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# ON THE RIGID BODY WITH TWO LINEAR CONTROLS

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Abstract. The rigid body dynamics with two particular linear controls is discussed and some of its geometrical properties are pointed out.

## 1. Introduction.

There has been a great deal of work over the past decade analyzing the geometry and the dynamics of the rigid body motion with $n$ controls, $0 \leq n \leq 2$. We can mention here the papers of Brockett [4], Aeyels [1], Aeyels and Szafranski [2], Bloch and Marsden [3], Holm and Marsden [6] and Puta [9], [10]. In general, in all the above papers the involved controls are of polynomial type of degree two. The goal of our paper is to consider the case of the linear controls. More precisely we present some geometrical properties of the rigid body with two particular linear controls about the minor and respectively the major axes.

## 2. The rigid body with two linear controls.

The rigid body equations with two controls about the minor and respectively major axes are given by

$$\begin{cases} \dot{m}_1 = a_1 m_2 m_3 + u_1, \\ \dot{m}_2 = a_2 m_1 m_3, \\ \dot{m}_3 = a_3 m_1 m_2 + u_3, \end{cases} \tag{2.1}$$

where

$$a_1 = \frac{1}{I_3} - \frac{1}{I_2}, a_2 = \frac{1}{I_1} - \frac{1}{I_3}, a_3 = \frac{1}{I_2} - \frac{1}{I_1}, \tag{2.2}$$

$I_1 > I_2 > I_3$ are the principal moments of inertia and $u_1, u_3 \in C^\infty(\mathbb{R}^3, \mathbb{R})$.

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Now employ the feedbacks

$$\begin{cases} u_1 = -km_3, \\ u_3 = km_1, \end{cases} \tag{2.3}$$

where $k \in \mathbb{R}$is the feedback gain parameter. We shall refer to the system (2.1)-(2.3) as the controlled system.

Let $SO(3)$ be the group of all linear orientation preserving orthogonal transformations of $\mathbb{R}^3$ onto itself. If we fix a basis of $\mathbb{R}^3$ then $SO(3)$ is the Lie group of all matrices $R$ of type 3x3 with real coefficients such that

$$R^t \cdot R = I_3 \tag{2.4}$$

and

$$\det(R) = 1. \tag{2.5}$$

Its Lie algebra $so(3)$ is the set of all sew-symmetric matrices of type 3x3, i.e.

$$so(3) = \left\{ \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \middle| a, b, c \in \mathbb{R} \right\}, \tag{2.6}$$

that can be identified with $\mathbb{R}^3$ via the mapping

$$v = (p, q, r) \in \mathbb{R}^3 \mapsto \hat{v} = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \in so(3). \tag{2.7}$$

The Lie bracket is then mapped to the cross-product in the sense that

$$[\hat{v}, \hat{w}] = \widehat{v \times w}. \tag{2.8}$$

Moreover, the dual of its Lie algebra, i.e. $so(3)^*$, may be identified with $so(3)$ via the Killing form. Then we can prove:

**Theorem 2.1.** *The controlled system* (2.1)-(2.3) *is a Hamiltonian-Poisson mechanical system with the phase space* $P = so(3)^* \cong \mathbb{R}^3$, *the Hamiltonian* $H$ *given by*

$$H(m_1, m_2, m_3) = \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\right) + km_2 \tag{2.9}$$

*and with respect to the minus-Lie-Poisson structure on* $so(3)^*$.

*Proof.* Firstly, let us remind ourselves that the minus-Lie-Poisson structure on $so(3)^*$ is in fact the Rigid-Body-Bracket on $so(3)^*$ and it is given by

$$\{f, g\}_-(m_1, m_2, m_3) = -(m_1, m_2, m_3) \cdot (\nabla f \times \nabla g), \tag{2.10}$$

where $\nabla f$ is the gradient of $f$. Now an easy computation shows us that

$$\dot{m}_i = \{m_i, H\}_-, i = 1, 2, 3,$$

which gives the result. □

**Remark 2.1.** It is easy to see that the function $C$ given by

$$C(m_1, m_2, m_3) = \frac{1}{2}(m_1^2 + m_2^2 + m_3^2) \tag{2.11}$$

is a Casimir of our configuration, i.e.

$$\{C, f\}_- = 0,$$

for each $f \in C^\infty(\mathbb{R}^3, \mathbb{R})$.

## 3. Stability and Stabilization.

It is a classical result that a rigid body rotates stably about its major and minor principal axes, but unstably about its intermediate axis.

We show here, via the Energy-Casimir method, that we can stabilize the rigid body equations about the intermediate axis of inertia by certain two linear controls about the minor and the major axes. More precisely we have:

**Theorem 3.1.** *The controlled system* (2.1)-(2.3) *may be stabilized about the equilibrium*

*(i)* $e_1 = (0, M, 0), M > 0$, *for* $k \in (-\infty, -a_3) \cup (a_1, \infty)$;
*(ii)* $e_2 = (0, M, 0), M < 0$, *for* $k \in (-\infty, -a_1) \cup (a_3, \infty)$.

*Proof.* (i) Without loss of generality, we can suppose that the equilibrium state is $e_1 = (0, 1, 0)$.

Consider first the system linearized about $(0, 1, 0)$. Its eigenvalues are given by the solutions of

$$\lambda[\lambda^2 + (k + a_3)(k - a_1)] = 0. \tag{3.1}$$

Hence for $k \in (-a_3, a_1)$, the system is unstable, but for $k \in (-\infty, -a_3) \cup (a_1, \infty)$ we have two imaginary and one zero eigenvalue. Is the system stable? We prove that it is via the Energy-Casimir method.

Recall that the Energy-Casimir method (see, e.g. Holm, Marsden, Ratiu and Weinstein [7]) requires finding a constant of the motion for the system, usually the energy $H$, and a family $\mathcal{C}$ of constants of the motion, such that for some $C \in \mathcal{C}$, $H + C$ has a critical point at the equilibrium of interest. Often the $C$'s are taken to be Casimir-functions that commute with all other functions under the Poisson bracket. Then, in the case of finite dimension, the definiteness of $\delta^2(H + C)$ at the critical point is sufficient to prove stability.

Now let us take the modified Energy-Casimir function

$$\begin{aligned} H_C(m_1, m_2, m_3) &= \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\right) \\ &\quad + km_2 + \varphi\left(\frac{1}{2}(m_1^2 + m_2^2 + m_3^2)\right), \end{aligned} \tag{3.2}$$

where $\varphi$ is an arbitrary smooth function. Then we have:

$$\begin{aligned}\delta(H_C) &= \delta\left(\frac{1}{2}\left(\frac{m_1^2}{I_1}+\frac{m_2^2}{I_2}+\frac{m_3^2}{I_3}\right)+km_2\right)\\ &+\delta\left(\varphi\left(\frac{1}{2}\left(m_1^2+m_2^2+m_3^2\right)\right)\right)\\ &=\frac{m_1}{I_1}\delta m_1+\frac{m_2}{I_2}\delta m_2+\frac{m_3}{I_3}\delta m_3+k\delta m_2\\ &+\dot\varphi(m_1\delta m_1+m_2\delta m_2+m_3\delta m_3).\end{aligned}$$

At the equilibrium of interest $(0,1,0)$ the first variation is zero if and only if

$$\dot\varphi(1/2) = -\frac{1}{I_2}+k. \tag{3.3}$$

Then

$$\begin{aligned}\delta^2(H_C) &= \frac{1}{I_1}(\delta m_1)^2+\frac{1}{I_2}(\delta m_2)^2+\frac{1}{I_3}(\delta m_3)^2\\ &+\ddot\varphi(m_1\delta m_1+m_2\delta m_2+m_3\delta m_3)^2+\dot\varphi[(\delta m_1)^2+(\delta m_2)^2+(\delta m_3)^2].\end{aligned}$$

At the equilibrium of interest $(0,1,0)$ we have:

$$\delta^2(H_C)(0,1,0) = -(k+a_3)(\delta m_1)^2-k(\delta m_2)^2+\ddot\varphi(\frac{1}{2})(\delta m_2)^2+(a_1-k)(\delta m_3)^2.$$

Hence for $k\in(-\infty,-a_3)$ [resp. $k\in(a_1,\infty)$] and choosing $\ddot\varphi>0$ [resp. $\ddot\varphi<0$] the second variation is positive [resp. negative] definite and we have nonlinear stability.

(ii) Without loss of generality, we can suppose that the equilibrium state is $e_2=(0,-1,0)$.

Consider first the system linearized about $(0,-1,0)$. Its eigenvalues are given by the solutions of

$$\lambda[\lambda^2+(k-a_3)(k+a_1)] = 0. \tag{3.4}$$

Hence for $k\in(-a_1,a_3)$, the system is unstable, but for $k\in(-\infty,-a_1)\cup(a_3,\infty)$ we have two imaginary and one zero eigenvalue. Is the system stable? We prove that it is using again the Energy-Casimir method.

Let us take the modified Energy-Casimir function (3.2). Then at the equilibrium of interest $(0,-1,0)$ the first variation is zero if and only if

$$\dot\varphi(1/2) = -\frac{1}{I_2}-k.$$

Then the second variation at the equilibrium of interest is:

$$\delta^2(H_C)(0,-1,0) = (k-a_3)(\delta m_1)^2+(k+a_1)(\delta m_3)^2+k(\delta m_2)^2+\ddot\varphi(\frac{1}{2})(\delta m_2)^2.$$

Hence for $k\in(-\infty,-a_1)$ [resp. $k\in(a_3,\infty)$] and choosing $\ddot\varphi<0$ [resp. $\ddot\varphi>0$] the second variation is negative [resp. positive] definite and we have nonlinear stability. $\square$

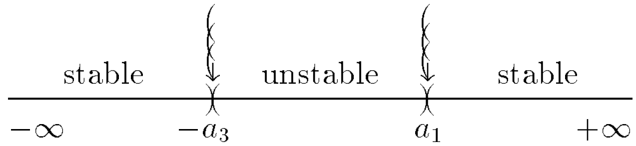


FIGURE 1. The global bifurcations for the equilibrium state $(0,1,0)$

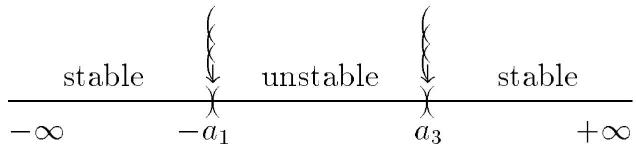


FIGURE 2. The global bifurcations for the equilibrium state $(0,-1,0)$

**Remark 3.1.** Let us observe that the gain parameter $k$ causes two global bifurcations where the equilibrium $(0,1,0)$ [resp. (0,-1,0)] changes its stability, see Fig. 1 [resp. Fig. 2].

## 4. Alternative Poisson structures.

In section 2 we have seen that our controlled system (2.1)-(2.3) is a Hamilton-Poisson mechanical system with the phase space $P=\mathbb{R}^3$, the Hamiltonian (2.9) and the Poisson structure (2.10). In this section we shall prove that it can be realized as a Hamiltonian-Poisson mechanical system in an infinite number of different ways.

For each $a,b\in\mathbb{R}$ let us define the following bracket

(4.1)

$$\{f,g\}_{a,b}(m_1,m_2,m_3)=\left((\frac{a}{I_1}+b)m_1,(\frac{a}{I_2}+b)m_2+ak,(\frac{a}{I_3}+b)m_3\right)\cdot(\nabla f\times\nabla g),$$

where $f,g\in C^\infty(\mathbb{R}^3,\mathbb{R})$. Then a very long but straightforward computation shows us that the bracket (4.1) is in fact a Poisson bracket on $\mathbb{R}^3$. Thus we have

**Theorem 4.1.** *The controlled system* (2.1)-(2.3) *is a Hamiltonian-Poisson mechanical system with the phase space* $\mathbb{R}^3$*, the Hamiltonian* $H'$ *given by*

$$H'(m_1,m_2,m_3)=\frac{c}{2}\left(\frac{m_1^2}{I_1}+\frac{m_2^2}{I_2}+\frac{m_3^2}{I_3}\right)+ckm_2+\frac{d}{2}(m_1^2+m_2^2+m_3^2), \tag{4.2}$$

*where* $c,d\in\mathbb{R}$, $ad-bc=1$, *and the Poisson structure* (4.1).

*Proof.* Indeed we have successively:

$$
\begin{aligned}
\{m_1, H'\}_{a,b} &= -\left(\frac{a}{I_2}+b\right)\left(\frac{c}{I_3}+d\right)m_2m_3 - ka\left(\frac{c}{I_3}+d\right)m_3 \\
&+\left(\frac{a}{I_3}+b\right)\left(\frac{c}{I_2}+d\right)m_2m_3 + kc\left(\frac{a}{I_3}+b\right)m_3 \\
&= -\frac{ac}{I_2I_3}m_2m_3 - \frac{bc}{I_3}m_2m_3 - \frac{ad}{I_2}m_2m_3 \\
&- bdm_2m_3 - k\frac{ac}{I_3}m_3 - kadm_3 \\
&+\frac{ac}{I_2I_3}m_2m_3 + \frac{bc}{I_2}m_2m_3 + \frac{ad}{I_3}m_2m_3 \\
&+ bdm_2m_3 + k\frac{ac}{I_3}m_3 + kbcm_3 \\
&= (ad-bc)\left(\frac{1}{I_3}-\frac{1}{I_2}\right)m_2m_3 - k(ad-bc)m_3 \\
&= a_1m_2m_3 - km_3 \\
&= \dot{m}_1
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\{m_2, H'\}_{a,b} &= \dot{m}_2, \\
\{m_3, H'\}_{a,b} &= \dot{m}_3,
\end{aligned}
$$

as required. □

**Remark 4.1.** The same richness of Hamiltonian-Poisson structures was found in the case of the rigid body system by Holm and Marsden [6], in the case of the rigid body system with one and respectively two quadratic controls by Holm and Marsden [6] and respectively by Puta [9], in the case of the rigid body system with one rotor and an internal torque by Puta [8], in the case of the Maxwell-Bloch system by David and Holm [5], and in the case of the Maxwell-Bloch system with one control by Puta [10].

We want to finish with the remark that for the particular case $k = 0$ we refined some results established by Holm and Marsden [6].

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