δ, μ^\pm, l, h . We study the case of δ , i.e. we study the behavior of the zero u^0 (where the function q has maximum). Let us note that if $g = g_0$ then $u^0 = a^0$ since $q(u) = r(u), u \in g_0$, (see (3.1)).

Proof of Theorem 2.4. By (3.5-7) and $2r(u) \leq l, u \in g_0$, we get

$$|\delta| \leq \frac{r^2(u^0)J(u^0)}{s(1+J(u^0))} \leq \frac{r^2(u^0)}{2s} \min\{1, J(u^0)\} = \frac{r(u^0)}{2s} \min\{r(u^0), h - r(u^0)\} \leq$$

$$\frac{l}{8s} \min\{l, 2h - l\} \leq \frac{l^2}{8s} \min\{1, J(u^0)\}.$$

and we have (2.13). We prove (2.14) for the case "+" , the proof for "-" is the same. Recall that $h^+ = \sqrt{l\mu^+}/2$. By (3.1-2), (3.6) and $r^2(u^0) + \delta^2 = (l/2)^2$ we get

$$|h - h^+| = |r(u^0)(1 + J(u^0)) - \frac{l}{2}(1 + J^+)| \leq$$

$$r(u^0)|J(u^0) - J^+| + |r(u^0) - \frac{l}{2}|(1 + J^+),$$

and by (3.4), (2.17), (3.7)

$$|h - h^+| \leq \frac{l^2}{2s}J^+ + \frac{\delta^2(1 + J^+)}{r(u^0) + (l/2)} \leq \frac{l^2J^0}{2s} + \frac{\delta(1 + J^+)}{(l/2)} \frac{l^2 \min(1, J^0)}{8s} \leq$$

$$\frac{l^2J^0}{2s} + \frac{\delta J^0}{2s} \leq \frac{l^2J^0}{2s} (1 + \frac{l \min\{1, J^0\}}{8s}).$$

and we have (2.14). We prove (2.15). By (3.2), (3.4) we obtain

$$2|\mu^+ + \mu^-| = l(2 + J^+ + J^-)|J^+ - J^-| \leq (2 + J^+ + J^-) \frac{l^2J^0}{s} \leq \frac{l}{s}(\sqrt{2l\mu^+} + \sqrt{2l\mu^-})J^0,$$

and by $2\mu^\pm \leq l(1 + J^0)^2$ we get (2.15). We prove (2.16). By (3.2), (2.14) we have

$$|\mu^+ - \mu^0| = |\frac{l}{2}(1 + J^+)^2 - \frac{2h^2}{l}| = (\frac{2}{l})|(\frac{l}{2})^2(1 + J^+)^2 - h^2| =$$

$$(\frac{2}{l})|(h^+)^2 - h^2| = (\frac{2}{l})|h^+ - h||h^+ + h| \leq |h^+ + h| \frac{lJ^0}{s} \{1 + \frac{l \min\{1, J^0\}}{8s}\},$$

and by $h + h^+ \leq l(1 + J^0)$ and by (3.7) we get

$$|\mu_+ - \mu_0| \leq \frac{l^2J^0(1 + J^0)}{s} \{1 + \frac{l \min\{1, J^0\}}{8s}\} \leq \frac{l^2J^0}{s} \{1 + (1 + \frac{l}{4s})J^0\}. QED$$

4 Estimates for the Hill operator

In this chapter we shall apply the previous results for the case both the Hill operator and the Dirac operator with periodic coefficients.

First we consider the Hill operator $H = -d^2/dx^2 + V(x)$ in $L^2(\mathbf{R})$ where V is 1-periodic real potential and $V \in L^1(0,1)$. It is important that in this case we have estimates

$$\frac{L_n}{2\sqrt{A_n^+}} \leq l_n = \frac{L_n}{a_n^+ + a_n^-} \leq \frac{L_n}{2ns}, \quad n \geq 1. \quad (4.1)$$

There are some estimates for l_n , h_n , μ_n^\pm , u_n , in Section 3. For the Hill operator we can rewrite these results more exact. The function $q(u)$, $u \in g_n$, has a maximum at the point $u_n \in g_n$, and we have $h_n = q(u_n)$. Recall that $\lambda_n = u_n^2$. Now we present

Proof of Theorem 2.2. By (2.13) we obtain

$$|u_n - a_n| \leq \frac{l_n^2}{8s} \min\{1, J_n^0\} \leq \frac{L_n^2}{8s(a_n^- + a_n^+)^2} \min\{1, J_n^0\},$$

and by (4.1) we get (2.5). Recall $A_n = \frac{1}{2}(A_n^+ + A_n^-)$. We have

$$4a_n^2 = (a_n^- + a_n^+)^2 = A_n^- + A_n^+ + 2a_n^- a_n^+ = 2(A_n^- + A_n^+) - (a_n^- - a_n^+)^2,$$

hence $a_n^2 = A_n - (l_n/2)^2 = A_n - (L_n^2/(16a_n^2))$. Then by (2.5) we obtain

$$|A_n - \lambda_n - \frac{l_n^2}{4}| = |a_n^2 - \lambda_n| = |a_n + \sqrt{\lambda_n}| |a_n - \sqrt{\lambda_n}| \leq |a_n + \sqrt{\lambda_n}| \frac{l_n^2}{s8} \min\{1, J_n^0\}$$

and by $2a_n l_n = L_n$ we obtain (2.6):

$$|A_n - \lambda_n - \frac{l_n^2}{4}| \leq \frac{3a_n l_n^2}{8} \min\{1, J_n^0\} = \frac{L_n^2}{8sa_n} \min\{1, J_n^0\} \leq \frac{L_n^2}{8s^2n} \min(1, J_n^0).$$

By (2.14) we get

$$|h_n - h_n^\pm| \leq \frac{l_n^2 J_n^0}{2s} \left\{1 + \frac{l_n}{8s} \min(1, J_n^0)\right\},$$

and by (4.1) we obtain (2.7). QED

We give the proof of the next theorem.

Proof of Theorem 2.3. We shall prove (2.8). By (2.15), (4.1-3) we get (2.8)

$$|\mu_n^+ + \mu_n^-| \leq \frac{l_n^2 J_n^0}{s} (1 + J_n^0) \leq \frac{L_n^2 J_n^0}{4s^3 n^2} (1 + J_n^0).$$

By (2.16) we get (2.9). Now we estimate the effective masses. By (2. 2) we have

$$2|M_n^+ + M_n^-| = \left| \frac{\mu_n^+}{a_n^+} + \frac{\mu_n^-}{a_n^-} \right| \leq \frac{l_n \mu_n^-}{a_n^- a_n^+} + \frac{|\mu_n^+ + \mu_n^-|}{a_n^+},$$

then by (3.2), (2.15) we have

$$2|M_n^+ + M_n^-| \leq l_n^2 \frac{1 + J_n^0}{a_n^+} \left\{ \frac{1 + J_n^0}{2a_n^-} + \frac{J_n^0}{s} \right\} \leq L_n^2 \frac{1 + J_n^0}{4a_n^2 a_n^+} \left\{ \frac{1 + J_n^0}{2a_n^-} + \frac{J_n^0}{s} \right\}.$$

Then we have (2.10). We prove (2.11) for "+" , the proof of "-" is the same. By (2.2) we get

$$2|M_n^+ - M_n^0| = \left| \frac{\mu_n^+}{a_n^+} - \frac{\mu_n^0}{a_n} \right| \leq \frac{l_n \mu_n^+}{a_n a_n^+} + \frac{|\mu_n^+ - \mu_n^0|}{a_n}$$

and by (3.2), (2.9) we have

$$2|M_n^+ - M_n^0| \leq l_n^2 \frac{(1 + J_n^0)^2}{2a_n a_n^+} + l_n^2 \frac{J_n^0}{s a_n} \left(1 + \frac{l_n}{2s}\right) (1 + J_n^0) = L_n^2 \frac{1 + J_n^0}{8a_n^3} \left\{ \frac{1 + J_n^0}{a_n^+} + \frac{J_n^0(2s + l_n)}{s^2} \right\}. \quad QED$$

Now we prove the identity.

Proof of Theorem 2.1. In [KK1] there are the equalities (2.3) and

$$Q_0 = \sum_{n \geq 1} (A_n^+ M_n^+ + A_n^- M_n^-), \quad \text{if } V \in L^2(0, 1), \quad (4.2)$$

where the series converges absolutely. Then we have to prove (4.2) in the case $V \in L^1(0, 1)$. Indeed by $L_n M_n^+ = a_n l_n \mu_n^+ / a_n^+ \leq l_n \mu_n^+$ we obtain $|A_n^+ M_n^+ + A_n^- M_n^-| \leq L_n M_n^+ + A_n^- |M_n^+ + M_n^-| \leq l_n \mu_n^+ + A_n^- |M_n^+ + M_n^-|$. Then by (2.10) and $J_n^0 \leq C n^{-2}, n \geq 1$, for some $C > 0$, (see Section 2) we have $|A_n^+ M_n^+ + A_n^- M_n^-| \leq C' l_n^2 / n$, for some C' . Hence we have proved (2.3). The prove for $V \in L^2(0, 1)$ is the same and so on. QED

Now we shall consider the Dirac operator H_D (with periodic coefficients) in the Hilbert space $\mathcal{H} = L^2(\mathbf{R}) \oplus L^2(\mathbf{R})$

$$H_D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} V_1(x) & 0 \\ 0 & V_2(x) \end{pmatrix}.$$

Later on we shall use the Dirac equation

$$f_2' + V_1 f_1 = w f_1, \quad -f_1' + V_2 f_2 = w f_2, \quad (4.3)$$

where V_1, V_2 are real 1-periodic functions in $t \in \mathbf{R}, V_1, V_2 \in L^1(1, 0)$. Let $f(x) = \{f_1(x), f_2(x)\}$ be a vector -function. The boundary value problem (4.3) with the boundary condition $f(0) = f(1)(f(0) = -f(1))$ is called periodic (antiperiodic). We denote the eigenvalues of the periodic problem by a_{2n}^\pm and the eigenvalues of the antiperiodic problem by $a_{2n+1}^\pm, n \in \mathbf{Z}$. It is well-known that $\dots < a_{2n-1}^- \leq a_{2n-1}^+ < a_{2n}^- \leq a_{2n}^+ < \dots$, and $a_n^\pm = n(\pi + o(1)), |n| \rightarrow \infty$. Let the vector-functions $\varphi(x, w) = (\varphi_1(x, w), \varphi_2(x, w))$, and $\theta(x, w) = (\theta_1(x, w), \theta_2(x, w))$ be the solutions of (4.21) satisfying $\varphi(0, w) = (0, 1), \theta(0, w) = (1, 0)$. We introduce the Lyapunov function for the Dirac equation $F_D(w) = \frac{1}{2}(\varphi_1(1, w) + \theta_2(1, w)), w \in \mathbf{C}$. The properties of the Lyapunov function for the Dirac operator and

for the Hill operator are similar, for example $F(a_n^\pm) = (-1)^n, n \in \mathbf{Z}$. But there is one exception. In general the function $F_D(w)$ is not even in $w \in \mathbf{C}$. The spectrum of H_D is purely absolutely continuous and is given by the set $\cup s_n$, where a interval $s_n = [a_{n-1}^+, a_n^-]$. These intervals are separated by gaps $g_n = (a_n^-, a_n^+)$. If a gap g_n is degenerate, i.e. $g_n = \emptyset$ then the corresponding segments s_n, s_{n+1} merge. Now we define the quasimomentum function $k(w) = \arccos F_D(w), w \in W = \mathbf{C} \setminus \bar{g}, g = \cup g_n$. The function $k(w)$ is analytic and moreover k is a conformal map from W onto the quasimomentum slit plane $K = \mathbf{C} \setminus \cup \Gamma_n$ where an excised slit is given by $\Gamma_n = \{\operatorname{Re} k = \pi n, |\operatorname{Im} k| \leq h_n\}, h_n \geq 0, n \in \mathbf{Z}$. A lot of estimates for the Dirac operator repeat corresponding estimates for the Hill operator.

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