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# Bands and Gaps for Periodic Magnetic Hamiltonians

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# Bands and gaps for periodic magnetic Hamiltonians.

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## 1. Introduction.

In this contribution, we wish to present some recent results and to discuss an open problem in the spectral theory of magnetic Hamiltonians

$$H(\vec{a}) = (-i\nabla - \vec{a}(x))^2 ,$$

acting in $L_2(\mathbf{R}^n)$, with $n \geq 2$, and $\vec{a} \in C^1(\mathbf{R}^n; \mathbf{R}^n)$. Here we focus on the cases where

(1) the field $B = d\vec{a}$ is periodic with respect to some discrete lattice $\Gamma$ spanning $\mathbf{R}^n$, e. g., $\Gamma = \mathbf{Z}^n$,

or where, more strongly,

(2) the vector potential $\vec{a}$ is periodic with respect to the lattice $\Gamma$.

In the second case, $H(\vec{a})$ commutes with translations $\in \Gamma$, and we may apply Floquet theory to obtain a direct fiber integral decomposition. In the first case, after going through any translation $\in \Gamma \setminus \{0\}$, the operator will in general differ from the original one by a non-trivial gauge transformation.

Periodicity of the vector potential $\vec{a}$ implies periodicity of the magnetic field, plus certain conditions on the magnetic flux. For example, it is easy to see that in dimension $n = 2$, the total flux through a period parallelogram vanishes. In general, it follows from some basic results in cohomology theory that periodicity of $B$ plus

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certain flux conditions are equivalent to the existence of an associated periodic vector potential; cf. Proposition 1.

Let us now first give some basic results on bands and gaps; Section 2 will be devoted to the problem of absolute continuity of $H(\vec{a})$.

It is well known that the magnetic Hamiltonian with constant field $B \neq 0$ in dimension $n = 2$ has pure point spectrum consisting of isolated eigenvalues of infinite multiplicity. One may think of this type of spectrum as a version of band – gap structure, with bands degenerated into points. However, constant fields in odd dimensions do not produce gaps.

There is a remarkable recent result due to Brüning and Sunada [BrS2] (cf. also [BrS1]) which says that for periodic field $B$ the spectrum of $H(\vec{a})$ has *band structure* in the sense that there are at most a finite number of gaps in any given bounded interval $[0, R]$.

In a recent paper, it was shown by Hempel and Herbst ([HH; Corollary 3.10]) that gaps can be produced in any dimension $n \geq 2$ by a periodic array of "magnetic barriers", with periodic vector potential or just periodic field. In order to describe this result (plus some generalizations) we again define for $\vec{a} \in C^1(\mathbf{R}^n; \mathbf{R}^n)$ the closed sets

$$M = \{x \in \mathbf{R}^n; B(x) = 0\}, \quad M_{\vec{a}} = \{x \in \mathbf{R}^n; \vec{a}(x) = 0\};$$

note that $M_{\vec{a}} \setminus M$ has measure zero while $M \setminus M_{\vec{a}}$ can be large. The condition that $M \setminus M_{\vec{a}}$ have measure zero is essential for most of the results in [HH]; Section 4 in [HH] gives a discussion of what this condition means.

**Theorem 1.** *Let $\vec{a}$ be periodic with respect to the lattice $\Gamma$ and assume that $M \setminus M_{\vec{a}}$ has measure zero. Suppose there exists a compact set $M_0$ with non-empty interior such that, writing $M_j = M_0 + j$ for $j \in \Gamma$, we have $M_j \cap M_k = \emptyset$, for all $k \neq j$ and $M = \cup_{j \in \Gamma} M_j$, up to a set of measure zero. Let $E_1 < E_2 < \ldots < E_k < \ldots$ denote the eigenvalues of the (Dirichlet) Laplacian $-\Delta_{M_0}$ on the closed set $M_0$, as defined in* [HH].

*Finally, let $\varepsilon > 0$ and $E > 0$, and suppose that $E_K < E < E_{K+1}$. Then there exists $m = m(\varepsilon, E)$ such that for $|\mu| \geq m$ the following holds:*

*(1) $\sigma(H(\mu\vec{a})) \cap (E_k - \varepsilon, E_k + \varepsilon) \neq \emptyset$, for $k = 1, \ldots, K$.*

*(2) $\sigma(H(\mu\vec{a})) \cap [0, E] \subset \cup_{k=1}^{K} (E_k - \varepsilon, E_k + \varepsilon)$.*

*(3) The spectrum of $H(\mu\vec{a})$ inside any of the intervals $(E_k - \varepsilon, E_k + \varepsilon)$, $k = 1, \ldots, K$, consists of a finite number of disjoint closed intervals (which may degenerate into points).*

*(4) $H(\mu\vec{a})$ has no singular continuous part.*

*(5) For all but a discrete set of $\mu$'s, $H(\mu\vec{a})$ is absolutely continuous in the interval $[0,E]$. In particular, there are no eigenvalues inside $[0,E]$ and the intervals of spectrum occuring in (3) are non-degenerate, for all but a discrete set of $\mu$'s.*

**Proof.** Since we assume that $M \setminus M_{\vec{a}}$ has measure zero, it follows from [HH] that $H(\mu\vec{a})$ converges in norm resolvent sense to the (Dirichlet) Laplacian $-\Delta_M$ on the set $M$, as $|\mu| \to \infty$. By periodicity, $-\Delta_M$ is nothing but an infinite direct sum of copies of $-\Delta_{M_0}$ and the spectrum of $-\Delta_M$ consists of the sequence of eigenvalues $E_k$, each eigenvalue having infinite multiplicity. Now, since norm resolvent convergence for self-adjoint operators implies convergence of spectra, statements (1) and (2) follow.

By Floquet theory, the spectrum of $H(\mu\vec{a})$ is given by the union of the ranges of the band functions, which are continuous functions on a fundamental cell of the dual lattice, and (3) follows (note that, by compactness, only a finite number of band functions have values in a given bounded interval, for any fixed $\mu$).

Claims (4) and (5) are a direct consequence of Theorems 2 and 4 in Section 2. ∎

As a corollary, we obtain a version of Theorem 1 where the assumptions are made in terms of the field $B$. Let $g_1, \ldots, g_n$ denote a set of linearly independent vectors in $\mathbf{R}^n$ spanning the lattice $\Gamma$. We first prepare an auxiliary result.

**Proposition 1.** *Suppose $B$ is a 2-form on $\mathbf{R}^n$ which is $C^1$ (or $C^\infty$) and $\Gamma$-periodic in the sense that $B(x+g_j) = B(x)$ for $j = 1, \ldots, n$ and for all $x \in \mathbf{R}^n$. Suppose $dB = 0$ and that the flux conditions*

$$\int_{C_{ij}} B = 0\,, \quad 1 \le i < j \le n\,, \tag{1}$$

*are satisfied, where $C_{ij} : [0,1] \times [0,1] \to \mathbf{R}^n$ is given by*

$$C_{ij}(t,s) = tg_i + sg_j\,. \tag{2}$$

*Then there is a $\Gamma$-periodic 1-form $\vec{a}$ of class $C^1$ (or of class $C^\infty$) such that $B = d\vec{a}$.*

This result can be proved without using cohomology theory, but the proof is somewhat lengthy and we refrain from reproducing it here. On the other hand, an application of de Rham's theorem to the case of a smooth closed 2-form on the $n$-torus $T^n$ reduces the proof of Proposition 1 to the problem of determining an explicit basis for the module of 2-cycles; it is easy to see that such a basis is conveniently given by the 2-tori defined via eqn. (2) (cf., e.g., [StZ; pp. 334 and 239]).

Combining Theorem 1 and Proposition 1, we now obtain the following corollary:

**Corollary 1.** *Let $B$ be a smooth $\Gamma$-periodic magnetic field (more precisely, a smooth, $\Gamma$-periodic, exact two-form on $\mathbf{R}^n$) satisfying the flux conditions described* *in eqns.* (1) *and* (2). *Furthermore, suppose that there exists a compact set* $M_0$ *as in Theorem 1 such that, in addition, the interior* $\mathrm{int}(M_0)$ *is a simply connected Lipschitz domain and* $M_0$ *is the closure of its interior.*

*Let* $-\Delta_M$ *and* $E_k$ *be as above. Then, for any given* $\varepsilon > 0$, $E > 0$ *and* $K$ *as in Theorem 1, there exists* $m = m(\varepsilon, E)$ *such that the statements* (1) – (5) *hold true for* $|\mu| \geq m$.

**Proof.** We have to produce a smooth vector potential $\vec{a}$ such that $d\vec{a} = B$, satisfying the conditions of Theorem 1. As a first step, Proposition 1 yields the existence of a periodic vector potential $\vec{a}_0 \in C^\infty(\mathbf{R}^n; \mathbf{R}^n)$ such that $d\vec{a}_0 = B$. In general, $\vec{a}_0$ will not vanish on $M_0$, however. Following the construction at the end of Section 4 in [HH] and using Stein [St; p. 181], we next find a function $f \in C^\infty(\mathbf{R}^n)$ such that $df(x) = \vec{a}_0(x)$ for $x \in \mathrm{int}(M_0)$. Pick a (real-valued) function $\varphi \in C_0^\infty(\mathbf{R}^n)$ which is 1 in a neighborhood of $M_0$, and which vanishes in a neighborhhood of each $M_j$, for all $j \in \Gamma$, $j \neq 0$. Define $F \in C^\infty(\mathbf{R}^n)$ by

$$F(x) = \sum_{j \in \Gamma} (\varphi f)(x - j), \quad x \in \mathbf{R}^n.$$

Then $\vec{a}(x) = \vec{a}_0(x) - dF(x)$ is periodic and satisfies $d\vec{a} = B$; furthermore $\vec{a}(x) = 0$ for almost every $x \in M$. ∎

# 2. On the absolute continuity of periodic Hamiltonians.

The central question of this note is to determine the nature of the spectrum separating gaps of $H(\vec{a})$ and of $H(\mu\vec{a})$, for $\mu \in \mathbf{R}$. Here the strongest result one might head for is to show that for periodic $\vec{a}$ the spectrum of $H(\vec{a})$ is absolutely continuous:

**Conjecture.** *Suppose* $\vec{a} \in C^1(\mathbf{R}^n; \mathbf{R}^n)$ *is periodic with respect to the lattice* $\Gamma$. *Then* $H(\vec{a})$ *is absolutely continuous.*

Below, we will present some evidence in support of this conjecture. We shall not persue the case of periodic field $B$ here; note, however, that the constant field case in 2-dimensions provides only a weak example, and one might still ask for the nature of the spectrum in cases where $B$ is periodic, but non-constant. In this direction, Iwatsuka [Iw] has announced some results on line-broadening.

Let us now sketch how far one can get in the question of absolute continuity by using the approach of L. Thomas [Th] (reproduced in [RS-IV]) or some more abstract results on the structure of the "Bloch-variety", the union of the graphs of the band-functions. We find that $H(\vec{a})$ has no singular continuous part, so the only obstacle in proving absolute contiunity is the possibility of eigenvalues. Furthermore, $H(\vec{a})$ is absolutely continuous for small $\vec{a}$, and $H(\mu\vec{a})$ is a. c. in any given interval $[0, R]$ for all but a discrete set of $\mu$'s (which we expect to be empty). We begin with results which exploit the fact that the Bloch variety associated with $H(\vec{a})$ can we viewed as the zero set of a suitable real-analytic function of several variables; cf. Gérard [G], Wilcox [W], Kuchment [Ku] for related methods and results in the case of periodic $-\Delta + V$.

The Floquet decomposition of $H(\vec{a})$ is fundamental to our analysis (cf., e. g., [RS-IV]): Let $Q$ denote a fundamental cell of the dual lattice spanned by the vectors $K_j$ (note that our $Q$ corresponds to $\tilde{Q}$ in [RS-IV; p. 305]). Defining a new Hilbert space $\mathcal{H}$ via a direct fiber integral of $\ell_2$-spaces,

$$\mathcal{H} = \int_Q^{\oplus} \ell_2(\mathbf{Z}^n) dk,$$

there exists a unitary operator $U : L_2(\mathbf{R}^n) \to \mathcal{H}$ such that

$$U H(\vec{a}) U^{-1} = \int_Q^{\oplus} H_k(\vec{a}) dk, \tag{3}$$

with suitable operators $H_k(\vec{a})$, acting in $\ell_2(\mathbf{Z}^n)$. We are now ready to state the following theorem:

**Theorem 2.** *Suppose that $\vec{a} \in C^1(\mathbf{R}^n; \mathbf{R}^n)$ is periodic with respect to the lattice $\Gamma$. Then:*

*(a) The singular continuous spectrum of $H(\vec{a})$ is empty.*

*(b) The eigenvalues of $H(\vec{a})$ form a discrete subset of $\mathbf{R}$.*

*(c) If $\lambda$ is an eigenvalue of $H(\vec{a})$, then $\lambda$ is an eigenvalue of $H_k(\vec{a})$ for all $k \in Q$.*

**Proof.** We start from the direct fiber decomposition (3). Using regularized determinants, one can construct a real analytic function $F(\lambda, k)$ on $\mathbf{R} \times Q$ with the property that $E$ is an eigenvalue of $H_k(\vec{a})$ iff $F(E, k) = 0$; cf. [G], [Ku], [W].

We first prove (b) and (c). By the direct fiber decomposition of $H(\vec{a})$, $\lambda_0$ is an eigenvalue of $H(\vec{a})$ iff there exists a set $M_0 \subset Q$ of positive measure such that $\lambda_0$ is an eigenvalue of $H_k(\vec{a})$ for all $k \in M_0$ iff $F(\lambda_0, k) = 0$ for all $k \in M_0$. It now follows from Theorem A that the set of eigenvalues is discrete.

Furthermore, if $\lambda_0$ is an eigenvalue of $H(\vec{a})$, then $F(\lambda_0, k) = 0$ for all $k \in Q$ (by Lemma A.1), so $\lambda_0$ is an eigenvalue of $H_k(\vec{a})$ for all $k \in Q$.

In order to prove that the singular continuous spectrum of $H(\vec{a})$ is empty, it is enough to show that for any Borel set $\Lambda \subset \mathbf{R} \setminus \sigma_{pp}(H(\vec{a}))$ of measure zero, and any $f \in L_2(\mathbf{R}^n)$, we have $\|P_\Lambda(H(\vec{a}))f\|^2 = 0$, where

$$P_\Lambda(H(\vec{a})) = \chi_\Lambda(H(\vec{a}))$$

is the spectral projection associated with $H(\vec{a})$ and $\Lambda$. By [RS-IV; Thm. XIII.85 (c)], we see that

$$\|P_\Lambda(H(\vec{a}))f\|^2 = \int_Q \|P_\Lambda(H_k(\vec{a}))f_k\|_{\ell_2}^2 \, dk.$$

The integrand on the RHS will vanish at $k$ unless $\Lambda$ contains an eigenvalue of $H_k(\vec{a})$. But the set

$$\{k \in Q; \Lambda \cap \sigma(H_k(\vec{a})) \neq \emptyset\} = \{k \in Q; F(\lambda, k) = 0 \text{ for some } \lambda \in \Lambda\}$$

has measure zero by Theorem A, and we are done. ∎

**Remark.** There is not much new in Theorem 2 beyond what can already be found in Kuchment [Ku], Gérard [G], or in Wilcox [W]. In particular, it is shown by Gérard [G] that the resolvent of a periodic Schrödinger operator $H = -\Delta + V$, with $V$ relatively bounded with respect to the Laplacian with bound $< 1$, can be analytically continued through the real axis (in a suitable sense), outside a discrete set of points. This establishes the absence of singular continuous spectrum, but leaves the possibility of a discrete set of eigenvalues. Below and in the Appendix, we give a self-contained introduction into the required machinery, keeping the use of algebra to a strict minimum. Note that Gérard seems to share our belief that there actually shouldn't be any such eigenvalues, for relatively bounded perturbations [G; p. 48].

L. Thomas' celebrated proof of the absolute continuity of periodic Schrödinger operators $-\Delta + V$ can be adapted to the magnetic case as long as the vector potential is not too large.

**Theorem 3.** ([HH; Thm. 3.12]) *For any given lattice $\Gamma$ there exists a constant $c > 0$ such that $H(\vec{a})$ is absolutely continuous for all $\Gamma$-periodic vector potentials $\vec{a} \in C^1(\mathbf{R}^n; \mathbf{R}^n)$ satisfying*

$$\sup_x |\vec{a}(x)| < c.$$

**Proof.** By Theorem 2(a), it is enough to find a constant $c > 0$ such that $H(\vec{a})$ has no eigenvalues if $\sup |\vec{a}(x)| < c$. By Theorem 2(c), $\lambda$ is an eigenvalue of $H(\vec{a})$ if and only if $\lambda$ is an eigenvalue of $H_k(\vec{a})$, for all $k = (k_1, \ldots, k_n) \in Q$ if and only if $\lambda$ is an eigenvalue of $H_k(\vec{a})$ for all $k \in \mathbf{C}^n$. We now take take Im $k_1$ to infinity, along suitable lines. As in the original proof of Thomas [Th], we find by a direct calculation that

$$\liminf_{\text{Im } k_1 \to \infty} \left\|(H_k(\vec{a}) + 1)^{-1}\right\| = 0,$$

provided $\sup |\vec{a}|$ is small. But then no eigenvalue of the family $H_k(\vec{a})$ can be constant. ∎

**Remark.** In the the case of periodic Hamiltonians $-\Delta+V$, with mild assumptions on $V$ ([RS-IV; p. 305 ff]), a similar calculation reveals that $\left\|(H_k+1)^{-1}\right\|$ tends to zero as $\operatorname{Im} k_1 \to \infty$. It is interesting to point out why a smallness conditions seems to be unavoidable in the magnetic case:

Let $P(k) = k + \sum m_j K_j$, acting in $\ell_2(\mathbf{Z}^n)$, where $m_j \in \mathbf{Z}$ and the vectors $K_j$, $j = 1, \ldots, n$ span the dual lattice. We then find that controlling the term coming from $\vec{a} \cdot \nabla$ requires control of $\left\|P(k)(P(k)^2+1)^{-1}\right\|$, for $\operatorname{Im} k_1 \to \infty$ on some line. It turns out that $\liminf_{\operatorname{Im} k_1 \to \infty} \left\|P(k)(P(k)^2+1)^{-1}\right\|$ is a certain finite number, which is independent of the lines along which $k_1$ tends to $\infty$, but, alas, this number is not zero.

We finally introduce a coupling $\mu \in \mathbf{R}$ and ask for the absolute continuity of $H(\mu\vec{a})$.

**Theorem 4.** *Suppose that $\vec{a} \in C^1(\mathbf{R}^n;\mathbf{R}^n)$ is periodic with respect to the lattice $\Gamma$. Then:*

*(a) For any fixed $R > 0$, the set of coupling constants $\mu \in \mathbf{R}$ for which $H(\mu\vec{a})$ is not absolutely continuous in the interval $[0, R]$ is discrete.*

*(b) The set of coupling constants $\mu \in \mathbf{R}$ for which $H(\mu\vec{a})$ is not absolutely continuous is at most countable.*

**Proof.** Let $R > 0$, and suppose for a contradiction that there exist sequences $\{\mu_j\} \subset \mathbf{R}$ with $\mu_j \to \mu_0 \in \mathbf{R}$, $\mu_j \neq \mu_k$ for $j \neq k$, and $\{\lambda_j\} \subset [0, R]$ such that $\lambda_j$ is an eigenvalue of $H(\mu_j\vec{a})$, for all $j \in \mathbf{N}$. Without restriction we may assume that the sequence $\{\lambda_j\}$ converges. Let $M \geq 0$ be such that $|\mu_j| \leq M$. We first conclude from Theorem 2(c) that $\lambda_j$ is an eigenvalue of $H_k(\mu_j\vec{a})$, for all $k \in Q$.

For each fixed $k \in Q$, $(H_k(\mu\vec{a}), \mu \in \mathbf{C})$ is a self-adjoint holomorphic family of type (A) in the sense of Kato. Furthermore, $H_k(\mu\vec{a})$ has compact resolvent for all $\mu$. By [K; Thm. VII-3.9], there exists a countable family of functions $\Lambda_m(k,\mu)$, $m \in \mathbf{N}$, real-analytic in $\mu \in \mathbf{R}$, which describe the eigenvalues of $H_k(\mu\vec{a})$. In addition, an easy compactness argument (using the standard bound given in [K;p. 391] for the derivatives $\frac{\partial}{\partial\mu}\Lambda(k,\mu)$) implies that for each fixed $k \in Q$ there exists at most a finite number of indices $m$ such that $\Lambda_m(k,\mu) \in [0, R]$ for some $|\mu| \leq M$.

We now pick any $k_0 \in Q$. Since $\lambda_j$ is an eigenvalue of $H_{k_0}(\mu_j\vec{a})$, for all $j \in \mathbf{N}$, and since only finitely many functions $\Lambda_m(k_0,\mu)$ meet the interval $[0, R]$, for $|\mu| \leq M$, there exists an index $m(k_0)$ such that

$$\lambda_j = \Lambda_{m(k_0)}(k_0, \mu_j) \tag{4}$$

for an infinite number of $j$'s. Without restriction, we may assume that (4) holds for all $j \in \mathbf{N}$. Write $\Lambda_0 = \Lambda_{m(k_0)}(k_0, 0)$. Suppose $k \in Q$. As above there exists an index $m(k)$ such that

$$\lambda_j = \Lambda_{m(k)}(k, \mu_j)$$

for infinitely many $j$. By unique continuation, $\Lambda_{m(k)}(k,\mu) = \Lambda_{m(k_0)}(k_0,\mu)$ for all $\mu$, in particular for $\mu = 0$. Thus for each $k \in Q$

$$\Lambda_{m(k)}(k,0) = \Lambda_0$$

so that $\Lambda_0$ is an eigenvalue of $H_k(0)$ for all $k \in Q$. But this implies that $H(0) = -\Delta$ has an eigenvalue, in contradiction with the absolute continuity of $-\Delta$. ∎

# Appendix.

**Theorem A.** *Suppose $I \subset \mathbf{R}$ is an open interval and $\mathcal{O} \subset \mathbf{R}^n$ is an open connected set. Suppose $f : I \times \mathcal{O} \to \mathbf{C}$ is real analytic and not identically zero. Define the measure $\mu$ on the Borel sets $\Lambda$ of $\mathbf{R}$ by*

$$\mu(\Lambda) = \text{meas}\,\{k \in \mathcal{O}; f(\lambda,k) = 0 \ \text{ for some } \ \lambda \in \Lambda\}).$$

*Then $\mu$ has no singular continuous component and, in addition, the set of pure points of $\mu$ has no accumulation point in $I$.*

We first prove two lemmas.

**Lemma A.1.** *Suppose $\mathcal{O} \subset \mathbf{R}^n$ is an open connected set and $h : \mathcal{O} \to \mathbf{C}$ is real analytic. Then if* $\text{meas}\,(h^{-1}(0)) > 0$ *it follows that $h \equiv 0$.*

**Proof.** Let $k_0$ be a point of density of the zero set of $h$. Then it easily follows by induction that all derivatives of $h$ vanish at $k_0$. By unique continuation, $h$ is zero. ∎

**Lemma A.2.** *Suppose $\mathcal{O} \subset \mathbf{R}^n$ is open and connected and the map $f : \mathcal{O} \to \mathbf{R}$ is real analytic. Then, if $f$ is not constant, it follows that if $\Lambda \subset \mathbf{R}$ is a Borel set with* $\text{meas}\,(\Lambda) = 0$ *we have* $\text{meas}\,(f^{-1}(\Lambda)) = 0$.

**Remark.** Lemma A.2 is a special case of Theorem A with $f(\lambda,k) = \lambda - f(k)$.

**Proof.** We can assume $\mathcal{O} = J_1 \times \cdots \times J_n$ where $J_\ell$ is an open interval. The lemma is easy to prove if $n = 1$ so we assume $n > 1$. Let

$$\Gamma_1 = \{(k_2,\ldots,k_n) \in J_2 \times \cdots \times J_n; f(k_1,k_2,\ldots,k_n) \,\text{is constant in the variable}\, k_1\}.$$

It is easy to see that $\Gamma_1$ is measurable. By Lemma A.1, if $\text{meas}\,(\Gamma_1) > 0$ it follows that for any $k_1$, $k_1' \in J_1$

$$f(k_1,k_2,\ldots,k_n) - f(k_1',k_2,\ldots,k_n) = 0$$

for all $(k_2,\ldots,k_n) \in J_2 \times \cdots \times J_n$, in other words, $f(\cdot,k_2,\ldots,k_n)$ is constant for all $(k_2,\ldots,k_n) \in J_2 \times \cdots \times J_n$. Similarly, we define $\Gamma_j$, $j = 2,\ldots,n$. If $\text{meas}\,(\Gamma_j) > 0$

for all $j$, it then easily follows that $f$ is constant. Without loss of generality we can therefore assume meas$(\Gamma_1) = 0$. Write $k = (k_1, k')$ and $f_{k'}(k_1) = f(k_1, k')$ with $k_1 \in J_1$, $k' \in J_2 \times \cdots \times J_n$. By Fubini's theorem we have

$$\begin{aligned} \text{meas}\,(f^{-1}(\Lambda)) &= \int_{f^{-1}(\Lambda)} dk = \int_{J_2 \times \cdots \times J_n} \left( \int_{k'=a,\ k \in f^{-1}(\Lambda)} dk_1 \right) da \\ &= \int_{J_2 \times \cdots \times J_n} \left( \int_{f_a^{-1}(\Lambda)} dk_1 \right) da. \end{aligned}$$

But for almost every $a \in J_2 \times \cdots \times J_n$, $f_a$ is non-constant so by the result for $n = 1$, $f_a^{-1}(\Lambda)$ has measure zero for almost every $a$. The result follows. ∎

**Proof of Theorem A.** If $\mu(\{\lambda_j\}) > 0$ for a sequence $\lambda_j \to \lambda \in I$ then by Lemma A.1, $f(\lambda_j, k) = 0$, for all $k \in \mathcal{O}$, and thus $f = 0$. This contradiction proves the second statement.

Let

$$M = \{k \in \mathcal{O}; f(\lambda, k) = 0 \text{ for all } \lambda \in I\}.$$

Then by Lemma A.1, $M$ is a set of measure zero, closed in the relative topology. Suppose $\Lambda \subset \mathbf{R}$ has Lebesgue measure zero and contains no pure points of $\mu$. It is enough to show that every point $(\lambda_0, k_0) \in I \times (\mathcal{O} \setminus M)$ has an open neighborhood $I_0 \times \mathcal{O}_0$ with $\mathcal{O}_0 \subset \mathcal{O} \setminus M$ such that

$$\{k \in \mathcal{O}_0; f(\lambda, k) = 0 \text{ for some } \lambda \in \Lambda \cap I_0\}$$

has measure zero. Clearly we can assume $f(\lambda_0, k_0) = 0$. For notational simplicity in what follows we take $(\lambda_0, k_0) = 0$. Then there is a neighborhood $U_1$ of 0 contained in $I \times (\mathcal{O} \setminus M)$ and functions $p_j$, $j = 1, \ldots, \ell$, real analytic on $U_1$, and integers $r_1, \ldots, r_\ell \geq 1$ so that on $U_1$

$$f = p_1^{r_1} \cdots p_\ell^{r_\ell}.$$

In addition, $p_j$ is prime ($j = 1, \ldots, \ell$) in the sense that if on any neighborhood of 0 we have $p_j = h_1 h_2$ with $h_1$ and $h_2$ real analytic and $h_1(0) = 0$ then $h_2(0) \neq 0$. This follows from the unique factorization property of the germs of analytic functions at 0 ([F; p. 116]). We can assume $U_1 = I_1 \times \mathcal{O}_1$ where $I_1$ is an interval of $\mathbf{R}$ and $\mathcal{O}_1$ is an open subset of $\mathbf{R}^n$. By the Weierstrass Preparation Theorem ([F; Thm. 3.23]) and the fact that $f(\lambda, 0) \not\equiv 0$, we can write (after shrinking $U_1$ perhaps)

$$f = h\tilde{p}_1^{r_1} \cdots \tilde{p}_\ell^{r_\ell} \qquad \text{on } U_1 = I_1 \times \mathcal{O}_1\,,$$

where $h$ is non-zero on $U_1$ and $\tilde{p}_j$ is a *Weierstrass polynomial*:

$$\tilde{p}_j(\lambda, k) = \lambda^{n_j} + a_{j1}(k)\lambda^{n_j - 1} + \ldots + a_{jn_j}(k),$$

with $a_{jm}(k)$ real analytic on $\mathcal{O}_1$, $a_{jm}(0) = 0$. In addition, $\tilde{p}_j$ is prime.

The discriminant $D_j(k)$ of $\tilde{p}_j$ is real analytic on $\mathcal{O}_1$ and is not identically zero ([vdW], [GrFr; Thm. 6.11]) because $\tilde{p}_j$ is prime. The set $M_j$ where $D_j$ is zero is a closed set of measure zero and $\tilde{p}_j$ has distinct roots on $\mathcal{O}_1 \setminus M_j$. It is enough to find for each $j$ and each point $(\lambda_0', k_0') \in I_1 \times (\mathcal{O}_1 \setminus M_j)$ a neighborhood $I_2 \times \mathcal{O}_2$ of $(\lambda_0', k_0')$ such that

$$\{k \in \mathcal{O}_2; \tilde{p}_j(\lambda, k) = 0 \text{ for some } \lambda \in I_2 \cap \Lambda\}$$

has measure zero. We can assume $\tilde{p}_j(\lambda_0', k_0') = 0$. By the Implicit Function Theorem there exists a neighborhood $I_2 \times \mathcal{O}_2$ of $(\lambda_0', k_0')$ and $n_j$ functions $\lambda_1, \ldots, \lambda_{n_j}$, real analytic on $\mathcal{O}_2$, such that for $(\lambda, k) \in I_2 \times \mathcal{O}_2$

$$\tilde{p}_j(\lambda, k) = 0 \iff \lambda = \lambda_i(k), \quad \text{for some } i \in \{1, \ldots, n_j\}.$$

We can assume that $\mathcal{O}_2$ is an open rectangle. We have

$$\{k \in \mathcal{O}_2; \tilde{p}_j(\lambda, k) = 0 \text{ for some } \lambda \in I_2 \cap \Lambda\} \\ = \cup_{i=1}^{n_j} \{k \in \mathcal{O}_2; \lambda = \lambda_i(k) \text{ for some } \lambda \in I_2 \cap \Lambda\}.$$

The set

$$\{k \in \mathcal{O}_2; \lambda = \lambda_i(k) \text{ for some } \lambda \in I_2 \cap \Lambda\} = \lambda_i^{-1}(I_2 \cap \Lambda)$$

will have non-zero measure only if $\operatorname{Im} \lambda_i(k) = 0$, for all $k \in \mathcal{O}_2$ (Lemma A.1) and if for some $\lambda_0 \in I_2 \cap \Lambda$, $\lambda_i(k) = \lambda_0$ for all $k \in \mathcal{O}_2$ (Lemma A.2). But the latter condition implies $f(\lambda_0, k) = 0$ for all $k \in \mathcal{O}_2$ and thus for all $k \in \mathcal{O}$. But $\Lambda$ was chosen to be free of pure points of $\mu$, so this is a contradiction. ∎

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