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ON AN EXTENSION OF THE 3-DIMENSIONAL TODA LATTICE

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Abstract. An extension of the 3-dimensional nonperiodic Toda lattice is proposed and some of its geometrical and dynamical properties are pointed out.

1. Introduction.

After the Flaschka transformation the dynamics of the 3-dimensional nonperiodic Toda lattice is given by the following system of differential equations:

$$(1.1) \quad \begin{cases} \dot{a}_1 = 2b_1^2 \\ \dot{a}_2 = 2(b_2^2 - b_1^2) \\ \dot{a}_3 = -2b_2^2 \\ \dot{b}_1 = b_1(a_2 - a_1) \\ \dot{b}_2 = b_2(a_3 - a_2) \\ b_1 > 0 \\ b_2 > 0 \end{cases}$$

If we drop the last two conditions we obtain a new system of differential equations which is a slight extension of the 3-dimensional non-periodic Toda lattice, namely:

$$(1.2) \quad \begin{cases} \dot{a}_1 = 2b_1^2 \\ \dot{a}_2 = 2(b_2^2 - b_1^2) \\ \dot{a}_3 = -2b_2^2 \\ \dot{b}_1 = b_1(a_2 - a_1) \\ \dot{b}_2 = b_2(a_3 - a_2) \end{cases}$$

The goal of this paper is to study some of its geometrical and dynamical properties.

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2. Lax representations.

Following Flaschka [1] rewrite (1.2) in the form

$$\frac{dL}{dt} = [L, B] = LB - BL, \tag{2.1}$$

where

$$L = \begin{bmatrix} a_1 & b_1 & 0 \\ b_1 & a_2 & b_2 \\ 0 & b_2 & a_3 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & -b_1 & 0 \\ b_1 & 0 & -b_2 \\ 0 & b_2 & 0 \end{bmatrix}$$

Then we can prove:

Theorem 2.1. *(Flaschka) The flow* (1.2) *is isospectral, i.e. the eigenvalues of* L *are independent of* t.

Proof. Let $V(t)$ be the solution of the matrix equation

$$\begin{cases} \dfrac{dV(t)}{dt} = -B(t) \cdot V(t) \\ V(0) = I_3. \end{cases} \tag{2.2}$$

Then one can easily check that

$$L(t) = V(t)L(0)V^{-1}(t).$$

Indeed, we have successively

$$\begin{aligned} \frac{d}{dt}[V^{-1}(t)L(0)V(t)] &= \frac{dV^{-1}(t)}{dt}L(t)V(t) + V^{-1}(t)\frac{d}{dt}[L(t)V(t)] \\ &= V^{-1}(t)B(t)L(t)V(t) + V^{-1}(t)\frac{dL(t)}{dt}V(t) \\ &+ V^{-1}(t)L(t)\frac{dV(t)}{dt} \\ &= V^{-1}(t)B(t)L(t)V(t) + V^{-1}(t)[L(t), B(t)]V(t) \\ &+ V^{-1}(t)L(t)[-B(t)V(t)] \\ &= V^{-1}(t)B(t)L(t)V(t) + V^{-1}(t)L(t)B(t)V(t) \\ &- V^{-1}(t)B(t)L(t)V(t) - V^{-1}(t)L(t)B(t)V(t) \\ &= 0 \end{aligned}$$

Hence

$$V^{-1}(t)L(t)V(t) = L(0)$$

or the equivalent

$$L(t) = V(t)L(0)V^{-1}(t)$$

as required. Now it is clear that the eigenvalues of L are independent on t, hence the flow (1.2) is isospectral. $\square$

3. The Lie-Poisson structure.

Let

$$\hat{e} = \{e_1 = (1,0,0,0,0), e_2 = (0,1,0,0,0),$$
$$e_3 = (0,0,1,0,0), e_4 = (0,0,0,1,0), e_5 = (0,0,0,0,1)\}$$

be the canonical basis of $\mathbb{R}^5$. If we define:

$$\begin{aligned}[e_1,e_4] &= -e_4; [e_2,e_4] = e_4 \\ [e_2,e_5] &= -e_5; [e_3,e_5] = e_5\end{aligned} \tag{3.1}$$

and all the other brackets are zero, then $(\mathbb{R}^5, [\cdot,\cdot])$ is a Lie-algebra. It follows that the minus - Lie-Poisson structure on $(\mathbb{R}^5)^* \simeq \mathbb{R}^5$ is given by the matrix:

$$\sqcap = \begin{bmatrix} 0 & 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & -b_1 & b_2 \\ 0 & 0 & 0 & 0 & -b_2 \\ -b_1 & b_1 & b_2 & 0 & 0 \end{bmatrix} \tag{3.2}$$

or equivalently:

$$\begin{aligned}\{f,g\}_{LP}^{-} &= \left[\frac{\partial f}{\partial a_1}, \frac{\partial f}{\partial a_2}, \frac{\partial f}{\partial a_3}, \frac{\partial f}{\partial b_1}, \frac{\partial f}{\partial b_2},\right] \cdot \sqcap \cdot \left[\frac{\partial g}{\partial a_1}, \frac{\partial g}{\partial a_2}, \frac{\partial g}{\partial a_3}, \frac{\partial g}{\partial b_1}, \frac{\partial g}{\partial b_2},\right]^t \\ &= b_1\left(\frac{\partial f}{\partial a_1}\frac{\partial g}{\partial b_1} - \frac{\partial f}{\partial b_1}\frac{\partial g}{\partial a_1}\right) - b_1\left(\frac{\partial f}{\partial a_2}\frac{\partial g}{\partial b_1} - \frac{\partial f}{\partial b_1}\frac{\partial g}{\partial a_2}\right) \\ &+ b_2\left(\frac{\partial f}{\partial a_2}\frac{\partial g}{\partial b_2} - \frac{\partial f}{\partial b_2}\frac{\partial g}{\partial a_2}\right) - b_2\left(\frac{\partial f}{\partial a_3}\frac{\partial g}{\partial b_2} - \frac{\partial f}{\partial b_2}\frac{\partial g}{\partial a_3}\right),\end{aligned}$$

for each $f,g \in C^\infty(\mathbb{R}^5,\mathbb{R})$. It follows that $(\mathbb{R}^5, \{\cdot,\cdot\}_{LP}^{-})$ is a Poisson manifold. Now we can prove:

Theorem 3.1. *The system* (1.2) *may be realized as a Hamilton-Poisson mechanical system with the phase space* $P = \mathbb{R}^5$, *the Hamiltonian* H *given by*

$$H = \frac{1}{2}[a_1^2 + a_2^2 + a_3^3] + b_1^2 + b_2^2 \tag{3.3}$$

and the Poisson structure (3.2).

Proof. Indeed,

$$\begin{aligned}\sqcap \cdot \nabla H &= \sqcap \cdot [a_1, a_2, a_3, 2b_1, 2b_2]^t \\ &= [2b_1^2, 2(b_2^2 - b_1^2), -2b_2^2, b_1(a_2 - a_1), b_2(a_3 - a_2)]^t \\ &= [\dot{a}_1, \dot{a}_2, \dot{a}_3, \dot{b}_1, \dot{b}_2]^t,\end{aligned}$$

as desired. □

Remark 3.1. If we write the above Poisson structure in the following more convenient form:

$$\{f,g\}_{LP}^{-} = \det \begin{bmatrix} -b_1 & -b_1 & 0 \\ \frac{\partial f}{\partial a_1} & \frac{\partial f}{\partial a_2} & \frac{\partial f}{\partial b_1} \\ \frac{\partial g}{\partial a_1} & \frac{\partial g}{\partial a_2} & \frac{\partial g}{\partial b_1} \end{bmatrix} + \det \begin{bmatrix} -b_2 & -b_2 & 0 \\ \frac{\partial f}{\partial a_2} & \frac{\partial f}{\partial a_3} & \frac{\partial f}{\partial b_2} \\ \frac{\partial g}{\partial a_2} & \frac{\partial g}{\partial a_3} & \frac{\partial g}{\partial b_2} \end{bmatrix}$$

for each $f, g \in C^{\infty}(\mathbb{R}^5, \mathbb{R})$ then it is easy to see that the function C given by

$$C = a_1 + a_2 + a_3 \tag{3.4}$$

is a Casimir of our configuration, i.e.

$$\{C, f\}_{LP}^{-} = 0,$$

for each $f \in C^{\infty}(\mathbb{R}^5, \mathbb{R})$.

4. Stability and stabilization.

It is clear that equilibrium states of the system (1.2) are:

$$(\alpha, \beta, \gamma, 0, 0); \alpha, \beta, \gamma \in \mathbb{R}. \tag{4.1}$$

Consider first the system (1.2) linearized about the equilibrium (4.1). Its eigenvalues are given by the solutions of

$$\lambda^3(\lambda + \alpha - \beta)(\lambda + \beta - \gamma) = 0. \tag{4.2}$$

If $\alpha, \beta, \gamma \in \mathbb{R}, \alpha < \beta$ or $\alpha, \beta, \gamma \in \mathbb{R}, \beta < \gamma$, or $\alpha, \beta, \gamma \in \mathbb{R}, \alpha < \beta < \gamma$ at least one root of the equation (4.2) is positive and so the corresponding equilibrium states are unstable.

If $\alpha, \beta, \gamma \in \mathbb{R}, \alpha \geq \beta \geq \gamma$ then the equilibrium states (4.1) are spectrally stable. Are they nonlinear stable? Using the energy-Casimir method we can prove the following statement:

Theorem 4.1. *The equilibrium states*

$$(\alpha, \alpha, \alpha, 0, 0), \alpha \in \mathbb{R} \tag{4.3}$$

are nonlinear stable.

Proof. Recall that the energy-Casimir method (see e.g. Holm, Marsden, Ratiu and Weinstein [2] or [3]) requires finding a constant of motion for the system, usually the energy H, and a family $\mathcal{C}$ of constants of motion such that for some $C \in \mathcal{C}, H + C$ has a critical point at the equilibrium of interest. Often the C's are taken to be Casimir - functions that commute with all other functions under the Poisson bracket. Then in the finite dimensional case, the definiteness of $\delta^2(H + C)$ at the critical point is sufficient to prove stability.

Now let us take the modified energy-Casimir function:

$$H + \varphi(C) = \frac{1}{2}[a_1^2 + a_2^2 + a_3^2] + b_1^2 + b_2^2 + \varphi(a_1 + a_2 + a_3), \tag{4.4}$$

where φ is a smooth arbitrary real valued function, i.e. $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$. Then we have:

$$\delta(H + \varphi(C)) = a_1\delta a_1 + a_2\delta a_2 + a_3\delta a_3 + 2b_1\delta b_1 + 2b_2\delta b_2 + \dot{\varphi}(\delta a_1 + \delta a_2 + \delta a_3).$$

At the equilibrium of interest (4.3) the first variation vanishes if and only if

$$\dot{\varphi}(3\alpha) = -\alpha \tag{4.5}$$

Then

$$\delta^2(H + \varphi(C)) = (\delta a_1)^2 + (\delta a_2)^2 + (\delta a_3)^2 + 2(\delta b_1)^2 + 2(\delta b_2)^2 + \ddot{\varphi}(\delta a_1 + \delta a_2 + \delta a_3)^2.$$

and at the equilibrium of interest (4.3) we have

$$\begin{aligned}\delta^2(H + \varphi(C))(\alpha, \alpha, \alpha, 0, 0) &= (\delta a_1)^2 + (\delta a_2)^2 + (\delta a_3)^2 + 2(\delta b_1)^2 + 2(\delta b_2)^2 \\ &\quad + \ddot{\varphi}(3\alpha)(\delta a_1 + \delta a_2 + \delta a_3)^2.\end{aligned}$$

If we choose φ such that

$$\ddot{\varphi}(3\alpha) \geq 0,$$

then the second variation at the equilibrium of interest is positive definite and we have nonlinear stability. □

Remark 4.1. It is an open problem to decide the stability or the instability of the equilibrium states which are not mentioned in our previous considerations.

5. Stabilization by one control.

In this section we shall prove that the equilibrium states:

$$(\alpha, \alpha, 0, 0, 0); \alpha \in \mathbb{R}, \alpha > 0 \tag{5.1}$$

$$(0, 0, \alpha, 0, 0); \alpha \in \mathbb{R}, \alpha < 0 \tag{5.2}$$

$$(0, \alpha, \alpha, 0, 0); \alpha \in \mathbb{R}, \alpha < 0 \tag{5.3}$$

of the system (1.2) may be nonlinear stabilized by a particular linear control applied to the axis $0b_2$ [resp. $0b_2$, resp. $0b_1$].

The system (1.2) with one control about the axis $0b_2$ can be written in the following form:

$$\begin{cases} \dot{a}_1 = 2b_1^2 \\ \dot{a}_2 = 2(b_2^2 - b_1^2) \\ \dot{a}_3 = -2b_2^2 \\ \dot{b}_1 = b_1(a_2 - a_1) \\ \dot{b}_2 = b_2(a_3 - a_2) + u \end{cases} \tag{5.4}$$

where $u \in C^\infty(\mathbb{R}^5, \mathbb{R})$.

In all that follows we shall employ the feedback:

$$u = kb_2, \tag{5.5}$$

where $k \in \mathbb{R}$ is the control parameter.

Theorem 5.1. *The controlled system* (5.4), (5.5) *is a Hamilton-Poisson mechanical system with the phase space* $P = \mathbb{R}^5$, *the Poisson structure* (3.2) *and the Hamiltonian* H *given by:*

$$H = \frac{1}{2}[a_1^2 + a_2^2 + a_3^2] + b_1^2 + b_2^2 + ka_3 \tag{5.6}$$

Proof. One can easily check that:

$$\dot{a}_i = \{a_i, H\}_{LP}^{-}; i = 1, 2, 3$$
$$\dot{b}_i = \{b_i, H\}_{LP}^{-}; i = 1, 2,$$

which prove the assertion. □

Using now the energy-Casimir method we can prove:

Theorem 5.2. *The controlled system* (5.4), (5.5) *may be nonlinear stabilized about the equilibrium states* (5.1) *for* $k = \alpha$.

Proof. Let us take the modified energy-Casimir function

$$H + \varphi(C) = \frac{1}{2}[a_1^2 + a_2^2 + a_3^2] + b_1^2 + b_2^2 + ka_3 + \varphi(a_1 + a_2 + a_3) \tag{5.7}$$

where $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$. Then we have:

$$\delta(H + \varphi(C)) = a_1\delta a_1 + a_2\delta a_2 + a_3\delta a_3 + 2b_1\delta b_1 + 2b_2\delta b_2 + k\delta a_3 + \dot{\varphi}(\delta a_1 + \delta a_2 + \delta a_3)$$

At the equilibrium of interest (5.1) the first variation vanishes if and only if

$$\begin{cases} \dot{\varphi}(2\alpha) = -\alpha \\ k = \alpha \end{cases} \tag{5.8}$$

Then

$$\delta^2(H + \varphi(C)) = (\delta a_1)^2 + (\delta a_2)^2 + (\delta a_3)^2 + 2(\delta b_1)^2 + 2(\delta b_2)^2 + \ddot{\varphi}(\delta a_1 + \delta a_2 + \delta a_3)^2,$$

and at the equilibrium of interest (5.1) it is positive definite if we choose φ such that:

$$\ddot{\varphi}(2\alpha) \geq 0.$$

Hence the equilibrium states (5.1) are nonlinear stable. □

If we employ now the feedback:

$$u = -kb_2 \tag{5.9}$$

then we can prove in a similar manner the following results:

Theorem 5.3. *The controlled system* (5.4), (5.8) *is a Hamilton-Poisson mechanical system with the phase space* $P = \mathbb{R}^5$*, the Poisson structures* (3.2) *and the Hamiltonian* H *given by:*

$$H = \frac{1}{2}[a_1^2 + a_2^2 + a_3^2] + b_1^2 + b_2^2 + ka_1 + ka_2 \tag{5.10}$$

Theorem 5.4. *The controlled system* (5.4), (5.8) *may be nonlinear stabilized about the equilibrium states* (5.3) *for* $k = \alpha$.

The system (1.2) with one control about the axis $0b_1$ can be written in the following form:

$$\begin{cases} \dot{a}_1 = 2b_1^2 \\ \dot{a}_2 = 2(b_2^2 - b_1^2) \\ \dot{a}_3 = -2b_2^2 \\ \dot{b}_1 = b_1(a_2 - a_1) + v \\ \dot{b}_2 = b_2(a_3 - a_2) \end{cases} \tag{5.11}$$

Let us employ now the feedback:

$$v = -kb_1 \tag{5.12}$$

where $k \in \mathbb{R}$ is the control parameter. Then we can prove:

Theorem 5.5. *The controlled system* (5.11), (5.12) *is a Hamilton-Poisson mechanical system with the phase space* $P = \mathbb{R}^5$*, the Poisson structure* (3.2) *and the Hamiltonian* H *given by:*

$$H = \frac{1}{2}[a_1^2 + a_2^2 + a_3^2] + b_1^2 + b_2^2 + ka_1 + ka_1. \tag{5.13}$$

Theorem 5.6. *The controlled system* (5.11), (5.12) *may be nonlinear stabilized about the equilibrium states* (5.2) *for* $k = \alpha$.

Remark 5.1. It is easy to see that the equilibrium states:

$$(\alpha, 0, \alpha, 0, 0), \alpha \in \mathbb{R}, \alpha \neq 0 \tag{5.14}$$

are all unstable.

6. Stabilization by two controls.

In this section we shall prove that the equilibrium states

$$(\alpha, 0, 0, 0, 0), \alpha \in \mathbb{R}, \alpha > 0 \tag{6.1}$$

of the system (1.2) may be nonlinear stabilized by two controls about the axes $0b_1$ and $0b_2$.

The system (1.2) with two controls about the axes $0b_1$ and $0b_2$ can be written in the following form:

$$\text{(6.2)}\qquad \begin{cases} \dot{a}_1 = 2b_1^2 \\ \dot{a}_2 = 2(b_2^2 - b_1^2) \\ \dot{a}_3 = -2b_2^2 \\ \dot{b}_1 = b_1(a_2 - a_1) + u \\ \dot{b}_2 = b_2(a_3 - a_2) + v \end{cases}$$

Let us employ now the feedback:

$$\text{(6.3)}\qquad \begin{aligned} u &= kb_1 \\ v &- kb_2 \end{aligned}$$

where $k \in \mathbb{R}$ is the control parameter.

Theorem 6.1. *The controlled system* (6.2), (6.3) *is a Hamilton-Poisson mechanical system with the phase space* $P = \mathbb{R}^5$, *the Poisson structure* (3.2) *and the Hamiltonian* H *given by:*

$$\text{(6.4)}\qquad H = \frac{1}{2}[a_1^2 + a_2^2 + a_3^2] + b_1^2 + b_2^2 + ka_2 + ka_3.$$

Proof. Indeed, we easily check that

$$\dot{a}_i = \{a_i, H\}_{LP}^-; i = 1, 2, 3$$
$$\dot{b}_i = \{b_i, H\}_{LP}^-; i = 1, 2$$

which give the result. □

Using now the energy-Casimir method we can prove:

Theorem 6.2. *The controlled system* (6.2), (6.3) *may be nonlinear stabilized about the equilibrium states* (6.1) *for* $k = \alpha$.

Proof. Let us take the modified energy-Casimir function

$$\text{(6.5)}\qquad H + \varphi(C) = \frac{1}{2}[a_1^2 + a_2^2 + a_3^2] + b_1^2 + b_2^2 + ka_2 + ka_3 + \varphi(a_1 + a_2 + a_3),$$

where $\varphi \in C^\infty(\mathbb{R}, \mathbb{R})$. Then we have:

$$\begin{aligned} \delta(H + \varphi(C)) = {} & a_1\delta a_1 + a_2\delta a_2 + a_3\delta a_3 + 2b_1\delta b_1 \\ & + 2b_2\delta b_2 + k\delta a_2 + k\delta a_3 + \dot{\varphi}(\delta a_1 + \delta a_2 + \delta a_3) \end{aligned}$$

At the equilibrium of interest (6.1) the first variation vanishes if and only if

$$\text{(6.6)}\qquad \begin{aligned} \dot{\varphi}(\alpha) &= -\alpha \\ k &= \alpha \end{aligned}$$

Then

$$\delta^2(H + \varphi(C)) = (\delta a_1)^2 + (\delta a_2)^2 + (\delta a_3)^2 + 2(\delta b_1)^2 + 2(\delta b_2)^2 + \ddot{\varphi}(\delta a_1 + \delta a_2 + \delta a_3)^2$$

and at the equilibrium of interest (6.1) it is positive definite if we choose φ such that:

$$\ddot{\varphi}(\alpha) \geq 0.$$

Therefore the equilibrium states (6.1) are nonlinear stable. □

Remark 6.1. It is easy to see that the equilibrium states

$$(0, \alpha, 0, 0, 0), \alpha \in \mathbb{R}, \alpha \neq 0$$

are all unstable.

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