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ON TWISTED TENSOR PRODUCTS OF ALGEBRAS

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Abstract. The problems considered in this paper are motivated by non–commutative geometry. Starting from two unital algebras A and B over a commutative ring $\mathbb{K}$ we describe all triples (C, i_A, i_B), where C is a unital algebra and i_A and i_B are inclusions of A and B into C such that the canonical linear map $(i_A, i_B) : A \otimes B \to C$ is a linear isomorphism. We discuss possibilities to construct differential forms and modules over C from differential forms and modules over A and B, and give a description of deformations of such structures using cohomological methods.

1. Introduction

Although the problems we consider are from pure algebra (and topological algebra), the motivation comes from non–commutative differential geometry: The simple question we started from is, given two algebras which are supposed to describe some "spaces", what is an appropriate representative of the product of the two "spaces"? Thinking of the commutative case one would be led to considering the (topological) tensor product of the two algebras. But in the non-commutative case this means that one assumes that functions on the two factors commute with each other, although the functions on the individual factors do not commute among themselves, and we see no reason to assume this. In this paper we study algebras, which are in a certain sense very close to the tensor product of the given ones, and in particular deformations of the tensor product.

It should also be remarked that special examples of such algebras, notably the non–commutative two tori and more generally crossed products of C^*–algebras by groups, already play an important role in non-commutative geometry.

The problem may as well be viewed as a question of decompositions of given algebras: Suppose that a unital algebra is, as a linear space, the tensor product of two subalgebras. What does this say about the algebra structure? From this point of view the analogous problems for discrete groups, Lie groups, Lie algebras and Hopf algebras have been studied (see e.g. [Majid, 1990], [Michor, 1990] and [Takeuchi, 1981]), often under the name of matched pairs or factorization of structures. In the study of the Hopf algebra case the basic conditions 2.4(1) for algebra structures have been obtained (c.f. [Majid, 1994, 7.2.3]). It turns out that the case of algebras

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is the most complicated one, since in all other cases the problem reduces to the study of mutual actions of the two factors on each other which are compatible in a certain sense, while in the algebra case such a reduction is not possible. Anyhow, for our work the point of view of decompositions is less important, since our main aim is the study of deformations.

We will study the problem without assuming that the algebras are endowed with topologies. In fact, all constructions can be carried out precisely in the same way in categories of vector spaces and linear maps, such that the Hom–functor $L(\ \ ,\ \ )$ lifts to the category, and which admit a tensor product $\hat{\otimes}$ such that there is a natural isomorphism $L(E\hat{\otimes}F,G) \cong L(E,L(F,G))$ (i.e. in monoidally closed categories). This is the case for example in the category of Banach spaces and continuous linear maps with the projective tensor product or, more generally, in the category of convenient vector spaces and bounded linear maps with the bornological tensor product (c.f. [Frölicher–Kriegl, 1988]).

2. Twisted tensor products

Throughout this paper we fix some commutative ring $\mathbb{K}$ with unit. Later on when we will study deformations we will specialize to $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We assume all algebras to be unital and all homomorphisms to preserve units.

2.1. Definition. Let A and B be algebras over $\mathbb{K}$. A *twisted tensor product* of A and B is an algebra C together with two injective algebra homomorphisms $i_A : A \to C$ and $i_B : B \to C$ such that the canonical linear map $(i_A, i_B)\colon A \otimes_{\mathbb{K}} B \to C$ defined by $(i_A, i_B)(a\otimes b) := i_A(a)\cdot i_B(b)$ is a linear isomorphism. An isomorphism of twisted tensor products is an isomorphism of algebras which respects the inclusions of A and B.

2.2. There is a simple way to construct candidates for twisted tensor products as follows: Let $\tau : B \otimes A \to A \otimes B$ be a $\mathbb{K}$-linear mapping, such that $\tau(b \otimes 1) = 1 \otimes b$ and $\tau(1 \otimes a) = a \otimes 1$. Then on $A \otimes B$ define a multiplication μ_τ by $\mu_\tau := (\mu_A \otimes \mu_B) \circ (A \otimes \tau \otimes B)$. We write $A \otimes \tau \otimes B$ for $id_A \otimes \tau \otimes id_B$. This is also justified by the fact that this is the functor $A \otimes\ \ \ \otimes B$ applied to the map τ.

Next define $i_A : A \to A\otimes B$ by $i_A(a) := a\otimes 1$ and likewise $i_B : B \to A\otimes B$. These are algebra homomorphisms by the conditions on τ. Obviously, if the multiplication μ_τ is associative, then $(A \otimes B, \mu_\tau)$ is a twisted tensor product of A and B.

Now the associativity of the multiplication μ_τ can be characterized in terms of τ as follows:

2.3. Proposition/Definition. *Suppose that $\tau(b\otimes 1) = 1\otimes b$ and $\tau(1\otimes a) = a\otimes 1$. Then the multiplication μ_τ is associative if and only if we have:*

$$\tau \circ (\mu_B \otimes \mu_A) = \mu_\tau \circ (\tau \otimes \tau) \circ (B \otimes \tau \otimes A)$$

A mapping τ which satisfies these conditions is called a twisting map for A and B, and we denote the algebra $(A \otimes B, \mu_\tau)$ by $A \otimes_\tau B$.

Proof. Let us first assume that μ_τ is associative. We also write $\cdot_\tau$ for the multiplication μ_τ. Since μ_τ is associative we get

$$\begin{aligned}(1\otimes b)\cdot_\tau(a_1\otimes b_1)\cdot_\tau(a\otimes 1) &=\\ ((1\otimes b)\cdot_\tau(a_1\otimes 1))\cdot_\tau((1\otimes b_1)\cdot_\tau(a\otimes 1)) &=\\ \tau(b\otimes a_1)\cdot_\tau\tau(b_1\otimes a).\end{aligned}$$

But using this we compute:

$$\begin{aligned}\tau(b_1b_2\otimes a_1a_2) &= (1\otimes b_1b_2)\cdot_\tau(a_1a_2\otimes 1) =\\ &= (1\otimes b_1)\cdot_\tau((1\otimes b_2)\cdot_\tau(a_1\otimes 1))\cdot_\tau(a_2\otimes 1) =\\ &= (1\otimes b_1)\cdot_\tau\tau(b_2\otimes a_1)\cdot_\tau(a_2\otimes 1) =\\ &= \mu_\tau\circ(\tau\otimes\tau)\circ(B\otimes\tau\otimes A)(b_1\otimes b_2\otimes a_1\otimes a_2),\end{aligned}$$

so τ indeed satisfies the condition.

So let us conversely assume that τ satisfies the conditions. Then

$$\begin{aligned}(1\otimes b)\cdot_\tau((a_1\otimes 1)\cdot_\tau(a\otimes 1)) &= (1\otimes b)\cdot_\tau(a_1a\otimes 1) =\\ &= \tau(b\otimes a_1a) = \mu_\tau\circ(\tau\otimes\tau)\circ(B\otimes\tau\otimes A)(b\otimes 1\otimes a_1\otimes a) =\\ &= \mu_\tau(\tau(b\otimes a_1)\otimes a\otimes 1) = \tau(b\otimes a_1)\cdot_\tau(a\otimes 1) =\\ &= ((1\otimes b)\cdot_\tau(a_1\otimes 1))\cdot_\tau(a\otimes 1).\end{aligned}$$

and similarly with $(a_1\otimes 1)$ replaced by $(1\otimes b_1)$. Next, from the definition of μ_τ it is obvious that μ_τ is a left A–module homomorphism for the canonical left actions of A and a right B–module homomorphism for the canonical right actions of B. Via the above computation this implies that associativity holds if the middle element is either of the form $(a\otimes 1)$ or of the form $(1\otimes b)$. But then we may compute as follows:

$$\begin{aligned}((a_0\otimes b_0)\cdot_\tau(a_1\otimes b_1))\cdot_\tau(a_2\otimes b_2) &=\\ &= ((a_0\otimes b_0)\cdot_\tau((a_1\otimes 1)\cdot_\tau(1\otimes b_1)))\cdot_\tau(a_2\otimes b_2) =\\ &= (((a_0\otimes b_0)\cdot_\tau(a_1\otimes 1))\cdot_\tau(1\otimes b_1))\cdot_\tau(a_2\otimes b_2) =\\ &= ((a_0\otimes b_0)\cdot_\tau(a_1\otimes 1))\cdot_\tau((1\otimes b_1)\cdot_\tau(a_2\otimes b_2)),\end{aligned}$$

and in the same way the last line is easily seen to be equal to

$$(a_0\otimes b_0)\cdot_\tau((a_1\otimes b_1)\cdot_\tau(a_2\otimes b_2)). \quad\square$$

2.4. Remarks. (1): The symmetric condition for being a twisting map used in 2.3 can be split into the two conditions:

$$\begin{aligned}\tau\circ(B\otimes\mu_A) &= (\mu_A\otimes B)\circ(A\otimes\tau)\circ(\tau\otimes A)\\ \tau\circ(\mu_B\otimes A) &= (A\otimes\mu_B)\circ(\tau\otimes B)\circ(B\otimes\tau)\end{aligned}$$

It is obvious that the condition in 2.3 implies these two conditions by setting appropriate entries equal to one. On the other hand, the condition in 2.3 can be easily deduced from successive applications of the two conditions above.
(2): Note that the multiplications on $A \otimes B$ defined by twisting maps are exactly those associative multiplications which are left A–module homomorphisms and right B–module homomorphisms for the canonical actions, and for which $1 \otimes 1$ is a unit. This can be seen as follows: Let μ be such a multiplication, and define τ by $\tau(b \otimes a) = \mu(1 \otimes b \otimes a \otimes 1)$. Then by the module homomorphism property $\mu = \mu_\tau$. Moreover, $\tau(b \otimes 1) = \mu(1 \otimes b \otimes 1 \otimes 1) = 1 \otimes b$, and in the same way $\tau(1 \otimes a) = a \otimes 1$. Thus from proposition 2.3 we see that τ is a twisting map.

2.5. Next we present an alternative characterization of twisting maps which will be very useful when dealing with differential forms. Let A and B be unital algebras and consider the space $L(A, A \otimes B)$ of linear maps. On this space we define a multiplication $*$ by $\varphi * \psi := (A \otimes \mu_B) \circ (\varphi \otimes B) \circ \psi$, where μ_B denotes the multiplication on B.

Proposition. *$(L(A, A \otimes B), *)$ is an associative unital algebra with unit given by the map $a \mapsto a \otimes 1$.*

Proof. We compute:

$$\begin{aligned}(\varphi * \psi) * \omega &= (A \otimes \mu_B) \circ ((\varphi * \psi) \otimes B) \circ \omega = \\ &= (A \otimes \mu_B) \circ (A \otimes \mu_B \otimes B) \circ (\varphi \otimes B \otimes B) \circ (\psi \otimes B) \circ \omega = \\ &= (A \otimes \mu_B) \circ (\varphi \otimes \mu_B) \circ (\psi \otimes B) \circ \omega = \varphi * (\psi * \omega),\end{aligned}$$

since the multiplication on B is associative. Obviously, $a \mapsto a \otimes 1$ is a unit for the multiplication $*$. □

Similarly, we define a multiplication on $L(B, A \otimes B)$ by $\varphi * \psi = (\mu_A \otimes B) \circ (A \otimes \psi) \circ \varphi$, and as above one easily proves that this is associative with unit $b \mapsto 1 \otimes b$.

2.6. Proposition. *A linear map $\tau : B \otimes A \to A \otimes B$ is a twisting map if and only if the two associated maps $B \to L(A, A \otimes B)$ and $A \to L(B, A \otimes B)$ are homomorphisms of unital algebras.*

Proof. The condition that the two associated maps preserve the units mean exactly that $\tau(1 \otimes a) = a \otimes 1$ and that $\tau(b \otimes 1) = 1 \otimes b$. Now let us write τ_b for the map $a \mapsto \tau(b \otimes a)$. Then the condition that the first associated map is an algebra homomorphism means that $\tau_{b_1 b_2} = \tau_{b_1} * \tau_{b_2}$, and by definition of the multiplication $*$ this means that $\tau_{b_1 b_2} = (A \otimes \mu_B) \circ (\tau_{b_1} \otimes B) \circ \tau_{b_2}$, and this is precisely the second condition of remark 2.4(1). In the same way the condition that the second associated map is an algebra homomorphism is easily seen to be the first condition of that remark, so we get the result. □

2.7. Proposition. *Let (C, i_A, i_B) be a twisted tensor product of A and B. Then there is a unique twisting map $\tau : B \otimes A \to A \otimes B$ such that C is isomorphic to $A \otimes_\tau B$ as a twisted tensor product.*

Proof. Let $\varphi : A \otimes B \to C$ be the $\mathbb{K}$–module isomorphism used in the definition of a twisted tensor product. Then we define $\tau : B \otimes A \to A \otimes B$ by $\tau(b \otimes a) :=$

$\varphi^{-1}(i_B(b)\cdot i_A(a))$. Then

$$\varphi((a\otimes 1)\cdot_\tau(a_1\otimes b_1))=\varphi(aa_1\otimes b_1)=$$
$$=i_A(aa_1)i_B(b_1)=i_A(a)i_A(a_1)i_B(b_1)=\varphi(a\otimes 1)\varphi(a_1\otimes b_1),$$

and likewise $\varphi((a_1\otimes b_1)\cdot_\tau(1\otimes b))=\varphi(a_1\otimes b_1)\varphi(1\otimes b)$. But by definition of μ_τ we have $(a_0\otimes b_0)\cdot_\tau(a_1\otimes b_1)=(a_0\otimes 1)\cdot_\tau \tau(b_0\otimes a_1)\cdot_\tau(1\otimes b_1)$, so φ is an algebra homomorphism since $\varphi(\tau(b\otimes a))=i_B(b)i_A(a)=\varphi(1\otimes b)\varphi(a\otimes 1)$.

Finally, uniqueness of τ is obvious since any algebra homomorphism $A\otimes_\tau B\to A\otimes_{\tau'}B$ which is compatible with the inclusions of A and B must be the identity. $\square$

2.8. Examples. (1): Let G be a discrete group which acts from the left by automorphisms on an algebra A, and let $\mathbb{K}[G]$ be the group algebra. Then there is a natural twisting map $\tau:\mathbb{K}[G]\otimes A\to A\otimes\mathbb{K}[G]$ induced by $\tau(g\otimes a):=(g\cdot a)\otimes g$. One immediately verifies by a direct computation that this is indeed a twisting map. Similarly, a right action of G on A induces a twisting map $A\otimes\mathbb{K}[G]\to\mathbb{K}[G]\otimes A$. This construction forms the basis for crossed products.

(2): Consider the algebra $A:=\mathbb{C}[z,z^{-1}]$ of complex Laurent–polynomials in one variable and let q be a complex number of modulus 1. Then define $\tau:A\otimes A\to A\otimes A$ by $\tau(z^k\otimes z^\ell):=q^{k\ell}z^\ell\otimes z^k$. Again a simple direct computation shows that this defines a twisting map.

In fact, this example is just a special instance of the first one, since we can identify A with the complex group ring of $\mathbb{Z}$ as well as with the algebra of trigonometric polynomials on S^1. Now the left action of $\mathbb{Z}$ on S^1 defined by $n\cdot z=q^nz$ induces an action on the algebra of trigonometric polynomials and applying the construction of (1) we get exactly the twisting map defined above.

Note that one can complete the algebra A to the algebra of Schwartz sequences, i.e. sequences which decay faster than any polynomial, and the above twisting map is still well defined and continuous for the natural Fréchet topology. Then this completion can be identified with the space of smooth functions on S^1 as well as with a smooth version of the group algebra of $\mathbb{Z}$, and the above construction leads to the smooth version of the famous non-commutative two–tori. It has also been shown in [Cap-Michor-Schichl, 1993] that on a non–commutative two–torus there is again a natural twisting map.

(3): The conditions 2.4(1) for being a twisting map can in several cases be obtained as a consequence of naturality conditions. Suppose that we have given a category of modules and module homomorphisms which is closed under the tensor product and equipped with a natural transformation τ between the tensor product and the opposite tensor product which is compatible with the tensor product in the sense that $\tau_{A\otimes B,C}=\tau_{A,C}\otimes B\circ A\otimes\tau_{B,C}$ and similarly for $\tau_{A,B\otimes C}$. Suppose further that A and B are algebras in this category, i.e. that there are associative multiplications $\mu_A:A\otimes A\to A$ and $\mu_B:B\otimes B\to B$ which are morphisms in the category. Then the value $\tau_{B,A}:B\otimes A\to A\otimes B$ of the natural transformation τ is automatically a twisting map since the conditions 2.4(1) are precisely the conditions defining a natural transformation applied to the maps to $B\otimes\mu_A$ and $\mu_B\otimes A$, respectively.

Nontrivial natural transformations as above have been constructed in several situations by S. Majid, for example on the category of all representations of a quasitriangular Hopf algebra. These twisting maps satisfy an additional condition thus leading to braided tensor categories, the algebras in which are also called braided groups, see e.g. [Majid, 1993].

3. Differential forms and modules over twisted tensor products

In this section we want to study the following problem: Suppose we have a twisting map $\tau : B \otimes A \to A \otimes B$ and the algebras A and B are equipped with graded differential algebras of differential forms, or with fixed modules. Can we construct in this situation differential forms or modules over $A \otimes_\tau B$, respectively?

3.1. Let A be a unital algebra and let $\mathcal{B}$ be a unital graded differential algebra with differential $d_{\mathcal{B}}$, and consider the algebra $L(A, A \otimes \mathcal{B})$ of linear maps, with multiplication $*$ as defined in 2.5. Obviously, this is a graded algebra with respect to the grading inherited from the grading of $\mathcal{B}$. Now we define a differential on this algebra by $d\varphi := (A \otimes d_{\mathcal{B}}) \circ \varphi$.

Proposition. *$(L(A, A \otimes \mathcal{B}), *, d)$ is a graded differential algebra.*

Proof. We only have to prove that d is a graded derivation with respect to $*$:

$$\begin{aligned} d(\varphi * \psi) &= (A \otimes d_{\mathcal{B}}) \circ (A \otimes \mu_{\mathcal{B}}) \circ (\varphi \otimes \mathcal{B}) \circ \psi = \\ &= (A \otimes \mu_{\mathcal{B}}) \circ (A \otimes d_{\mathcal{B}} \otimes \mathcal{B} + A \otimes \varepsilon \otimes d_{\mathcal{B}}) \circ (\varphi \otimes \mathcal{B}) \circ \psi, \end{aligned}$$

where ε denotes the grading of $\mathcal{B}$, i.e. it is given by multiplication with $(-1)^k$ on $\mathcal{B}_k$. Now the first term in the sum equals

$$(A \otimes \mu_{\mathcal{B}}) \circ (d\varphi \otimes \mathcal{B}) \circ \psi = d\varphi * \psi,$$

while in the second for homogeneous φ, ε is just multiplication by $(-1)^{|\varphi|}$, and this term becomes

$$(-1)^{|\varphi|}(A \otimes \mu_{\mathcal{B}}) \circ (\varphi \otimes d_{\mathcal{B}}) \circ \psi = (-1)^{|\varphi|}\varphi * d\psi. \qquad \square$$

3.2. Now start with two graded differential algebras $\mathcal{A}$ and $\mathcal{B}$, a twisting map $\tau : B \otimes A \to A \otimes B$, where $A = \mathcal{A}_0$ and $B = \mathcal{B}_0$ and let us denote by $\Omega(\quad)$ the functor which assigns to a unital algebra the graded differential algebra of universal differential forms (c.f. [Karoubi 1982,1983], and [Cap–Kriegl–Michor–Vanzura 1993] for a construction in a topological setting). Consider the map $B \to L(A, A \otimes \mathcal{B})$ associated to τ. By 2.6 this is an algebra homomorphism to the zero component of a graded differential algebra, so directly by the universal property of the universal differential forms this prolongs to a homomorphism of graded differential algebras $\Omega(B) \to L(A, A \otimes \mathcal{B})$.

Now let us assume that $\mathcal{B} = \Omega(B)$. Then we claim that the associated map $\tilde{\tau} \colon \Omega(B) \otimes A \to A \otimes \Omega(B)$ is again a twisting map, which is moreover compatible with the grading and with the differential $A \otimes d$, i.e. $\tilde{\tau}(d\omega \otimes a) = (A \otimes d)(\tilde{\tau}(\omega \otimes a))$. In fact, the latter condition is clear from the fact that we had a homomorphism of graded differential algebras. So by 2.6 we just have to show that the mapping $\check{\tilde{\tau}} : A \to L(\Omega(B), A \otimes \Omega(B))$ associated to $\tilde{\tau}$ is a homomorphism of unital algebras.

Let us first show that this map preserves the unit, i.e. that $\tilde{\tau}(\omega \otimes 1) = 1 \otimes \omega$. Now if $\omega \in \Omega_0(B)$ this is clear since τ is a twisting map. Moreover if it is true for ω then it is true for $d\omega$ since by construction $\tilde{\tau}(d\omega \otimes a) = (A \otimes d)(\tilde{\tau}(\omega \otimes a))$. Finally, it is true for $\omega_1\omega_2$ if it is true for each ω_i by the algebra homomorphism property. Thus, the result follows since the elements of the form $b_0 db_1 \dots db_n$ span the space $\Omega(B)$.

It remains to show that

$$\check{\tilde{\tau}}(a_1 a_2) = \check{\tilde{\tau}}(a_1) * \check{\tilde{\tau}}(a_2) \in L(\Omega(B), A \otimes \Omega(B)), \tag{1}$$

so we have to show that this holds when evaluating at any $\omega \in \Omega(B)$. For any $a \in A$ the map $\check{\tilde{\tau}}(a) : \Omega(B) \to A \otimes \Omega(B)$ satisfies $\check{\tilde{\tau}}(a)(d\omega) = (A \otimes d)(\check{\tilde{\tau}}(a)(\omega))$. But now one easily verifies that also $\check{\tilde{\tau}}(a_1) * \check{\tilde{\tau}}(a_2)$ is compatible with the differential in the above sense. Thus it follows that if (1) holds when evaluating on ω then it also holds when evaluating on $d\omega$.

Now (1) holds when evaluating on ω if and only if $\tilde{\tau}_\omega \circ \mu_A = (\mu_A \otimes \Omega(B)) \circ (A \otimes \tilde{\tau}) \circ (\tilde{\tau}_\omega \otimes A)$, where we write $\tilde{\tau}_\omega$ for the map $a \mapsto \tilde{\tau}(\omega \otimes a)$. On the other hand, since the map $\Omega(B) \to L(A, A \otimes \Omega(B))$ is an algebra homomorphism we see that $\tilde{\tau} \circ (\mu_{\Omega(B)} \otimes A) = (A \otimes \mu_{\Omega(B)}) \circ (\tilde{\tau} \otimes \Omega(B)) \circ (\Omega(B) \otimes \tilde{\tau})$ or written after evaluation in the $\Omega(B)$ factor: $\tilde{\tau}_{\omega_1\omega_2} = (A \otimes \mu_{\Omega(B)}) \circ (\tilde{\tau}_{\omega_1} \otimes \Omega(B)) \circ \tilde{\tau}_{\omega_2}$. Now assuming that (1) holds when evaluating at ω_1 and when evaluating at ω_2 we compute:

$$\begin{aligned}
\tilde{\tau}_{\omega_1\omega_2} \circ \mu_A &= (A \otimes \mu_{\Omega(B)}) \circ (\tilde{\tau}_{\omega_1} \otimes \Omega(B)) \circ \tilde{\tau}_{\omega_2} \circ \mu_A = \\
&= (A \otimes \mu_{\Omega(B)}) \circ (\tilde{\tau}_{\omega_1} \otimes \Omega(B)) \circ (\mu_A \otimes \Omega(B)) \circ (A \otimes \tilde{\tau}) \circ (\tilde{\tau}_{\omega_2} \otimes A) = \\
&= (A \otimes \mu_{\Omega(B)}) \circ (\mu_A \otimes \Omega(B) \otimes \Omega(B)) \circ (A \otimes \tilde{\tau} \otimes \Omega(B)) \circ \\
&\qquad\qquad \circ (\tilde{\tau}_{\omega_1} \otimes A \otimes \Omega(B)) \circ (A \otimes \tilde{\tau}) \circ (\tilde{\tau}_{\omega_2} \otimes A) = \\
&= (\mu_A \otimes \Omega(B)) \circ (A \otimes A \otimes \mu_{\Omega(B)}) \circ (A \otimes \tilde{\tau} \otimes \Omega(B)) \circ \\
&\qquad\qquad \circ (A \otimes \Omega(B) \otimes \tilde{\tau}) \circ (\tilde{\tau}_{\omega_1} \otimes \Omega(B) \otimes A) \circ (\tilde{\tau}_{\omega_2} \otimes A) = \\
&= (\mu_A \otimes \Omega(B)) \circ (A \otimes \tilde{\tau}) \circ (A \otimes \mu_{\Omega(B)} \otimes A) \circ \\
&\qquad\qquad \circ (\tilde{\tau}_{\omega_1} \otimes \Omega(B) \otimes A) \circ (\tilde{\tau}_{\omega_2} \otimes A) = \\
&= (\mu_A \otimes \Omega(B)) \circ (A \otimes \tilde{\tau}) \circ (\tilde{\tau}_{\omega_1\omega_2} \otimes A)
\end{aligned}$$

Thus we see that the space of all $\omega \in \Omega(B)$ such that (1) holds when evaluating in ω is closed under multiplication and under the differential, and since $\tilde{\tau}$ extends τ it contains $B = \Omega_0(B)$, so it must be all of $\Omega(B)$. Thus $\tilde{\tau} : A \otimes \Omega(B) \to \Omega(B) \otimes A$ is again a twisting map.

3.3. Now consider the space $L^0(\Omega(B), \Omega(A) \otimes \Omega(B))$ of all linear maps which are homogeneous of degree zero with respect to the grading of $\Omega(B)$. Recall that on this space we have the structure of a unital algebra with multiplication defined by $f * g = (\mu_{\Omega(A)} \otimes \Omega(B)) \circ (\Omega(A) \otimes g) \circ f$. With this multiplication we get the structure of a graded algebra with respect to the grading induced from $\Omega(A)$. We define a differential on this space by $f \mapsto (d_A \otimes \varepsilon_B) \circ f$, where d_A denotes the differential on $\Omega(A)$ and ε_B denotes the grading of $\Omega(B)$. As above, one easily verifies that this is a graded derivation with respect to the multiplication $*$. Since $\tilde{\tau}$ is a twisting map, the associated mapping $A \to L^0(\Omega(B), \Omega(A) \otimes \Omega(B))$ is a homomorphism of unital algebras, and thus by the universal property of $\Omega(A)$ there is an induced homomorphism of graded differential algebras $\Omega(A) \to L^0(\Omega(B), \Omega(A) \otimes \Omega(B))$. As above one verifies that the corresponding map $\Omega(B) \otimes \Omega(A) \to \Omega(A) \otimes \Omega(B)$ is again a twisting map.

Theorem. *A twisting map $\tau : B \otimes A \to A \otimes B$ extends to a unique twisting map $\tilde{\tau} : \Omega(B) \otimes \Omega(A) \to \Omega(A) \otimes \Omega(B)$ which satisfies $\tilde{\tau} \circ (d_B \otimes \Omega(A)) = (\varepsilon_A \otimes d_B) \circ \tilde{\tau}$ and $\tilde{\tau} \circ (\Omega(B) \otimes d_A) = (d_A \otimes \varepsilon_B) \circ \tilde{\tau}$. Moreover $\Omega(A) \otimes_{\tilde{\tau}} \Omega(B)$ is a graded differential algebra with differential $d(\varphi \otimes \omega) = d_A\varphi \otimes \omega + (-1)^{|\varphi|}\varphi \otimes d_B\omega$.*

Proof. Note first that $\tilde{\tau}$ is uniquely determined by its restriction to $A \otimes B$ and the compatibility with the two differentials because the behavior on products is determined by the fact that it is a twisting map. So it suffices to show that the twisting map we have constructed above has all properties listed in the theorem.

The compatibility with d_A is clear from the second step of our construction. Moreover, from the first step of the construction it is obvious that the compatibility with d_B is satisfied for elements of the form $\omega \otimes a$ with $a \in A$ and $\omega \in \Omega(B)$. Now suppose that for some $\varphi \in \Omega(A)$ and all $\omega \in \Omega(B)$ we have $\tilde{\tau}(d_B\omega \otimes \varphi) = (\varepsilon_A \otimes d_B)(\tilde{\tau}(\omega \otimes \varphi))$. Then we compute:

$$\begin{aligned}\tilde{\tau}(d_B\omega \otimes d_A\varphi) &= (d_A \otimes \varepsilon_B)(\tilde{\tau}(d_B\omega \otimes \varphi)) = \\ &= (-1)^{|\varphi|}(d_A \otimes \varepsilon_B) \circ (\Omega(A) \otimes d_B)(\tilde{\tau}(\omega \otimes \varphi)) = \\ &= (-1)^{|\varphi|+1}(\Omega(A) \otimes d_B) \circ (d_A \otimes \varepsilon_B)(\tilde{\tau}(\omega \otimes \varphi)) = \\ &= (-1)^{|\varphi|+1}(\Omega(A) \otimes d_B)(\tilde{\tau}(\omega \otimes d_A\varphi)).\end{aligned}$$

Furthermore, if the compatibility with d_B is satisfied for two elements φ_1 and φ_2 it is easily shown using the fact that $\tilde{\tau}$ is a twisting map that it is also satisfied for their product. Consequently, the compatibility with d_B must hold in general. (Another way to prove this is to show that the maps in $L^0(\Omega(B), \Omega(A) \otimes \Omega(B))$ which satisfy this compatibility condition form a subalgebra.)

Finally, we have to show that the differential d acts as a graded derivation with respect to the multiplication $\mu_{\tilde{\tau}}$. First note that by definition of d for $\varphi \in \Omega(A)$ and $\omega \in \Omega(B)$ we have

$$\begin{aligned}d((\varphi \otimes 1_B) \cdot_{\tilde{\tau}} (1_A \otimes \omega)) &= d(\varphi \otimes \omega) = d_A\varphi \otimes \omega + (-1)^{|\varphi|}\varphi \otimes d_B\omega = \\ &= d(\varphi \otimes 1_B) \cdot_{\tilde{\tau}} (1_A \otimes \omega) + (-1)^{|\varphi|}(\varphi \otimes 1_B) \cdot_{\tilde{\tau}} d(1_A \otimes \omega).\end{aligned}$$

Direct computations easily show that

$$d((\varphi_1 \otimes 1_B) \cdot_{\tilde{\tau}} (\varphi_2 \otimes \omega)) = d(\varphi_1 \otimes 1_B) \cdot_{\tilde{\tau}} (\varphi_2 \otimes \omega) + (-1)^{|\varphi_1|}(\varphi_1 \otimes 1_B) \cdot_{\tilde{\tau}} d(\varphi_2 \otimes \omega)$$

and similarly for $(\varphi \otimes \omega_1) \cdot_{\tilde{\tau}} (1_A \otimes \omega_2)$. Next we compute:

$$\begin{aligned}d((1_A \otimes \omega) \cdot_{\tilde{\tau}} (\varphi \otimes 1_B)) &= d(\tilde{\tau}(\omega \otimes \varphi)) = \\ &= (d_A \otimes \Omega(B) + \varepsilon_A \otimes d_B)(\tilde{\tau}(\omega \otimes \varphi)) = \\ &= (-1)^{|\omega|}\tilde{\tau}(\omega \otimes d_A\varphi) + \tilde{\tau}(d_B\omega \otimes \varphi) = \\ &= d(1_A \otimes \omega) \cdot_{\tilde{\tau}} (\varphi \otimes 1_B) + (-1)^{|\omega|}(1_A \otimes \omega) \cdot_{\tilde{\tau}} d(\varphi \otimes 1_B)\end{aligned}$$

Now the general result immediately follows from the fact that

$$(\varphi_1 \otimes \omega_1) \cdot_{\tilde{\tau}} (\varphi_2 \otimes \omega_2) = (\varphi_1 \otimes 1_B) \cdot_{\tilde{\tau}} (\tilde{\tau}(\omega_1 \otimes \varphi_2)) \cdot_{\tilde{\tau}} (1_A \otimes \omega_2). \quad \square$$

3.4. Let us return to the case of general differential forms, so assume we have given graded differential algebras $\mathcal{A}$ and $\mathcal{B}$ with $A = \mathcal{A}_0$ and $B = \mathcal{B}_0$ and a twisting map $\tau : B \otimes A \to A \otimes B$. Considering $\mathcal{A}$ and $\mathcal{B}$ as algebras of differential forms it is a very reasonable assumption that they are quotients of $\Omega(A)$ and $\Omega(B)$. Algebraically this just means that they are generated as differential algebras by the zero components, while in the topological case it also implies that they do not have a too coarse topology. But assuming this there is an obvious procedure to determine whether τ induces a twisting map (which is then clearly unique) on the level of these differential forms: First consider the map $B \to L(A, A\otimes\mathcal{B})$ associated to τ. From 3.2 we see that this induces a homomorphism of graded differential algebras $\Omega(B) \to L(A, A \otimes \mathcal{B})$, and we just have to check whether this factors to a map $\mathcal{B} \to L(A, A \otimes \mathcal{B})$. If this is the case then as in 3.2 one shows that it again corresponds to a twisting map. Then as before we take the corresponding map $A \to L^0(\mathcal{B}, \mathcal{A} \otimes \mathcal{B})$ which induces a homomorphism of graded differential algebras $\Omega(A) \to L^0(\mathcal{B}, \mathcal{A}\otimes\mathcal{B})$, and again we have to check whether this factors to $\mathcal{A}$. If this is the case then as in 3.3 one proves that one gets a twisting map $\tilde{\tau} : \mathcal{B}\otimes\mathcal{A} \to \mathcal{A}\otimes\mathcal{B}$.

3.5. Example. As an example for the procedure described in 3.4 we show that the twisting maps which define the non–commutative 2–tori induce twisting maps on the level of Kähler differentials and (in the smooth case) on smooth differential forms. The computations for the smooth case are precisely as the ones in the algebraic case which we carry out here, one just has to check continuity for the natural Fréchet topologies at several points, which is quite elementary.

First we have to discuss differential forms on the algebra $A = \mathbb{C}[u, u^{-1}]$ of trigonometric polynomials on the unit circle. Let us start with the universal forms. By the derivation property of the differential we have $d(u^2) = udu + (du)u$ and inductively we get $d(u^n) = \sum_{i=0}^{n-1} u^i(du)u^{n-i-1}$ for $n \in \mathbb{N}$. Moreover, since $0 = d(1) = d(uu^{-1}) = ud(u^{-1}) + (du)u^{-1}$ we see that $d(u^{-1}) = -u^{-1}(du)u^{-1}$. Now again using the derivation property and induction one can compute $d(u^{-n})$ for $n \in \mathbb{N}$. We will only need the fact that any element of $\Omega_1(A)$ can be written as a sum of elements of the form $u^i(du)u^j$ for $i, j \in \mathbb{Z}$. This implies that any element of $\Omega_n(A)$ can be written as a sum of elements of the form $u^{i_0}(du)u^{i_1}(du)\dots(du)u^{i_n}$.

Next let us turn to the Kähler differentials $\Lambda(A)$ over A (cf. [Kunz, 1986]). From the fact that $\Lambda(A)$ is the universal graded commutative differential algebra with zero component A it is clear that $\Lambda(A)$ is just the graded abelization of $\Omega(A)$, i.e. the quotient of $\Omega(A)$ by the ideal generated by all graded commutators. From the above description of $\Omega(A)$ it is then clear that $\Lambda(A) = A \oplus A \cdot du$, while all $\Lambda_k(A)$ for $k \geq 2$ are zero. Moreover, in $\Lambda(A)$ we have the usual relations $d(u^n) = nu^{n-1}du$ for all $n \in \mathbb{Z}$. This description in the smooth case also shows that for the unit circle the smooth differential forms coincide with a topological version of the Kähler differentials, or, more precisely, that the smooth differential forms on the unit circle are the universal complete locally convex graded commutative differential algebra with zero component the smooth functions on the circle.

Now let $B = \mathbb{C}[v, v^{-1}]$, fix a complex number q of modulus one, and consider the twisting map $\tau : B \otimes A \to A \otimes B$ given by $\tau(v^k \otimes u^\ell) = q^{k\ell}u^\ell \otimes v^k$. As in 3.2 we get a homomorphism of graded differential algebras $\Omega(B) \to L(A, A \otimes \Lambda(B))$, which is characterized by

$$v \mapsto (u^k \mapsto q^k u^k \otimes v) \quad \text{and} \quad dv \mapsto (u^k \mapsto q^k u^k \otimes dv)$$

Next one computes directly that since v and dv commute in $\Lambda(B)$ that the images of vdv and $d(v)v$ under this homomorphism coincide. Since the homomorphism preserves the grading and $L(A, A \otimes \Lambda(B))$ has nonzero components only in degree zero and one this implies that the homomorphism factors over $\Lambda(B)$.

Next we have to consider the corresponding homomorphism

$$\Omega(A) \to L^0(\Lambda(B), \Lambda(A) \otimes \Lambda(B)).$$

From the construction in 3.3 we see that the image of u under this homomorphism is characterized by

$$v^k \mapsto q^k u \otimes v^k \quad \text{and} \quad v^k dv \mapsto q^{k+1} u \otimes v^k dv,$$

while the image of du is characterized by

$$v^k \mapsto q^k du \otimes v^k \quad \text{and} \quad v^k dv \mapsto -q^{k+1} du \otimes v^k dv.$$

Again an easy direct computation shows that the images of udu and $(du)u$ coincide and from compatibility with the grading we conclude that this homomorphism factors to $\Lambda(A)$, so indeed we get an induced twisting map $\tilde{\tau} : \Lambda(B) \otimes \Lambda(A) \to \Lambda(A) \otimes \Lambda(B)$.

3.6. Modules over twisted tensor products. The problem we start to discuss here is, given two unital algebras A and B, a left A–module M and a left B–module N, and a twisting map $\tau : B \otimes A \to A \otimes B$, can we make $M \otimes N$ into a left $A \otimes_\tau B$ module in a way which is compatible with the inclusion of A, i.e. such that $(a \otimes 1_B) \cdot (m \otimes n) = (a \cdot m) \otimes n$? Clearly the idea we follow is that we consider an exchange map $\tau_M : B \otimes M \to M \otimes B$ and define the action $\lambda_{\tau_M} : A \otimes_\tau B \otimes M \otimes N \to M \otimes N$ by $(\lambda_A \otimes \lambda_B) \circ (A \otimes \tau_M \otimes N)$, where λ_A and λ_B denote the left actions of A on M and of B on N, respectively.

Definition. The mapping $\tau_M : B \otimes M \to M \otimes B$ is called a *(left) module twisting map* if and only if $\tau_M(1_B \otimes m) = m \otimes 1_B$ for all $m \in M$, and

$$\tau_M \circ (\mu_B \otimes \lambda_A) = (\lambda_A \otimes \mu_B) \circ (A \otimes \tau_M \otimes B) \circ (\tau \otimes \tau_M) \circ (B \otimes \tau \otimes M).$$

3.7. As in the case of twisting maps for algebras the symmetric condition defining a module twisting map can be split into two conditions, namely

$$\tau_M \circ (\mu_B \otimes M) = (M \otimes \mu_B) \circ (\tau_M \otimes B) \circ (B \otimes \tau_M)$$
$$\tau_M \circ (B \otimes \lambda_A) = (\lambda_A \otimes B) \circ (A \otimes \tau_M) \circ (\tau \otimes M).$$

These conditions follow by applying the above one to $b_1 \otimes b_2 \otimes 1_A \otimes m$ and $b \otimes 1_B \otimes a \otimes m$, respectively. Conversely, the condition from 3.6 can be deduced by iterated application of the two conditions here.

3.8. Theorem. *If $\tau_M : B \otimes M \to M \otimes B$ is a module twisting map, then the map λ_{τ_M} defined above is a left action which is compatible with the inclusion of A for any B–module N.*

Conversely, if M is a projective $\mathbb{K}$–module and for one effective B–module N the map λ_{τ_M} defines a left action which is compatible with the inclusion of A then τ_M is a module twisting map.

Proof. Let us first assume that τ_M is a module twisting map. Clearly the condition that $\tau_M(1_B \otimes m) = m \otimes 1_B$ ensures compatibility of the action λ_{τ_M} with the inclusion of A. Using the conditions of 3.7 we compute:

$$\begin{aligned}
&\lambda_{\tau_M} \circ (\mu_\tau \otimes M \otimes N) = \\
&\quad = (\lambda_A \otimes \lambda_B) \circ (A \otimes \tau_M \otimes N) \circ (\mu_A \otimes \mu_B \otimes M \otimes N) \circ (A \otimes \tau \otimes B \otimes M \otimes N) = \\
&\quad = (\lambda_A \otimes \lambda_B) \circ (\mu_A \otimes M \otimes \mu_B \otimes N) \circ (A \otimes A \otimes \tau_M \otimes B \otimes N) \circ \\
&\qquad\qquad\qquad\qquad \circ (A \otimes \tau \otimes \tau_M \otimes N) = \\
&\quad = (\lambda_A \otimes \lambda_B) \circ (A \otimes \lambda_A \otimes B \otimes \lambda_B) \circ (A \otimes A \otimes \tau_M \otimes B \otimes N) \circ \\
&\qquad\qquad\qquad\qquad \circ (A \otimes \tau \otimes \tau_M \otimes N) = \\
&\quad = (\lambda_A \otimes \lambda_B) \circ (A \otimes \tau_M \otimes N) \circ (A \otimes B \otimes \lambda_A \otimes \lambda_B) \circ (A \otimes B \otimes A \otimes \tau_M \otimes N) = \\
&\quad = \lambda_{\tau_M} \circ (A \otimes B \otimes \lambda_{\tau_M})
\end{aligned}$$

Thus having given a module twisting map we get a module structure for any N.

Conversely, let us assume that M is $\mathbb{K}$-projective and λ_{τ_M} defines a left module structure, which is compatible with the inclusion of A, for one effective B–module N. Effectivity of N means that the algebra homomorphism $B \to L(N, N)$ which defines the action of B on N is injective. Since M is projective over $\mathbb{K}$ this implies that the induced map $M \otimes B \to M \otimes L(N, N)$ is injective. Next we claim that the latter space maps injectively to $L(N, M \otimes N)$. Let us first assume that M is a free $\mathbb{K}$-module, so $M = \mathbb{K}^{(\alpha)}$, a direct sum of copies of $\mathbb{K}$ Then $M \otimes L(N, N) \simeq L(N, N)^{(\alpha)}$, which maps injectively to $L(N, N)^\alpha$, the direct product. The latter space is isomorphic to $L(N, N^\alpha)$. Thus the composition of the natural map $\mathbb{K}^{(\alpha)} \otimes L(N, N) \to L(N, \mathbb{K}^{(\alpha)} \otimes N)$ with the inclusion of the latter space into $L(N, N^\alpha)$ is an injection, so the claim holds in this case. In general, if M is a direct summand in some $\mathbb{K}^{(\alpha)}$, we get an injection $M \otimes L(N, N) \to \mathbb{K}^{(\alpha)} \otimes L(N, N) \to L(N, N^{(\alpha)}$, and this map is just the composition of the map $M \otimes L(N, N) \to L(N, M \otimes N)$ with the obvious map from the latter space to $L(N, \mathbb{K}^{(\alpha)} \otimes N) \simeq L(N, N^{(\alpha)})$, so the result holds in this case, too.

Thus we see that for any $\mathbb{K}$-module V the induced mapping $L(V, M \otimes B) \to L(V, L(N, M \otimes N)) \cong L(V \otimes N, M \otimes N)$ is injective. This map is given by mapping φ to $v \otimes n \mapsto (M \otimes \lambda_B)(\varphi(v) \otimes n)$.

First the compatibility of the action λ_{τ_M} with the inclusion of A shows that the map $M \otimes N \to M \otimes N$ given by $m \otimes n \mapsto (M \otimes \lambda_B)(\tau_M(1_B \otimes m) \otimes n)$ is the identity. From the injectivity result above we see that thus $\tau_M(1_B \otimes m) = m \otimes 1_B$.

Next, the condition that $(1 \otimes b_1 b_2) \cdot (m \otimes n) = (1 \otimes b_1) \cdot ((1 \otimes b_2) \cdot (m \otimes n))$ immediately implies that

$$\begin{aligned}
&(M \otimes \lambda_B) \circ (\tau_M \otimes N) \circ (\mu_B \otimes M \otimes N) = \\
&\qquad = (M \otimes \lambda_B) \circ (M \otimes \mu_B \otimes N) \circ (\tau_M \otimes B \otimes N) \circ (B \otimes \tau_M \otimes N),
\end{aligned}$$

and by the injectivity result above this immediately gives the first condition of 3.7.

On the other hand, we must have $\tau(b\otimes a)\cdot(m\otimes n)=(1\otimes b)\cdot((a\otimes 1)\cdot(m\otimes n))$. This gives:

$$(M\otimes\lambda_B)\circ(\lambda_A\otimes M\otimes N)\circ(A\otimes\tau_M\otimes N)\circ(\tau\otimes M\otimes N)=\\ (M\otimes\lambda_B)\circ(\tau_M\otimes N)\circ(B\otimes\lambda_A\otimes N),$$

and again by the injectivity result above this implies the second condition of 3.7. □

Remark. The condition used in the converse part of theorem 3.8 is just one possibility. Also, in the case of topological algebras and topological tensor products this condition is not sufficient in general. The main point is that one has to ensure the injectivity of the map $L(V,M\otimes B)\to L(V\otimes N,M\otimes N)$ constructed in the proof for any V. An example of a condition which ensures this, even in the topological case, is that λ_{τ_M} defines a left action compatible with the inclusion of A for one B–module N which contains a free submodule of rank one as a direct summand. In this case it is easy to explicitly reconstruct an element of $L(V,M\otimes B)$ from its image in $L(V\otimes N,M\otimes N)$.

3.9. Next we give a characterization of module twisting maps which is analogous to the characterization of twisting maps in 2.6: First consider the space $L(M,M\otimes B)$. As in 2.5 we see that this space is a unital associative algebra with multiplication defined by $\varphi*\psi:=(M\otimes\mu_B)\circ(\varphi\otimes B)\circ\psi$ and unit $m\mapsto m\otimes 1_B$. On the other hand consider the space $L(B,M\otimes B)$. On this space we define a left action of A by $a\cdot\varphi:=(\lambda_A\otimes B)\circ(A\otimes\varphi)\circ\tau_a$, where $\tau_a:B\to A\otimes B$ is given by $\tau_a(b):=\tau(b\otimes a)$.

3.10. Proposition. *The action defined in 3.9 makes $L(B,M\otimes B)$ into a left A–module.*

Proof. Consider the first condition from 2.4(1):

$$\tau\circ(B\otimes\mu_A)=(\mu_A\otimes B)\circ(A\otimes\tau)\circ(\tau\otimes A)$$

Evaluating with elements of A this reads as $\tau_{a_1a_2}=(\mu_A\otimes B)\circ(A\otimes\tau_{a_2})\circ\tau_{a_1}$. Then we compute:

$$\begin{aligned}(a_1a_2)\cdot\varphi&=(\lambda_A\otimes B)\circ(A\otimes\varphi)\circ\tau_{a_1a_2}=\\&=(\lambda_A\otimes B)\circ(A\otimes\varphi)\circ(\mu_A\otimes B)\circ(A\otimes\tau_{a_2})\circ\tau_{a_1}=\\&=(\lambda_A\otimes B)\circ(A\otimes\lambda_A\otimes B)\circ(A\otimes A\otimes\varphi)\circ(A\otimes\tau_{a_2})\circ\tau_{a_1}=\\&=(\lambda_A\otimes B)\circ(A\otimes(a_2\cdot\varphi))\circ\tau_{a_1}=a_1\cdot(a_2\cdot\varphi)\end{aligned}$$

Moreover, since $\tau(b\otimes 1)=1\otimes b$ it is obvious that 1_A acts as the identity. □

3.11. Proposition. *A linear map $\tau_M:B\otimes M\to M\otimes B$ is a module twisting map if and only if the associated map $B\to L(M,M\otimes B)$ is a homomorphism of unital algebras and the associated map $M\to L(B,M\otimes B)$ is a homomorphism of left A–modules.*

Proof. First the condition that the associated map $B\to L(M,M\otimes B)$ preserves the units means exactly that $\tau_M(1\otimes m)=m\otimes 1$. Next as in proof of 2.6 one sees that this map being an algebra homomorphism is precisely the first condition of 3.7.

On the other hand, the condition that the associated map $M \to L(B, M \otimes B)$ is a homomorphism of left A–modules means just that $\tau_M(b \otimes (a \cdot m)) = (\lambda_A \otimes B) \circ (A \otimes \tau_M)(\tau(b \otimes a) \otimes m)$ which is precisely the second condition of 3.7 evaluated on $b \otimes a \otimes m$. $\square$

3.12. Right modules. What we have done above for left modules can be developed completely analogous for right modules. For completeness we list here the corresponding conditions. We start with a twisting map $\tau : B \otimes A \to A \otimes B$, a right A–module M and a right B–module N, and we are looking for a right $A \otimes_\tau B$–module structure on $M \otimes N$ such that $(m \otimes n) \cdot (1 \otimes b) = m \otimes (n \cdot b)$. Thus we need an exchange map $\tau_N : N \otimes A \to A \otimes N$, and define then $\rho_{\tau_N} := (\rho_A \otimes \rho_B) \circ (M \otimes \tau_N \otimes B)$, where the ρ's denote the given right action. We call τ_N a *(right) module twisting map* if and only if

$$\tau_N \circ (\rho_B \otimes \mu_A) = (\mu_A \otimes \rho_B) \circ (A \otimes \tau_N \otimes B) \circ (\tau_N \otimes \tau) \circ (N \otimes \tau \otimes A),$$

and the obvious analog of theorem 3.8 holds. The analogs of the conditions of 3.7 look as

$$\begin{aligned} \tau_N \circ (N \otimes \mu_A) &= (\mu_A \otimes N) \circ (A \otimes \tau_N) \circ (\tau_N \otimes A) \\ \tau_N \circ (\rho_B \otimes A) &= (A \otimes \rho_B) \circ (\tau_N \otimes B) \circ (N \otimes \tau). \end{aligned}$$

Next as in 3.9 and 3.10 we get a unital associative algebra structure on $L(N, A \otimes N)$ via $\varphi * \psi := (\mu_A \otimes N) \circ (A \otimes \psi) \circ \varphi$, and a right B–module structure on $L(A, A \otimes N)$ via $\varphi \cdot b := (A \otimes \rho_B) \circ (\varphi \otimes B) \circ \tau_b$, where $\tau_b : A \to A \otimes B$ is the map $a \mapsto \tau(b \otimes a)$, and the obvious analog of proposition 3.11 holds.

3.13. Bimodules. Again, let us start from a twisting map $\tau \colon B \otimes A \to A \otimes B$, and suppose that M is an A–bimodule and N is a B–bimodule. Moreover, suppose we have given a left module twisting map $\tau_M : B \otimes M \to M \otimes B$ and a right module twisting map $\tau_N : N \otimes A \to A \otimes N$. Thus we have a left and a right $A \otimes_\tau B$–module structure on $M \otimes N$, which we denote by λ_{τ_M} and ρ_{τ_N}, respectively.

Proposition. *$M \otimes N$ is an $A \otimes_\tau B$–bimodule with respect to the structures from above if and only if*

$$\begin{aligned} (\rho_A \otimes N) \circ (M \otimes \tau_N) \circ (M \otimes \lambda_B \otimes A) \circ (\tau_M \otimes N \otimes A) &= \\ = (A \otimes \lambda_B) \circ (\tau_M \otimes N) \circ (B \otimes \rho_A \otimes N) \circ (B \otimes M \otimes \tau_N). \end{aligned}$$

Proof. The condition above is precisely the translation of the fact that $((1 \otimes b) \cdot (m \otimes n)) \cdot (a \otimes 1) = (1 \otimes b) \cdot ((m \otimes n) \cdot (a \otimes 1))$. By the compatibility of λ_{τ_M} with the left action of A and of ρ_{τ_N} with the right action of B this condition is equivalent to $M \otimes N$ being a bimodule. $\square$

4. Cohomology for twisted tensor products and deformations.

In this section we construct an analog of the Hochschild cohomology of an algebra with coefficients in the algebra for twisted tensor products. We show that the relation of this cohomology to (formal) deformations (in the sense of twisted tensor products) is similar as in the classical case. In particular, we consider the case of deformations of the ordinary multiplication on the tensor product. As we indicated before, from now on we put $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$.

4.1. From 2.7 we see that we may reduce the study of twisted tensor products to the study of multiplications on the tensor product which are defined by a twisting map like in 2.2. Now for multiplication maps on a fixed module there is a conceptual approach to Hochschild cohomology via a certain graded Lie algebra, which is probably due to [Gerstenhaber, 1953]. This approach is probably better known in the case of Lie algebras, see [Nijenhuis-Richardson, 1967]. A multigraded version both for associative and Lie algebras is developed in [Lecomte-Michor-Schicketanz, 1992]. Here we give a short outline in the associative case: Let V be a $\mathbb{K}$-vector space and for $n\in\mathbb{N}$ put $M^n(V):=L^{n+1}(V,V)$, the space of all $n+1$–linear maps from V^{n+1} to V. Then for $L_i\in M^{\ell_i}(V)$ define $j(L_1)L_2$ and $[L_1,L_2]$ in $M^{\ell_1+\ell_2}(V)$ by:

$$\begin{aligned}(j(L_1)L_2)&(v_0,\dots,v_{\ell_1+\ell_2}):=\\&=\sum_{i=0}^{\ell_2}(-1)^{\ell_1 i}L_2(v_0,\dots,L_1(v_i,\dots,v_{i+\ell_1}),\dots,v_{\ell_1+\ell_2})\\ [L_1,L_2]:=&j(L_1)L_2-(-1)^{\ell_1\ell_2}j(L_2)L_1.\end{aligned}$$

Then it turns out that this bracket defines a graded Lie algebra structure on $M(V)=\bigoplus_n M^n(V)$. Moreover, an element $\mu\in M^1(V)$, i.e. a bilinear map $V\times V\to V$ is an associative multiplication if and only if $[\mu,\mu]=0$. But if this is the case then by the graded Jacobi–identity the mapping $d_\mu: M^n(V)\to M^{n+1}(V)$ defined by $d_\mu(L)=[\mu,L]$ is a differential, i.e. $d_\mu\circ d_\mu=0$ and the cohomology of $(M(V),d_\mu)$ is exactly the Hochschild cohomology of the algebra (V,μ) with coefficients in the bimodule V.

4.2. Let us now adapt this construction for twisted tensor products. Thus we have $V=A\otimes B$. Now $M^n(A\otimes B)=L((A\otimes B)^{\otimes n+1},A\otimes B)$ and both $(A\otimes B)^{\otimes n+1}$ and $A\otimes B$ have canonical structures of a left A–module and a right B–module, by acting on the leftmost A component and the rightmost B-component, respectively. By definition, any multiplication of the form μ_τ as defined in 2.2 is a homomorphism for both these module structures. Moreover, from the definition of the operator j above it is obvious that the subspace $\tilde M(A\otimes B)=\bigoplus\tilde M^n(A\otimes B)$, where $\tilde M^n$ denotes the set of linear maps $(A\otimes B)^{\otimes n+1}\to A\otimes B$, which respect both module structures, is a Lie subalgebra of $M(A\otimes B)$. Since A and B are unital we may identify $\tilde M^0(A\otimes B)$ with $A\otimes B$ and $\tilde M^n(A\otimes B)$ with $L(B\otimes(A\otimes B)^{\otimes n-1}\otimes A,A\otimes B)$. In particular, under this identification $\tilde M^1(A\otimes B)=L(B\otimes A,A\otimes B)$ and one easily checks that the element in $\tilde M^1(A\otimes B)$ corresponding to a multiplication μ_τ as in

2.2 is exactly the map τ. Let us again denote by $[\quad,\quad]$ the induced bracket on $\tilde{M}(A\otimes B)$. Then from the above we see that μ_τ is associative if and only if $[\tau,\tau]=0$.

Since we are interested in twisting maps τ, we have to take into account the additional condition that $\tau(1\otimes a)=a\otimes 1$ and $\tau(b\otimes 1)=1\otimes b$. We define $C^n(A\otimes B)$ to be the set of all $\sigma\in\tilde{M}^n(A\otimes B)$ which satisfy the following condition: $\sigma(b\otimes a_1\otimes b_1\otimes\cdots\otimes a_{n-1}\otimes b_{n-1}\otimes a)=0$ if either $b=1$ or $a=1$ or $a_i=1$ and $b_i=1$ for some $i=1,\dots,n-1$. The motivation for this definition is that elements in $C^1(A\otimes B)$ should be candidates for infinitesimal deformations of twisting maps and in order to remain in the realms of twisting maps they have to satisfy $\sigma(1\otimes a)=\sigma(b\otimes 1)=0$, and the following result, the proof of which also shows that $C(A\otimes B)$ is just the natural analog of the normalized Hochschild complex.

4.3. Proposition. *The space $C(A\otimes B)=\bigoplus_n C^n(A\otimes B)$ is a graded Lie subalgebra of $\tilde{M}(A\otimes B)$ and for any twisting map τ it is closed under the differential $d_\tau=[\tau,\quad]$.*

Proof. Take $\sigma_i\in C^{n_i}(A\otimes B)$ for $i=1,2$. To compute the bracket $[\sigma_1,\sigma_2]$ we proceed as follows: To σ_i we associate $L_i\in M^{n_i}(A\otimes B)$ defined by

$$L_i(a_0\otimes b_0\otimes\cdots\otimes a_{n_i}\otimes b_{n_i})=a_0\cdot\sigma_i(b_0\otimes\cdots\otimes a_{n_i})\cdot b_{n_i},$$

and then

$$[\sigma_1,\sigma_2](b\otimes a_1\otimes\cdots\otimes b_{n_1+n_2-1}\otimes a)=[L_1,L_2](1\otimes b\otimes\cdots\otimes a\otimes 1).$$

This shows that $[\sigma_1,\sigma_2]\in C(A\otimes B)$ is equivalent to $[L_1,L_2](v_0,\dots,v_{n_1+n_2})=0$ if one of the v_i equals 1 assuming that the L_i have this property. (Here we write v_i for elements of $A\otimes B$). But obviously each individual summand occurring in the definition of $j(L_1)L_2$ vanishes under these conditions, so $C(A\otimes B)$ is indeed a subalgebra.

Now suppose that $\sigma\in C^n(A\otimes B)$ and that τ is a twisting map. As above we form L_σ and consider the bracket $[\mu_\tau,L_\sigma](1\otimes b\otimes\cdots\otimes a\otimes 1)$. Now we have to show that $[\mu_\tau,L_\sigma](v_0,\dots,v_{n+1})$ vanishes if any of the $v_i=1$ assuming that L_σ has this property and that 1 is really the multiplicative unit for μ_τ. This is exactly the classical fact that the normalized Hochschild cochains form a subcomplex. Explicitly this can be seen as follows: Consider first the case where $v_i=1$ for some $i\neq 0,n+1$. Then in the sum defining $j(\mu_\tau)L_\sigma$ all terms vanish obviously, but the two in which v_i goes into the μ_τ. But these two give the same result with opposite signs since $\mu_\tau\in M^1(A\otimes B)$ and 1 is a multiplicative unit. On the other hand, in both summands of $j(L_\sigma)\mu_\tau$ this v_i must go into the L_σ since $i\neq 0,n+1$, so these vanish, too.

Next suppose that $v_0=1$. Then in $j(\mu_\tau)L_\sigma$ the only surviving summand equals $L_\sigma(v_1,\dots,v_{n+1})$, while from $j(L_\sigma)\mu_\tau$ we get $-(-1)^n(-1)^{n\cdot 1}L_\sigma(v_1,\dots,v_{n+1})$, so these two terms cancel. Similarly one proves the result if $v_{n+1}=1$, and thus $C(A\otimes B)$ is indeed closed under the differential d_τ for any twisting map τ. $\square$

4.4. Definition. We now define the cohomology $H^*(\tau)$ to be the cohomology of the complex $(C^*(A\otimes B),d_\tau)$. Note that $C^0(A\otimes B)=0$, so $H^0(\tau)=0$ and

$H^1(\tau) = \mathrm{Ker}(d_\tau) \subset C^1(\tau)$. This is true since $\tilde{M}^0(A \otimes B) \cong A \otimes B$, with $(\alpha \otimes \beta)(a \otimes b) = a\alpha \otimes \beta b$ and such a map vanishes on $1 \otimes 1$ only if $\alpha \otimes \beta = 0$.

Nevertheless, we do not renumber the cohomology groups so that we get the usual correspondences between cohomology groups and deformations.

Note that by the graded Jacobi identity the differential d_τ acts as a graded derivation with respect to the graded Lie bracket, which thus induces a graded Lie algebra structure on the cohomology space $H^*(\tau)$.

4.5. Formal Deformations. To study deformations of twisted tensor products we have to study deformations of twisting maps. So let τ_0 be a twisting map and consider a formal power series $\tau = \sum_{k\geq 0} \tau_k t^k$, where each τ_k is a linear mapping $\tau_k : B \otimes A \to A \otimes B$. To have a chance that at least for small t this power series defines a twisting map we obviously have to assume that for each k, a and b we have $\tau_k(b \otimes 1) = \tau_k(1 \otimes a) = 0$, i.e. that each τ_k is in fact an element of $C^1(A \otimes B)$. Next let $[\tau, \tau]$ be the formal power series $\sum_k [\tau, \tau]_k t^k$ with $[\tau, \tau]_k := \sum_{i+j=k} [\tau_i, \tau_j]$, so we have

$$[\tau, \tau] = 0 + 2[\tau_0, \tau_1]t + (2[\tau_0, \tau_2] + [\tau_1, \tau_1])t^2 + \dots$$

Now we call τ a *formal deformation* of τ_0 if and only if $[\tau, \tau] = 0$. Clearly, if the power series τ converges for $|t| < t_0$ for some $t_0 > 0$ then this condition is equivalent to $\tau(t)$ being a twisting map for all $|t| < t_0$.

4.6. Now it is quite easy to relate formal deformations to the cohomology. Assume that we have given a formal power series $\tau = \sum_k \tau_k t^k$ as above. Then the first term in the expansion of $[\tau, \tau]$ is just $2d_{\tau_0}(\tau_1)$, so the first condition for being a formal deformation is that $\tau_1 \in H^1(\tau_0)$. In particular, if $H^1(\tau_0) = 0$ then $\tau_1 = 0$ and the next equation reads just as $2d_{\tau_0}(\tau_2) = 0$, thus $\tau_2 = 0$ and inductively one gets that all τ_k must be zero. Thus if $H^1(\tau_0) = 0$ there is no formal deformation of τ_0, and we call τ_0 *formally rigid* in this case.

On the other hand if $H^1(\tau_0) \neq 0$ let us fix some τ_1. Then the next term in the expansion of $[\tau, \tau]$ is $2d_{\tau_0}(\tau_2) + [\tau_1, \tau_1]$. Since d_{τ_0} acts as a graded derivation with respect to the bracket we see that $[\tau_1, \tau_1]$ is always a cocycle, and thus the obstruction against the existence of a $\tau_2 \in C^1(A \otimes B)$ which solves the equation $2d_{\tau_0}(\tau_2) + [\tau_1, \tau_1] = 0$ is exactly the cohomology class of $[\tau_1, \tau_1]$ in $H^2(\tau_0)$. Moreover, if this class vanishes then this element τ_2 is determined up to elements of $H^1(\tau_0)$.

Now let us inductively assume that for $k < N$ we have found elements $\tau_k \in C^1(A \otimes B)$ such that $0 = 2d_{\tau_0}(\tau_k) + \sum_{i,j>0;i+j=k} [\tau_i, \tau_j]$ for all k. Clearly, this implies that

$$[\tau, \tau] = t^N (2d_{\tau_0}(\tau_N) + \sum_{i,j>0;i+j=N} [\tau_i, \tau_j]) + \text{higher order terms.}$$

Now applying d_{τ_0} to $\sum_{i,j>0;i+j=N} [\tau_i, \tau_j]$ we get:

$$\sum_{\substack{i,j>0\\ i+j=N}} ([d_{\tau_0}\tau_i, \tau_j] - [\tau_i, d_{\tau_0}\tau_j]) = 2 \sum_{\substack{i,j>0\\ i+j=N}} [d_{\tau_0}\tau_i, \tau_j]$$

which by our assumptions equals

$$2\sum_{\substack{i,j>0\\ i+j=N}}\sum_{\substack{m,n>0\\ m+n=i}}[[\tau_m,\tau_n],\tau_j]=2\sum_{\substack{i,j,k>0\\ i+j+k=N}}[[\tau_i,\tau_j],\tau_k].$$

If in this sum there is a term in which all three indices are equal than this term vanishes by the graded Jacobi identity. Next, the sum of all terms in which exactly two indices are equal can be written as

$$2\sum_{\substack{i,j>0\\ 2i+j=N}}([[\tau_i,\tau_i],\tau_j]+[[\tau_i,\tau_j],\tau_i]+[[\tau_j,\tau_i],\tau_i]),$$

and each of these summands vanishes by the graded Jacobi identity and the symmetry of the bracket for elements of degree one. Finally, again using this symmetry the sum of the terms in which all three indices are different can be rewritten as

$$4\sum_{\substack{i>j>k>0\\ i+j+k=N}}([[\tau_i,\tau_j],\tau_k]+[[\tau_i,\tau_k],\tau_j]+[[\tau_j,\tau_k],\tau_i]),$$

and as above each of these summands vanishes.

So again the obstruction against finding $\tau_N\in C^1(A\otimes B)$ such that $2d_{\tau_0}(\tau_N)+\sum_{i,j>0;i+j=N}[\tau_i,\tau_j]=0$ is the cohomology class of $\sum_{i,j>0;i+j=N}[\tau_i,\tau_j]$ in $H^2(\tau_0)$, and if this vanishes the choice of τ_N is unique up to elements of $H^1(\tau_0)$. Together we see that if we try to extend a cocycle $\tau_1\in H^1(\tau_0)$ to a formal deformation, in each step there is an obstruction in $H^2(\tau_0)$ and if this vanishes the extension is unique up to elements of $H^1(\tau_0)$.

4.7. Definition. In order to proceed towards the computation of the cohomology of a twisting map τ, we have to consider some module structures depending on τ. From 3.9 we get a left A–module structure on $L(B,A\otimes B)$ given by $a\cdot\sigma:=(\mu_A\otimes B)\circ(A\otimes\sigma)\circ\tau_a$, where $\tau_a:B\to A\otimes B$ is the map given by $\tau_a(b):=\tau(a\otimes b)$. Moreover, we define a right action of A on this space by $(\sigma\cdot a)(b):=\sigma(b)\cdot_\tau(a\otimes 1)$. Similarly from 3.12 we get a right B–module structure on $L(A,A\otimes B)$ given by $\sigma\cdot b:=(A\otimes\mu_B)\circ(\sigma\otimes B)\circ\tau_b$ and we define a left action of B on this space by $(b\cdot\sigma)(a):=(1\otimes b)\cdot_\tau\sigma(a)$.

4.8. Proposition. *The actions from 4.7 make $L(B,A\otimes B)$ into an A–bimodule and $L(A,A\otimes B)$ into a B–bimodule.*

Proof. We prove this only for the actions of A, the proof for the actions of B is completely analogous.

In 3.10 we have shown that the left action of A is indeed an action. Next by definition the τ–multiplication with $a\otimes 1$ from the right is just given by $(\mu_A\otimes B)\circ(A\otimes\tau_a)$. Thus we get:

$$\begin{aligned}(a_1\cdot\sigma)\cdot a_2&=(\mu_A\otimes B)\circ(A\otimes\tau_{a_2})\circ(a_1\cdot\sigma)=\\ &=(\mu_A\otimes B)\circ(A\otimes\tau_{a_2})\circ(\mu_A\otimes B)\circ(A\otimes\sigma)\circ\tau_{a_1},\end{aligned}$$

while on the other hand

$$a_1 \cdot (\sigma \cdot a_2) = (\mu_A \otimes B) \circ (A \otimes (\sigma \cdot a_2)) \circ \tau_{a_1} = \\ = (\mu_A \otimes B) \circ (A \otimes \mu_A \otimes B) \circ (A \otimes A \otimes \tau_{a_2}) \circ (A \otimes \sigma) \circ \tau_{a_1},$$

and again by associativity of μ_A this equals the above expression. □

4.9. Remarks. (1) If V is an arbitrary $\mathbb{K}$–vector space then the module structures from above induce an A–bimodule structure on $L(B \otimes V, A \otimes B)$, since this space is canonically isomorphic to $L(V, L(B, A \otimes B))$ by the universal property of the tensor product. In the same way, one gets a B–bimodule structure on $L(V \otimes A, A \otimes B)$ for any V.

A short computation shows that these structures are given by $(\sigma \cdot a)(b \otimes v) = \sigma(b \otimes v) \cdot_\tau (a \otimes 1)$ and $a \cdot \sigma = (\mu_A \otimes B) \circ (A \otimes \sigma) \circ (\tau_a \otimes V)$, and likewise for the actions of B.

(2) To any mapping $\sigma \in L(B \otimes V, A \otimes B)$ we can associate a linear map $L_\sigma : A \otimes B \otimes V \to A \otimes B$, which is a homomorphism for the left A–module structures given by left multiplication. Now the nontrivial module structures defined above can be conveniently expressed using L_σ as $(a \cdot \sigma)(b \otimes v) = L_\sigma(\tau(b \otimes a) \otimes v)$. Similarly one can express the right actions of B. This follows directly from the definitions.

4.10. Consider the space $C^1(A \otimes B) \subset L(B \otimes A, A \otimes B)$. This can be canonically identified with $L_0(B, L_0(A, A \otimes B))$, where L_0 denotes the space of those linear maps which vanish on 1. Thus we can consider the Hochschild differential ∂_B^τ with respect to B and the B–bimodule structure on $L_0(A, A \otimes B)$ constructed in 4.7 above (obviously $L_0(A, A \otimes B)$ is a sub–bimodule of $L(A, A \otimes B)$ for this structure). ∂_B^τ has then values in the space $L(B \otimes B, L_0(A, A \otimes B))$ which can be canonically identified with a subspace of $L(B \otimes B \otimes A, A \otimes B)$. Since the Hochschild differential respects the normalized Hochschild complex the values are in fact in the subspace of those maps which vanish if one entry is equal to one.

Similarly, identifying $C^1(A \otimes B)$ with $L_0(A, L_0(B, A \otimes B))$ we get a Hochschild differential ∂_A^τ which has values in (a subspace of) $L(B \otimes A \otimes A, A \otimes B)$.

For later use let us compute these differentials explicitly:

For $\sigma \in L_0(B, L_0(A, A \otimes B))$ we have by definition $\partial_B^\tau \sigma(b_1 \otimes b_2) = b_1 \cdot (\sigma(b_2)) - \sigma(b_1 b_2) + (\sigma(b_1)) \cdot b_2$. Reinterpreting σ as a map from $B \otimes A$ to $A \otimes B$ we thus get

$$\partial_B^\tau \sigma(b_1 \otimes b_2 \otimes a) = (1 \otimes b_1) \cdot_\tau (\sigma(b_2 \otimes a)) - \sigma(b_1 b_2 \otimes a) + L_\sigma(b_1 \otimes \tau(b_2 \otimes a)).$$

Here $L_\sigma : B \otimes A \otimes B \to A \otimes B$ denotes the homomorphism of right B–modules induced by σ (cf. 4.9(2)).

Similarly one computes

$$\partial_A^\tau \sigma(b \otimes a_1 \otimes a_2) = L_\sigma(\tau(b \otimes a_1) \otimes a_2) - \sigma(b \otimes a_1 a_2) + (\sigma(b \otimes a_1)) \cdot_\tau (a_2 \otimes 1).$$

In this case $L_\sigma : A \otimes B \otimes A \to A \otimes B$ denotes the homomorphism of left A–modules induced by σ as in 4.9(2).

4.11. Next consider the map $\Psi_1 : C^2(A \otimes B) \to L(B \otimes B \otimes A, A \otimes B)$ given by $(\Psi_1 \sigma)(b_1 \otimes b_2 \otimes a) := \sigma(b_1 \otimes 1 \otimes b_2 \otimes a)$. Obviously, this is a homomorphism of A–bimodules and of B–bimodules for the structures defined in 4.8 for any twisting map τ.

Similarly, we get such a homomorphism $\Psi_2 : C^2(A \otimes B) \to L(B \otimes A \otimes A, A \otimes B)$.

Theorem. *For a fixed twisting map τ let $Z^2_\tau(A\otimes B)$ denote the space of two–cocycles in $C^2(A\otimes B)$ with respect to d_τ. Then the induced map*

$$\Psi=(\Psi_1,\Psi_2):Z^2_\tau(A\otimes B)\to L(B\otimes B\otimes A,A\otimes B)\oplus L(B\otimes A\otimes A,A\otimes B)$$

is injective and we have $\Psi\circ d_\tau=(-\partial^\tau_B,\partial^\tau_A)$. Moreover, Ψ_1 has values in the subspace of normalized Hochschild two–cocycles on B with values in $L_0(A,A\otimes B)$ and Ψ_2 has values in the subspace of normalized Hochschild two–cocycles on A with values in $L_0(B,A\otimes B)$.

Proof. Let us first compute the two–cocycle equation for $\sigma\in C^2(A\otimes B)$. This reads as:

$$\begin{aligned}0=&[\mu_\tau,L_\sigma](1\otimes b_0\otimes a_1\otimes b_1\otimes a_2\otimes b_2\otimes a_3\otimes 1)=\\&=L_\sigma((1\otimes b_0)\cdot_\tau(a_1\otimes b_1)\otimes a_2\otimes b_2\otimes a_3\otimes 1)-\\&-L_\sigma(1\otimes b_0\otimes(a_1\otimes b_1)\cdot_\tau(a_2\otimes b_2)\otimes a_3\otimes 1)+\\&+L_\sigma(1\otimes b_0\otimes a_1\otimes b_1\otimes(a_2\otimes b_2)\cdot_\tau(a_3\otimes 1))-\\&-(1\otimes b_0)\cdot_\tau L_\sigma(a_1\otimes b_1\otimes a_2\otimes b_2\otimes a_3\otimes 1)-\\&-L_\sigma(1\otimes b_0\otimes a_1\otimes b_1\otimes a_2\otimes b_2)\cdot_\tau(a_3\otimes 1)\end{aligned}$$

Now apply this equation with $b_1=1$ and $a_2=1$. Then the last two terms vanish since $\sigma(1\otimes\dots)=\sigma(\dots\otimes 1)=0$ and since $(1\otimes b)\cdot_\tau(a\otimes 1)=\tau(b\otimes a)$ and $(a\otimes 1)\cdot_\tau(1\otimes b)=a\otimes b$ we get:

$$0=(a_1\cdot\sigma)(b_0\otimes 1\otimes b_2\otimes a_3)-\sigma(b_0\otimes a_1\otimes b_2\otimes a_3)+(\sigma\cdot b_2)(b_0\otimes a_1\otimes 1\otimes a_3),$$

and since Ψ_1 and Ψ_2 are bimodule homomorphisms this implies injectivity of Ψ.

Next applying the cocycle equation with $a_1=a_2=1$ we get:

$$\begin{aligned}0=\sigma(b_0b_1\otimes 1\otimes b_2\otimes a_3)-\sigma(b_0\otimes 1\otimes b_1b_2\otimes a_3)+&\\+(\sigma\cdot b_2)(b_0\otimes 1\otimes b_1\otimes a_3)-(b_0\cdot\sigma)(b_1\otimes 1\otimes b_2\otimes a_3),&\end{aligned}$$

which exactly means that $\Psi_1(\sigma)$ is a Hochschild two–cocycle. Similarly, the cocycle equation for $b_1=b_2=1$ shows that $\Psi_2(\sigma)$ is a Hochschild two–cocycle.

Finally, for $\sigma\in C^1(A\otimes B)$ we have:

$$\begin{aligned}d_\tau(\sigma)(b_0\otimes a_1\otimes b_1\otimes a_2)&=[\mu_\tau,L_\sigma](1\otimes b_0\otimes a_1\otimes b_1\otimes a_2\otimes 1)=\\&=L_\sigma((1\otimes b_0)\cdot_\tau(a_1\otimes b_1)\otimes a_2\otimes 1)-L_\sigma(1\otimes b_0\otimes(a_1\otimes b_1)\cdot_\tau(a_2\otimes 1))-\\&-(1\otimes b_0)\cdot_\tau L_\sigma(a_1\otimes b_1\otimes a_2\otimes 1)+L_\sigma(1\otimes b_0\otimes a_1\otimes b_1)\cdot_\tau(a_2\otimes 1)\end{aligned}$$

Putting in this equation $a_1=1$ we get

$$\Psi_1(d_\tau\sigma)(b_0\otimes b_1\otimes a_2)=\sigma(b_0b_1\otimes a_2)-(\sigma\cdot b_1)(b_0\otimes a_2)-(b_0\cdot\sigma)(b_1\otimes a_2),$$

and this is just $-\partial^\tau_B\sigma(b_0\otimes b_1\otimes a_2)$, while putting $b_1=1$ we get

$$\Psi_2(d_\tau\sigma)(b_0\otimes a_1\otimes a_2)=(a_1\cdot\sigma)(b_0\otimes a_2)-\sigma(b_0\otimes a_1a_2)+(\sigma\cdot a_2)(b_0\otimes a_1),$$

which is just $\partial^\tau_A\sigma(b_0\otimes a_1\otimes a_2)$. $\square$

4.12. The case of the trivial twisting map. We investigate the case of the trivial twisting map $s : B \otimes A \to A \otimes B$, $s(b \otimes a) = a \otimes b$, i.e. of deformations of the ordinary tensor product. First note that in this case the bimodule structures defined in 4.7 simplify considerably: By definition, the right action of A on $L(B, A \otimes B)$ and the left action of B on $L(A, A \otimes B)$ are just given by right respectively left multiplication on the values of the maps. On the other hand the left action of A on $L(B, A \otimes B)$ is defined by $a \cdot \sigma := (\mu_A \otimes B) \circ (A \otimes \sigma) \circ s_a$ and this is the map $b \mapsto a \otimes b \mapsto a \otimes \sigma(b) \mapsto (a \otimes 1)(\sigma(b))$, so the left action reduces to left multiplication on the values and similarly for the right action of B.

Using this fact we can now give a nice description of $H^1(s)$: Consider a map $\sigma : B \otimes A \to A \otimes B$. By theorem 4.11 σ is in $H^1(s)$ if and only if $\partial^s_B(\sigma) = \partial^s_A(\sigma) = 0$. From the equations in 4.10 one immediately reads off that in this case these conditions just mean that

$$\sigma(b_1 b_2 \otimes a) = \sigma(b_1 \otimes a)(1 \otimes b_2) + (1 \otimes b_1)(\sigma(b_2 \otimes a))$$
$$\sigma(b \otimes a_1 a_2) = \sigma(b \otimes a_1)(a_2 \otimes 1) + (a_1 \otimes 1)(\sigma(b \otimes a_2)).$$

Now viewing σ as an element of $L(B, L(A, A \otimes B))$ the second equation just means that the values are in the subspace $\mathrm{Der}(A, A \otimes B)$ of derivations, while the first condition means that the map itself is a derivation with respect to the bimodule structure on $\mathrm{Der}(A, A \otimes B)$ given by multiplication on the values. Thus $H^1(s) \cong \mathrm{Der}(B, \mathrm{Der}(A, A \otimes B)) \cong \mathrm{Der}(A, \mathrm{Der}(B, A \otimes B))$, since the compatibility with the units is automatically satisfied by derivations.

4.13. Examples. Still in the case of the trivial twisting map, suppose that $\varphi \in \mathrm{Der}(A, A)$ and $\psi \in \mathrm{Der}(B, B)$ are derivations. Consider the map $b \otimes a \mapsto \varphi(a) \otimes \psi(b)$. Obviously this is an element of $H^1(s)$. Now consider the formal power series $exp(t\varphi \otimes \psi) := s + \sum_{k \geq 1} \frac{t^k}{k!}(\varphi \otimes \psi)^k$, where $(\varphi \otimes \psi)^k(b \otimes a) := \varphi^k(a) \otimes \psi^k(b)$. We claim that this is always a formal deformation of s. By theorem 4.11 we have to show that for any $N \in \mathbb{N}$ we have

$$\tfrac{2}{N!}\partial^s_B((\varphi \otimes \psi)^N) = \sum_{i,j>0; i+j=N} \tfrac{1}{i!j!}\Psi_1([(\varphi \otimes \psi)^i, (\varphi \otimes \psi)^j])$$

and

$$-\tfrac{2}{N!}\partial^s_A((\varphi \otimes \psi)^N) = \sum_{i,j>0; i+j=N} \tfrac{1}{i!j!}\Psi_2([(\varphi \otimes \psi)^i, (\varphi \otimes \psi)^j]).$$

By the product rule for powers of derivations we have $(\varphi \otimes \psi)^N(b_1 b_2 \otimes a) = \sum_{i=0}^N \binom{N}{i}\varphi^N(a) \otimes \psi^i(b_1)\psi^{N-i}(b_2)$. Thus, using the formula for ∂^τ_B derived in 4.10 we see that

$$\partial^s_B((\varphi \otimes \psi)^N)(b_1 \otimes b_2 \otimes a) = \sum_{i=1}^{N-1} \binom{N}{i} \varphi^N(a) \otimes \psi^i(b_1)\psi^{N-i}(b_2).$$

On the other hand, writing L_i for the extension of $(\varphi \otimes \psi)^i$ to $A \otimes B \otimes A \otimes B$ as a left A–module and a right B–module homomorphism we compute:

$$\begin{aligned}
&[(\varphi\otimes\psi)^i,(\varphi\otimes\psi)^j](b_1\otimes 1\otimes b_2\otimes a)=\\
&\qquad = L_j(L_i(1\otimes b_1\otimes 1\otimes b_2)\otimes a\otimes 1)-L_j(1\otimes b_1\otimes L_i(1\otimes b_2\otimes a\otimes 1))+\\
&\qquad + L_i(L_j(1\otimes b_1\otimes 1\otimes b_2)\otimes a\otimes 1)-L_i(1\otimes b_1\otimes L_j(1\otimes b_2\otimes a\otimes 1))=\\
&\qquad = 0-\varphi^{i+j}(a)\otimes\psi^j(b_1)\psi^i(b_2)+0-\varphi^{i+j}(a)\otimes\psi^i(b_1)\psi^j(b_2)
\end{aligned}$$

Using this it is easy to see that the first of the two above equations holds. The second one is proved similarly.

In fact, this is closely related to a result of [Mourre 1990], who has shown that if A is an algebra and D_1 and D_2 are commuting derivations on A, then $a*b := \sum_{k=0}^{\infty}\frac{t^k}{k!}D_1^k(a)D_2^k(b)$ is a formal deformation of A.

Let us carry this out in the two simplest situations: First put $A=B=\mathbb{C}[x]$ viewed as the algebra of complex valued polynomials on the real line, and consider the formal deformation $exp(it\frac{d}{dx}\otimes\frac{d}{dx})$. Calling the variable in B p instead of x we can write the resulting commutation relation as $px=xp+it$ or $[p,x]=it$, so we get exactly the Heisenberg uncertainty relation. It should be remarked that clearly this is a true deformation on the level of polynomials. The extension of this deformation to bigger subalgebras of all complex valued functions on the real line is a more subtle (topological) problem.

Second put $A=\mathbb{C}[u,u^{-1}]$, $B=\mathbb{C}[v,v^{-1}]$ viewed as two copies of the algebra of trigonometric polynomials on the unit circle, and consider the formal deformation $exp(it\partial_u\otimes\partial_v)$, where ∂u is the derivation given by $\partial_u(u^n)=inu^n$, and similarly for ∂_v. We then get the commutation relation $vu=uv\sum_{k\geq 0}\frac{(-it)^k}{k!}=e^{-it}uv$, which is exactly the non–commutative two torus (or irrational rotation algebra) with parameter $q=e^{-it}$ or in terms of physics the Weyl relations. In this case this is not only a true deformation in the case of polynomials but it is also well known that this extends to a true deformation for smooth functions and even to continuous and essentially bounded functions.

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