

**Finitary Galois Extensions
over Noncommutative Bases**

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FINITARY GALOIS EXTENSIONS OVER NONCOMMUTATIVE BASES

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ABSTRACT. We study Galois extensions $M^{(\text{co-})H} \subset M$ for H -(co)module algebras M if H is a Frobenius Hopf algebroid. The relation between the action and coaction pictures is analogous to that found in Hopf-Galois theory for finite dimensional Hopf algebras over fields. So we obtain generalizations of various classical theorems of Kreimer-Takeuchi, Doi-Takeuchi and Cohen-Fischman-Montgomery. We find that the Galois extensions $N \subset M$ over some Frobenius Hopf algebroid are precisely the balanced depth 2 Frobenius extensions. We prove that the Yetter-Drinfeld categories over H are always braided and their braided commutative algebras play the role of noncommutative scalar extensions by a slightly generalized Brzeziński-Militaru Theorem. Contravariant "fiber functors" are used to prove an analogue of Ulbrich's Theorem and to get a monoidal embedding of the module category \mathbf{M}_E of the endomorphism Hopf algebroid $E = \text{End}_N M_N$ into ${}_N \mathbf{M}_N^{\text{op}}$.

1. INTRODUCTION

The problem of extending Hopf Galois theory to quantum groupoids has been attracting some attention in recent years. That this theory should possess interesting new applications even for finite quantum groupoids is manifest already from the pioneering work of D. Nikshych and L. Vainerman [19]. A pure algebraic Galois theory for weak Hopf algebras has been proposed by S. Caenepeel and E. de Groot [8]. As a different generalization, which maintains finiteness of the total algebra over the base but lets the base algebra to be unrestricted, this paper is devoted to developing a Galois theory for Frobenius Hopf algebroids. These quantum groupoids can be thought as the analogues of finite dimensional Hopf algebras over a field or Frobenius Hopf algebras over a commutative ring k in which k is replaced with a noncommutative base ring R . Therefore it is not surprising, but also not trivial, that we obtain generalizations of the classical theorems of Kreimer-Takeuchi [16], Doi-Takeuchi [11] and Cohen-Fischman-Montgomery [10] (see Theorems 3.3 and 3.6). Our results partly overlap with those of the recent paper [2] by G. Böhm who studies Galois theory for general Hopf algebroids using previous results from the theory of corings [5, 7]. In our approach the double algebraic structure [25] of Frobenius Hopf algebroids is particularly useful e. g. in proving that Yetter-Drinfeld categories are braided (Proposition 4.9), in this way generalizing a result of [9], or in obtaining an intrinsic characterization of Galois extensions as being the depth 2, balanced, Frobenius extensions (Theorem 3.7).

As far as bialgebroids are concerned, their Galois theory is not so well understood. There exist generalizations of certain notions and theorems of Hopf-Galois theory and, therefore, hints toward the proper definition of Galois bialgebroid

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[15, 24, 14]. For example, the depth 2 balanced extensions $N \subset M$ of algebras seem to be the Galois extensions in the very noncommutative sense. Unfortunately, in the absence of antipode, even in the finitely generated projective case, many results of classical and Hopf Galois theory are far from reach. Nevertheless, certain results, e. g. [13, Theorem 5.1], indicate that the theory of Galois corings will prove to be useful in developing such a theory in the future.

In the rest of this Introduction we would like to concentrate on two topics that are crucial for the present paper. At first we discuss comodules over bialgebroids and then we summarize the basic properties of distributive double algebras, the structure that is always present in a Frobenius Hopf algebroid if a Frobenius integral is chosen.

1.1. Modules and comodules over bialgebroids. Let k be a commutative ring. We choose the category $\mathbf{M} = \mathbf{M}_k$ of k -modules as our base category. This means that all objects and morphisms we use have an underlying k -module or k -module morphism, respectively. In particular, algebras are always meant to be k -algebras and unadorned \otimes means tensor product in \mathbf{M}_k .

Let T be an algebra and let $T^e := T^{\text{op}} \otimes T$ be its enveloping algebra. A right bialgebroid over T consists of

- an algebra A
- a T^e -ring structure on A , i.e., an algebra morphism $t_r \otimes s_r : T^e \rightarrow A$
- and a T -coring structure $\langle A_{T^e}, \Delta_T, \varphi_T \rangle$

subject to axioms, see e.g. [15].

If A is a right bialgebroid over T then a right module over A is the same thing as a right module over the k -algebra A and a right A -module map is defined accordingly. The T^e -ring structure $T^e \rightarrow A$ endows the category \mathbf{M}_A of right A -modules with a (monadic) forgetful functor $U : \mathbf{M}_A \rightarrow {}_T\mathbf{M}_T$ by identifying \mathbf{M}_{T^e} with ${}_T\mathbf{M}_T$. The coring structure of A serves to make \mathbf{M}_A a monoidal category. The monoidal product of the A -modules V and W is the k -module $V \otimes_T W$ together with the right A -action $(v \otimes_T w) \triangleleft a := (v \triangleleft a^{(1)}) \otimes_T (w \triangleleft a^{(2)})$. In this way the forgetful functor U becomes strict monoidal.

Left bialgebroids and their category of left modules can be defined by passing to the opposite algebra in all occurrences of an algebra in the definition of a right bialgebroid and their right modules. So let B be an algebra which stands for T^{op} and let $B^e := B \otimes B^{\text{op}}$. Then a left bialgebroid over B consists of

- an algebra A
- a B^e -ring structure on A , i.e., an algebra morphism $s_l \otimes t_l : B^e \rightarrow A$
- and a B -coring structure $\langle {}_{B^e}A, \Delta_B, \varphi_B \rangle$.

The category of left A -modules has a monoidal product $V \otimes_B W$ such that the forgetful functor ${}_A V \mapsto {}_{B^e} V \equiv {}_B V_B$ is strict monoidal.

Right comodules can be defined for both left and right bialgebroids as follows. Let A be a right bialgebroid over T . Then a right A -comodule consists of

- a right T -module X
- a right T -module map $\delta : X \rightarrow X \otimes_T A$

such that

$$\begin{aligned} (\delta \otimes_T A) \circ \delta &= (X \otimes_T \Delta_T) \circ \delta \\ (X \otimes_T \varphi_T) &= X \end{aligned}$$

suppressing the coherence isomorphisms of ${}_T\mathbf{M}_T$. A morphism of comodules $\tau : \langle X, \gamma \rangle \rightarrow \langle Y, \delta \rangle$ is a right T -module map $\tau : X \rightarrow Y$ satisfying $(\tau \otimes_T A) \circ \gamma = \delta \circ \tau$.

The category of right A -comodules is denoted \mathbf{M}^A .

The above definition of comodules disguises the fact that \mathbf{M}^A is monoidal with a strict monoidal forgetful functor $\mathbf{M}^A \rightarrow {}_T\mathbf{M}_T$. Notice that although M is not a left T -module, $M \otimes_T A$ is by setting $t \cdot (x \otimes_T a) = x \otimes_T s_r(t)a$.

Proposition 1.1. *Let $\langle X, \delta \rangle$ be a right comodule over the right bialgebroid A . Then X has a unique left T -module structure such that δ is a left T -module map. With this left module structure*

- (1) X is a T - T -bimodule,
- (2) δ is a T - T -bimodule map,
- (3) $\delta(X) \subset X \times_T A$,
- (4) and every arrow $\tau \in \mathbf{M}^A$ is a T - T -bimodule map.

In (3) we used Takeuchi's \times -product which is defined by

$$X \times_T A := \left\{ \sum_i x_i \otimes_T a_i \in X \otimes_T A \mid \sum_i t \cdot x_i \otimes_T a_i = \sum_i x_i \otimes_T t_r(t)a_i \forall t \in T \right\}.$$

Proof. If X is a left T -module and δ is a left T -module map then

$$\begin{aligned} t \cdot x &= (t \cdot x)^{(0)} \cdot \varphi_T((t \cdot x)^{(1)}) \\ &= x^{(0)} \cdot \varphi_T(s_r(t)x^{(1)}) \end{aligned}$$

This proves uniqueness. If we use the above formula to define $t \cdot x$ then we find that it is a left action because $s_r : T \rightarrow A$ is an algebra homomorphism. It commutes with the right T -action

$$t \cdot (x \cdot t') = x^{(0)} \cdot \varphi_T(s_r(t)as_r(t')) = (t \cdot x) \cdot t'$$

so X is a T - T -bimodule and the coaction is a bimodule map,

$$\delta(t \cdot x \cdot t') = x^{(0)} \otimes_T s_r(t)x^{(1)}s_r(t').$$

Now the Takeuchi property (3) holds automatically,

$$\begin{aligned} t \cdot x^{(0)} \otimes_T x^{(1)} &= x^{(0)} \cdot \varphi_T(s_r(t)x^{(1)}) \otimes_T x^{(2)} \\ &= x^{(0)} \otimes_T \varphi_T(t_r(t)x^{(1)}) \cdot x^{(2)} \\ &= x^{(0)} \otimes_T t_r(t)x^{(1)}. \end{aligned}$$

If $\tau : X \rightarrow Y$ is a comodule morphism then

$$\begin{aligned} \tau(t \cdot x) &= \tau(x^{(0)}) \cdot \varphi_T(s_r(t)x^{(1)}) = \tau(x)^{(0)} \cdot \varphi_T(s_r(t)\tau(x)^{(1)}) \\ &= t \cdot \tau(x). \end{aligned}$$

□

The tensor product of right comodules X and Y can now be defined as $X \otimes_T Y$ with coaction

$$(1.1) \quad (x \otimes_T y)^{(0)} \otimes_T (x \otimes_T y)^{(1)} = (x^{(0)} \otimes_T y^{(0)}) \otimes_T x^{(1)} y^{(1)}.$$

This makes the category of right A -comodules \mathbf{M}^A monoidal and the forgetful functor $\mathbf{M}^A \rightarrow {}_T\mathbf{M}_T$ strict monoidal.

For left bialgebroids A over B a right comodule is an arrow $\delta_A : M \rightarrow M \otimes_B A \in \mathbf{M}_B$ satisfying coassociativity and counitality. A right comodule carries a left B -module structure such that δ_A is a B - B -bimodule map and such that \mathbf{M}^A is a monoidal category with strict monoidal forgetful functor to ${}_B\mathbf{M}_B$. The monoidal product of two right comodules X and Y has coaction

$$(1.2) \quad (x \otimes_B y)_{(0)} \otimes_B (x \otimes_B y)_{(1)} = (x_{(0)} \otimes_B y_{(0)}) \otimes_B y_{(1)} x_{(1)}.$$

Note the different order compared to (1.1).

1.2. Double algebras. A double algebra is a k -module A equipped with two associative unital multiplications: the vertical multiplication, denoted $a \circ a'$, has unit element e and the horizontal multiplication, denoted $a \star a'$, has unit element i . So we have the horizontal and vertical algebras $H = \langle A, \star, i \rangle$ and $V = \langle A, \circ, e \rangle$, respectively. The multiplications with the wrong unit, i.e.,

$$\begin{aligned} \varphi_L(a) &:= a \star e & \varphi_R(a) &:= e \star a \\ \varphi_B(a) &:= a \circ i & \varphi_T(a) &:= i \circ a \end{aligned}$$

map onto subalgebras L and R of V and B and T of H . Assuming for $X = L, R, B, T$ that the algebra extensions $X \subset A$ are Frobenius with Frobenius homomorphism φ_X we obtain the notion of Frobenius DA's. In this way A has Frobenius algebra structures in all the bimodule categories ${}_X\mathbf{M}_X$ for $X = L, R, B, T$ which implies four comultiplications

$$\begin{aligned} \langle A, \Delta_B, \varphi_B \rangle &\text{ is a comonoid in } {}_B\mathbf{M}_B, \text{ where } \Delta_B(a) \equiv a_{(1)} \otimes_B a_{(2)} = a \star u_k \otimes_B v_k, \\ \langle A, \Delta_L, \varphi_L \rangle &\text{ is a comonoid in } {}_L\mathbf{M}_L, \text{ where } \Delta_L(a) \equiv a_{[1]} \otimes_L a_{[2]} = a \circ x_j \otimes_L y_j, \\ \langle A, \Delta_T, \varphi_T \rangle &\text{ is a comonoid in } {}_T\mathbf{M}_T, \text{ where } \Delta_T(a) \equiv a^{(1)} \otimes_T a^{(2)} = a \star u^k \otimes_T v^k, \\ \langle A, \Delta_R, \varphi_R \rangle &\text{ is a comonoid in } {}_R\mathbf{M}_R, \text{ where } \Delta_R(a) \equiv a^{[1]} \otimes_R a^{[2]} = a \circ x^j \otimes_R y^j. \end{aligned}$$

where note the special notation for the dual bases of the base homomorphisms φ_X . It turns out [25, Proposition 3.2] that vertical multiplication with the horizontal type of comultiplications Δ_B and Δ_T obey bialgebroid like relations. However, if we also postulate the distributivity rules

$$(1.3) \quad a \circ (a' \star a'') = (a_{(1)} \circ a') \star (a_{(2)} \circ a'')$$

$$(1.4) \quad a \star (a' \circ a'') = (a_{[1]} \star a') \circ (a_{[2]} \star a'')$$

$$(1.5) \quad (a' \star a'') \circ a = (a' \circ a^{(1)}) \star (a'' \circ a^{(2)})$$

$$(1.6) \quad (a' \circ a'') \star a = (a' \star a^{[1]}) \circ (a'' \star a^{[2]})$$

in which case we say that $\langle A, \circ, e, \star, i \rangle$ is a distributive double algebra (DDA), then V and H become Hopf algebroids [3] in duality. The underlying left bialgebroids are

$$\langle V, B, \varphi_L|_B, \varphi_R|_B, \Delta_B, \varphi_B \rangle \quad \text{and} \quad \langle H, L, \varphi_B|_L, \varphi_T|_L, \Delta_L, \varphi_L \rangle$$

and the right bialgebroids are

$$\langle V, T, \varphi_R|_T, \varphi_L|_T, \Delta_T, \varphi_T \rangle \quad \text{and} \quad \langle H, R, \varphi_T|_R, \varphi_B|_R, \Delta_R, \varphi_R \rangle$$

The notation means e.g. that V over T has source map $s_r : t \mapsto \varphi_R(t)$, target map $t_r : t \mapsto \varphi_L(t)$ and counit φ_T . Or, H over R has source map $s_r : r \mapsto \varphi_T(r)$, target map $t_r : r \mapsto \varphi_B(r)$, and counit φ_R . The antipode of V – called the antipode of the double algebra – is an antiautomorphism S which is also an antiautomorphism of H but the antipode of H is S^{-1} . (There is a regrettable mistake in [25, Theorem 7.4] where H was claimed to have antipode also S ; see [arXiv: math.QA/0402151 v2](#) for the corrected version.) The vertical Hopf algebroid has Frobenius integral i and H has e .

2. MODULES AND COMODULES OVER DDA'S

2.1. Modules. Let $\langle A, \circ, e, \star, i \rangle$ be a double algebra. A *right A -module* is a k -module M together with an associative unital action $M \otimes_R H \rightarrow M$ of the horizontal algebra $H = \langle A, \star, i \rangle$ denoted $m \otimes_R h \mapsto m \triangleleft h$.

Equivalently, a right A -module can be formulated in the category $\mathbf{M}_{B \otimes T}$ as an object $M_{B \otimes T}$ and an arrow $M \otimes_{B \otimes T} A \rightarrow M$ satisfying associativity and unitality w.r.t the algebra H in ${}_{B \otimes T} \mathbf{M}_{B \otimes T}$. The T and B -actions are denoted by $m \cdot t$ and $m \cdot b$, respectively.

Analogously one can define *left A -modules* as left modules over H and *bottom* and *top A -modules* as "left", respectively "right", modules over the vertical algebra $V = \langle A, \circ, e \rangle$.

2.2. Comodules. A *right A -comodule* over a Frobenius double algebra consists of an object M and two arrows $\delta_M : M \rightarrow M \otimes_B A$, $\delta^M : M \rightarrow M \otimes_T A$ in $\mathbf{M}_{B \otimes T}$ such that

- $\langle M_B, \delta_M \rangle$ is a right comodule over the left bialgebroid V over B ,
- $\langle M_T, \delta^M \rangle$ is a right comodule over the right bialgebroid V over T
- and the two coactions satisfy the mixed coassociativity conditions

$$(2.1) \quad m^{(0)}_{(0)} \otimes_B m^{(0)}_{(1)} \otimes_T m^{(1)} = m^{(0)} \otimes_B m^{(1)}_{(1)} \otimes_T m^{(1)}_{(2)}$$

$$(2.2) \quad m^{(0)}_{(0)} \otimes_T m^{(0)}_{(1)} \otimes_B m^{(1)} = m^{(0)} \otimes_T m^{(1)}_{(1)} \otimes_B m^{(1)}_{(2)}$$

where we used the notation

$$\delta_M(m) = m^{(0)} \otimes_B m^{(1)}$$

$$\delta^M(m) = m^{(0)} \otimes_T m^{(1)}$$

for $m \in M$.

A *right A -comodule morphism* $\tau : X \rightarrow Y$ is a right $B \otimes T$ -module map which is a right comodule morphism for both the left bialgebroid V_B and the right bialgebroid V_T . The category of right A -comodules is denoted by \mathbf{M}^V . The occurrence of two compatible coactions in the definition of an A -comodule is precisely what we need to identify \mathbf{M}^V and \mathbf{M}_H in case of DDA's.

Lemma 2.1. *Let A be a DDA and let δ_M and δ^M be two coactions of V_B , respectively V_T , on M . They then determine two right H -actions on M ,*

$$(2.3) \quad m \underset{B}{\triangleleft} h = m_{(0)} \cdot \varphi_B(m_{(1)} \star h)$$

$$(2.4) \quad m \underset{T}{\triangleleft} h = m^{(0)} \cdot \varphi_T(m^{(1)} \star h).$$

The two actions coincide if and only if the two coactions satisfy the mixed coassociativity condition (2.1) and (2.2).

Proof. The inverses of (2.3) and (2.4) can be given in terms of the dual bases of φ_B and φ_T as

$$(2.5) \quad m^{(0)} \underset{T}{\otimes} m^{(1)} = m \underset{T}{\triangleleft} u^k \underset{T}{\otimes} v^k$$

$$(2.6) \quad m_{(0)} \underset{B}{\otimes} m_{(1)} = m \underset{B}{\triangleleft} u_k \underset{T}{\otimes} v_k$$

Therefore if $\underset{B}{\triangleleft} = \underset{T}{\triangleleft}$ then

$$\begin{aligned} m^{(0)}_{(0)} \underset{B}{\otimes} m^{(0)}_{(1)} \underset{T}{\otimes} m^{(1)} &= (m \underset{T}{\triangleleft} u^k) \underset{B}{\triangleleft} u_l \underset{B}{\otimes} v_l \underset{T}{\otimes} v^k = m \underset{B}{\triangleleft} (u^k \star u_l) \underset{B}{\otimes} v_l \underset{T}{\otimes} v^k \\ &= m \underset{B}{\triangleleft} u_l \underset{B}{\otimes} v_l \star u^k \underset{T}{\otimes} v^k = m_{(0)} \underset{B}{\otimes} m_{(1)}^{(1)} \underset{T}{\otimes} m_{(1)}^{(2)} \end{aligned}$$

and similarly for (2.2). On the other hand, if mixed coassociativity holds then

$$\begin{aligned} m \underset{T}{\triangleleft} h &= (m \underset{T}{\triangleleft} h)_{(0)} \cdot \varphi_B((m \underset{T}{\triangleleft} h)_{(1)}) = m^{(0)}_{(0)} \cdot \varphi_B(m^{(0)}_{(1)} \star \varphi_T(m^{(1)} \star h)) \\ &= m_{(0)} \cdot \varphi_B(m_{(1)}^{(1)} \star \varphi_T(m_{(1)}^{(2)} \star h)) = m_{(0)} \cdot \varphi_B(m_{(1)} \star h) \\ &= m \underset{B}{\triangleleft} h \end{aligned}$$

□

If M is a right module over the DDA A then it is a right V -comodule M^V and a right H -module M_H at the same time. The invariants of M_H ,

$$(2.7) \quad M^H := \{n \in M \mid n \underset{B}{\triangleleft} h = n \underset{B}{\triangleleft} \varphi_T \varphi_R(h), \quad h \in H\}$$

$$(2.8) \quad = \{n \in M \mid n \underset{B}{\triangleleft} h = n \underset{B}{\triangleleft} \varphi_B \varphi_R(h), \quad h \in H\}$$

and the coinvariants of M^V ,

$$(2.9) \quad \begin{aligned} M^{\text{co-}V} &:= \{n \in M \mid n^{(0)} \underset{T}{\otimes} n^{(1)} = n \underset{T}{\otimes} e\} \\ &= \{n \in M \mid n_{(0)} \underset{B}{\otimes} n_{(1)} = n \underset{B}{\otimes} e\}, \end{aligned}$$

yield one and the same k -submodule of M . This is an instance of the more general identification between the categories of H -modules, V_B -comodules, and V_T -comodules. Since φ_T and φ_B restrict to algebra isomorphisms $R \rightarrow T$ and $R^{\text{op}} \rightarrow B$, respectively [25, Lemma 2.2], the identifications between H -modules and V -comodules provide a monoidal category isomorphism $\mathbf{M}^{V_T} \cong \mathbf{M}_H$ and the antimonoidal category isomorphism $\mathbf{M}^{V_B} \cong \mathbf{M}_H$. We can use these isomorphisms to introduce $\underset{R}{\otimes}$ both in \mathbf{M}^{V_T} and \mathbf{M}^{V_B} as the monoidal product while keeping $\underset{T}{\otimes}$ and $\underset{B}{\otimes}$ to appear in the coactions. One advantage of this convention is that the difference between (1.2) and (1.1) disappears, viz. (2.10) and (2.11). Now the R becomes a monoidal unit in three senses: As a right ideal in H it is the trivial right H -module,

$r \triangleleft h = r \star h$. But it is also a right comodule over V_T via $r^{(0)} \otimes_T r^{(1)} = \epsilon \otimes_T r$ and a right comodule over V_B via $r_{(0)} \otimes_B r_{(1)} = \epsilon \otimes_B r$.

2.3. Module (co)algebras. Comodule algebras over V are monoids in \mathbf{M}^V and therefore they are the same as monoids in \mathbf{M}_H , i.e., module algebras over H .

Hence a right H -module algebra M consists of an algebra map $\eta : R \rightarrow M$ inducing the bimodule structure ${}_R M_R$ and a bimodule map $\mu : M \otimes_R M \rightarrow M$, $m \otimes_R m' \mapsto mm'$, satisfying $(mm') \triangleleft h = (m \triangleleft h^{[1]})(m \triangleleft h^{[2]})$. In the language of the V -coactions (2.5), (2.6) these correspond to the right comodule algebra relations

$$(2.10) \quad (mm')^{(0)} \otimes_T (mm')^{(1)} = m^{(0)} m'^{(0)} \otimes_T m^{(1)} \circ m'^{(1)} \quad 1^{(0)} \otimes_T 1^{(1)} = 1 \otimes_T \epsilon$$

$$(2.11) \quad (mm')_{(0)} \otimes_B (mm')_{(1)} = m_{(0)} m'_{(0)} \otimes_B m_{(1)} \circ m'_{(1)} \quad 1_{(0)} \otimes_B 1_{(1)} = 1 \otimes_B \epsilon$$

respectively. Just as in the case of Hopf algebras the invariants of a module algebra form a subalgebra. More precisely we have the following

Lemma 2.2. *For any right H -module M there is a unique k -module map $\text{Hom}_H(R, M) \rightarrow M^H$ that makes the diagram*

$$\begin{array}{ccc} \text{Hom}_H(R, M) & \xrightarrow{\text{Hom}(\varphi_R, M)} & \text{Hom}_H(H, M) \\ \downarrow & & \downarrow f \mapsto f(i) \\ M^H & \xrightarrow{\subset} & M \end{array}$$

commutative. This k -module map is an isomorphism. If M_H is a module algebra then the diagram is in the category of k -algebras. In particular, $M^H \subset M$ is a subalgebra which is isomorphic to the convolution algebra $\text{Hom}_H(R, M)$.

The smash product $H \# M$ for a right H -module algebra is defined to be to k -module $H \otimes_R M$ equipped with multiplication

$$(2.12) \quad (h \# m)(h' \# m') = h \star h'^{[1]} \# (m \triangleleft h'^{[2]}) m'$$

and unit element $i \# 1$.

Next we consider extensions. Let $N \rightarrow M^H \subset M$ be an algebra map. Then we have left actions λ of N and λ of M^H on M . Denoting $\mathcal{E} := \text{End}({}_N M)$ we have an algebra map $H \# M \rightarrow \mathcal{E}$ by

$$(2.13) \quad m' \cdot (h \# m) := (m' \triangleleft h) m$$

so that M becomes an N - $(H \# M)$ -bimodule. We have the inclusions

$$(2.14) \quad \lambda(N) \subset \text{End}_{\mathcal{E}}(M) \subset \text{End}_{H \# M}(M) = \lambda(M^H)$$

where the last equality can be proven exactly as in the Hopf algebra case [18, 8.3.2].

Definition 2.3. An algebra homomorphism $\eta : N \rightarrow M$ is called a right A -extension for some DDA A if M is a right module algebra over A and η factorizes through $M^H \subset M$ via an algebra isomorphism $N \xrightarrow{\sim} M^H$.

Later on an A -extension will be meant in the narrower sense that $N = M^H$ but sometimes, as in Section 5 we need this more categorical definition.

Lemma 2.4. *Let A be a DDA and $N \rightarrow M$ be a right A -extension. Then*

- (1) ${}_N M$ is balanced, i.e., $\text{BiEnd}({}_N M) = \lambda(N)$ and
- (2) ${}_N M_{H\#M}$ is faithfully balanced iff the canonical map $H\#M \rightarrow \mathcal{E}$ given by (2.13) is an isomorphism.

Proof. Both statements are immediate consequences of the fact that all the inclusions in (2.14) reduce to equalities in case of A -extensions. \square

3. GALOIS EXTENSIONS

3.1. The coaction picture. Let M be a right comodule algebra over the Hopf algebroid V and let $N := M^{\text{co-}V}$. Then the maps

$$(3.1) \quad \gamma^M : M \otimes_N M \rightarrow M \otimes_T V, \quad m \otimes_N m' \mapsto mm'^{(0)} \otimes_T m'^{(1)}$$

$$(3.2) \quad \gamma_M : M \otimes_N M \rightarrow M \otimes_B V, \quad m \otimes_N m' \mapsto m_{(0)}m' \otimes_B m_{(1)}$$

are M - M -bimodule maps if we endow $M \otimes_T V$ and $M \otimes_B V$ with the structure

$$(3.3) \quad m' \cdot (m \otimes_T v) \cdot m'' = m' mm''^{(0)} \otimes_T v \circ m''^{(1)},$$

$$(3.4) \quad m' \cdot (m \otimes_B v) \cdot m'' = m'_{(0)} mm'' \otimes_B m'_{(1)} \circ v,$$

respectively. They are also right V -comodule maps, i.e., belong to \mathbf{M}^V , because they can be written as composites of μ_M and δ^M , respectively μ_M and δ_M .

Lemma 3.1. *Let M be a right V -comodule algebra over the Hopf algebroid V . Then γ^M is epimorphism iff γ_M is and γ^M is isomorphism iff γ_M is.*

Proof. Let ϕ denote the composite

$$(3.5) \quad M \otimes_T V \xrightarrow{\delta_M \otimes V} M \otimes_B V \otimes_T V \xrightarrow{M \otimes_B V \otimes S} M \otimes_B V \otimes_R V \xrightarrow{M \otimes_B \mu_V} M \otimes_B V$$

$$m \otimes_T v \mapsto m_{(0)} \otimes_B m_{(1)} S(v)$$

where S is the antipode of the Hopf algebroid V . Then ϕ has inverse

$$\phi^{-1}(m \otimes_B v) = m^{(0)} \otimes_T S^{-1}(v) m^{(1)}.$$

and one obtains that $\phi \circ \gamma^M = \gamma_M$. \square

The next result is an immediate generalization of [18, Theorem 8.3.1].

Proposition 3.2. *Assume that V is a Frobenius Hopf algebroid and M is a right V -comodule algebra with coinvariant subalgebra N . Then γ^M being epi implies that γ^M is an isomorphism and M_N is finitely generated projective.*

Proof. Let V and H be the vertical and horizontal Hopf algebroid of a distributive double algebra $\langle A, \circ, e, \star, i \rangle$. Then M is a right H -module algebra and e , the unit of V , is an integral for H , therefore $m \triangleleft e \in N$, $m \in M$. By the hypothesis there exists $\sum_j m_j \otimes_N m'_j \in M \otimes_N M$ such that

$$\sum_j m_j m'_j{}^{(0)} \otimes_T m'_j{}^{(1)} = 1 \otimes_T i.$$

Therefore we can write for arbitrary $m \in M$ that

$$\begin{aligned}
 \sum_j m_j((m'_j m) \triangleleft e) &= \sum_j m_j(m'_j \triangleleft e^{[1]})(m \triangleleft e^{[2]}) \\
 &= \sum_j m_j \left(m'^{(0)}_j \cdot \varphi_T(m'^{(1)}_j \star e^{[1]}) \right) (m \triangleleft e^{[2]}) \\
 &= (1 \triangleleft (i \circ e^{[1]}))(m \triangleleft e^{[2]}) = (1 \triangleleft i^{[1]})(m \triangleleft i^{[2]}) \\
 &= m \triangleleft i = m
 \end{aligned}$$

proving that $(m'_j -) \triangleleft e$ is a dual basis of m_j for M_N , thus M_N is fgp.

Next we show that γ_M is mono. Suppose $\sum_i z_i \otimes_N w_i \in \text{Ker } \gamma_M$. Then

$$\sum_i z_{i(0)} w_i \otimes_B z_{i(1)} = 0.$$

Using the dual bases for M_N we find that

$$\begin{aligned}
 \sum_i z_i \otimes_N w_i &= \sum_i \sum_j m_j((m'_j z_i) \triangleleft e) \otimes_N w_i \\
 &= \sum_j m_j \otimes_N \sum_i \left(m'_{j(0)} z_{i(0)} \cdot \varphi_B((m'_{j(1)} \circ z_{i(1)}) \star e) \right) w_i \\
 &= \sum_j m_j \otimes_N \sum_i m'_{j(0)} z_{i(0)} w_i \cdot \varphi_B(m'_{j(1)} \circ z_{i(1)}) \\
 &= 0.
 \end{aligned}$$

Therefore γ_M is mono. But it is also epi because γ^M is. Therefore γ_M is iso, and so is γ^M . \square

3.2. The action picture. For a right bialgebroid H over R and an H -module algebra M there are canonical maps

$$(3.6) \quad \Gamma^M : M \otimes_R H \rightarrow \text{End}(M_N) \quad m \otimes_R h \mapsto \{m' \mapsto m(m' \triangleleft h)\}$$

$$(3.7) \quad \Gamma_M : H \otimes_R M \rightarrow \text{End}({}_N M) \quad h \otimes_R m \mapsto \{m' \mapsto (m' \triangleleft h)m\}$$

being algebra maps from the smash products $M \# H^{\text{op}}$ and $H \# M$, respectively, where in the latter case $\text{End}({}_N M)$ is considered with multiplication that arises from its natural right action on M .

Note that if H is the horizontal Hopf algebroid of a DDA and the right H action arises from a right V -coaction as in Lemma 2.1 then M being a left H^{op} -module algebra is in complete agreement with the familiar Hopf algebraic situation since it is H^{op} which is the dual of V .

Theorem & Definition 3.3. *Let A be a distributive double algebra and M a right H -module algebra, equivalently a right V -comodule algebra, over the horizontal, resp. vertical Hopf algebroid of A . Let $N = M^H \cong M^{\text{co-}V}$. Then $N \subset M$ is called an A -Galois extension if any one of the following equivalent conditions hold:*

- (1) γ^M is epi.
- (2) γ_M is epi.
- (3) γ^M is iso.
- (4) γ_M is iso.
- (5) Γ^M is iso and M_N is fgp.
- (6) Γ_M is iso and ${}_N M$ is fgp.

Proof. Equivalence of the first four conditions follows from Proposition 3.2 and Lemma 3.1.

(3) \Rightarrow (5) Considering it as a right M -module map, γ^M induces the isomorphism (of left M -modules)

$$\gamma^{M*} : \text{Hom}_{-M}(M \otimes_T V, M) \xrightarrow{\sim} \text{Hom}_{-M}(M \otimes_N M, M).$$

If $\chi \in \text{Hom}_{-M}(M \otimes_T V, M)$ then $\chi(1 \otimes_T -) \in \text{Hom}(V_T, M_T)$ because

$$\begin{aligned} \chi(1 \otimes_T v \star t) &= \chi(1 \otimes_T v \circ \varphi_R(t)) \\ &= \chi\left(j(\varphi_R(t))^{(0)} \otimes_T v \circ j(\varphi_R(t))^{(1)}\right) \\ &= \chi(1 \otimes_T v)j(\varphi_R(t)). \end{aligned}$$

Thus we have a well defined map (of left M -modules)

$$(3.8) \quad \begin{aligned} \text{Hom}_{-M}(M \otimes_T V, M) &\rightarrow \text{Hom}(V_T, M_T) \\ \chi &\mapsto \chi(1 \otimes_T -) \end{aligned}$$

We claim that this map is an isomorphism with inverse

$$\kappa \mapsto \{m \otimes_T v \mapsto \kappa(v \circ S^{-1}(m_{(1)}))m_{(0)}\}$$

This follows from the computation

$$\begin{aligned} \chi(m \otimes_T v) &= \chi(m^{(0)} \otimes_T \varphi_T(m^{(1)}) \star v) = \chi(m^{(0)} \otimes_T v \circ \varphi_L \varphi_T(m^{(1)})) \\ &= \chi(m^{(0)} \otimes_T v \circ S^{-1}(m^{(1)}_{(2)}) \circ m^{(1)}_{(1)}) \\ &= \chi(m_{(0)}^{(0)} \otimes_T v \circ S^{-1}(m_{(1)}) \circ m_{(0)}^{(1)}) \\ &= \chi(1 \otimes_T v \circ S^{-1}(m_{(1)})) m_{(0)} \end{aligned}$$

on the one hand and on the other hand from $\delta_M(1) = 1 \otimes_B e$. Composing the map

(3.8) with the isomorphism

$$(3.9) \quad \begin{aligned} \text{Hom}(V_T, M_T) &\rightarrow M \otimes_R H \\ \kappa &\mapsto \kappa(x^j) \otimes_R y^j \end{aligned}$$

where $x^j \otimes_R y^j \equiv \Delta_R(e)$ is the dual basis of φ_R , we obtain the left vertical arrow in the diagram

$$(3.10) \quad \begin{array}{ccc} \text{Hom}_{-M}(M \otimes_T V, M) & \xrightarrow{\gamma^{M*}} & \text{Hom}_{-M}(M \otimes_N M, M) \\ \downarrow & & \downarrow \\ M \otimes_R H & \xrightarrow{\Gamma^M} & \text{End}(M_N) \end{array}$$

The vertical arrow on the right is the isomorphism $\sigma \mapsto \sigma(- \otimes_N 1)$ therefore the composite along the top and right is $\chi \mapsto \chi(- \otimes_T e)$. The other two compose to give

$$\chi \mapsto \chi(1 \otimes_T x^j) \otimes_R y^j \mapsto \chi(1 \otimes_T x^j)(- \triangleleft y^j).$$

In order to see commutativity of the diagram we need a calculation.

$$\begin{aligned}
 \chi(1 \otimes_T x^j)(m \triangleleft y^j) &= \chi(1 \otimes_T x^j) m^{(0)} \cdot \varphi_T(m^{(1)} \star y^j) \\
 &= \chi(m^{(0)} \otimes_T x^j \circ m^{(1)} \circ \varphi_R \varphi_T(m^{(2)} \star y^j)) \\
 &= \chi(m^{(0)} \otimes_T x^j \circ (m^{(1)} \star \varphi_T(m^{(2)} \star y^j))) \\
 &= \chi(m^{(0)} \otimes_T x^j \circ (m^{(1)} \star y^j)) \\
 &= \chi(m^{(0)} \otimes_T \varphi_L \varphi_T(m^{(1)})) \\
 &= \chi(m^{(0)} \cdot \varphi_T(m^{(1)}) \otimes_T e) = \chi(m \otimes_T e),
 \end{aligned}$$

where in the fifth equality we used [25, Equation (4.16)]. So (3.10) is commutative and therefore Γ^M is an isomorphism.

The proof of (4) \Rightarrow (6) goes similarly by proving commutativity of the diagram

$$\begin{array}{ccc}
 \text{Hom}_{M-}(M \otimes_B V, M) & \xrightarrow{\gamma_M^*} & \text{Hom}_{M-}(M \otimes_N M, M) \\
 \downarrow & & \downarrow \\
 H \otimes_R M & \xrightarrow{\Gamma_M} & \text{End}({}_N M)
 \end{array} \tag{3.11}$$

with the left hand side arrow being the isomorphism $\chi \mapsto x^j \otimes_R \chi(1 \otimes_B y^j)$ and the one on the right hand side being $\sigma \mapsto \sigma(1 \otimes_N -)$.

(5) \Rightarrow (4) Consider the diagram

$$\begin{array}{ccc}
 M \otimes_N M & \xrightarrow{\gamma_M} & M \otimes_B V \\
 \downarrow & & \uparrow \\
 \text{Hom}_{M-}(\text{End}(M_N), M) & \xrightarrow{\Gamma_M^*} & \text{Hom}_{M-}(M \otimes_R H, M)
 \end{array} \tag{3.12}$$

The lower horizontal arrow is an isomorphism since Γ^M is. The vertical arrow on the left, mapping $m \otimes_N m'$ to the homomorphism $\alpha \mapsto \alpha(m)m'$, is an isomorphism because M_N is fgp. The other vertical arrow is the composite of two maps,

$$\text{Hom}_{M-}(M \otimes_R H, M) \longrightarrow \text{Hom}({}_R H, {}_R M) \longrightarrow M \otimes_B V$$

where the second one is the isomorphism $\kappa \mapsto \kappa(u^k) \otimes_B v^k$ with $u^k \otimes_B v^k \equiv \Delta_B(i)$ denoting the dual basis of φ_B . The first one, $\chi \mapsto \chi(1 \otimes_R -)$, is obviously invertible (in contrast to the similar map in the (3) \Rightarrow (5) part) because the left M -module structure of $M \otimes_R H$ we need here is the trivial one. It remains to show commutativity of (3.12). So we compute the action of the lower three arrows,

$$\begin{aligned}
 m \otimes_N m' &\mapsto \{\alpha \mapsto \alpha(m)m'\} \mapsto \{m'' \otimes_R h \mapsto m''(m \triangleleft h)m'\} \\
 &\mapsto \{h \mapsto (m \triangleleft h)m'\} \mapsto (m \triangleleft u^k)m' \otimes_B v^k
 \end{aligned}$$

which is indeed γ_M if we compare the right H -action with the right V -coaction δ_M . This proves that γ_M is invertible.

The proof of the implication (6) \Rightarrow (3) can be done similarly by using the diagram

$$(3.13) \quad \begin{array}{ccc} M \otimes_N M & \xrightarrow{\gamma^M} & M \otimes_T V \\ \downarrow & & \uparrow \\ \mathrm{Hom}_{-M}(\mathrm{End}({}_N M), M) & \xrightarrow{\Gamma_M^*} & \mathrm{Hom}_{-M}(H \otimes_R M, M) \end{array}$$

where on the left hand side we have the map $m \otimes m' \mapsto \{\alpha \mapsto m\alpha(m')\}$ which is an isomorphism because ${}_N M$ is fgp. \square

Remark 3.4. The terminology "right A -Galois extension" where A is a distributive double algebra does not, by any means, imply that the choice of the integral i in the vertical Hopf algebroid V plays any role. This is clear from the coaction picture that uses γ^M alone. Therefore we might as well call it "right V -Galois extensions" which would then be in complete agreement with the Hopf-Galois terminology. Saying " A -Galois" we try to put the coaction and action pictures on equal footing. For example, "bottom A -Galois" and "top A -Galois" extensions correspond to the left and right H -Galois extensions in the Hopf-Galois language if H denotes the horizontal Hopf algebroid of A .

3.3. Weak and strong structure theorems. For a Frobenius Hopf algebroid A let M be a right H -module algebra and $N = M^H$. The category $(\mathbf{M}_H)_M$ of right M -modules in \mathbf{M}_H (by the identification $\mathbf{M}_H = \mathbf{M}^V$ being the analogue of relative Hopf modules) is nothing but the category of right modules over the smash product,

$$(3.14) \quad (\mathbf{M}_H)_M = \mathbf{M}_{H\#M}.$$

Indeed, for any action $X \otimes_R M \rightarrow X$, $x \otimes m \mapsto x \cdot m$ in \mathbf{M}_H one has the smash product action

$$X \otimes (H\#M) \rightarrow X, \quad x \otimes (h\#m) \mapsto (x \triangleleft h) \cdot m.$$

Vice versa, any $H\#M$ -module is an H -module and an M -module and the M -action is an H -module map. Considering M as an N - $H\#M$ bimodule, it defines an adjoint pair $F \dashv U$ of functors

$$\begin{aligned} F : \mathbf{M}_N &\rightarrow \mathbf{M}_{H\#M} & X &\mapsto X \otimes_N M \\ U : \mathbf{M}_{H\#M} &\rightarrow \mathbf{M}_N & Y &\mapsto \mathrm{Hom}_{H\#M}(M, Y) \end{aligned}$$

with counit and unit

$$\begin{aligned} \zeta_Y : \mathrm{Hom}_{H\#M}(M, Y) \otimes_N M &\rightarrow M & \chi \otimes_N m &\mapsto \chi(m) \\ \vartheta_X : X &\rightarrow \mathrm{Hom}_{H\#M}(M, X \otimes_N M) & x &\mapsto \{m \mapsto x \otimes_N m\} \end{aligned}$$

We note that UY is isomorphic to the submodule of invariants via

$$(3.15) \quad \mathrm{Hom}_{H\#M}(M, Y) \xrightarrow{\sim} \mathrm{Hom}_H(R, Y) \xrightarrow{\sim} Y^H.$$

Lemma 3.5. *For any A -extension $N \subset M$*

- (1) *if $M_{H\#M}$ is fgp then ϑ is invertible,*
- (2) *if ${}_N M$ is fgp and Γ_M is invertible then ζ is invertible.*

Proof. (1) Apply [1, 20.10] to the last arrow in the decomposition of ϑ_X

$$X \xrightarrow{\sim} X \otimes_N M \xrightarrow{\sim} X \otimes_N \text{Hom}_{H\#M}(M, M) \longrightarrow \text{Hom}_{H\#M}(M, X \otimes_N M)$$

(2) Apply [1, 20.11] to the first arrow in the decomposition of ζ_Y

$$\begin{array}{c} \text{Hom}_{H\#M}(M, Y) \otimes_N M \\ \downarrow \\ \text{Hom}_{H\#M}(\text{Hom}_{N-}(M, M), Y) \xrightarrow{\text{Hom}(\Gamma_M, Y)} \text{Hom}_{H\#M}(H\#M, Y) \xrightarrow{\sim} Y \end{array}$$

□

Theorem 3.6. *Let A be a distributive double algebra.*

- (1) *For an A -extension $N \subset M$ the following conditions are equivalent:*
 - (a) $\zeta : FU \rightarrow M_{H\#M}$ is an isomorphism.
 - (b) $N \subset M$ is A -Galois.
 - (c) ${}_N M$ is fgp and ${}_N M_{H\#M}$ is faithfully balanced.
 - (d) $M_{H\#M}$ is a generator.
- (2) *For an A -Galois extension $N \subset M$ the following conditions are equivalent:*
 - (a) $F \dashv U$ is an adjoint equivalence.
 - (b) $\vartheta : {}_N M \rightarrow UF$ is an isomorphism.
 - (c) ${}_N M_{H\#M}$ is a Morita equivalence bimodule.
 - (d) $M_{H\#M}$ is fgp.
 - (e) ${}_N M$ is a generator.
 - (f) ${}_N N \subset {}_N M$ is a direct summand.

Proof. (1a) \Leftrightarrow (1b): The \Leftarrow follows from Lemma 3.5 (2). As for the \Rightarrow direction consider ζ_Y for $Y = M \otimes_T V$ which is a $H\#M$ -module via

$$(m' \otimes v) \cdot (h\#m) := m' m^{(0)} \otimes (v \star h) \circ m^{(1)}.$$

This is a well-defined action due to

$$(m \triangleleft h)^{(0)} \otimes (m \triangleleft h)^{(1)} = m^{(0)} \otimes m^{(1)} \star h.$$

Now consider the map

$$(3.16) \quad Y^H \rightarrow M \quad \sum_j m_j \otimes_T w_j \mapsto \sum_j m_j \cdot \varphi_T(w_j)$$

which has inverse $m \mapsto m \otimes_T e$. As a matter of fact,

$$\begin{aligned} m \cdot \varphi_T(e) &= m \eta \varphi_R \varphi_T(e) = m \eta(e) = m \\ \sum_j m_j \cdot \varphi_T(w_j) \otimes_T e &= \sum_j m_j \otimes_T \varphi_L \varphi_T(w_j) = \sum_j m_j \otimes_T (i \circ w_j) \star e \\ &= \sum_j m_j \otimes_T (i \star e^{[1]}) \circ (w_j \star e^{[2]}) \\ &= \sum_j m_j \otimes_T e^{[1]} \circ (w_j \star \varphi_B \varphi_R(e^{[2]})) \\ &= \sum_j m_j \otimes_T e^{[1]} \circ \varphi_R(e^{[2]}) \circ w_j = \sum_j m_j \otimes_T w_j. \end{aligned}$$

Composing ζ_Y with the inverses of (3.15) and (3.16) we obtain the mapping

$$m \otimes_N m' \mapsto (m \otimes_T \epsilon) \otimes_N m' \mapsto (m \otimes_T \epsilon) \cdot m' = mm'^{(0)} \otimes_T m'^{(1)} = \gamma^M(m \otimes_N m')$$

Therefore γ^M is invertible.

(1b) \Leftrightarrow (1c): This is Lemma 2.4 (2) together with the Theorem 3.3 (6).

(1c) \Leftrightarrow (1d): Since $N \subset M$ is an extension, ${}_N M$ is balanced by Lemma 2.4 (1). So (c) is equivalent to that ${}_N M$ is fgp and $M_{H\#M}$ is faithful and balanced. But these are the necessary and sufficient conditions for (d) by [1, Theorem 17.8].

(2a) \Leftrightarrow (2b): This is clear from the equivalence of (1a) and (1b).

(2b) \Leftrightarrow (2c): Consider the composite

$$\begin{array}{ccc} \text{Hom}_N(N, X) & & \text{Hom}_N(M \underset{H\#M}{\otimes} \text{Hom}({}_N M, {}_N N), X) \\ \downarrow \iota & & \uparrow \iota \\ X & \xrightarrow{\vartheta} & \text{Hom}_{H\#M}(M, X \underset{N}{\otimes} M) \end{array}$$

of natural isomorphisms where the last isomorphism exists because ${}_N M$ is fgp. By the Yoneda lemma this determines an isomorphism

$$M \underset{H\#M}{\otimes} \text{Hom}({}_N M, {}_N N) \rightarrow N \in {}_N M$$

which is nothing but the evaluation associated to the right dual of the bimodule ${}_N M_{H\#M}$. Postulating the usual right N -module structure on $\text{Hom}({}_N M, {}_N N)$ it becomes in fact an N - N -bimodule isomorphism. Another hom-tensor relation for fgp ${}_N M$ and the isomorphism Γ_M compose to give

$$\text{Hom}({}_N M, {}_N N) \underset{N}{\otimes} M \xrightarrow{\sim} \text{Hom}({}_N M, {}_N M) \xrightarrow{\sim} H\#M \in {}_{H\#M} M_{H\#M}.$$

Thus $\text{Hom}({}_N M, {}_N N)$ is the inverse equivalence of ${}_N M_{H\#M}$. It follows from Morita theory that both ${}_N M$ and $M_{H\#M}$ are progenerators which prove that (2c) \Rightarrow (2d) and (2c) \Rightarrow (2e).

(2d) \Leftrightarrow (2b) follows from Lemma 3.5 (1).

In order to show (2e) \Rightarrow (2f) we use that an N -module M is a generator iff a finite direct sum of M 's contains the regular object as a summand, i.e., there exist N -module maps $N \xrightarrow{\iota_k} M \xrightarrow{\pi_k} N$ such that $\sum_k \pi_k \circ \iota_k = \text{id}_N$. In this case $\{m \mapsto \sum_k \pi_k(m \iota_k(1))\} \in \text{Hom}({}_N M, {}_N N)$ splits the inclusion $N \subset M$. The implication (2f) \Rightarrow (2e) is now obvious.

Finally (2e) \Rightarrow (2d) follows from that ${}_N M_{H\#M}$ is faithfully balanced by Lemma 2.4. \square

The second part of the Theorem has also a formulation in terms of Doi's total integral. By the isomorphism $M^V \cong M_H$ a total integral is an H -module map $\phi : A \rightarrow M$ such that $\phi(e) = 1$. By cyclicity of A_H such ϕ 's are uniquely determined by the "total element" $m = \phi(i) \in M$ satisfying $m \triangleleft e = 1$. As we shall see in the next subsection the map $\psi : M \rightarrow N$, $m \mapsto m \triangleleft e$ is a Frobenius homomorphism. Therefore one can extend the list of equivalent conditions in part (2) of Theorem 3.6 with two more:

- (g) There is a total integral ϕ on M .
- (h) $N_N \subset M_N$ is a direct summand.

In the special case when the total element m is in the centralizer M^N we obtain that ${}_N N_N \subset {}_N M_N$ is a direct summand, i.e., the extension is split. In the more special case $m = \eta(r)$ for some $r \in R$ we have

$$\begin{aligned} 1 &= m \triangleleft e = 1 \triangleleft \varphi_T(r) \star e = 1 \triangleleft \varphi_L \varphi_T(r) = 1 \triangleleft \varphi_T \varphi_R \varphi_L \varphi_T(r) \\ &= \eta(r \star e). \end{aligned}$$

Therefore if η is mono (e.g. if M_R is faithful) then we conclude from [25, Theorem 4.2] that the k -algebra H is a separable extension of B or, equivalently, of T .

3.4. An intrinsic characterization of finitary Galois extensions.

Theorem 3.7. *For an algebra extension $N \subset M$ the following conditions are equivalent.*

- (1) *There is a Frobenius Hopf algebroid V and a coaction of V on M such that $N \subset M$ is V -Galois.*
- (2) *$N \subset M$ is of depth 2 and Frobenius and M_N is balanced.*

Proof. (1) \Rightarrow $N \subset M$ is Frobenius: Consider the composite

$$(3.17) \quad M \otimes_N M \xrightarrow{\gamma^M} M \otimes_T V \xrightarrow{M \otimes S} M \otimes_R H \xrightarrow{\Gamma^M} \text{End}(M_N)$$

where the middle arrow is meaningful in the double algebraic picture because V and H have the same underlying k -module A and $S(t \star a) = S(a) \star \varphi_B \varphi_R(t) = \varphi_R(t) \circ a$ holds for all $a \in A$, $t \in T$, see [25, Lemma 5.4]. Computing the value of the map (3.17) on $m \otimes_N m'$ we obtain

$$\begin{aligned} mm'^{(0)}(m'' \triangleleft S(m'^{(1)})) &= mm'^{(0)} m''^{(0)} \cdot \varphi_T(m''^{(1)} \star S(m'^{(1)})) \\ &= mm'^{(0)} m''^{(0)} \cdot \varphi_T \varphi_L(m'^{(1)} \circ m''^{(1)}) \\ &= m(m' m'')^{(0)} \cdot \varphi_T((m' m'')^{(1)} \star e) \\ &= m((m' m'') \triangleleft e) \end{aligned}$$

Therefore (3.17) has the familiar form $m \otimes_N m' \mapsto m \psi m'$ in terms of the N - N -bimodule map $\psi = _ \triangleleft e$ from M into N . Since (3.17) is isomorphism it follows that ψ is a Frobenius homomorphism with dual basis obtained from id_M by applying the inverse of (3.17).

(1) \Rightarrow $N \subset M$ is D2: Since ${}_T V$ is fgp and γ^M provides an M - N -bimodule isomorphism $M \otimes_N M \xrightarrow{\sim} ({}_M M_N) \otimes_T V$, it follows that $N \subset M$ is right D2. Similarly, the existence of the isomorphism γ_M and the ${}_B V$ being fgp imply that $N \subset M$ is left D2.

(1) \Rightarrow $N \subset M$ is balanced: This follows from that every V -extension is balanced, see Lemma 2.4.

(2) \Rightarrow (1): The endomorphism algebra $H^{\text{op}} := \text{End}({}_N M_N)$ has a natural structure of a Frobenius Hopf algebroid, see [25, Subsection 8.6] or [4]. Moreover, the natural action of H^{op} on M makes it a left H^{op} -module algebra and the corresponding smash product $M \# H^{\text{op}}$ is isomorphic to $\text{End}(M_N)$ via Γ^M by [15, Corollary 4.5]. So $N \subset M$ will be V -Galois, for V the dual of H^{op} , provided $N = M^H$. But this is equivalent to M_N being balanced. \square

Note that in the presence of the Frobenius condition left D2 is equivalent to right D2 and in the presence of the D2 Frobenius condition M_N is balanced iff ${}_N M$ is balanced.

4. NONCOMMUTATIVE SCALAR EXTENSIONS

The Hopf algebra V making a given algebra extension V -Galois is highly nonunique. This phenomenon can be observed already for Hopf Galois extensions. As Greither and Pareigis have shown [12] certain separable field extensions can be H -Galois for two different Hopf algebras H and H' . By an appropriate extension $k \subset K$ of the scalars, however, they become isomorphic, $K \otimes H \cong K \otimes H'$, as K -Hopf algebras. The k -Hopf algebras H, H' for which such a (commutative, faithfully flat) k -algebra K exists are called forms of each other [20].

If we admit Hopf algebraoids to appear in place of Hopf algebras then an interesting generalization of scalar extension is provided by the Brzeziński-Militaru theorem [6] constructing a Hopf algebraoid structure on the smash product $M \# H$ if M is a braided commutative algebra in the Yetter-Drinfeld category ${}^H \mathcal{YD}^H$ over the Hopf algebra H . As we shall see the Brzeziński-Militaru theorem holds also for H a bialgebroid or Frobenius Hopf algebraoid. Since the base algebra of $M \# H$ is just M , the braided commutative algebras (BCA's) play the role of (noncommutative) scalars.

If $N \subset M$ is a Galois extension for some Frobenius Hopf algebraoid H then the center $C = M^N$ of the extension is a BCA over H (Corollary 4.5) and the scalar extension $H \# C$ is the endomorphism Hopf algebraoid E (Proposition 4.12). Therefore all Frobenius Hopf algebraoids H for which $N \subset M$ is H -Galois are forms of each other.

4.1. Braided commutative algebras. Yetter-Drinfeld modules over bialgebroids have been introduced in [23]. They form a prebraided monoidal category, the weak center of the category of modules over the bialgebroid. In this subsection we adapt the weak center construction to the double algebraic notation and describe the (braided) center $\mathcal{Z}(\mathbf{M}_H)$ as 'double' Yetter-Drinfeld modules ${}^H \mathcal{YD}_H^H$ with two related coactions.

For a right bialgebroid H over R the weak center $\overrightarrow{\mathcal{Z}}(\mathbf{M}_H)$ is defined as follows. The objects $\langle Z, \theta \rangle$ are H -modules equipped with a natural transformation $\theta_Y : Z \otimes_R Y \rightarrow Y \otimes_R Z$ satisfying

$$(4.1) \quad \theta_{X \otimes_R Y} = (X \otimes_R \theta_Y) \circ (\theta_X \otimes_R Y) \quad \text{and} \quad \theta_R = Z$$

where the coherence isomorphisms are not written out explicitly. An arrow $\langle Z, \theta \rangle \rightarrow \langle Z', \theta' \rangle$ is an H -module map $\alpha : Z \rightarrow Z'$ such that

$$(4.2) \quad (Y \otimes_R \alpha) \circ \theta_Y = \theta'_Y \circ (\alpha \otimes_R Y)$$

for all objects $Y \in \mathbf{M}_H$. This category has a monoidal product which is defined for objects by

$$\langle Z, \theta \rangle \otimes_R \langle Z', \theta' \rangle = \langle Z \otimes_R Z', (\theta_- \otimes_R Z') \circ (Z \otimes_R \theta'_-) \rangle$$

and for arrows by taking the ordinary tensor product in \mathbf{M}_H . The category $\overrightarrow{\mathcal{Z}}(\mathbf{M}_H)$ is prebraided with

$$\beta_{\langle Z, \theta \rangle, \langle Z', \theta' \rangle} = \theta_{Z'}.$$

Given an object $\langle Z, \theta \rangle \in \overrightarrow{\mathcal{Z}}(\mathbf{M}_H)$ one can introduce

$$(4.3) \quad \tau : Z \rightarrow H \otimes_R Z, \quad \tau(z) := \theta_H(z \otimes i) = z^{(-1)} \otimes_R z^{(0)}$$

which, as being the composite

$$(4.4) \quad Z \xrightarrow{\sim} Z \otimes_R R \xrightarrow{Z \otimes_R \varphi_B} Z \otimes_R H \xrightarrow{\theta_H} H \otimes_R Z,$$

preserves the left R -module structures inherited from \mathbf{M}_H . By naturality of θ , the τ determines θ_X for all X by the formula

$$(4.5) \quad \theta_X(z \otimes x) = x \triangleleft z^{(-1)} \otimes_R z^{(0)}.$$

Using this formula it is easy to show that (4.1) implies that τ is coassociative and counital, thereby making Z a left H -comodule. We not only have Takeuchi's centrality property

$$(4.6) \quad \begin{aligned} \varphi_T(r) \star z^{(-1)} \otimes_R z^{(0)} &= \varphi_T(r) \triangleleft z^{(-1)} \otimes_R z^{(0)} = \theta_H(z \otimes \varphi_T(r)) \\ &= \theta_H((z \otimes i) \triangleleft \varphi_T(r)) = \tau(z) \triangleleft \varphi_T(r) \\ &= z^{(-1)} \otimes_R z^{(0)} \triangleleft \varphi_T(r) \quad r \in R, z \in Z \end{aligned}$$

but also

$$(4.7) \quad \begin{aligned} \varphi_B(r) \star z^{(-1)} \otimes_R z^{(0)} &= \varphi_B(r) \triangleleft z^{(-1)} \otimes_R z^{(0)} = \theta_H(z \otimes \varphi_B(r)) \\ &= \theta_H(z \triangleleft \varphi_T(r) \otimes i) = \tau(z \triangleleft \varphi_T(r)) = \tau(z \cdot r). \end{aligned}$$

The latter means that the right R -action we could construct from the left R -action - in analogy with the left action we had in Proposition 1.1 for right comodules - would be the same as the original right R -module structure inherited from (4.4). In other words, the requirement for (4.4) to be an R - R -bimodule map defines a right R -action on $H \otimes_R Z$ which is conveyed by naturality of θ and not by θ_H being an arrow in ${}_R\mathbf{M}_R$.

Given a left H -comodule Z which is also a right H -module (with the same underlying R - R -bimodule structure) the condition for (4.5) to determine an H -module map is precisely the Yetter-Drinfeld condition given below.

Summarizing, one has a prebraided monoidal isomorphism $\overrightarrow{\mathcal{Z}}(\mathbf{M}_H) \cong {}^H\mathcal{YD}_H$ with the following Yetter-Drinfeld category:

Definition 4.1. For a right bialgebroid $\langle H, \star, i, R, \varphi_T, \varphi_B, \Delta_R, \varphi_R \rangle$ the category ${}^H\mathcal{YD}_H$ has objects $\langle Z, \triangleleft, \tau \rangle$ where

- (1) $\langle Z, \triangleleft \rangle$ is a right H -module, hence also an R - R -bimodule via $r \cdot z \cdot r' = z \triangleleft (\varphi_B(r) \star \varphi_T(r'))$.
- (2) $\langle Z, \tau \rangle$ is a left H -coaction, that is to say,
 - (a) $\tau : Z \rightarrow H \otimes_R Z$ is an R - R -bimodule map in the sense of

$$(4.8) \quad (r \cdot z \cdot r')^{(-1)} \otimes_R (r \cdot z \cdot r')^{(0)} = \varphi_B(r') \star z^{(-1)} \star \varphi_B(r) \otimes_R z^{(0)},$$

(b) τ is coassociative and counital,

$$\begin{aligned} z^{(-1)} \otimes_R z^{(0)\langle -1 \rangle} \otimes_R z^{(0)\langle 0 \rangle} &= z^{(-1)\langle 1 \rangle} \otimes_R z^{(-1)\langle 2 \rangle} \otimes_R z^{(0)} \\ \varphi_R(z^{(-1)}) \cdot z^{(0)} &= z \end{aligned}$$

(c) τ factorizes through $H \times Z \subset H \otimes_R Z$, i.e., (4.6) holds.

(3) The action and coaction satisfy the Yetter-Drinfeld condition

$$h^{[2]} \star (z \triangleleft h^{[1]})^{\langle -1 \rangle} \otimes_R (z \triangleleft h^{[1]})^{\langle 0 \rangle} = z^{\langle -1 \rangle} \star h^{[1]} \otimes_R z^{\langle 0 \rangle} \triangleleft h^{[2]}.$$

The arrows are the H -module H -comodule maps $Z \rightarrow Z'$. The monoidal product of two Yetter-Drinfeld modules Z and Z' is $Z \otimes_R Z'$ equipped with

$$\begin{aligned} (z \otimes_R z') \triangleleft h &= (z \triangleleft h^{[1]}) \otimes_R (z' \triangleleft h^{[2]}) \\ (z \otimes_R z')^{\langle -1 \rangle} \otimes_R (z \otimes_R z')^{\langle 0 \rangle} &= z'^{\langle -1 \rangle} \star z^{\langle -1 \rangle} \otimes_R (z^{\langle 0 \rangle} \otimes_R z'^{\langle 0 \rangle}) \end{aligned}$$

The monoidal unit is R with $r \triangleleft h = r \star h$ and $r^{\langle -1 \rangle} \otimes_R r^{\langle 0 \rangle} = \varphi_B(r) \otimes_R \epsilon$. The prebraiding is defined by

$$\beta_{Z, Z'} : Z \otimes_R Z' \rightarrow Z' \otimes_R Z, \quad z \otimes_R z' \mapsto z' \triangleleft z^{\langle -1 \rangle} \otimes_R z^{\langle 0 \rangle}.$$

There is a coopposite version $\overleftarrow{\mathcal{Z}}(\mathbf{M}_H) = \overrightarrow{\mathcal{Z}}(\mathbf{M}_H^{\text{coop}}) = \overrightarrow{\mathcal{Z}}(\mathbf{M}_{H^{\text{coop}}})$ of the left weak center, called the right weak center, in which an object $\langle Z, \bar{\theta} \rangle$ has natural transformation $\bar{\theta}_Y : Y \otimes_R Z \rightarrow Z \otimes_R Y$ satisfying $\bar{\theta}_{X \otimes_R Y} = (\bar{\theta}_X \otimes_R Y) \circ (X \otimes_R \bar{\theta}_Y)$. This determines a right coaction

$$\bar{\tau} : Z \rightarrow Z \otimes_R H, \quad z \mapsto z^{\langle 0 \rangle} \otimes_R z^{\langle 1 \rangle} = \bar{\theta}_H(i \otimes z)$$

and is determined by this coaction,

$$(4.9) \quad \bar{\theta}_Y(y \otimes z) = z^{\langle 0 \rangle} \otimes_R y \triangleleft z^{\langle 1 \rangle}.$$

The center $\mathcal{Z}(\mathbf{M}_H)$ is the full subcategory of $\overrightarrow{\mathcal{Z}}(\mathbf{M}_H)$ in which the objects $\langle Z, \theta \rangle$ have invertible θ . For such objects $\langle Z, \theta^{-1} \rangle$ is an object in $\overleftarrow{\mathcal{Z}}(\mathbf{M}_H)$ in which $\bar{\theta}$ is invertible. The center is braided monoidal. In the language of Yetter-Drinfeld modules the objects of the center are two-sided Yetter-Drinfeld modules $\langle Z, \triangleleft, \tau, \bar{\tau} \rangle \in {}^H \mathcal{YD}_H^H$ in which the two coactions are inverse to each other, i.e.,

$$(4.10) \quad z^{\langle 0 \rangle \langle 0 \rangle} \otimes_R z^{\langle -1 \rangle} \star z^{\langle 0 \rangle \langle 1 \rangle} = z \otimes_R i$$

$$(4.11) \quad z^{\langle 1 \rangle} \star z^{\langle 0 \rangle \langle -1 \rangle} \otimes_R z^{\langle 0 \rangle \langle 0 \rangle} = i \otimes_R z.$$

Definition 4.2. For a right bialgebroid H the commutative monoids in $\mathcal{Z}(\mathbf{M}_H)$ are called BCA's (braided commutative algebras) over H . The commutative monoids in $\overrightarrow{\mathcal{Z}}(\mathbf{M}_H)$ and $\overleftarrow{\mathcal{Z}}(\mathbf{M}_H)$ are called left and right pre-BCA's over H , respectively.

Therefore a left pre-BCA consists of an algebra Q with an algebra map $\eta : R \rightarrow Q$ and a Yetter-Drinfeld module structure $\langle Q, \triangleleft, \tau \rangle \in {}^H\mathcal{YD}_H$ such that

$$(4.12) \quad \eta(r) q \eta(r') = r \cdot q \cdot r'$$

$$(4.13) \quad (qq') \triangleleft h = (q \triangleleft h^{[1]})(q' \triangleleft h^{[2]})$$

$$(4.14) \quad 1 \triangleleft h = \eta \varphi_R(h)$$

$$(4.15) \quad (qq')^{(-1)} \otimes_R (qq')^{(0)} = q'^{(-1)} \star q^{(-1)} \otimes_R q^{(0)} q'^{(0)}$$

$$(4.16) \quad \eta(r)^{(-1)} \otimes_R \eta(r)^{(0)} = \varphi_B(r) \otimes_R 1$$

and the prebraided commutativity

$$(4.17) \quad (q' \triangleleft q^{(-1)})q^{(0)} = qq'$$

holds. If Q is a BCA then there exists also a right coaction $\bar{\tau}$ with which $\langle Q, \triangleleft, \bar{\tau} \rangle \in \mathcal{YD}_H^H$ and which is inverse to τ in the sense of equations (4.10), (4.11).

We note that the ground ring R of the bialgebroid is always a BCA with the structure $\langle R, \mu_R, R \rangle$ that comes from R being the monoidal unit of $\vec{\mathcal{Z}}(\mathbf{M}_H)$.

4.2. The centralizer of a Galois extension. Interesting examples for BCA's are obtained from considering centralizers M^N of Galois extensions.

Proposition 4.3. *Let M be a monoid in \mathbf{M}_H over the right bialgebroid H and let $N = M^H$. Assume that H_R is fgp and that the canonical map $\Gamma_M : H \# M \rightarrow \text{End}({}_N M)$ is an isomorphism. Then the centralizer $M^N = \{c \in M \mid nc = cn, n \in N\}$ of the extension $N \subset M$ is a left pre-BCA over H with H -module algebra structure inherited from $M^N \subset M$ (the Miyashita-Ulbrich action) and with left coaction $\tau(c) := \Gamma_M^{-1}(\lambda_M(c))$ where $\lambda_M(c) = \{m \mapsto cm\}$.*

Proof. For each $h \in H$ the action $- \triangleleft h$ is an N - N -bimodule map. Therefore $M^H \subset M$ is a sub- H -module algebra. As such the unit $\eta : R \rightarrow M$ has image in M^H . Since Γ_M is an N - N -bimodule map, it restricts to an isomorphism $(H \otimes_R M)^N \xrightarrow{\sim} \text{End}({}_N M_N)$ between the centralizers. The H_R being fgp we have $(H \otimes_R M)^N = H \otimes_R M^N$. Since $\lambda(c)$ for $c \in M^N$ belongs to $\text{End}({}_N M_N)$, the τ is a map $M^N \rightarrow H \otimes_R M^N$. The τ is uniquely determined by the equation

$$(4.18) \quad (m \triangleleft c^{(-1)})c^{(0)} = cm, \quad m \in M$$

from which the bimodule property (4.8) and the centrality (4.6) easily follow. The calculation

$$\begin{aligned} c(mm') &= ((mm') \triangleleft c^{(-1)})c^{(0)} = (m \triangleleft c^{(-1)}^{(1)})(m' \triangleleft c^{(-1)}^{(2)})c^{(0)} \\ (cm)m' &= (m \triangleleft c^{(-1)})c^{(0)}m' = (m \triangleleft c^{(-1)})(m' \triangleleft c^{(0)}{}^{(-1)})c^{(0)}{}^{(0)} \end{aligned}$$

will imply coassociativity after verifying the next

Lemma 4.4. *Under the assumptions of the Proposition and with the notations $E := \text{End}({}_N M_N)$, $C := M^N$ the maps*

$$(4.19) \quad \begin{aligned} E \otimes_C E &\rightarrow \text{Hom}_{N-N}(M \otimes_N M, M) \\ \alpha \otimes_C \alpha' &\mapsto \{m \otimes_N m' \mapsto \alpha(m)\alpha'(m')\} \end{aligned}$$

$$(4.20) \quad \begin{aligned} H \otimes_R H \otimes_R C &\rightarrow \text{Hom}_{N-N}(M \otimes_N M, M) \\ h \otimes_R h' \otimes_R c &\mapsto \{m \otimes_N m' \mapsto (m \triangleleft h)(m' \triangleleft h')c\} \end{aligned}$$

are isomorphisms.

Proof. Using both the isomorphism Γ_M and its restriction $H \# C \xrightarrow{\sim} E$ we have a sequence of isomorphisms

$$\begin{aligned} E \otimes_C E &\xrightarrow{\sim} (H \otimes_R C) \otimes_C E \xrightarrow{\sim} H \otimes_R E = H \otimes_R \text{Hom}_{N-N}(M, M) \\ &\xrightarrow{\sim} \text{Hom}_{N-N}(M, H \otimes_R M) \xrightarrow{\sim} \text{Hom}_{N-N}(M, \text{Hom}_N(M, M)) \\ &\xrightarrow{\sim} \text{Hom}_{N-N}(M \otimes_N M, M) \end{aligned}$$

The action of these isomorphisms can be computed by inserting $\alpha = (- \triangleleft h)c$ and $\alpha' = (- \triangleleft h')c'$:

$$\begin{aligned} \alpha \otimes_C \alpha' &\mapsto (h \otimes_R c) \otimes_C \alpha' \mapsto h \otimes_R c \alpha'(-) \mapsto \{m \mapsto h \otimes_R c \alpha'(m)\} \\ &\mapsto \{m \mapsto \{m' \mapsto \alpha(m')\alpha'(m)\}\} \mapsto \{m' \otimes_N m \mapsto \alpha(m')\alpha'(m)\} \end{aligned}$$

This proves that (4.19) is an isomorphism. The map in (4.20) is the composite

$$\begin{array}{ccc} H \otimes_R H \otimes_R C & & \text{Hom}_{N-N}(M \otimes_N M, M) \\ H \otimes_R \Gamma \downarrow & & \uparrow \cong \\ H \otimes_R E & \xrightarrow{\cong} & H \otimes_R C \otimes_C E \xrightarrow{\Gamma_C^{\otimes E}} & E \otimes_C E \end{array}$$

of isomorphisms. □

Returning to the proof of the Proposition counitality of τ can be seen as

$$\varphi_R(c^{(-1)}) \cdot c^{(0)} = (1 \triangleleft c^{(-1)})c^{(0)} = c1 = c.$$

As for the Yetter-Drinfeld compatibility condition it suffices to verify the equality

$$\begin{aligned} (m \triangleleft c^{(-1)} \star h^{(1)})(c^{(0)} \triangleleft h^{(2)}) &= \left((m \triangleleft c^{(-1)})c^{(0)} \right) \triangleleft h = (cm) \triangleleft h \\ &= (c \triangleleft h^{(1)})(m \triangleleft h^{(2)}) = \left(m \triangleleft h^{(2)} \star (c \triangleleft h^{(1)})^{(-1)} \right) (c \triangleleft h^{(1)})^{(0)} \end{aligned}$$

In order to see compatibility of τ with multiplication and unit in C it suffices to check

$$c'cm = c'(m \triangleleft c^{(-1)})c^{(0)} = (m \triangleleft c^{(-1)} \star c'^{(-1)})c'^{(0)}c^{(0)}.$$

Finally, braided commutativity $(c' \triangleleft c^{(-1)})c^{(0)} = cc'$ follows from the more general relation (4.18). □

Corollary 4.5. *If $N \subset M$ is a right A -Galois extension for a distributive double algebra A then M^N is a BCA over the horizontal Hopf algebroid H .*

Proof. It suffices to prove that the prebraiding is invertible. Define the right coaction $\bar{\tau}(c) := (\Gamma^M)^{-1}(\rho_M(c))$ where ρ_M is right multiplication on M . This is equivalent to $\bar{\tau}(c) = c^{(0)} \otimes_R c^{(1)}$ satisfying

$$(4.21) \quad c^{(0)}(m \triangleleft c^{(1)}) = mc, \quad m \in M.$$

Applying (4.18) to (4.21) we obtain

$$(m \triangleleft i)c = mc = (m \triangleleft c^{(-1)} \star c^{(0)^{(-1)}}) c^{(0)^{(0)}}$$

from which equation (4.11) follows. Equation (4.10) can be seen similarly. \square

Notice that this proof does not use very much from the Hopf algebroid structure. Therefore the Corollary holds true for any right bialgebroid for which both H_R and ${}_R H$ are fgp and for all extensions for which both Γ^M and Γ_M are invertible.

4.3. Extensions by BCA's. For any H -module algebra Q over the right bialgebroid H the category $\mathbf{M}_{H\#Q}$ of modules over the smash product can be identified with the category of (internal) Q -modules $(\mathbf{M}_H)_Q$ in \mathbf{M}_H .

If Q is also a pre-BCA then every right Q -module in \mathbf{M}_H is also a left Q -module by pre-braided commutativity. This defines an embedding of categories

$$(4.22) \quad \mathbf{M}_{H\#Q} = (\mathbf{M}_H)_Q \hookrightarrow {}_Q(\mathbf{M}_H)_Q$$

into the monoidal category of internal Q - Q -bimodules. Since the Q - Q -bimodule tensor product of diagonal bimodules $X, Y \in (\mathbf{M}_H)_Q$ is again diagonal due to one of the hexagons, this embedding is actually strong monoidal. Composing (4.22) with the strong monoidal forgetful functor ${}_Q(\mathbf{M}_H)_Q \rightarrow {}_Q\mathbf{M}_Q$ we obtain a strong monoidal functor

$$(4.23) \quad \mathbf{M}_{H\#Q} = (\mathbf{M}_H)_Q \rightarrow {}_Q\mathbf{M}_Q.$$

This functor is precisely the forgetful functor associated to the algebra map

$$(4.24) \quad Q^{\text{op}} \otimes Q \rightarrow H\#Q, \quad q \otimes q' \mapsto q^{(-1)} \# q^{(0)} q'$$

therefore, by a theorem of Schauenburg [22], there is a unique bialgebroid structure on $H\#Q$ such that the given monoidal structure of $\mathbf{M}_{H\#Q}$ is that of the module category of a bialgebroid. This is the Brzeziński-Militaru Theorem in disguise. More precisely this is the "only if" part of [6, Theorem 4.1] generalized to bialgebroids H .

Theorem 4.6. *Let H be a right bialgebroid over R and let Q be a left pre-BCA over H . Then the smash product $G := H\#Q$ is a right bialgebroid over Q with structure maps*

$$(4.25) \quad s_G(q) = i\#q$$

$$(4.26) \quad t_G(q) = q^{(-1)} \# q^{(0)}$$

$$(4.27) \quad \Delta_G(h\#q) = (h^{[1]}\#1) \otimes_Q (h^{[2]}\#q)$$

$$(4.28) \quad \varepsilon_G(h\#q) = \eta(\varepsilon_H(h))q$$

where $\eta : R \rightarrow Q$ is the unit of Q . Moreover, $h \mapsto h\#1$ is a bialgebroid map $\iota : H \rightarrow G$.

If H is a Frobenius Hopf algebroid with Frobenius integral e then G is also a Frobenius Hopf algebroid with $e_G = \iota(e)$ a Frobenius integral.

Proof. The observation made before the formulation of the Theorem, in particular equation (4.24) implies the formulae for s_G and t_G . In order to obtain the expressions for Δ_G and ε_G at once, and also to prove the Frobenius Hopf algebroid case, the next Proposition, however simple, is very useful.

Proposition 4.7. *If H is a right bialgebroid over R and Q is a left pre-BCA over H then the functor $-\otimes_R Q : \mathbf{M}_H \rightarrow (\mathbf{M}_H)_Q$ is strong monoidal.*

Proof. The natural transformation

$$\begin{aligned} (Y \otimes_R Q) \otimes_Q (Y' \otimes_R Q) &\rightarrow (Y \otimes_R Y') \otimes_R Q \\ (y \otimes_R q) \otimes_Q (y' \otimes_R q') &\mapsto (y \otimes_R y' \triangleleft q^{(-1)}) \otimes_R q^{(0)} q' \end{aligned}$$

has inverse $(y \otimes_R y') \otimes_R q \mapsto (y \otimes_R 1) \otimes_Q (y' \otimes_R q)$. The $H\#Q$ -module map

$$Q \rightarrow R \otimes_R Q, \quad q \mapsto e \otimes_R q$$

is the unit part of the monoidal structure and is obviously invertible. \square

Continuing the proof of the Theorem we take the comonoid $\langle H, \Delta_H, \varepsilon_H \rangle$ in \mathbf{M}_H and apply the strong monoidal functor $-\otimes_R Q$. It is easy to check that the result is precisely $\langle G, \Delta_G, \varepsilon_G \rangle$ which is then necessarily a comonoid in \mathbf{M}_G . This comonoid is obviously strong [26] proving that $\langle G, Q, s_G, t_G, \Delta_G, \varepsilon_G \rangle$ is a bialgebroid. It is straightforward to verify that the pair $\langle \iota, \eta \rangle$ satisfies the four axioms [24, 26] for a bialgebroid map $H \rightarrow G$.

If H is a Frobenius Hopf algebroid then it has a distributive double algebra structure [25]. Therefore we may assume that H is the horizontal Hopf algebroid of $\langle A, \circ, e, \star, i \rangle$. Then $\langle H, \Delta_R, \varphi_R, \circ, R \hookrightarrow H \rangle$ is a Frobenius algebra in \mathbf{M}_H , so it is mapped by the strong monoidal functor of Proposition 4.7 to a Frobenius algebra in \mathbf{M}_G . The comonoid part of this Frobenius algebra has already been determined to be $\langle G, \Delta_G, \varepsilon_G \rangle$. The monoid part will provide a convolution product with unit on G which, together with the smash product algebra structure, will make G a distributive double algebra. This convolution product (vertical multiplication) is obtained as the composite

$$(h\#q) \otimes_Q (h'\#q') \mapsto (h \otimes_R h' \star q^{(-1)}) \otimes_R q^{(0)} q' \mapsto h \circ (h' \star q^{(-1)}) \# q^{(0)} q'$$

and its unit element e_G is the image of $1 \in Q$ under the map

$$Q \xrightarrow{\sim} R \otimes_R Q \rightarrow H\#Q.$$

So $e_G = e\#1$ is a two-sided Frobenius integral in G . \square

Remark 4.8. The construction of a vertical multiplication on $H\#Q$ suggests the new interpretation of the smash product as a double algebraic one. If $\langle A, \circ, e, \star, i \rangle$ is a DDA and Q is a BCA over the bialgebroid H over R then there is a smash product double algebra $A\#Q$ with

- underlying k -module $A \otimes_R Q$,
- horizontal multiplication $(a\#q) \star (a'\#q') = a \star a'^{[1]} \# (q \triangleleft a'^{[2]}) q'$,
- horizontal unit $i\#1$,
- vertical multiplication $(a\#q) \circ (a'\#q') = a \circ (a' \star q^{(-1)}) \# q^{(0)} q'$,

- and vertical unit $\epsilon \# 1$.

As a biproduct of the double algebraic picture we obtain the following result.

Proposition 4.9. *For Frobenius Hopf algebroids H the prebraiding of the left weak center $\overrightarrow{\mathcal{Z}}(\mathbf{M}_H)$ is a braiding. Therefore $\overrightarrow{\mathcal{Z}}(\mathbf{M}_H) = \mathcal{Z}(\mathbf{M}_H) = \overleftarrow{\mathcal{Z}}(\mathbf{M}_H)$ and every pre-BCA is a BCA over H .*

Proof. We claim that the inverse braiding encoded in the right coaction $\bar{\tau}$ by (4.9) is given by

$$(4.29) \quad q^{(0)} \otimes_R q^{(1)} = \eta_Q \varphi_R \varphi_T (x^j \star q^{(-1)}) q^{(0)} \otimes_R y^j .$$

The proof is motivated by the double algebraic structure on $H \# Q$ given in the above Remark but we do not use that the given structure maps satisfy the axioms of a DDA. Let us compute the would-be φ_R of $H \# Q$. It is

$$\Phi_R(h \# q) := (\epsilon \# 1) \star (h \# q) = \epsilon \# \eta_Q \varphi_R(h) q .$$

One conjectures $(x^j \# 1) \otimes_{\mathcal{Q}} (y^j \# 1)$ to be its dual basis. Instead of proving that we prove its special case

$$\begin{aligned} \Phi_R((i \# q) \circ (x^j \# 1)) \circ (y^j \# 1) &= (\epsilon \# \eta_Q \varphi_R \varphi_T (x^j \star q^{(-1)}) q^{(0)}) \circ (y^j \# 1) \\ &= y^j \star (\varphi_R \varphi_T (x^j \star q^{(-2)}) \circ q^{(-1)}) \# q^{(0)} \\ &= y^j \star q^{(-1)} \star \varphi_B \varphi_R \varphi_T (x^j \star q^{(-2)}) \# q^{(0)} \\ &= \varphi_R(i \circ \varphi_R(q^{(-1)})^{[1]}) \circ \varphi_R(q^{(-1)})^{[2]} \# q^{(0)} \\ &= i \circ \varphi_R(q^{(-1)}) \# q^{(0)} \\ &= i \# q . \end{aligned}$$

Comparing the first row with the Ansatz (4.29) and then using the vertical multiplication of $H \# Q$ we arrive at

$$\begin{aligned} i \# q &= (\epsilon \# q^{(0)}) \circ (q^{(1)} \# 1) \\ &= q^{(1)} \star q^{(0) \langle -1 \rangle} \# q^{(0) \langle 0 \rangle} \end{aligned}$$

which is equation (4.11). The verification of (4.10) is a bit longer,

$$\begin{aligned} q^{(0) \langle 0 \rangle} \otimes_R q^{(-1)} \star q^{(0) \langle 1 \rangle} &= \eta_Q \varphi_R \varphi_T (x^j \star q^{(-1)}) q^{(0)} \otimes_R q^{(-2)} \star y^j \\ &= \eta_Q \varphi_R \varphi_T (S^{-1}(q^{(-2)}) \star x^j \star q^{(-1)}) q^{(0)} \otimes_R y^j \\ &= \eta_Q \varphi_R \varphi_T (S^{-1}(x^k) \star x^j \star (y^k \circ q^{(-1)})) q^{(0)} \otimes_R y^j \\ &= \eta_Q \varphi_R \varphi_T ((S^{-1}(y^k) \circ (S^{-1}(x^k) \star x^j)) \star q^{(-1)}) q^{(0)} \otimes_R y^j \\ &= \eta_Q \varphi_R \varphi_T ((x_k \circ (y_k \star x^j)) \star q^{(-1)}) q^{(0)} \otimes_R y^j \\ &= \eta_Q \varphi_R \varphi_T (\varphi_R \varphi_T (x^j) \star q^{(-1)}) q^{(0)} \otimes_R y^j \end{aligned}$$

$$\begin{aligned}
&= \eta_Q \varphi_R (\varphi_T(x^j) \star q^{(-1)}) q^{(0)} \otimes_R y^j \\
&= \eta_Q \varphi_R (q^{(-1)}) q^{(0)} \triangleleft \varphi_T(x^j) \otimes_R y^j \\
&= q \otimes_R \varphi_R \varphi_T(x^j) \circ y^j \\
&= q \otimes_R i.
\end{aligned}$$

□

Extensions of quantum groupoids by BCA's are transitive in the following sense.

Proposition 4.10. *For a bialgebroid H if Q is a pre-BCA over H and P is a pre-BCA over $H \# Q$ then P is a pre-BCA over H , too. Furthermore, $(H \# Q) \#_Q P \cong H \#_R P$ as bialgebroids.*

Proof. Composing the units of Q and P we obtain the algebra map $\eta = \eta^P \circ \eta^Q : R \rightarrow P$ which is going to be the unit of P as a monoid in $\vec{\mathcal{Z}}(\mathbf{M}_H)$. The H -module structure on P is defined by restricting the $H \# Q$ -action, i.e., $p \triangleleft h := p \triangleleft (h \# 1)$. The more complicated piece of structure is the H -comodule ${}^H P$ given by

$$p^{\{-1\}} \otimes_R p^{\{0\}} := (H \otimes_R \eta^P)(p^{\{-1\}}) p^{\{0\}} \cong p^{\{-1\}H} \otimes_R p^{\{-1\}Q} \cdot p^{\{0\}}$$

where $p^{\{-1\}} \otimes_Q p^{\{0\}}$ denotes the given $H \# Q$ -coaction on P and we introduced the notation $g^H \otimes_R g^Q$ for elements $g \in H \# Q$.

Counitality:

$$\begin{aligned}
\varepsilon_H(p^{\{-1\}}) \cdot p^{\{0\}} &= \varepsilon_H(p^{\{-1\}H}) \cdot (p^{\{-1\}Q} \cdot p^{\{0\}}) \\
&= (\varepsilon_H(p^{\{-1\}H}) \cdot p^{\{-1\}Q}) \cdot p^{\{0\}} = \varepsilon_{H \# Q}(p^{\{-1\}}) \cdot p^{\{0\}} = p.
\end{aligned}$$

Coassociativity:

$$\begin{aligned}
&p^{\{-1\}} \otimes_R p^{\{0\}\{-1\}} \otimes_R p^{\{0\}\{0\}} = \\
&= p^{\{-1\}H} \otimes_R (p^{\{0\}\{-1\}} t_{H \# Q}(p^{\{-1\}Q}))^H \otimes_R (p^{\{0\}\{-1\}} t_{H \# Q}(p^{\{-1\}Q}))^Q \cdot p^{\{0\}\{0\}} \\
&= p^{\{-1\}H} \otimes_R p^{\{0\}\{-1\}H} \star p^{\{-1\}Q\{-1\}} \otimes_R p^{\{-1\}Q\{0\}} p^{\{0\}\{-1\}Q} \cdot p^{\{0\}\{0\}} \\
&= p^{\{-2\}H} \otimes_R p^{\{-1\}H} \star p^{\{-2\}Q\{-1\}} \otimes_R p^{\{-2\}Q\{0\}} p^{\{-1\}Q} \cdot p^{\{0\}} \\
&= p^{\{-2\}H} \otimes_R p^{\{-2\}Q} \cdot (p^{\{-1\}H} \otimes_R p^{\{-1\}Q}) \cdot p^{\{0\}} \\
&= p^{\{-1\}H[1]} \otimes_R p^{\{-1\}H[2]} \otimes_R p^{\{-1\}Q} \cdot p^{\{0\}} \\
&= p^{\{-1\}[1]} \otimes_R p^{\{-1\}[2]} \otimes_R p^{\{0\}}.
\end{aligned}$$

Takeuchi property:

$$\begin{aligned}
p^{\{-1\}} \otimes_R p^{\{0\}} \triangleleft s_H(r) &= p^{\{-1\}H} \otimes_R p^{\{-1\}Q} \cdot (p^{\{0\}} \triangleleft (s_H(r) \# 1)) \\
&= p^{\{-1\}H} \otimes_R p^{\{-1\}Q} \cdot (p^{\{0\}} \triangleleft s_{H \# Q}(\eta^Q(r))) \\
&= s_H(r) \star p^{\{-1\}} \otimes_R p^{\{0\}}.
\end{aligned}$$

Yetter-Drinfeld condition:

$$\begin{aligned}
 & h^{[2]} \star (p \triangleleft h^{[1]})^{\{-1\}} \otimes_R (p \triangleleft h^{[1]})^{\{0\}} = \\
 & = h^{[2]} \star (p \triangleleft (h^{[1]} \# 1))^{\{-1\}H} \otimes_R (p \triangleleft (h^{[1]} \# 1))^{\{-1\}Q} \cdot (p \triangleleft (h^{[1]} \# 1))^{\{0\}} \\
 & = p^{\{-1\}H} \star h^{[1]} \otimes_R (p^{\{-1\}Q} \triangleleft h^{[2]}) \cdot (p^{\{0\}} \triangleleft h^{[3]}) \\
 & = p^{\{-1\}} \star h^{[1]} \otimes_R p^{\{0\}} \triangleleft h^{[2]}.
 \end{aligned}$$

Hence we have P as an object in $\overrightarrow{\mathcal{Z}}(\mathbf{M}_H)$ and it remains to show that its k -algebra structure induces a commutative monoid structure in the weak center.

Bimodule property:

$$\begin{aligned}
 r \cdot p \cdot r' & = p \triangleleft (t_H(r) \star s_H(r') \# 1) = p \triangleleft t_{H\#Q}(\eta^Q(r)) s_{H\#Q}(\eta^Q(r')) \\
 & = \eta^P(\eta^Q(r)) p \eta^P(\eta^Q(r')) = \eta(r) p \eta(r').
 \end{aligned}$$

The P is clearly an H -module algebra. The multiplicativity of the coaction can be seen as

$$\begin{aligned}
 (pp')^{\{-1\}} \otimes_R (pp')^{\{0\}} & = p'^{\{-1\}H} \star p^{\{-1\}H[1]} \otimes_R (p'^{\{-1\}Q} \triangleleft p^{\{-1\}H[2]}) p^{\{-1\}Q} \cdot p^{\{0\}} p'^{\{0\}} \\
 & = p'^{\{-1\}H} \star p^{\{-2\}H} \otimes_R p^{\{-2\}Q} (p'^{\{-1\}Q} \triangleleft p^{\{-1\}H}) p^{\{-1\}Q} \cdot p^{\{0\}} p'^{\{0\}} \\
 & = p'^{\{-1\}H} \star p^{\{-1\}H} \otimes_R p^{\{-1\}Q} \cdot (\eta^P(p'^{\{-1\}Q} \triangleleft p^{\{0\}\{-1\}}) p^{\{0\}\{0\}}) p'^{\{0\}} \\
 & = p'^{\{-1\}H} \star p^{\{-1\}H} \otimes_R \eta^P(p^{\{-1\}Q}) p^{\{0\}} \eta^P(p'^{\{-1\}Q}) p'^{\{0\}} \\
 & = p'^{\{-1\}} \star p^{\{-1\}} \otimes_R p^{\{0\}} p'^{\{0\}}
 \end{aligned}$$

while its unitality, $1^{\{-1\}} \otimes_R 1^{\{0\}} = i \otimes_R 1$, and prebraided commutativity

$$(p' \triangleleft p^{\{-1\}}) p^{\{0\}} = (p' \triangleleft p^{\{-1\}}) p^{\{0\}} = pp',$$

are obvious. Now one can easily see that $h \otimes_R q \otimes_P p \mapsto h \otimes_R q \cdot p$ is a map of bialgebroids from the iterated smash product to $H\#P$ and it is an isomorphism. \square

A sort of converse to the previous Proposition is the following:

Proposition 4.11. *If $\eta : Q \rightarrow P$ is a monoid morphism in $\overrightarrow{\mathcal{Z}}(\mathbf{M}_H)$ between commutative monoids (i.e., pre-BCA's over H) then there is a unique pre-BCA structure on P over $H\#Q$ which returns the original pre-BCA over H when Proposition 4.10 is applied to it.*

Proof. Uniqueness: The unique $H\#Q$ -action on P which restricts to the given H -action and which can be an $H\#Q$ -module algebra with unit η is

$$(4.30) \quad p \triangleleft (h \# q) = (p \triangleleft h) \eta(q).$$

The unique left $H\#Q$ -coaction on P which projects to the given $p^{\{-1\}} \otimes_R p^{\{0\}}$ is

$$(4.31) \quad p^{\{-1\}} \otimes_Q p^{\{0\}} = (p^{\{-1\}} \otimes_R 1_Q) \otimes_Q p^{\{0\}}.$$

As a matter of fact, if $p^{(-1)H} \otimes_R p^{(-1)Q} \cdot p^{(0)} = p^{\{-1\}} \otimes_R p^{\{0\}}$ then

$$(p^{\{-1\}} \# 1_Q) \otimes_{\mathbb{Q}} p^{\{0\}} = (p^{(-1)H} \# p^{(-1)Q}) \otimes_{\mathbb{Q}} p^{(0)} = p^{(-1)} \otimes_{\mathbb{Q}} p^{(0)}.$$

Existence: That (4.30) is associative and unital is clear from that η is an H -module map and an algebra homomorphism. In order to show that (4.31) is a left $H \# Q$ -coaction we proceed as follows.

Bimodule property: At first we show that the bimodule structure induced from the $H \# Q$ -action is the same as the one induced by η . For the right action, $p \triangleleft s_{H \# Q}(q) = p\eta(q)$, is obvious. For the left action we use braided commutativity in \mathbf{M}_H to get

$$\eta(q)p = (p \triangleleft \eta(q)^{\{-1\}})\eta(q)^{\{0\}} = (p \triangleleft q^{(-1)})\eta(q^{(0)}) = p \triangleleft t_{H \# Q}(q).$$

Now the bimodule property (4.8) can be obtained as

$$\begin{aligned} & (q \cdot p \cdot q')^{(-1)} \otimes_{\mathbb{Q}} (q \cdot p \cdot q')^{(0)} = \\ & = (\eta(q) p \eta(q'))^{\{-1\}} \# 1 \otimes_{\mathbb{Q}} (\eta(q) p \eta(q'))^{\{0\}} \\ & = \eta(q')^{\{-1\}} \star p^{\{-1\}} \star \eta(q)^{\{-1\}} \# 1 \otimes_{\mathbb{Q}} \eta(q)^{\{0\}} p^{\{0\}} \eta(q')^{\{0\}} \\ & = q'^{(-1)} \star p^{\{-1\}} \star q^{(-1)} \# 1 \otimes_{\mathbb{Q}} \eta(q^{(0)}) p^{\{0\}} \eta(q'^{(0)}) \\ & = q'^{(-1)} \star p^{\{-1\}[1]} \star q^{(-1)} \# 1 \otimes_{\mathbb{Q}} \eta(q^{(0)}) (\eta(q'^{(0)}) \triangleleft p^{\{-1\}[2]}) p^{\{0\}} \\ & = q'^{(-1)} \star p^{\{-1\}[1]} \star q^{(-1)} \# q^{(0)} (q'^{(0)} \triangleleft p^{\{-1\}[2]}) \otimes_{\mathbb{Q}} p^{\{0\}} \\ & = q'^{(-1)} \star p^{\{-1\}[1]} \star q^{(-2)} \# (q'^{(0)} \triangleleft p^{\{-1\}[2]} \star q^{(-1)}) q^{(0)} \otimes_{\mathbb{Q}} p^{\{0\}} \\ & = (q'^{(-1)} \# q'^{(0)}) (p^{\{-1\}} \# 1) (q^{(-1)} \# q^{(0)}) \otimes_{\mathbb{Q}} p^{\{0\}} \\ & = t_{H \# Q}(q') p^{(-1)} t_{H \# Q}(q) \otimes_{\mathbb{Q}} p^{(0)}. \end{aligned}$$

Coassociativity and counitality of (4.31) are consequences of the special form of the coalgebra structure (4.27) and (4.28) of the smash product. The Takeuchi property follows as

$$\begin{aligned} & p^{(-1)} \otimes_{\mathbb{Q}} p^{(0)} \triangleleft (i \# q) = p^{\{-1\}} \# 1 \otimes_{\mathbb{Q}} p^{\{0\}} \eta(q) = p^{\{-2\}} \# 1 \otimes_{\mathbb{Q}} (\eta(q) \triangleleft p^{\{-1\}}) p^{\{0\}} \\ & = p^{\{-1\}[1]} \# q \triangleleft p^{\{-1\}[2]} \otimes_{\mathbb{Q}} p^{\{0\}} = (i \# q) p^{(-1)} \otimes_{\mathbb{Q}} p^{(0)} \end{aligned}$$

and the Yetter-Drinfeld condition as

$$\begin{aligned} & (h^{[2]} \# q) (p \triangleleft h^{[1]})^{(-1)} \otimes_{\mathbb{Q}} (p \triangleleft h^{[1]})^{(0)} = \\ & = h^{[2]} \star (p \triangleleft h^{[1]})^{\{-2\}} \# q \triangleleft (p \triangleleft h^{[1]})^{\{-1\}} \otimes_{\mathbb{Q}} (p \triangleleft h^{[1]})^{\{0\}} = \\ & = h^{[2]} \star (p \triangleleft h^{[1]})^{\{-1\}} \# 1 \otimes_{\mathbb{R}} (p \triangleleft h^{[1]})^{\{0\}} \eta(q) = \\ & = p^{\{-1\}} \star h^{[1]} \# 1 \otimes_{\mathbb{Q}} (p^{\{0\}} \triangleleft h^{[2]}) \eta(q) = \\ & = p^{(-1)} (h \# q)^{[1]} \otimes_{\mathbb{Q}} p^{(0)} \triangleleft (h \# q)^{[2]}. \end{aligned}$$

Thus P is a Yetter-Drinfeld module over $H\#Q$. The multiplication of P is an $H\#Q$ -module map since

$$(p \triangleleft (h^{[1]}\#1))(p' \triangleleft (h^{[2]}\#q)) = (p \triangleleft h^{[1]})(p' \triangleleft h^{[2]})\eta(q)$$

and an $H\#Q$ -comodule map since

$$\begin{aligned} (pp')^{(-1)} \otimes_{\mathcal{Q}} (pp')^{(0)} &= (pp')^{\{-1\}} \#1 \otimes_{\mathcal{Q}} (pp')^{\{0\}} \\ &= p^{\{-1\}} \star p^{\{-1\}} \#1 \otimes_{\mathcal{Q}} p^{\{0\}} p'^{\{0\}} \\ &= p'^{\{-1\}} p^{(-1)} \otimes_{\mathcal{Q}} p^{(0)} p'^{(0)}. \end{aligned}$$

The unit η is easily seen to be both a module and a comodule map and prebraided commutativity holds. \square

The above results reduce the study of (successive) scalar extensions of a given H to the study of commutative monoids in the weak center of \mathbf{M}_H .

The next proposition shows that for a given Galois extension of algebras there is always a maximal quantum groupoid with respect to which the extension is Galois.

Proposition 4.12. *Let $N \subset M$ be a Galois extension over the Frobenius Hopf algebroid H . Then the restriction of the Galois map Γ_M provides an isomorphism of Hopf algebroids $H\#C \cong E$ where E is the endomorphism Hopf algebroid of the extension.*

Proof. The structure maps (4.25), (4.26), (4.27) and (4.28) of the smash product are mapped by Γ_M to

$$(4.32) \quad s_E : C \rightarrow E \quad c \mapsto \{m \mapsto mc\}$$

$$(4.33) \quad t_E : C^{\text{op}} \rightarrow E \quad c \mapsto \{m \mapsto cm\}$$

$$(4.34) \quad \Delta_E : E \rightarrow E \otimes_{\mathcal{C}} E \quad \text{such that } \alpha^{[1]}(m)\alpha^{[2]}(m') = \alpha(mm')$$

$$(4.35) \quad \varepsilon_E : E \rightarrow C \quad \alpha \mapsto \alpha(1)$$

respectively, where note that multiplicativity of Δ_E uniquely fixes it by Lemma 4.4, (4.19). Now it is easy to check that $\Gamma_M : H\#C \rightarrow E$ satisfies the axioms of bialgebroid maps. \square

5. CONTRAVARIANT FIBER FUNCTORS

In this section we study functors from the module category of a Hopf algebroid A that correspond to A -Galois extensions of a given algebra N . In this sense we study generalizations of Ulbrich's Theorem [27] relating Hopf-Galois extensions to fiber functors. Technically speaking, however, the functors we study here are very different from the usual fiber functors. They are contravariant hom-functors $\text{Hom}_H(-, M)$ from \mathbf{M}_H to ${}_N\mathbf{M}_N$. So they are colimit preserving but rarely faithful and exact. Still they have some properties that are worthy of discussion. As a preparation we prove

Lemma 5.1. *For H a Frobenius Hopf algebroid the full subcategory $\mathbf{M}_H^{\text{fgp}}$ of \mathbf{M}_H the objects of which are the finitely generated projective H -modules is a monoidal subcategory.*

Proof. It suffices to show that $H \otimes_R H$, the tensor square of the regular object in \mathbf{M}_H is a fgp module. This in turn follows from the existence of the isomorphism [25, (4.1)]

$$\Gamma_{RB} : H \otimes_R H \xrightarrow{\sim} A \otimes_B H, \quad a \otimes_R a' \mapsto a_{(1)} \otimes_B a_{(2)} \circ a'$$

which happens to be an H -module map,

$$\begin{aligned} \Gamma_{RB}(a \triangleleft h^{[1]} \otimes_R a' \triangleleft h'^{[2]}) &= a \star h^{[1]} \star u_k \otimes_B v_k \circ (a' \star h'^{[2]}) \\ &= a \star u_k \otimes_B (v_k \circ a') \star h \end{aligned}$$

thanks to right distributivity in A . Since A_B is fgp, the statement is proven. \square

Theorem 5.2. *Let N be an algebra and A a distributive double algebra. As usual, H denotes the horizontal Hopf algebroid of A .*

(1) *The mappings*

$$M \mapsto F = \text{Hom}_H(-, M) \quad \text{respectively} \quad F \mapsto M = F(H_H)$$

provide mutually inverse category equivalences between the following two categories.

- *The category of H -module algebras M equipped with an algebra map $\hat{\eta} : N \rightarrow M^H \subset M$. The arrows from $\langle M, \hat{\eta} \rangle$ to $\langle M', \hat{\eta}' \rangle$ are the H -module algebra maps $\alpha : M \rightarrow M'$ for which $\alpha \circ \hat{\eta} = \hat{\eta}'$.*
- *The category of opmonoidal functors $F : \mathbf{M}_H^{\text{fgp}} \rightarrow {}_N\mathbf{M}_N^{\text{op}}$ as objects and monoidal natural transformations as arrows.*

(2) *M is an A -extension of N iff F is normal opmonoidal.*

(3) *M is an A -Galois extension of N iff F is strong (op)monoidal.*

(4) *The full subcategories $A\text{-Gal}(N)$ and $\mathcal{F}(\mathbf{M}_H^{\text{fgp}}, {}_N\mathbf{M}_N^{\text{op}})$ of those in (1) selected by the conditions of (3), respectively, are groupoids.*

In this way (1) and (3) establish a category equivalence $A\text{-Gal}(N) \sim \mathcal{F}(\mathbf{M}_H^{\text{fgp}}, {}_N\mathbf{M}_N^{\text{op}})$ between A -Galois extensions of N and strong monoidal functors $\mathbf{M}_H^{\text{fgp}} \rightarrow {}_N\mathbf{M}_N^{\text{op}}$.

Proof. The construction of the functor $M \mapsto F$ goes as follows. Given $\langle M, \hat{\eta} \rangle$ the M is an N^e - H -bimodule so $\text{Hom}_H(-, M)$ is a contravariant functor from H -modules to ${}_N\mathbf{M}_N$. The monoid structure $\langle M, \mu, \eta \rangle$ defines an opmonoidal structure on this functor¹

$$\begin{aligned} \text{Hom}_H(Y, M) \otimes_N \text{Hom}_H(Y', M) &\rightarrow \text{Hom}_H(Y \otimes_R Y', M) & \xi \otimes_N \xi' &\mapsto \mu \circ (\xi \otimes_R \xi') \\ N &\rightarrow \text{Hom}_H(R, M) & n &\mapsto \{r \mapsto n\eta(r)\} \end{aligned}$$

An arrow α is mapped to the monoidal natural transformation $\text{Hom}_H(-, \alpha)$.

Now we construct the functor $F \mapsto M$. Any opmonoidal functor $\langle F, F^2, F^0 \rangle : \mathbf{M}_H^{\text{fgp}} \rightarrow {}_N\mathbf{M}_N^{\text{op}}$ maps comonoids to comonoids. Therefore it maps $\langle H_H, \Delta_R, \varphi_R \rangle$ to a monoid $M := F(H_H)$ in ${}_N\mathbf{M}_N$. The unit of this monoid is the composite

$$\hat{\eta} := N \xrightarrow{F^0} FR \xrightarrow{F(\varphi_R)} M$$

which becomes a k -algebra map by prolongation of the multiplication

$$\hat{\mu} := M \otimes_N M \xrightarrow{F^{H,H}} F(H \otimes_R H) \xrightarrow{F(\Delta_R)} M$$

¹The arrows between N - N -bimodules are always considered in ${}_N\mathbf{M}_N$ and never in ${}_N\mathbf{M}_N^{\text{op}}$.

to a k -algebra multiplication. The M also inherits a right H -module structure from left multiplication $\lambda_h = h \star_-$ via $H \xrightarrow{\lambda} \text{End}(H_H) \xrightarrow{F} \text{End}({}_N M_N)$. We have

$$\begin{aligned} F\lambda_h \circ \hat{\eta} &= F(\varphi_R \circ \lambda_h) \circ F^0 = F(\varphi_R \circ \lambda_{\varphi_T \varphi_R(h)}) \circ F^0 \\ &= F\lambda_{\varphi_T \varphi_R(h)} \circ \hat{\eta} \end{aligned}$$

implying that $\hat{\eta}$ factors uniquely through the inclusion $M^H \subset M$. For each h the action $F\lambda_h$ is an N - N bimodule map which makes M an N^e - H -bimodule. By means of the isomorphism $\iota : m \mapsto \{h \mapsto m \triangleleft h\}$ the monoid $\langle {}_N M_N, \hat{\mu}, \hat{\eta} \rangle$ becomes isomorphic to the convolution monoid $\text{Hom}_H(H, M)$ associated to an H -module algebra $\langle M, \mu, \eta \rangle$ structure on M . Of course, the monoid $\langle M, \mu, \eta \rangle$ arises from the k -algebra structure of M just as the monoid $\langle M, \hat{\mu}, \hat{\eta} \rangle$ does.

$$(5.1) \quad \begin{array}{ccc} FH \otimes FH & \xrightarrow{\iota \otimes \iota} & \text{Hom}_H(H, M) \otimes \text{Hom}_H(H, M) \\ \downarrow & & \downarrow \\ FH \otimes_N FH & \xrightarrow{\iota \otimes_N \iota} & \text{Hom}_H(H, M) \otimes_N \text{Hom}_H(H, M) \\ F^{H,H} \downarrow & & \downarrow \mu \circ (- \otimes_R -) \\ F(H \otimes_R H) & & \text{Hom}_H(H \otimes_R H, M) \\ F\Delta_R \downarrow & & \downarrow - \circ \Delta_R \\ FH & \xrightarrow{\iota} & \text{Hom}_H(H, M) \end{array}$$

This yields the object map of the functor $F \mapsto M$. As for the arrow map take any monoidal natural transformation $\nu : F \rightarrow F'$ and define $\alpha := \nu_H : M \rightarrow M'$. Then by the multiplicativity constraint for F the α is an H -module algebra morphism and the unit constraint implies that

$$\begin{array}{ccccc} N & \xrightarrow{F^0} & FR & \xrightarrow{F\varphi_R} & M \\ \parallel & & \downarrow \nu_R & & \downarrow \alpha \\ N & \xrightarrow{F'^0} & F'R & \xrightarrow{F'\varphi_R} & M' \end{array}$$

is commutative, i.e., $\alpha \circ \hat{\eta} = \hat{\eta}'$.

Now we construct a natural isomorphism ν from the identity functor $F \mapsto F$ to the composite $F \mapsto M \mapsto F$. Choosing a direct summand diagram $Y \xrightarrow{\pi_k} H \xrightarrow{\sigma_k} Y$ for each fgp H -module Y the isomorphism $\iota : M \rightarrow \text{Hom}_H(H, M)$ for $M = FH$ extends to a natural isomorphism $\nu : F \rightarrow \text{Hom}_H(-, M)$ by

$$\begin{array}{ccccc} FY & \xrightarrow{F\sigma_k} & FH & \xrightarrow{F\pi_k} & FY \\ \nu_Y \downarrow & & \downarrow \iota & & \downarrow \nu_Y \\ \text{Hom}_H(Y, M) & \xrightarrow{- \circ \sigma_k} & \text{Hom}_H(H, M) & \xrightarrow{- \circ \pi_k} & \text{Hom}_H(Y, M) \end{array}$$

This natural isomorphism will then be automatically monoidal due to the interplay between the multiplications $\hat{\mu}$ and μ seen on the diagram (5.1).

The natural isomorphism from the identity functor $M \mapsto M$ to the composite $M \mapsto F \mapsto M$ is just $\iota : M \rightarrow \text{Hom}_H(H, M)$ viewed as a map of monoids in

${}_N\mathbf{M}_N$. So in particular $\iota \circ \hat{\eta}$ is equal to $F\varphi_R \circ F^0$ for the opmonoidal functor $F = \text{Hom}_H(-, M)$. This completes the proof of the equivalence in (1).

By Lemma 2.2 the unique arrow $N \rightarrow M^H$ factorizing $\hat{\eta}$ is an isomorphism iff F^0 is an isomorphism, i.e., iff F is normal. This proves (2).

Strong (op)monoidality of F is equivalent to invertibility of F^0 and $F^{H,H}$. By the natural isomorphism $F \cong \text{Hom}_H(H, M)$ the latter is equivalent to invertibility of the left vertical arrow in the next diagram.

$$\begin{array}{ccc} \text{Hom}_H(H, M) \otimes_N \text{Hom}_H(H, M) & \xleftarrow{\sim} & M \otimes_N M \\ \mu \circ (- \otimes_R -) \downarrow & & \downarrow \gamma^M \\ \text{Hom}_H(H \otimes_R H, M) & \xrightarrow{\sim} & M \otimes_T A \end{array}$$

where the lower horizontal arrow is given by a composition of isomorphisms

$$\text{Hom}_H(H \otimes_R H, M) \xrightarrow{\sim} \text{Hom}_H(H, M \otimes_R H) \xrightarrow{\sim} M \otimes_R H \xrightarrow{M \otimes S^{-1}} M \otimes_T V$$

performing the mappings

$$\chi \mapsto \chi(- \otimes_R x^j) \otimes_R y^j \mapsto \chi(i \otimes_R x^j) \otimes_R y^j \mapsto \chi(i \otimes_R u^j) \otimes_T v^j.$$

Commutativity of the diagram now follows from the simple calculation

$$(m \triangleleft i)(m' \triangleleft u^k) \otimes_T v^k = mm'^{(0)} \otimes_T m'^{(1)} = \gamma^M(m \otimes_N m')$$

Therefore γ^M is invertible iff $F^{H,H}$ is invertible. Adding the condition that $N \subset M$ is an A -extension we obtain (3).

Since H_H is a Frobenius algebra, it is a selfdual object in \mathbf{M}_H . Therefore any monoidal natural transformation between strong monoidal functors from $\mathbf{M}_H^{\text{fgp}}$ is invertible at H_H [21] and therefore it is invertible everywhere. This proves (4). \square

Corollary 5.3. *The map $M \mapsto \text{Hom}_H(-, M)$ is a category equivalence between the category $A\text{-Gal}(N)$ of A -Galois extensions of N and the category $\mathcal{F}(\mathbf{M}_H, {}_N\mathbf{M}_N^{\text{op}})$ of colimit preserving opmonoidal functors the restrictions of which to $\mathbf{M}_H^{\text{fgp}}$ is strong (op)monoidal.*

Proof. If $F : \mathbf{M}_H \rightarrow {}_N\mathbf{M}_N^{\text{op}}$ is colimit preserving then the corresponding $F^{\text{op}} : \mathbf{M}_H^{\text{op}} \rightarrow {}_N\mathbf{M}_N$ is limit preserving and H_H is a cogenerator for \mathbf{M}_H^{op} . The conditions for the special adjoint functor theorem [17] hold, so F^{op} has a left adjoint. It follows that F^{op} is a hom-functor, $F \cong \text{Hom}_{\mathbf{M}_H^{\text{op}}}(M, -)$, i.e., $F \cong \text{Hom}_H(-, M)$. Now Theorem 5.2 implies that F has a strong restriction to the fgp modules precisely when $N \subset M$ is A -Galois. Vice versa, every Galois extension M gives rise to a colimit preserving opmonoidal functor $\text{Hom}_H(-, M)$ the restriction of which to $\mathbf{M}_H^{\text{fgp}}$ is strong. \square

As an application of the strong monoidal functor $\text{Hom}(-, M)$ we present here another characterization of Galois extensions over DDA's. In order to understand the terminology "left distributivity" let us look at multiplication of M as a vertical one and the right H -action \triangleleft as a partially defined horizontal multiplication between M and H .

Proposition 5.4. *Let A be a DDA and M be a right A -module algebra with $N = M^H$. Then $N \subset M$ is A -Galois if and only if $\psi = _ \triangleleft e : M \rightarrow N$ is a Frobenius homomorphism and the "left distributivity" rule*

$$m \triangleleft (a \circ a') = (m_{[1]} \triangleleft a)(m_{[2]} \triangleleft a')$$

holds for all $m \in M$ and $a, a' \in A$. Here $m_{[1]} \otimes_N m_{[2]}$ is the coproduct associated to the Frobenius structure on $N \subset M$ defined by ψ .

Note that "right distributivity" $(mm') \triangleleft a = (m \triangleleft a^{[1]})(m' \triangleleft a^{[2]})$ holds for all right module algebras. Note also that \circ for H plays the role of convolution product while the ordinary product is \star .

Proof. Necessity: Consider the contravariant functor $\text{Hom}_H(_, M) : \mathbf{M}_H \rightarrow {}_N\mathbf{M}_N$. It is strong monoidal, so maps monoids to comonoids, comonoids to monoids, and Frobenius algebras to Frobenius algebras. Therefore it maps $\langle A, \Delta_R, \varphi_R, \mu_V, R \hookrightarrow A \rangle$ to some Frobenius algebra structure on $\text{Hom}_H(A, M) \cong M \in {}_N\mathbf{M}_N$. Since a Frobenius algebra structure in ${}_N\mathbf{M}_N$ is uniquely determined by the algebra structure and by the Frobenius homomorphism, the counit, it is sufficient to check that the image of $\langle A, \Delta_R, \varphi_R \rangle$ is the convolution algebra $\text{Hom}_H(A, M)$ and the image of the unit $R \hookrightarrow A$ is ψ . Then the coproduct must have the form

$$\Delta_M(m) \equiv m_{[1]} \otimes_N m_{[2]} = \sum_i m e_i \otimes_N f_i$$

where $\sum_i e_i \otimes_N f_i$ is the dual basis of ψ . This means that the composite

$$(5.2) \quad \begin{array}{ccc} \text{Hom}_H(A, M) & \xrightarrow{\text{Hom}(\mu_V, M)} & \text{Hom}_H(H \otimes_R H, M) \\ & & \downarrow [\mu \circ (- \otimes_R -)]^{-1} \\ & & \text{Hom}_H(A, M) \otimes_N \text{Hom}_H(A, M) \end{array}$$

must be the map

$$(m \triangleleft _) \mapsto (m_{[1]} \triangleleft _) \otimes_N (m_{[2]} \triangleleft _)$$

Applying $\mu \circ (- \otimes_R -)$ we obtain left distributivity.

Sufficiency: Consider the map $M \otimes_T A \rightarrow M \otimes_N M$ defined by $m \otimes_T a \mapsto m_{[1]} \otimes_N m_{[2]} \triangleleft a$. Then

$$\begin{aligned} \gamma^M(m_{[1]} \otimes_N m_{[2]} \triangleleft a) &= m_{[1]}(m_{[2]} \triangleleft a)^{(0)} \otimes_T (m_{[2]} \triangleleft a)^{(1)} \\ &= (m_{[1]} \triangleleft i)(m_{[2]} \triangleleft u^k) \otimes_T v^k \star a = (m \triangleleft (i \circ u^k)) \otimes_T v^k \star a \\ &= m \otimes_T \varphi_T(u^k) \star v^k \star a = m \otimes_T a \end{aligned}$$

proves that γ^M is epi. □

6. A MONOIDAL DUALITY

Given a right bialgebroid H over R , an H -module algebra M and an algebra map $N \rightarrow M^H$ we can look for a duality between - full subcategories of $\mathbf{-M}_H$ and

${}_N\mathbf{M}_N$ in the following form. The M being an N^e - H -bimodule, it determines two functors

$$(6.1) \quad J : {}_N\mathbf{M}_N^{\text{op}} \rightarrow \mathbf{M}_H \quad X \mapsto \text{Hom}_{N^e}(X, M)$$

$$(6.2) \quad K : \mathbf{M}_H \rightarrow {}_N\mathbf{M}_N^{\text{op}} \quad Y \mapsto \text{Hom}_H(Y, M),$$

the M -dual functors, that are in adjunction $K \dashv J$. The counit and unit of the adjunction are just the natural homomorphism to the double dual,

$$\begin{aligned} \sigma_X : X &\rightarrow \text{Hom}_H(\text{Hom}_{N^e}(X, M), M) && \in {}_N\mathbf{M}_N \\ \sigma_Y : Y &\rightarrow \text{Hom}_{N^e}(\text{Hom}_H(Y, M), M) && \in \mathbf{M}_H \end{aligned}$$

By definition they are isomorphisms precisely for the M -reflexive modules [1]. Either one of the M -dual functors map reflexive modules to reflexive ones, so the restriction of J and K to the M -reflexive modules provides an adjoint equivalence

$$(6.3) \quad \mathbf{M}_H^{M\text{-ref}} \sim ({}_N\mathbf{M}_N^{M\text{-ref}})^{\text{op}},$$

that is to say, a duality between the reflexive modules themselves.

Since M has monoid structures both in \mathbf{M}_H and ${}_N\mathbf{M}_N$, the functor J is monoidal and K is opmonoidal,

$$\left. \begin{aligned} J_{X, X'} : JX \otimes_R JX' &\rightarrow J(X \otimes_N X') && (\xi \otimes_R \xi') \mapsto \hat{\mu} \circ (\xi \otimes_N \xi') \\ J_0 : R &\rightarrow JN && r \mapsto \{n \mapsto \hat{\eta}(n) \triangleleft r\} \end{aligned} \right\} \in \mathbf{M}_H$$

$$\left. \begin{aligned} K^{Y, Y'} : KY \otimes_N KY' &\rightarrow K(Y \otimes_R Y') && (\beta \otimes_N \beta') \mapsto \mu \circ (\beta \otimes_R \beta') \\ K^0 : N &\rightarrow KR && n \mapsto \{r \mapsto n \cdot \eta(r)\} \end{aligned} \right\} \in {}_N\mathbf{M}_N$$

They are mates under the given adjunction $K \dashv J$, that is to say,

$$(6.4) \quad K^{JX, JX'} \circ (\sigma_X \otimes_N \sigma_{X'}) = KJ_{X, X'} \circ \sigma_{X \otimes_N X'}$$

$$(6.5) \quad K^0 = KJ_0 \circ \sigma_N$$

$$(6.6) \quad J_{KY, KY'} \circ (\sigma_Y \otimes_R \sigma_{Y'}) = JK^{Y, Y'} \circ \sigma_{Y \otimes_R Y'}$$

$$(6.7) \quad J_0 = JK^0 \circ \sigma_R$$

These equations are simple consequences of the fact that the two monoid structures on M come from the same k -algebra structure,

$$\begin{array}{ccc} M \otimes M & \longrightarrow & M \otimes_R M & & k & \longrightarrow & R \\ & & \downarrow \mu & & \downarrow & & \downarrow \eta \\ & & M & & N & \xrightarrow{\hat{\eta}} & M \end{array}$$

We have, as in Theorem 5.2 (2), that K is normal iff the map $N \rightarrow M^H$ is an isomorphism and J is normal iff the map $R \rightarrow M^N$ is an isomorphism.

In order to find monoidal subcategories in \mathbf{M}_H and ${}_N\mathbf{M}_N$ that become monoidally dual under (6.3) we have to make further assumptions. Assume that the right bialgebroid H is that of the horizontal Hopf algebroid of a distributive double algebra A and assume that $N \subset M$ is A -Galois. We know from Lemma 5.1 and Theorem 5.2 that $\mathbf{M}_H^{\text{fgp}}$ is a full monoidal subcategory and the restriction of K to this subcategory is strong opmonoidal. Therefore the restriction of K will provide a monoidal equivalence iff all the fgp H -modules are M -reflexive. Since the class

of reflexive modules is closed under taking direct summands and finite direct sums, this happens precisely when the regular object H_H is M -reflexive. Now

$$\sigma_H : H \rightarrow \text{Hom}_{N_e}(\text{Hom}_H(H, M), M) \xrightarrow{\sim} \text{End}_N M_N$$

being just the canonical embedding to the endomorphism Hopf algebroid E we are left with considering the case when H is E and acts canonically on M .

Theorem 6.1. *Let E be the endomorphism Hopf algebroid associated to the balanced depth 2 Frobenius extension $N \subset M$. Then the functor*

$$\text{Hom}_E(-, M) : \mathbf{M}_E^{\text{fgp}} \rightarrow {}_N \mathbf{M}_N^{\text{op}}$$

provides a monoidal duality between the categories of all fgp E -modules and those N - N -bimodules that are direct summands of finite direct sums of M 's.

Proof. E_E is M -reflexive by construction. Thus $\mathbf{M}_E^{\text{fgp}}$ is a full subcategory of the category of reflexive modules and (6.3) restricts to a category equivalence F . Since $\mathbf{M}_E^{\text{fgp}}$ is generated by direct sums and direct summands from E_E , the same holds for the N - N -bimodules in the image of F and for $F(E_E) \cong M$. The extension $N \subset M$ is E -Galois therefore $\text{Hom}_E(-, M)$ is strong monoidal on fgp modules. \square

Depending on the applications the content of the theorem varies from trivialities to nontrivial statements. For example, if $k \subset K$ is a separable field extension - including the case of classical Galois field extensions - then $E = \text{End } K_k$ is the Hopf algebroid version of the weak Hopf algebra constructed in [24] and its representation category is trivial: The theorem reduces to the statement that the category of finite dimensional k -vector spaces is selfdual.

If N is a strongly G -graded k -algebra for a finite group G and $N_e \subset N$ has centralizer $k \cdot 1$ then choosing $M = N \# (kG)^*$ we obtain that E is the group algebra kG acting canonically on the smash product M .

If N is the observable algebra in rational quantum field theory and M is the algebra of charge carrying fields then ${}_N M$ is freely generated by finitely many fields $f_q^i \in M$ each of them implementing a localized endomorphism ρ_q of N , i.e., $f_q^i n = \rho_q(n) f_q^i$, $n \in N$, $i = 1, \dots, I_q$. The Doplicher-Haag-Roberts category $\text{DHR}(N)$ is the full subcategory of $\text{End } N$ the objects of which are finite direct sums of ρ_q 's and is a monoidal category by composition of endomorphisms. One has a contravariant monoidal equivalence between $\text{DHR}(N)$ and the category of N - N -bimodules that are finite direct sums of the bimodules $N f_q^i$. Hence Theorem 6.1 gives a monoidal equivalence $\mathbf{M}_E^{\text{fgp}} \simeq \text{DHR}(N)$ and therefore the Hopf algebroid E can be interpreted as the global gauge symmetry of the superselection sectors.

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