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ON THE GEOMETRY OF CHAINS

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Abstract. The chains studied in this paper generalize Chern–Moser chains for CR structures. They form a distinguished family of one dimensional submanifolds in manifolds endowed with a parabolic contact structure. Both the parabolic contact structure and the system of chains can be equivalently encoded as Cartan geometries (of different types). The aim of this paper is to study the relation between these two Cartan geometries for Lagrangean contact structures and partially integrable almost CR structures.

We develop a general method for extending Cartan geometries which generalizes the Cartan geometry interpretation of Fefferman's construction of a conformal structure associated to a CR structure. For the two structures in question, we show that the Cartan geometry associated to the family of chains can be obtained in that way if and only if the original parabolic contact structure is torsion free. In particular, the procedure works exactly on the subclass of (integrable) CR structures.

This tight relation between the two Cartan geometries leads to an explicit description of the Cartan curvature associated to the family of chains. On the one hand, this shows that the homogeneous models for the two parabolic contact structures give rise to examples of non–flat path geometries with large automorphism groups. On the other hand, we show that one may (almost) reconstruct the underlying torsion free parabolic contact structure from the Cartan curvature associated to the chains. In particular, this leads to a very conceptual proof of the fact that chain preserving contact diffeomorphisms are either isomorphisms or anti–isomorphisms of parabolic contact structures.

1. Introduction

Parabolic contact structures are a class of geometric structures having an underlying contact structure. They admit a canonical normal Cartan connection corresponding to a contact grading of a simple Lie algebra. The best known examples of such structures are non–degenerate partially integrable almost CR structures of hypersurface type. The construction of the canonical Cartan connection is due to Chern and Moser ([8]) for the subclass of CR structures, and to Tanaka ([16]) in general.

In the approach of Chern and Moser, a central role is played by a canonical class of unparametrized curves called *chains*. For each point x and each direction ξ at x, which is transverse to the contact distribution, there is a unique chain through x in direction ξ. In addition, each chain comes with a projective class of distinguished parametrizations. The notion of chains easily generalizes to arbitrary parabolic contact structures, and the chains are easy to describe in terms of the Cartan connection.

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A *path geometry* on a smooth manifold M is given by a smooth family of unparametrized curves on M such that for each $x \in M$ and each direction ξ at x there is a unique curve through x in direction ξ. The best way to encode this structure is to pass to the projectivized tangent bundle $\mathcal{P}TM$, the space of all lines in TM. Then a path geometry is given by a line subbundle in the tangent bundle of $\mathcal{P}TM$ with certain properties, see [9] and [11] for a modern presentation. It turns out that these structures are equivalent to regular normal Cartan geometries of a certain type, which fall under the general concept of parabolic geometries, see section 4.7 of [3].

In the description as a Cartan geometry, path geometries immediately generalize to open subsets of the projectivized tangent bundle. In particular, given a manifold M endowed with a parabolic contact structure, the chains give rise to a path geometry on the open subset $\mathcal{P}_0TM \subset \mathcal{P}TM$ formed by all lines transversal to the contact subbundle. The general question addressed in this paper is how to describe the resulting Cartan geometry on $\mathcal{P}_0TM$ in terms of the original Cartan geometry on M. We study this in detail in the case of Lagrangean contact structures and, in the end, briefly indicate how to deal with partially integrable almost CR structures, which can be viewed as a different real form of the same complex geometric structure.

The first observation is that $\mathcal{P}_0TM$ can be obtained as a quotient of the Cartan bundle $\mathcal{G} \to M$ obtained from the parabolic contact structure. More precisely, there is a subgroup $Q \subset P$ such that $\mathcal{P}_0TM \cong \mathcal{G}/Q$. In particular, $\mathcal{G}$ is a principal Q–bundle over $\mathcal{P}_0TM$ and the canonical Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ associated to the parabolic contact structure can be also viewed as a Cartan connection on $\mathcal{G} \to \mathcal{P}_0TM$. The question then is whether the canonical Cartan geometry $(\tilde{\mathcal{G}} \to \mathcal{P}_0TM, \tilde{\omega})$ determined by the path geometry of chains can be constructed directly from $(\mathcal{G} \to \mathcal{P}_0TM, \omega)$.

To attack this problem, we study a class of extension functors mapping Cartan geometries of some type (G, Q) to Cartan geometries of another type $(\tilde{G}, \tilde{P})$. These functors have the property that there is a homomorphism between the two Cartan bundles, which relates the two Cartan connections. We show that in order to obtain such a functor, one needs a homomorphism $i : Q \to \tilde{P}$ (which we assume to be infinitesimally injective) and a linear map $\alpha : \mathfrak{g} \to \tilde{\mathfrak{g}}$ which satisfy certain compatibility conditions. There is a simple notion of equivalence for such pairs and equivalent pairs lead to naturally isomorphic extension functors.

There is a particular simple source of pairs (i, α) leading to extension functors as above. Namely, one may start from a homomorphism $G \to \tilde{G}$ and take i the restriction to Q and α the induced homomorphism of Lie algebras. In a special case, this leads to the Cartan geometry interpretation of Fefferman's construction of a canonical conformal structure on a circle bundle over a CR manifold.

One can completely describe the effect of the extension functor associated to a pair (i, α) on the curvature of the Cartan geometries. Apart from the curvature of the original geometry, also the deviation from α being a homomorphism of Lie algebras enters into the curvature of the extended Cartan geometry.

An important feature of the special choice for $(\tilde{G}, \tilde{P})$ that we are concerned with, is a uniqueness result for such extension functors. We show (see Theorem 3.4) that if the extension functor associated to a pair (i, α) maps locally flat geometries of type (G, Q) to regular normal geometries of type $(\tilde{G}, \tilde{P})$, then the pair (i, α) is already determined uniquely up to equivalence. For the two parabolic contact structures studied in this paper, we show that there exist appropriate pairs (i, α) in 3.5 and 5.2.

In both cases, the resulting extension functor does *not* produce the canonical Cartan geometry associated to the path geometry of chains in general. We show that the canonical Cartan connection is obtained if and only if the original parabolic contact geometry is torsion free. For a Lagrangean contact structure this means that the two Lagrangean subbundles are integrable, while it is the usual integrability condition for CR structures. This ties in nicely with the Fefferman construction, where one obtains a conformal structure for arbitrary partially integrable almost CR structures, but the normal Cartan connection is obtained by equivariant extension if and only if the structure is integrable (and hence CR).

Finally, we discuss applications of our construction, which are based on an analysis of the curvature of the canonical Cartan connection associated to the path geometry of chains. We show that chains never are geodesics of a connection, and they give rise to a torsion free path geometry if and only if the original parabolic contact structure is locally flat. Then we show that the underlying parabolic contact structure can be almost reconstructed from the harmonic curvature of the path geometry of chains. In particular, this leads to a very conceptual proof of the fact that a contact diffeomorphism which maps chains to chains must (essentially) preserve the original torsion free parabolic contact structure.

2. Parabolic contact structures, chains, and path geometries

In this section, we will discuss the concepts of chains and the associated path geometry for a parabolic contact structure, focusing on the example of Lagrangean contact structures. We only briefly indicate the changes needed to deal with general parabolic contact structures.

2.1. Lagrangean contact structures. The starting point to define a parabolic contact structure is a simple Lie algebra $\mathfrak{g}$ endowed with a *contact grading*, i.e. a vector space decomposition $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, $\mathfrak{g}_{-2}$ has real dimension one, and the bracket $\mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is non–degenerate. It is known that such a grading is unique up to an inner automorphism and it exists for each non–compact non–complex real simple Lie algebra except $\mathfrak{sl}(n, \mathbb{H})$, $\mathfrak{so}(n, 1)$, $\mathfrak{sp}(p, q)$, one real form of E_6 and one of E_7, see section 4.2 of [19].

Here we will mainly be concerned with the contact grading of $\mathfrak{g} = \mathfrak{sl}(n + 2, \mathbb{R})$, corresponding to the following block decomposition with blocks of size 1, n, and 1:

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1^L & \mathfrak{g}_2 \\ \mathfrak{g}_{-1}^L & \mathfrak{g}_0 & \mathfrak{g}_1^R \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1}^R & \mathfrak{g}_0 \end{pmatrix}.$$

We have indicated the splittings $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^L \oplus \mathfrak{g}_{-1}^R$ respectively $\mathfrak{g}_1 = \mathfrak{g}_1^L \oplus g_1^R$, which are immediately seen to be $\mathfrak{g}_0$–invariant. Further, the subspaces $\mathfrak{g}_{-1}^L$ and $\mathfrak{g}_{-1}^R$ of $\mathfrak{g}_{-1}$ are isotropic for $[\ ,\ ] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$.

Put $G := PGL(n + 2, \mathbb{R})$, the quotient of $GL(n + 2, \mathbb{R})$ by its center. We will view G as the quotient of the group of matrices whose determinant has modulus one by the two element subgroup generated by $\pm\,\mathrm{id}$ and work with representative matrices. The group G always has Lie algebra $\mathfrak{g}$. For odd n, one can identify G with $SL(n + 2, \mathbb{R})$. For even n, G has two connected components, and the component containing the identity is $PSL(n + 2, \mathbb{R})$.

By $G_0 \subset P \subset G$ we denote the subgroups formed by matrices which are block diagonal respectively block upper triangular with block sizes 1, n, and 1. Then the Lie algebras of G_0 and P are $\mathfrak{g}_0$ respectively $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. For $g \in G_0$, the map $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ preserves the grading while for $g \in P$ one obtains $\mathrm{Ad}(g)(\mathfrak{g}_i) \in \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_2$ for $i = -1, \ldots, 2$. This can be used as an alternative characterization of the two subgroups. The reason for the choice of the specific group G with Lie algebra $\mathfrak{g}$ is that the adjoint action identifies G_0 with the group of all automorphisms of the graded Lie algebra $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ which in addition preserve the decomposition $\mathfrak{g}_{-1} = \mathfrak{g}^L_{-1} \oplus \mathfrak{g}^R_{-1}$.

Let M be a smooth manifold of dimension $2n+1$ and let $H \subset TM$ be a subbundle of corank one. The Lie bracket of vector fields induces a tensorial map $\mathcal{L} : \Lambda^2 H \to TM/H$, and that H is called a *contact structure* on M if this map is non–degenerate. A *Lagrangean contact structure* on M is a contact structure $H \subset TM$ together with a fixed decomposition $H = L \oplus R$ such that each of the subbundles is isotropic with respect to $\mathcal{L}$. This forces the two bundles to be of rank n, and $\mathcal{L}$ induces isomorphisms $R \cong L^* \otimes (TM/H)$ and $L \cong R^* \otimes (TM/H)$.

In view of the description of G_0 above, the following result is a special case of general prolongation procedures [17, 14, 4], see [15] and section 4.1 of [3] for more information on this specific case.

Theorem. *Let $H = L \oplus R$ be a Lagrangean contact structure on a manifold M of dimension $2n + 1$. Then there exists a principal P–bundle $p : \mathcal{G} \to M$ endowed with a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ such that $L = Tp(\omega^{-1}(\mathfrak{g}^L_{-1} \oplus \mathfrak{p}))$ and $R = Tp(\omega^{-1}(\mathfrak{g}^R_{-1} \oplus \mathfrak{p}))$. The pair $(\mathcal{G}, \omega)$ is uniquely determined up to isomorphism provided that one in addition requires the curvature of ω to satisfy a normalization condition discussed in 3.6.*

Similarly, for any contact grading of a simple Lie algebra $\mathfrak{g}$ and a choice of a Lie group G with Lie algebra $\mathfrak{g}$, one defines a subgroup $P \subset G$ with Lie algebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. One then obtains an equivalence of categories between regular normal parabolic geometries of type (G, P) and underlying geometric structures, which in particular include a contact structure.

The second case of such structures we will be concerned with in this paper, is partially integrable almost CR structures of hypersurface type, see section 5.

2.2. Chains. Let $(p : \mathcal{G} \to M, \omega)$ be the canonical Cartan geometry determined by a parabolic contact structure. Then one obtains an isomorphism $TM \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$ such that $H \subset TM$ corresponds to $(\mathfrak{g}_{-1} \oplus \mathfrak{p})/\mathfrak{p} \subset \mathfrak{g}/\mathfrak{p}$. Of course, we may identify $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ as a vector space with $\mathfrak{g}/\mathfrak{p}$ and use this to carry over the natural P–action to $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. Let $Q \subset P$ be the stabilizer of the line $\mathfrak{g}_{-2}$ under this action. By definition, this is a closed subgroup of P. Let us denote by $G_0 \subset P$ the closed subgroup consisting of all elements whose adjoint action respects the grading of $\mathfrak{g}$. Then G_0 has Lie algebra $\mathfrak{g}_0$ and by Proposition 2.10 of [4], any element $g \in P$ can be uniquely written in the form $g_0 \exp(Z_1) \exp(Z_2)$ for $g_0 \in G_0$, $Z_1 \in \mathfrak{g}_1$, and $Z_2 \in \mathfrak{g}_2$.

Lemma. *(1) An element $g = g_0 \exp(Z_1) \exp(Z_2) \in P$ lies in the subgroup $Q \subset P$ if and only if $Z_1 = 0$. In particular, $\mathfrak{q} = \mathfrak{g}_0 \oplus \mathfrak{g}_2$ and for $g \in Q$ we have $\mathrm{Ad}(g)(\mathfrak{g}_{-2}) \subset \mathfrak{g}_{-2} \oplus \mathfrak{q}$.*
(2) Let $(p : \mathcal{G} \to M, \omega)$ be the canonical Cartan geometry determined by a parabolic contact structure. Let $x \in M$ be a point and $\xi \in T_xM \setminus H_x$ a tangent vector transverse to the contact subbundle.

Then there is a point $u \in p^{-1}(x) \subset \mathcal{G}$ and a unique lift $\tilde{\xi} \in T_u\mathcal{G}$ of ξ such that $\omega(u)(\tilde{\xi}) \in \mathfrak{g}_{-2}$. The point u is unique up to the principal right action of an element $g \in Q \subset P$.

Proof. (1) We first observe that for a nonzero element $X \in \mathfrak{g}_{-2}$, the map $Z \mapsto [Z, X]$ is a bijection $\mathfrak{g}_1 \to \mathfrak{g}_{-1}$. This is easy to verify directly for the examples discussed in 2.1 and 5.1. For general contact gradings it follows from the fact that $[\mathfrak{g}_{-2}, \mathfrak{g}_2]$ consists of all multiples of the grading element, see section 4.2 of [19].

By definition, $g \in Q$ if and only if $\mathrm{Ad}(g)(\mathfrak{g}_{-2}) \subset \mathfrak{g}_{-2} \oplus \mathfrak{p}$. Now from the expression $g^{-1} = \exp(-Z_2)\exp(-Z_1)g_0^{-1}$ one immediately concludes that $\mathrm{Ad}(g^{-1})(X)$ is congruent to $-[Z_1, X] \in \mathfrak{g}_{-1}$ modulo $\mathfrak{g}_{-2} \oplus \mathfrak{p}$. Hence we see that $g \in Q$ if and only if $Z_1 = 0$, and the rest of (1) evidently follows.

(2) Choose any point $v \in p^{-1}(x)$. Since the vertical bundle of $\mathcal{G} \to M$ equals $\omega^{-1}(\mathfrak{p})$, there is a unique lift $\eta \in T_v\mathcal{G}$ of ξ such that $\omega(v)(\eta) \in \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. The assumption that ξ is transverse to H_x means that $\omega(v)(\eta) \notin \mathfrak{g}_{-1}$. For an element $g \in P$ we can consider $v \cdot g$ and $T_v r^g \cdot \eta \in T_{v\cdot g}\mathcal{G}$, where $v \cdot g = r^g(v)$ denotes the principal right action of g on v. Evidently, $T_v r^g \cdot \eta$ is again a lift of ξ and equivariancy of ω implies that $\omega(v \cdot g)(T_v r^g \cdot \eta) = \mathrm{Ad}(g^{-1})(\omega(v)(\eta))$.

Writing $\omega(v)(\eta) = X_{-2} + X_{-1}$ we have $X_{-2} \neq 0$, so from above we see that there is an element $Z \in \mathfrak{g}_1$ such that $[Z, X_{-2}] = X_{-1}$. Putting $g = \exp(Z) \in P$ we conclude that $\omega(v\cdot g)(T_v r^g \cdot \eta) \in \mathfrak{g}_{-2} \oplus \mathfrak{p}$. Hence putting $u = v \cdot g$ and subtracting an appropriate vertical vector from $T_v r^g \cdot \eta$, we have found a couple $(u, \tilde{\xi})$ as required.

Any other choice of a preimage of x has the form $u \cdot g$ for some $g \in P$. Any lift of ξ in $T_{u\cdot g}\mathcal{G}$ is of the form $T_u r^g \cdot \tilde{\xi} + \zeta$ for some vertical vector ζ. Clearly, there is a choice for ζ such that $\omega(T_u r^g \cdot \tilde{\xi} + \zeta) \in \mathfrak{g}_{-2}$ if and only if $\omega(T_u r^g \cdot \tilde{\xi}) \in \mathfrak{g}_{-2} \oplus \mathfrak{p}$ and equivariancy of ω implies that this is equivalent to $g \in Q$. □

This lemma immediately leads us to chains: Fix a nonzero element $X \in \mathfrak{g}_{-2}$. For a point $x \in M$ and a line ℓ in T_xM which is transverse to H_x, we can find a point $u \in \mathcal{G}$ such that $T_u p \cdot \omega_u^{-1}(X) \in \ell$. Denoting by $\tilde{X}$ the "constant vector field" $\omega^{-1}(X)$ we can consider the flow of $\tilde{X}$ through u and project it onto M to obtain a (locally defined) smooth curve through x whose tangent space at x is ℓ. In section 4 of [6] it has been shown that, as an unparametrized curve, this is uniquely determined by x and ℓ, and it comes with a distinguished projective family of parametrizations.

The lemma also leads us to a nice description of the space of all transverse directions: For a point $u \in \mathcal{G}$, we obtain a line in $T_{p(u)}M$ which is transverse to $H_{p(u)}$, namely $T_p(\omega_u^{-1}(\mathfrak{g}_{-2}))$. This defines a smooth map $\mathcal{G} \to \mathcal{P}TM$, where $\mathcal{P}TM$ denotes the projectivized tangent bundle of M. Since P acts freely on $\mathcal{G}$ so does Q and hence $\mathcal{G}/Q$ is a smooth manifold. By the lemma, we obtain a diffeomorphism from $\mathcal{G}/Q$ to the open subset $\mathcal{P}_0TM \subset \mathcal{P}TM$ formed by all lines which are transverse to the contact distribution H.

2.3. Path geometries. Classically, path geometries are associated to certain families of unparametrized curves in a smooth manifold. Suppose that in a manifold Z we have a smooth family of curves such that through each point of Z there is exactly one curve in each direction. Let $\mathcal{P}TZ$ be the projectivized tangent bundle of Z, i.e. the space of all lines through the origin in tangent spaces of Z. Given a line ℓ in T_xZ, we can choose the unique curve in the family which goes through x in direction ℓ. Choosing a local regular parametrization $c : I \to Z$ of this curve we obtain a lift $\tilde{c} : I \to \mathcal{P}TZ$ by defining $\tilde{c}(t)$ to be the line in $T_{c(t)}Z$ generated by $c'(t)$. Choosing a different regular parametrization, we just obtain a reparametrization of $\tilde{c}$, so the submanifold $\tilde{c}(I) \subset \mathcal{P}TZ$ is independent of all choices. These curves foliate $\mathcal{P}TZ$, and their tangent spaces give rise to a line subbundle $E \subset T\mathcal{P}TZ$.

This subbundle has a special property: Similarly to the tautological line bundle on a projective space, a projectivized tangent bundle carries a tautological subbundle $\Xi \subset T\mathcal{P}TZ$ of rank $\dim(Z)$. By definition, given a line $\ell \subset T_zZ$, a tangent vector $\xi \in T_\ell\mathcal{P}TZ$ lies in Ξ_ℓ if and only if its image under the tangent map of the projection $\mathcal{P}TZ \to Z$ lies in the line ℓ. By construction, the line subbundle E associated to a family of curves as above always is contained in Ξ and is transverse to the vertical subbundle V of $\mathcal{P}TZ \to Z$. Hence we see that $\Xi = E \oplus V$.

Conversely, having given a decomposition $\Xi = E \oplus V$ of the tautological bundle, we can project the leaves of the foliation of $\mathcal{P}TZ$ defined by E to the manifold Z to obtain a smooth family of curves in Z with exactly one curve through each point in each direction. Hence one may use the decomposition $\Xi = E \oplus V$ as an alternative definition of such a family of curves, and this decomposition is usually referred to as a *path geometry* on Z. It is easy to verify that the Lie bracket of vector fields induces an isomorphism $E \otimes V \to T\mathcal{P}TZ/\Xi$.

It turns out that path geometries also admit an equivalent description as regular normal parabolic geometries. Putting $m := \dim(Z) - 1$ we consider the Lie algebra $\tilde{\mathfrak{g}} := \mathfrak{sl}(m+2, \mathbb{R})$ with the $|2|$–grading obtained by a block decomposition

$$\begin{pmatrix} \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_1^E & \tilde{\mathfrak{g}}_2 \\ \tilde{\mathfrak{g}}_{-1}^E & \tilde{\mathfrak{g}}_0 & \tilde{\mathfrak{g}}_1^V \\ \tilde{\mathfrak{g}}_{-2} & \tilde{\mathfrak{g}}_{-1}^V & \tilde{\mathfrak{g}}_0 \end{pmatrix}.$$

as in 2.1, but this time with blocks of size 1, 1, and m. Hence $\tilde{\mathfrak{g}}_{\pm 1}^E$ has dimension 1 while $\tilde{\mathfrak{g}}_{\pm 1}^V$ and $\tilde{\mathfrak{g}}_{\pm 2}$ are all m–dimensional. Put $\tilde{G} := PGL(m+2, \mathbb{R})$ and let $\tilde{G}_0 \subset \tilde{P} \subset \tilde{G}$ be the subgroups formed by matrices which are block diagonal respectively block upper triangular with block sizes 1, 1, and m. Then $\tilde{G}_0$ and $\tilde{P}$ have Lie algebras $\tilde{\mathfrak{g}}_0$ respectively $\tilde{\mathfrak{p}} := \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{g}}_1 \oplus \tilde{\mathfrak{g}}_2$, where $\tilde{\mathfrak{g}}_1 = \tilde{\mathfrak{g}}_1^E \oplus \tilde{\mathfrak{g}}_1^V$.

The adjoint action identifies $\tilde{G}_0$ with the group of automorphisms of the graded Lie algebra $\tilde{\mathfrak{g}}_{-2} \oplus \tilde{\mathfrak{g}}_{-1}$ which in addition preserve the decomposition $\tilde{\mathfrak{g}}_{-1} = \tilde{\mathfrak{g}}_{-1}^E \oplus \tilde{\mathfrak{g}}_{-1}^V$. Hence the following result is a special case of the general prolongation procedures [17, 14, 4], see section 4.7 of [3] for this specific case.

Theorem. *Let $\tilde{Z}$ be a smooth manifold of dimension $2m+1$ endowed with transversal subbundles E and V in $T\tilde{Z}$ of rank 1 and m, respectively, and put $\Xi := E \oplus V \subset T\tilde{Z}$. Suppose that the Lie bracket of two sections of V is a section of Ξ and that the tensorial map $E \otimes V \to T\tilde{Z}/\Xi$ induced by the Lie bracket of vector fields is an isomorphism.*

Then there exists a principal bundle $\tilde{p} : \tilde{\mathcal{G}} \to \tilde{Z}$ with structure group $\tilde{P}$ endowed with a Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ such that $E = T\tilde{p}(\tilde{\omega}^{-1}(\tilde{\mathfrak{g}}_{-1}^E \oplus \tilde{\mathfrak{p}}))$ and $V = T\tilde{p}(\tilde{\omega}^{-1}(\tilde{\mathfrak{g}}_{-1}^V \oplus \tilde{\mathfrak{p}}))$. The pair $(\tilde{\mathcal{G}}, \tilde{\omega})$ is uniquely determined up to isomorphism provided that $\tilde{\omega}$ is required to satisfy a normalization condition discussed in 3.6.

In particular, a family of paths on Z as before gives rise to a Cartan geometry on $\mathcal{P}TZ$. This immediately generalizes to the case of an open subset of $\mathcal{P}TZ$, i.e. the case where paths are only given through each point in an open set of directions.

It turns out that for $m \neq 2$, the assumptions of the theorem already imply that the subbundle $V \subset T\tilde{Z}$ is involutive. Then $\tilde{Z}$ is automatically locally diffeomorphic to a projectivized tangent bundle in such a way that V is mapped to the vertical subbundle and Ξ to the tautological subbundle. Hence for $m \neq 2$, the geometries discussed in the theorem are locally isomorphic to path geometries.

2.4. The path geometry of chains. From 2.2 we see that for a manifold M endowed with a parabolic contact structure the chains give rise to a path geometry on the open subset $\tilde{M} := \mathcal{P}_0TM$ of the projectivized tangent bundle of M. We can easily describe the corresponding configuration of bundles explicitly: Denoting by $(p : \mathcal{G} \to M, \omega)$ the Cartan geometry induced by the parabolic contact structure, we know from 2.2 that $\tilde{M} = \mathcal{G}/Q$, where $Q \subset P$ denotes the stabilizer of the line in $\mathfrak{g}/\mathfrak{p}$ corresponding to $\mathfrak{g}_{-2} \subset \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. In particular, $\mathcal{G}$ is a Q–principal bundle over $\tilde{M}$ and ω is a Cartan connection on $\mathcal{G} \to \tilde{M}$. This implies that $T\tilde{M} = \mathcal{G} \times_Q \mathfrak{g}/\mathfrak{q}$, and the tangent map to the projection $\pi : \tilde{M} \to M$ corresponds to the obvious projection $\mathfrak{g}/\mathfrak{q} \to \mathfrak{g}/\mathfrak{p}$. In particular, the vertical bundle $V = \ker(T\pi)$ corresponds to $\mathfrak{p}/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$. From the construction of the isomorphism $\mathcal{G}/Q \to \tilde{M}$ in 2.2, it is evident that the tautological bundle Ξ corresponds to $(\mathfrak{g}_{-2} \oplus \mathfrak{p})/\mathfrak{q}$. By part (1) of Lemma 2.2, the subspace $(\mathfrak{g}_{-2} \oplus \mathfrak{q})/\mathfrak{q} \subset \mathfrak{g}/\mathfrak{q}$ is Q–invariant, thus it gives rise to a line subbundle E in Ξ, which is complementary to V. By construction, this exactly describes the path geometry determined by the chains.

If $\dim(M) = 2n+1$, then the dimension of $\tilde{M}$ is $4n+1$. Put $\tilde{G} := PGL(2n+2, \mathbb{R})$ and let $\tilde{P} \subset \tilde{G}$ be the subgroup described in 2.3. Then by Theorem 2.3 the path geometry on $\tilde{M}$ gives rise to a canonical principal bundle $\tilde{\mathcal{G}} \to \tilde{M}$ with structure group $\tilde{P}$ endowed with a canonical normal Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$. The main question now is whether there is a direct relation between the Cartan geometries $(\mathcal{G} \to \tilde{M}, \omega)$ and $(\tilde{\mathcal{G}} \to \tilde{M}, \tilde{\omega})$.

The only reasonable way to relate these two Cartan geometries is to consider a morphism $j : \mathcal{G} \to \tilde{\mathcal{G}}$ of principal bundles and compare the pull–back $j^*\tilde{\omega}$ to ω. This means that j is equivariant, so we first have to choose a group homomorphism $i : Q \to \tilde{P}$ and require that $j(u \cdot g) = j(u) \cdot i(g)$ for all $g \in Q$. Having chosen i and j, we have $j^*\tilde{\omega} \in \Omega^1(\mathcal{G}, \tilde{\mathfrak{g}})$ and the only way to directly relate this to $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is to have $j^*\tilde{\omega} = \alpha \circ \omega$ for some linear map $\alpha : \mathfrak{g} \to \tilde{\mathfrak{g}}$. If we have such a relation, then we can immediately recover $\tilde{\mathcal{G}}$ from $\mathcal{G}$: Consider the map $\Phi : \mathcal{G} \times \tilde{P} \to \tilde{\mathcal{G}}$ defined by $\Phi(u, \tilde{g}) := j(u) \cdot \tilde{g}$. Equivariancy of j immediately implies that $\Phi(u \cdot g, \tilde{g}) = \Phi(u, i(g)\tilde{g})$, so Φ descends to a bundle map $\mathcal{G} \times_Q \tilde{P} \to \tilde{\mathcal{G}}$, where the left action of Q on $\tilde{P}$ is defined via i. This is immediately seen to be an isomorphism of principal bundles, so $\tilde{\mathcal{G}}$ is obtained from $\mathcal{G}$ by an extension of structure group. Under this isomorphism, the given morphism $j : \mathcal{G} \to \tilde{\mathcal{G}}$ corresponds to the natural inclusion $\mathcal{G} \to \mathcal{G} \times_Q \tilde{P}$ induced by $u \mapsto (u, e)$.

3. Induced Cartan connections

In this section, we study the problem of extending Cartan connections. We derive the basic results in the setting of general Cartan geometries, and then specialize to the case of parabolic contact structures and, in particular, Lagrangean contact structures. Some of the developments in 3.1 and 3.3 below are closely related to [12, 18].

3.1. Extension functors for Cartan geometries. Motivated by the last observations in 2.4, let us consider the following problem: Suppose we have given Lie groups G and $\tilde{G}$ with Lie algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$, closed subgroups $Q \subset G$ and $\tilde{P} \subset \tilde{G}$, a homomorphism $i : Q \to \tilde{P}$ and a linear map $\alpha : \mathfrak{g} \to \tilde{\mathfrak{g}}$. We will assume throughout that i is infinitesimally injective, i.e. $i' : \mathfrak{q} \to \tilde{\mathfrak{p}}$ is injective.

Given a Cartan geometry $(p : \mathcal{G} \to N, \omega)$ of type (G, Q), we put $\tilde{\mathcal{G}} := \mathcal{G} \times_Q \tilde{P}$ and denote by $j : \mathcal{G} \to \tilde{\mathcal{G}}$ the canonical map. Since i is infinitesimally injective, this is an immersion, i.e. $T_u j$ is injective for all $u \in \mathcal{G}$. We want to understand whether there is a Cartan connection $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ such that $j^*\tilde{\omega} = \alpha \circ \omega$, and if so, whether $\tilde{\omega}$ is uniquely determined.

Proposition. *There is a Cartan connection $\tilde{\omega}$ on $\tilde{\mathcal{G}}$ such that $j^*\tilde{\omega} = \alpha \circ \omega$ if and only if the pair (i, α) satisfies the following conditions:*

(1) $\alpha \circ \operatorname{Ad}(g) = \operatorname{Ad}(i(g)) \circ \alpha$ *for all* $g \in Q$.
(2) *On the subspace* $\mathfrak{q} \subset \mathfrak{g}$*, the map* α *restricts to the derivative* i' *of* $i : Q \to \tilde{P}$.
(3) *The map* $\underline{\alpha} : \mathfrak{g}/\mathfrak{q} \to \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ *induced by* α *is a linear isomorphism.*

If these conditions are satisfied, then $\tilde{\omega}$ is uniquely determined.

Proof. Let us first assume that there is a Cartan connection $\tilde{\omega}$ on $\tilde{\mathcal{G}}$ such that $j^*\tilde{\omega} = \alpha \circ \omega$. For $u \in \mathcal{G}$, the tangent space $T_{j(u)}\tilde{\mathcal{G}}$ is spanned by $T_u j(T_u \mathcal{G})$ and the vertical subspace $V_{j(u)}\tilde{\mathcal{G}}$. The behavior of $\tilde{\omega}$ on the first subspace is determined by the fact that $j^*\tilde{\omega} = \alpha \circ \omega$, while on the second subspace $\tilde{\omega}$ has to reproduce the generators of fundamental vector fields. Hence the restriction of $\tilde{\omega}$ to $j(\mathcal{G})$ is determined by the fact that $j^*\tilde{\omega} = \alpha \circ \omega$. By definition of $\tilde{\mathcal{G}}$, any point $\tilde{u} \in \tilde{\mathcal{G}}$ can be written as $j(u) \cdot \tilde{g}$ for some $u \in \mathcal{G}$ and some $\tilde{g} \in \tilde{P}$, so uniqueness of $\tilde{\omega}$ follows from equivariancy.

Still assuming that $\tilde{\omega}$ exists, condition (1) follows from equivariancy of j, ω, and $\tilde{\omega}$. Equivariancy of j also implies that for $A \in \mathfrak{q}$ and the corresponding fundamental vector field ζ_A we get $Tj \circ \zeta_A = \zeta_{i'(A)}$. Thus condition (2) follows from the fact that both ω and $\tilde{\omega}$ reproduce the generators of fundamental vector fields. Let $p : \mathcal{G} \to N$ and $\tilde{p} : \tilde{\mathcal{G}} \to N$ be the bundle projections, so $\tilde{p} \circ j = p$. For $\xi \in T_u\mathcal{G}$ we have $\alpha(\omega(\xi)) = \tilde{\omega}(T_u j \cdot \xi)$, so if this lies in $\tilde{\mathfrak{p}}$ then $T_u j \cdot \xi$ is vertical. But then ξ is vertical and hence $\omega(\xi) \in \mathfrak{q}$. Therefore, the map $\underline{\alpha}$ is injective, and since both $\mathcal{G}$ and $\tilde{\mathcal{G}}$ admit a Cartan connection, we must have $\dim(\mathfrak{g}/\mathfrak{q}) = \dim(N) = \dim(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})$, so (3) follows.

Conversely, suppose that (1)–(3) are satisfied for (i, α) and ω is given. For $\tilde{u} \in \tilde{\mathcal{G}}$ and $\tilde{\xi} \in T_{\tilde{u}}\tilde{\mathcal{G}}$ we can find elements $u \in \mathcal{G}$, $\xi \in T_u\mathcal{G}$, $A \in \tilde{\mathfrak{p}}$, and $\tilde{g} \in \tilde{P}$ such that $\tilde{u} = j(u)\cdot\tilde{g}$ and $\tilde{\xi} = Tr^{\tilde{g}}\cdot(Tj\cdot\xi + \zeta_A)$. Then we define $\tilde{\omega}(\tilde{\xi}) := \mathrm{Ad}(\tilde{g})^{-1}(\alpha(\omega(\xi)) + A)$. Using properties (1) and (2) one verifies that this is independent of all choices. By (3), it defines a linear isomorphism $T_{\tilde{u}}\tilde{\mathcal{G}} \to \tilde{\mathfrak{g}}$, and the remaining properties of a Cartan connection are easily verified directly. $\square$

Any pair (i, α) which satisfies the properties (1)–(3) of the proposition gives rise to an extension functor from Cartan geometries of type (G, Q) to Cartan geometries of type $(\tilde{G}, \tilde{P})$: Starting from a geometry $(p : \mathcal{G} \to N, \omega)$ of type (G, Q), one puts $\tilde{\mathcal{G}} := \mathcal{G} \times_Q \tilde{P}$ (with Q acting on $\tilde{P}$ via i) and defines $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}})$ to be the unique Cartan connection on $\tilde{\mathcal{G}}$ such that $j^*\tilde{\omega} = \alpha \circ \omega$, where $j : \mathcal{G} \to \tilde{\mathcal{G}}$ is the canonical map. For a morphism $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ between geometries of type (G, Q), we can consider the principal bundle map $\Phi : \tilde{\mathcal{G}}_1 \to \tilde{\mathcal{G}}_2$ induced by $\varphi \times \mathrm{id}_{\tilde{P}}$. By construction, this satisfies $\Phi \circ j_1 = j_2 \circ \varphi$ and we obtain

$$j_1^*\Phi^*\tilde{\omega}_2 = \varphi^* j_2^*\tilde{\omega}_2 = \varphi^*(\alpha \circ \omega_2) = \alpha \circ \varphi^*\omega_2 = \alpha \circ \omega_1.$$

But $\tilde{\omega}_1$ is the unique Cartan connection whose pull–back along j_1 coincides with $\alpha \circ \omega_1$, which implies that $\Phi^*\tilde{\omega}_2 = \tilde{\omega}_1$, and hence Φ is a morphism of Cartan geometries of type $(\tilde{G}, \tilde{P})$.

There is a simple notion of equivalence for pairs (i, α): We call (i, α) and $(\hat{i}, \hat{\alpha})$ equivalent and write $(i, \alpha) \sim (\hat{i}, \hat{\alpha})$ if and only if there is an element $\tilde{g} \in \tilde{P}$ such that $\hat{i}(g) = \tilde{g}^{-1} i(g)\tilde{g}$ and $\hat{\alpha} = \mathrm{Ad}(\tilde{g}^{-1}) \circ \alpha$. Notice that if (i, α) satisfies conditions (1)–(3) of the proposition, then so does any equivalent pair. In order to distinguish between different extension functors, for a geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, Q) we will often denote the geometry of type $(\tilde{G}, \tilde{P})$ obtained using (i, α) by $(\mathcal{G} \times_i \tilde{P}, \tilde{\omega}_\alpha)$.

Lemma. *Let (i, α) and $(\hat{i}, \hat{\alpha})$ be equivalent pairs satisfying conditions (1)–(3) of the proposition. Then the resulting extension functors for Cartan geometries are naturally isomorphic.*

Proof. By assumption, there is an element $\tilde{g} \in \tilde{P}$ such that $\hat{i}(g) = \tilde{g}^{-1} i(g)\tilde{g}$ and $\hat{\alpha} = \mathrm{Ad}(\tilde{g}^{-1}) \circ \alpha$. Let $j : \mathcal{G} \to \mathcal{G} \times_i \tilde{P}$ and $\hat{j} : \mathcal{G} \to \mathcal{G} \times_{\hat{i}} \tilde{P}$ be the natural inclusions, and consider the map $r^{\tilde{g}} \circ j : \mathcal{G} \to \mathcal{G} \times_i \tilde{P}$. Evidently, we have $j(u\cdot g)\cdot\tilde{g} = j(u)\cdot\tilde{g}\cdot\hat{i}(g)$. Hence, by the last observation in 2.4, we obtain an isomorphism $\Psi : \mathcal{G} \times_{\hat{i}} \tilde{P} \to \mathcal{G} \times_i \tilde{P}$ such that $\Psi \circ \hat{j} = r^{\tilde{g}} \circ j$. Now we compute

$$\hat{j}^*\Psi^*\tilde{\omega}_\alpha = j^*(r^{\tilde{g}})^*\tilde{\omega}_\alpha = \mathrm{Ad}(\tilde{g}^{-1}) \circ j^*\tilde{\omega}_\alpha = \hat{\alpha} \circ \omega.$$

By uniqueness, $\Psi^*\tilde{\omega}_\alpha = \tilde{\omega}_{\hat{\alpha}}$, so Ψ is a morphism of Cartan geometries. It is clear from the construction that this defines a natural transformation between the two extension functors and an inverse can be constructed in the same way using $\tilde{g}^{-1}$ rather than $\tilde{g}$. $\square$

3.2. The relation to the Fefferman construction. There is a simple source of pairs (i,α) which satisfy conditions (1)–(3) of Proposition 3.1: Suppose that $\varphi : G \to \tilde{G}$ is an infinitesimally injective homomorphism of Lie groups such that $\varphi(Q) \subset \tilde{P}$. Then $i := \varphi|_Q : Q \to \tilde{P}$ is an infinitesimally injective homomorphism and $\alpha := \varphi' : \mathfrak{g} \to \tilde{\mathfrak{g}}$ is a Lie algebra homomorphism. Then condition (2) of Proposition 3.1 is satisfied by construction, while condition (1) easily follows from differentiating the equation $\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g)^{-1}$. Hence the only nontrivial condition is (3). Note that if (i,α) is obtained from φ in this way, than any pair equivalent to (i,α) is obtained in the same way from the map $g \mapsto \tilde{g}\varphi(g)\tilde{g}^{-1}$ for some $\tilde{g} \in \tilde{G}$. The main feature of such pairs is that α is a homomorphism of Lie algebras.

In this setting, one may actually go one step further: Suppose we have fixed an infinitesimally injective $\varphi : G \to \tilde{G}$ and a closed subgroup $\tilde{P} \subset \tilde{G}$. Then we put $Q := \varphi^{-1}(\tilde{P}) \subset G$ to obtain a pair $(i := \varphi|_Q, \alpha := \varphi')$ and hence an extension functor from Cartan geometries of type (G,Q) to geometries of type $(\tilde{G},\tilde{P})$. For a closed subgroup $P \subset G$ with $Q \subset P$, one gets a functor from geometries of type (G,P) to geometries of type (G,Q) as described in 2.2: Given a geometry $(p : \mathcal{G} \to M, \omega)$ of type (G,P), one defines $\tilde{M} := \mathcal{G}/Q = \mathcal{G} \times_P (P/Q)$ and $(\mathcal{G} \to \tilde{M}, \omega)$ is a geometry of type (G,Q). Combining with the above, one gets a functor from geometries of type (G,P) to geometries of type $(\tilde{G},\tilde{P})$.

The most important example of this is the Cartan geometry interpretation of Fefferman's construction of a Lorentzian conformal structure on the total space of a certain circle bundle over a CR manifold, see [10]. In this case $G = SU(n+1,1)$, $\tilde{G} = SO(2n+2,2)$, and φ is the evident inclusion. Putting $\tilde{P}$ the stabilizer of a real null line $\ell \subset \mathbb{R}^{2n+4}$ in $\tilde{G}$, the group $Q = G \cap \tilde{P}$ is the stabilizer of ℓ in G. Evidently, this is contained in the stabilizer $P \subset G$ of the complex null line spanned by ℓ, and $P/Q \cong \mathbb{R}P^1 \cong S^1$. Hence the above procedure defines a functor, which to a parabolic geometry of type (G,P) on M associates a parabolic geometry of type $(\tilde{G},\tilde{P})$ on the total space $\tilde{M}$ of a circle bundle over M. More details about this can be found in [2].

3.3. The effect on curvature. We next discuss the effect of extension functors of the type discussed in 3.1 on the curvature of Cartan geometries. This will show specific features of the special case discussed in 3.2.

For a Cartan connection ω on a principal P–bundle $\mathcal{G} \to M$ with values in $\mathfrak{g}$, one initially defines the curvature $K \in \Omega^2(\mathcal{G},\mathfrak{g})$ by $K(\xi,\eta) := d\omega(\xi,\eta) + [\omega(\xi),\omega(\eta)]$. This measures the amount to which the Maurer–Cartan equation fails to hold. The defining properties of a Cartan connection immediately imply that K is horizontal and P–equivariant. In particular, $K(\xi,\eta) = 0$ for all η provided that ξ is vertical or, equivalently, that $\omega(\xi) \in \mathfrak{p}$.

Using the trivialization of $T\mathcal{G}$ provided by ω, one can pass to the curvature function $\kappa : \mathcal{G} \to L(\Lambda^2(\mathfrak{g}/\mathfrak{p}),\mathfrak{g})$, which is characterized by

$$\kappa(u)(X+\mathfrak{p}, Y+\mathfrak{p}) := K(u)(\omega^{-1}(X),\omega^{-1}(Y)).$$

This is well defined by horizontality of K, and equivariancy of K easily implies that κ is equivariant for the natural P–action on the space $L(\Lambda^2(\mathfrak{g}/\mathfrak{p}),\mathfrak{g})$, which is induced from the adjoint action on all copies of $\mathfrak{g}$.

Using the setting of 3.1, suppose that $(i : Q \to \tilde{P}, \alpha : \mathfrak{g} \to \tilde{\mathfrak{g}})$ is a pair satisfying the conditions (1)–(3) of Proposition 3.1. Consider the map $\mathfrak{g} \times \mathfrak{g} \to \tilde{\mathfrak{g}}$ defined by $(X,Y) \mapsto [\alpha(X),\alpha(Y)]_{\tilde{\mathfrak{g}}} - \alpha([X,Y]_{\mathfrak{g}})$, which measures the deviation from α

being a homomorphism of Lie algebras. This map is evidently skew symmetric. By condition (1), $\alpha\circ\mathrm{Ad}(g)=\mathrm{Ad}(i(g))\circ\alpha$ for all $g\in Q$, which infinitesimally implies that $\alpha\circ\mathrm{ad}(X)=\mathrm{ad}(i'(X))\circ\alpha$ for all $X\in\mathfrak{q}$, and by condition (2) we have $i'(X)=\alpha(X)$ in this case. Hence this map vanishes if one of the entries is from $\mathfrak{q}\subset\mathfrak{g}$, and we obtain a well defined linear map $\Lambda^2(\mathfrak{g}/\mathfrak{q})\to\tilde{\mathfrak{g}}$. By condition (3), α induces a linear isomorphism $\underline{\alpha}:\mathfrak{g}/\mathfrak{q}\to\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$, and we conclude that we obtain a well defined map $\Psi_\alpha:\Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})\to\tilde{\mathfrak{g}}$ by putting

$$\Psi_\alpha(\tilde{X}+\tilde{\mathfrak{p}},\tilde{Y}+\tilde{\mathfrak{p}})=[\alpha(X),\alpha(Y)]-\alpha([X,Y]),$$

where $\alpha(X)+\tilde{\mathfrak{p}}=\tilde{X}+\tilde{\mathfrak{p}}$ and $\alpha(Y)+\tilde{\mathfrak{p}}=\tilde{Y}+\tilde{\mathfrak{p}}$.

Proposition. *Let (i,α) be a pair satisfying conditions (1)–(3) of Proposition 3.1. Let $(p:\mathcal{G}\to N,\omega)$ be a Cartan geometry of type (G,Q), let $(\mathcal{G}\times_i\tilde{P},\tilde{\omega}_\alpha)$ be the geometry of type $(\tilde{G},\tilde{P})$ obtained using the extension functor associated to (i,α), and let $j:\mathcal{G}\to\mathcal{G}\times_i\tilde{P}$ be the natural map.*

Then the curvature functions κ and $\tilde{\kappa}$ of the two geometries satisfy

$$\tilde{\kappa}(j(u))(\tilde{X},\tilde{Y})=\alpha(\kappa(u)(\underline{\alpha}^{-1}(\tilde{X}),\underline{\alpha}^{-1}(\tilde{Y})))+\Psi_\alpha(\tilde{X},\tilde{Y}),$$

for any $\tilde{X},\tilde{Y}\in\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$, and this completely determines $\tilde{\kappa}$.

In particular, if ω is flat, then $\tilde{\omega}$ is flat if and only if α is a homomorphism of Lie algebras.

Proof. By definition, $j^*\tilde{\omega}_\alpha=\alpha\circ\omega$, and hence $j^*d\tilde{\omega}_\alpha=\alpha\circ d\omega$. This immediately implies that for the curvatures K and $\tilde{K}$ and $\xi,\eta\in\mathfrak{X}(\mathcal{G})$ we get

$$\tilde{K}(j(u))(Tj\cdot\xi,Tj\cdot\eta)=\alpha(d\omega(u)(\xi,\eta))+[\alpha(\omega(u)(\xi)),\alpha(\omega(u)(\eta))].$$

On the other hand, we get

$$\alpha(K(u)(\xi,\eta))=\alpha(d\omega(u)(\xi,\eta))+\alpha([\omega(u)(\xi),\omega(u)(\eta)]).$$

Now the formula for $\tilde{\kappa}(j(u))$ follows immediately from the definition of the curvature functions. Since $\tilde{\kappa}$ is $\tilde{P}$–equivariant, it is completely determined by its restriction to $j(\mathcal{G})$. The final claim follows directly, since Ψ_α vanishes if and only if α is a homomorphism of Lie algebras. □

3.4. Uniqueness. A crucial fact for the further development is that, passing from parabolic contact structures to the associated path geometries of chains, there is actually no freedom in the choice of the pair (i,α) up to equivalence as introduced in 3.1 above. This result certainly is valid in a more general setting but it seems to be difficult to give a nice formulation for conditions one has to assume.

Therefore we return to the setting of section 2, i.e. G is semisimple, $P\subset G$ is obtained from a contact grading, Q is the subgroup described in 2.2, and $\tilde{G}$ and $\tilde{P}$ correspond to path geometries in the appropriate dimension as in 2.3. In this setting we can now prove:

Theorem. *Let (i,α) and $(\hat{i},\hat{\alpha})$ be pairs satisfying conditions (1)–(3) of Proposition 3.1. Suppose that there is a Cartan geometry $(p:\mathcal{G}\to M,\omega)$ of type (G,Q) such that there is an isomorphism between the geometries of type $(\tilde{G},\tilde{P})$ obtained using (i,α) and $(\hat{i},\hat{\alpha})$, which covers the identity on M. Then (i,α) and $(\hat{i},\hat{\alpha})$ are equivalent.*

Proof. Using the notation of the proof of Lemma 3.1, suppose that we have an isomorphism $\Psi:\mathcal{G}\times_i\tilde{P}\to\mathcal{G}\times_{\hat{i}}\tilde{P}$ of principal bundles which covers the identity on M and has the property that $\Psi^*\tilde{\omega}_{\hat{\alpha}}=\tilde{\omega}_\alpha$. Let us denote by j and $\hat{j}$ the natural inclusions of $\mathcal{G}$ into the two extended bundles. Since Ψ covers the identity on M, there must be a smooth function $\varphi:\mathcal{G}\to\tilde{P}$ such that $\Psi(j(u))=\hat{j}(u)\cdot\varphi(u)$.

By construction we have $j(u\cdot g)=j(u)\cdot i(g)$ and $\hat{j}(u\cdot g)=\hat{j}(u)\cdot\hat{i}(g)$, and using the fact that Ψ is $\tilde{P}$–equivariant we obtain $\hat{i}(g)=\varphi(u)i(g)\varphi(u\cdot g)^{-1}$. On the other hand, differentiating the equation $\Psi(j(u))=\hat{j}(u)\cdot\varphi(u)$, we obtain

$$(T\Psi\circ Tj)\cdot\xi=(Tr^{\varphi(u)}\circ T\hat{j})\cdot\xi+\zeta_{\delta\varphi(u)(\xi)}(\Psi(j(u)))$$

where $\delta\varphi\in\Omega^1(\mathcal{G},\tilde{\mathfrak{p}})$ denotes the left logarithmic derivative of $\varphi:\mathcal{G}\to\tilde{P}$. Applying $\tilde{\omega}_{\hat{\alpha}}$ to the left hand side of this equation, we simply get

$$(j^*\Psi^*\tilde{\omega}_{\hat{\alpha}})(\xi)=(j^*\tilde{\omega}_{\alpha})(\xi)=\alpha(\omega(\xi)).$$

Applying $\tilde{\omega}_{\hat{\alpha}}$ to the right hand side, we obtain

$$\begin{aligned}(\hat{j}^*(r^{\varphi(u)})^*\tilde{\omega}_{\hat{\alpha}})(\xi)+\delta\varphi(u)(\xi)&=\\ &\mathrm{Ad}(\varphi(u)^{-1})((\hat{j}^*\tilde{\omega}_{\hat{\alpha}})(\xi))+\delta\varphi(u)(\xi)=\\ &\qquad\mathrm{Ad}(\varphi(u)^{-1})(\hat{\alpha}(\omega(\xi)))+\delta\varphi(u)(\xi),\end{aligned}$$

and we end up with the equation

$$(*)\qquad \alpha(\omega(\xi))=\mathrm{Ad}(\varphi(u)^{-1})(\hat{\alpha}(\omega(\xi)))+\delta\varphi(u)(\xi)$$

for all $\xi\in T\mathcal{G}$. Together with the relation between i and $\hat{i}$ derived above, this shows that it suffices to show that $\varphi(u)$ is constant to prove that $(i,\alpha)\sim(\hat{i},\hat{\alpha})$.

By construction, $\delta\varphi(u)$ has values in $\tilde{\mathfrak{p}}$, so projecting equation $(*)$ to $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ implies that

$$\alpha(\omega(\xi))+\tilde{\mathfrak{p}}=\underline{\mathrm{Ad}}(\varphi(u)^{-1})(\hat{\alpha}(\omega(\xi))+\tilde{\mathfrak{p}}),$$

for all $\xi\in T_u\mathcal{G}$, where $\underline{\mathrm{Ad}}$ is the action of $\tilde{P}$ on $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ induced by the adjoint action. By property (3) from Proposition 3.1 this implies that $\underline{\alpha}=\underline{\mathrm{Ad}}(\varphi(u)^{-1})\circ\underline{\hat{\alpha}}$, so we see that $\underline{\mathrm{Ad}}(\varphi(u)^{-1})$ must be independent of u. Hence we must have $\varphi(u)=\tilde{g}_1\varphi_1(u)$ for some element $\tilde{g}_1\in\tilde{P}$ and a smooth function $\varphi_1:\mathcal{G}\to\tilde{P}$ which has values in the kernel of $\underline{\mathrm{Ad}}$. As in 2.2, any element of $\tilde{P}$ can be uniquely written in the form $\tilde{g}_0\exp(\tilde{Z}_1)\exp(\tilde{Z}_2)$ with $\tilde{g}_0\in\tilde{G}_0$ and $\tilde{Z}_i\in\tilde{\mathfrak{g}}_i$, and such an element lies in the kernel of $\underline{\mathrm{Ad}}$ if and only if $\mathrm{Ad}(\tilde{g}_0)$ restricts to the identity on $\tilde{\mathfrak{g}}_-$ and $\tilde{Z}_1=0$. Since $\tilde{\mathfrak{p}}_+$ is dual to $\tilde{\mathfrak{g}}_-$ and $\tilde{\mathfrak{g}}_0$ injects into $L(\tilde{\mathfrak{g}}_-,\tilde{\mathfrak{g}}_-)$ the first condition implies that $\mathrm{Ad}(\tilde{g}_0)=\mathrm{id}_{\tilde{\mathfrak{g}}}$. Since $\tilde{G}=PGL(k,\mathbb{R})$ for some k, this implies that $\tilde{g}_0$ is the identity.

Hence φ_1 has values in $\exp(\tilde{\mathfrak{g}}_2)$ and therefore $\delta\varphi(u)$ has values in $\tilde{\mathfrak{g}}_2$. Projecting equation $(*)$ to $\tilde{\mathfrak{g}}/\tilde{\mathfrak{g}}_2$, we obtain

$$\alpha(\omega(\xi))+\tilde{\mathfrak{g}}_2=\underline{\mathrm{Ad}}(\varphi(u)^{-1})(\hat{\alpha}(\omega(\xi))+\tilde{\mathfrak{g}}_2),$$

where this time $\underline{\mathrm{Ad}}$ denotes the natural action on $\tilde{\mathfrak{g}}/\tilde{\mathfrak{g}}_2$. But by [19, Lemma 3.2] an element of $\tilde{\mathfrak{g}}_2$ vanishes provided that all brackets with elements of $\tilde{\mathfrak{g}}_{-1}$ vanish, and this easily implies that $\varphi_1(u)$ is the identity and so φ is constant. □

This result has immediate consequences on the problem of describing the path geometry of chains associated to a parabolic contact structure: If we start with the homogeneous model G/P for a parabolic contact geometry, the induced path geometry of chains is defined on the homogeneous space G/Q. To obtain this by an extension functor as described in 3.1, we need a homomorphism $i:Q\to\tilde{P}$ and a linear map $\alpha:\mathfrak{g}\to\tilde{\mathfrak{g}}$, where $(\tilde{G},\tilde{P})$ gives rise to path geometries in the appropriate dimension. The pair (i,α) has to satisfy conditions (1)–(3) of Proposition 3.1 in order to give rise to an extension functor. The only additional condition is that the extended geometry $(G\times_i\tilde{P},\tilde{\omega}_\alpha)$ obtained from $(G\to G/Q,\omega^{MC})$ is regular and normal. By Theorem 2.3, a regular normal parabolic geometry of type $(\tilde{G},\tilde{P})$ is uniquely determined by the underlying path geometry, which is encoded into $(G\to G/Q,\omega^{MC})$, see 2.4.

The theorem above then implies that (i, α) is uniquely determined up to equivalence. In view of Lemma 3.1, the extension functor obtained from (i, α) is (up to natural isomorphism) the only extension functor of the type discussed in 3.1 which produces the right result for the homogeneous model (and hence for locally flat geometries).

The final step is then to study under which conditions on a geometry of type (G, P), the extension functor associated to (i, α) produces a regular normal geometry of type $(\tilde{G}, \tilde{P})$.

3.5. Let us return to the case of Lagrangean contact structures as discussed in 2.1. By definition, we have $G = PGL(n+2, \mathbb{R})$ and $P \subset G$ is the subgroup of all matrices which are block upper triangular with blocks of sizes 1, n, and 1. From part (1) of Lemma 2.2 one immediately concludes that $Q \subset P$ is the subgroup formed by all matrices of the block form

$$\begin{pmatrix} p & 0 & s \\ 0 & R & 0 \\ 0 & 0 & q \end{pmatrix},$$

such that $|pq\det(R)| = 1$. Since the corresponding manifolds have dimension $2n+1$, the right group for the path geometry defined by the chains is $\tilde{G} = PGL(2n+2, \mathbb{R})$. The subgroup $\tilde{P} \subset \tilde{G}$ is given by the classes of those matrices which are block upper triangular with blocks of sizes 1, 1, $2n$. In the sequel, we will always further split the last block into two blocks of size n.

Consider the (well defined) smooth map $i : Q \to \tilde{P}$ and the linear map $\alpha : \mathfrak{g} \to \tilde{\mathfrak{g}}$ defined by

$$i\begin{pmatrix} p & 0 & s \\ 0 & R & 0 \\ 0 & 0 & q \end{pmatrix} := \begin{pmatrix} \operatorname{sgn}(\frac{q}{p})\sqrt{|\frac{p}{q}|} & \operatorname{sgn}(\frac{q}{p})\frac{s}{p}\sqrt{|\frac{p}{q}|} & 0 & 0 \\ 0 & \sqrt{|\frac{q}{p}|} & 0 & 0 \\ 0 & 0 & q^{-1}\sqrt{|\frac{q}{p}|}R & 0 \\ 0 & 0 & 0 & p\sqrt{|\frac{q}{p}|}(R^{-1})^t \end{pmatrix},$$

$$\alpha\begin{pmatrix} a & u & d \\ x & B & v \\ z & y & c \end{pmatrix} := \begin{pmatrix} \frac{a-c}{2} & d & \frac{1}{2}u & \frac{1}{2}v^t \\ z & \frac{c-a}{2} & \frac{1}{2}y & -\frac{1}{2}x^t \\ x & v & B - \frac{a+c}{2}\operatorname{id} & 0 \\ y^t & -u^t & 0 & -B^t + \frac{a+c}{2}\operatorname{id} \end{pmatrix},$$

where id denotes the $n \times n$ identity matrix.

Proposition. *The map $i : Q \to \tilde{P}$ is an injective group homomorphism and the pair (i, α) satisfies conditions (1)–(3) of Proposition 3.1. Hence it gives rise to an extension functor from Cartan geometries of type (G, Q) to Cartan geometries of type $(\tilde{G}, \tilde{P})$.*

Proof. All these facts are verified by straightforward computations, some of which are a little tedious. □

3.6. Regularity and normality. We next have to discuss the conditions on the curvature of a Cartan connection which were used in Theorems 2.1 and 2.3. If G is a semisimple group and $P \subset G$ is parabolic, then one can identify $(\mathfrak{g}/\mathfrak{p})^*$ with $\mathfrak{p}_+$, the sum of all positive grading components, via the Killing form, see [19, Lemma 3.1]. Hence we can view the curvature function defined in 3.3 as having values in $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$. Via the gradings of $\mathfrak{p}_+$ and $\mathfrak{g}$, this space is naturally graded, and the Cartan connection ω is called *regular* if its curvature function has values in the part of positive homogeneity. Otherwise put, if $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$, then $\kappa(u)(X + \mathfrak{p}, Y + \mathfrak{p}) \in \mathfrak{g}_{i+j+1} \oplus \cdots \oplus \mathfrak{g}_k$.

Recall that a Cartan geometry is torsion free, if and only if κ has values in $\Lambda^2\mathfrak{p}_+\otimes\mathfrak{p}$. Since elements of $\mathfrak{p}_+$ have strictly positive homogeneity, this subspace is contained in the part of positive homogeneity, and any torsion free Cartan geometry is automatically regular. Hence regularity should be viewed as a condition which avoids particularly bad types of torsion.

On the other hand, there is a natural map $\partial^*:\Lambda^2\mathfrak{p}_+\otimes\mathfrak{g}\to\mathfrak{p}_+\otimes\mathfrak{g}$ defined by

$$\partial^*(Z\wedge W\otimes A):=-W\otimes[Z,A]+Z\otimes[W,A]-[Z,W]\otimes A$$

for decomposable elements. This is the differential in the standard complex computing the Lie algebra homology of $\mathfrak{p}_+$ with coefficients in the module $\mathfrak{g}$. This map is evidently equivariant for the natural P–action, so in particular, $\ker(\partial^*)\subset\Lambda^2\mathfrak{p}_+\otimes\mathfrak{g}$ is a P–submodule. The Cartan connection ω is called *normal* if and only if its curvature has values in this submodule.

To proceed with the program set out in the end of 3.4 we next have to analyze the map $\Psi_\alpha:\Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})\to\tilde{\mathfrak{g}}$ introduced in 3.3 in the special case of the pair (i,α) from 3.5. As a linear space, we may identify $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ with $\tilde{\mathfrak{g}}_-=\tilde{\mathfrak{g}}^E_{-1}\oplus\tilde{\mathfrak{g}}^V_{-1}\oplus\tilde{\mathfrak{g}}_{-2}$. Note that using brackets in $\tilde{\mathfrak{g}}$, we may identify $\tilde{\mathfrak{g}}^V_{-1}$ with $\tilde{\mathfrak{g}}^E_1\otimes\tilde{\mathfrak{g}}_{-2}$ if necessary. We will view $\tilde{\mathfrak{g}}_{-2}$ as $\mathbb{R}^{2n}=\mathbb{R}^n\oplus\mathbb{R}^n$ and correspondingly write $X\in\tilde{\mathfrak{g}}_{-2}$ as (X_1,X_2). By $\langle\ ,\ \rangle$ we denote the standard inner product on $\mathbb{R}^n$.

Lemma. *Viewing Ψ_α as an element of $\Lambda^2(\tilde{\mathfrak{g}}_-)^*\otimes\tilde{\mathfrak{g}}$, it lies in the subspace $(\tilde{\mathfrak{g}}^V_{-1})^*\wedge(\tilde{\mathfrak{g}}_{-2})^*\otimes\tilde{\mathfrak{g}}_0$. Denoting by $W_0\in\tilde{\mathfrak{g}}^E_1$ the element whose unique nonzero entry is equal to 1, the trilinear map $\tilde{\mathfrak{g}}_{-2}\times\tilde{\mathfrak{g}}_{-2}\times\tilde{\mathfrak{g}}_{-2}\to\tilde{\mathfrak{g}}_{-2}$ defined by $(X,Y,Z)\mapsto[\Psi_\alpha(X,[Y,W_0]),Z]$ is (up to a nonzero multiple) the complete symmetrization of the map $(X,Y,Z)\mapsto\langle X_1,Y_2\rangle\begin{pmatrix}Z_1\\-Z_2\end{pmatrix}$.*

Proof. Let $x\in\tilde{\mathfrak{g}}^E_{-1}$ be the element whose unique nonzero entry is equal to 1. Then an arbitrary element of $\tilde{\mathfrak{g}}_-$ can be written uniquely as $X+[Y,W_0]+ax$ for $X,Y\in\tilde{\mathfrak{g}}_{-2}$ and $a\in\mathbb{R}$. From the definition of α in 3.5 we obtain

$$\alpha\begin{pmatrix}0&-Y_2^t&0\\X_1&0&Y_1\\a&X_2^t&0\end{pmatrix}=\begin{pmatrix}0&0&-\frac12Y_2^t&\frac12Y_1^t\\a&0&\frac12X_2^t&-\frac12X_1^t\\X_1&Y_1&0&0\\X_2&Y_2&0&0\end{pmatrix},$$

so this is congruent to $X+[Y,W_0]+ax$ modulo $\tilde{\mathfrak{p}}$. Using this, one can now insert into the defining formula for Ψ_α from 3.3 and compute directly that the result always has values in $\tilde{\mathfrak{g}}_0$, and indeed only in the lower right $2n\times2n$ block. Moreover, all the entries in that block are made up from bilinear expressions involving one entry from $\tilde{\mathfrak{g}}_{-2}$ and one entry from $\tilde{\mathfrak{g}}^V_{-1}$, so we see that $\Psi_\alpha\in(\tilde{\mathfrak{g}}_{-2})^*\wedge(\tilde{\mathfrak{g}}^V_{-1})^*\otimes\tilde{\mathfrak{g}}_0$.

For $X,Y\in\tilde{\mathfrak{g}}_{-2}$, one next computes that the only nonzero block in $\Psi_\alpha(X,[Y,W_0])$ (which is a $2n\times2n$–matrix) is explicitly given by

$$\frac12\begin{pmatrix}X_1Y_2^t+Y_1X_2^t+(Y_2^tX_1+X_2^tY_1)\,\mathrm{id}&X_1Y_1^t+Y_1X_1^t\\-X_2Y_2^t-Y_2X_2^t&-Y_2X_1^t-X_2Y_1^t-(Y_2^tX_1+X_2^tY_1)\,\mathrm{id}\end{pmatrix}.$$

To obtain $[\Psi_\alpha(X,[Y,W_0]),Z]\in\tilde{\mathfrak{g}}_{-2}$ for another element $Z\in\tilde{\mathfrak{g}}_{-2}$, we now simply have to apply this matrix to $\begin{pmatrix}Z_1\\Z_2\end{pmatrix}$. Taking into account that $\langle v,w\rangle=v^tw=w^tv$ for $v,w\in\mathbb{R}^n$ we obtain half the sum of all cyclic permutations of

$$(\langle X_1,Y_2\rangle+\langle Y_1,X_2\rangle)\begin{pmatrix}Z_1\\-Z_2\end{pmatrix},$$

which is three times the total symmetrization of $(X,Y,Z)\mapsto\langle X_1,Y_2\rangle\begin{pmatrix}Z_1\\-Z_2\end{pmatrix}$. $\square$

Using this we can now complete the first part of the program outlined in the end of 3.4:

Theorem. *The extension functor associated to the pair (i, α) from 3.5 maps locally flat Cartan geometries of type (G, Q) to torsion free (and hence regular), normal parabolic geometries of type $(\tilde{G}, \tilde{P})$.*

Proof. Let $(p : \mathcal{G} \to N, \omega)$ be a locally flat Cartan geometry of type (G, Q). This means that ω has trivial curvature, so by Proposition 3.3, the curvature function $\tilde{\kappa}$ of the parabolic geometry $(\mathcal{G} \times_i \tilde{P}, \tilde{\omega}_\alpha)$ has the property that

$$\tilde{\kappa}(j(u)) = \Psi_\alpha : \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}) \to \tilde{\mathfrak{g}},$$

where $j : \mathcal{G} \to \mathcal{G} \times_i \tilde{P}$ is the natural map. By the lemma above, $\tilde{\kappa}(j(u))$ has values in $\tilde{\mathfrak{g}}_0 \subset \tilde{\mathfrak{p}}$, and since having values in $\tilde{\mathfrak{p}}$ is a $\tilde{P}$–invariant property, torsion freeness follows.

Similarly, since $\ker(\partial^*)$ is a $\tilde{P}$–submodule in $\Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$, it suffices to show that $\partial^*(\Psi_\alpha) = 0$ to complete the proof of the theorem. This may be checked by a direct computation, but there is a more conceptual argument: Tracefree matrices in the lower right $2n \times 2n$ block of $\tilde{\mathfrak{g}}_0$ form a Lie subalgebra isomorphic to $\mathfrak{sl}(2n, \mathbb{R})$ which acts on each of the spaces $\Lambda^k(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$. Hence we may decompose each of them into a direct sum of irreducible representations. Since ∂^* is a $\tilde{P}$–homomorphism, it is equivariant for this action of $\mathfrak{sl}(2n, \mathbb{R})$, and hence it can be nonzero only between isomorphic irreducible components.

In the proof of the lemma we have noted that $\tilde{\mathfrak{g}}_{-2}$ is the standard representation of $\mathfrak{sl}(2n, \mathbb{R})$, so the explicit formula for Ψ_α shows that it sits in a component isomorphic to $S^3\mathbb{R}^{2n*} \otimes \mathbb{R}^{2n}$. There is a unique trace from this representation to $S^2\mathbb{R}^{2n*}$, and the kernel of this is well known to be irreducible. One immediately checks that $(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$ cannot contain an irreducible component isomorphic to the kernel of this trace. Hence we can finish the proof by showing that Ψ_α lies in the kernel of that trace, which is a simple direct computation. □

This has a nice immediate application:

Corollary. *Consider the homogeneous model $G \to G/P$ of Lagrangean contact structures. Then the resulting path geometry of chains is non–flat and hence not locally isomorphic to $\tilde{G}/\tilde{P}$, but its automorphism group contains G. In particular, for each $n \geq 1$, we obtain an example of a non–flat torsion free path geometry on a manifold of dimension $2n + 1$ whose automorphism group has dimension at least $n^2 + 4n + 3$.*

Remark. (1) In [11], the author directly constructed a torsion free path geometry from the homogeneous model of three–dimensional Lagrangean contact structures. This construction was one of the motivations for this paper and one of the guidelines for the right choice of the pair (i, α). The other main guideline for this choice are the computations needed to show that Ψ_α has values in $\tilde{\mathfrak{g}}_0$.
(2) We shall see later that in the situation of the corollary, the dimension of the automorphism group actually equals the dimension of G. In particular, for $n = 1$, one obtains a non–flat path geometry on a three manifold with automorphism group of dimension 8. To our knowledge, this is the maximal possible dimension for the automorphism group of a non–flat path geometry in this dimension.

Via the interpretation of path geometries in terms of systems of second order ODE's, we obtain examples of nontrivial systems of such ODE's with large automorphism groups.

3.7. More on curvatures of regular normal geometries. We have completed half of the program outlined in the end of 3.4 at this point: Theorem 3.6 shows that the extension functor associated to the pair (i, α) defined in 3.5 produces the regular normal parabolic geometry determined by the path geometry of chains for locally flat Lagrangean contact structures. In view of Theorem 3.4 and Lemma 3.1 this pins down the pair (i, α) up to equivalence and hence the associated extension functor up to isomorphism.

Hence it only remains to clarify under which conditions on a Lagrangean contact structure this extension procedure produces a regular normal parabolic geometry. This then tells us the most general situation in which a direct relation (as discussed in 2.4 and 3.1) between the two parabolic geometries can exist. As it can already be expected from the case of the Fefferman construction (see [2]) this is a rather subtle question. Moreover, the result cannot be obtained by algebraically comparing the two normalization conditions, but one needs more information on the curvature of regular normal and torsion free normal geometries. In particular, the proof of part (2) of the Lemma below needs quite a lot of deep machinery for parabolic geometries.

As discussed in 3.6, the curvature function of a parabolic geometry of type (G, P) has values in $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$. Since both $\mathfrak{p}_+$ and $\mathfrak{g}$ are graded, there is a natural notion of homogeneity on this space. While being of some fixed homogeneity is not a P–invariant property, the fact that all nonzero homogeneous components have at least some given homogeneity is P–invariant. This is used in the definition of regularity in 3.6, which simply says that all nonzero homogeneous components are in positive homogeneity.

The map ∂^* used in the definition of normality in 3.6 actually extends to a family of maps $\partial^* : \Lambda^\ell\mathfrak{p}_+ \otimes \mathfrak{g} \to \Lambda^{\ell-1}\mathfrak{p}_+ \otimes \mathfrak{g}$. These are the differentials in the standard complex computing the Lie algebra homology $H_*(\mathfrak{p}_+, \mathfrak{g})$. By definition, the curvature function κ of a normal parabolic geometry of type (G, P) has values in $\ker(\partial^*) \subset \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$. Hence we can naturally project to the quotient to obtain a function κ_H with values in $\ker(\partial^*)/\operatorname{im}(\partial^*) = H_2(\mathfrak{p}_+, \mathfrak{g})$. Equivariancy of κ implies that κ_H can be viewed as a smooth section of the bundle $\mathcal{G} \times_P H_2(\mathfrak{p}_+, \mathfrak{g})$. This section is called the *harmonic curvature* of the normal parabolic geometry. It turns out (see [5]) that P_+ acts trivially on $H_*(\mathfrak{p}_+, \mathfrak{g})$, so this bundle admits a direct interpretation in terms of the underlying structure. As we shall see below, this bundle is algorithmically computable.

Now from 3.6 we know that $\mathfrak{p}_+ \cong (\mathfrak{g}/\mathfrak{p})^*$ as a P–module, and since $\mathfrak{g}_- \subset \mathfrak{g}$ is a complementary subspace (and G_0–module) to $\mathfrak{p} \subset \mathfrak{g}$ we can identify $\mathfrak{p}_+$ with $(\mathfrak{g}_-)^*$ as a G_0–module. Hence we can also view the spaces $\Lambda^\ell\mathfrak{p}_+ \otimes \mathfrak{g}$ as $L(\Lambda^\ell\mathfrak{g}_-, \mathfrak{g})$, which are the chain spaces in the standard complex computing the Lie algebra cohomology of $\mathfrak{g}_-$ with coefficients in $\mathfrak{g}$. The differentials $\partial : L(\Lambda^\ell\mathfrak{g}_-, \mathfrak{g}) \to L(\Lambda^{\ell+1}\mathfrak{g}_-, \mathfrak{g})$ in that complex turn out to be adjoint to the maps ∂^* with respect to a certain inner product.

Hence we obtain an algebraic Hodge theory on each of the spaces $\Lambda^\ell\mathfrak{p}_+ \otimes \mathfrak{g}$, with algebraic Laplacian $\Box = \partial^* \circ \partial + \partial \circ \partial^*$. This construction is originally due to Kostant (see [13]), whence $\Box$ is usually called the Kostant Laplacian. The kernel of $\Box$ is a G_0–submodule called the *harmonic subspace* of $\Lambda^\ell\mathfrak{p}_+ \otimes \mathfrak{g}$. Kostant's version of the Bott–Borel–Weil theorem in [13] gives a complete algorithmic description of the G_0–module $\ker(\Box)$. By the Hodge decomposition, $\ker(\Box)$ is isomorphic to the homology group of the appropriate dimension.

We will need two general facts about the curvature of regular normal respectively torsion free normal parabolic geometries in the sequel:

Lemma. *Let $(p : \mathcal{G} \to M, \omega)$ be a regular normal parabolic geometry of type (G, P) with curvature functions $\kappa : \mathcal{G} \to \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$ and $\kappa_H : \mathcal{G} \to H_2(\mathfrak{p}_+, \mathfrak{g})$. Then we have:*
(1) The lowest nonzero homogeneous component of κ has values in the subset $\ker(\Box) \subset \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$.
(2) Suppose that $(p : \mathcal{G} \to M, \omega)$ is torsion free and that $E_0 \subset \ker(\Box) \subset \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$ is a G_0–submodule such that κ_H has values in the image of E_0 under the natural isomorphism $\ker(\Box) \to H_2(\mathfrak{p}_+, \mathfrak{g})$ (induced by projecting $\ker(\Box) \subset \ker(\partial^*)$ to the quotient). Then κ has values in the P–submodule of $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$ generated by E_0.

Proof. (1) is an application of the Bianchi identity, which goes back to [17], see also [4, Corollary 4.10]. (2) is proved in [3, Corollary 3.2]. □

The final bit of information we need is the explicit form of $\ker(\Box)$ for the pairs $(\mathfrak{g}, \mathfrak{p})$ and $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{p}})$ corresponding to Lagrangean contact structures on manifolds of dimension $2n+1$ respectively path geometries in dimension $4n+1$. Obtaining the explicit description of the irreducible components of these submodules is an exercise in the application of Kostant's results from [13] and the algorithms from the book [1], see also [3]. The results are listed in the tables below. The first column contains the homogeneity of the component and the second column contains the subspace that it is contained in. The actual component is always the highest weight part in that subspace, so in particular, it lies in the kernel of all traces one can form.

$(\mathfrak{g}, \mathfrak{p})$, $n = 1$

homog.	contained in
4	$\mathfrak{g}_1^R \wedge \mathfrak{g}_2 \otimes \mathfrak{g}_1^R$
4	$\mathfrak{g}_1^L \wedge \mathfrak{g}_2 \otimes \mathfrak{g}_1^L$

$(\mathfrak{g}, \mathfrak{p})$, $n > 1$

homog.	contained in
2	$\mathfrak{g}_1^L \wedge \mathfrak{g}_1^R \otimes \mathfrak{g}_0$
1	$\Lambda^2\mathfrak{g}_1^L \otimes \mathfrak{g}_{-1}^R$
1	$\Lambda^2\mathfrak{g}_1^R \otimes \mathfrak{g}_{-1}^L$

$(\tilde{\mathfrak{g}}, \tilde{\mathfrak{p}})$, $n = 1$

homog.	contained in
3	$\tilde{\mathfrak{g}}_1^V \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_0$
2	$\tilde{\mathfrak{g}}_1^E \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_{-1}^V$
1	$\Lambda^2\tilde{\mathfrak{g}}_1^V \otimes \tilde{\mathfrak{g}}_{-1}^E$

$(\tilde{\mathfrak{g}}, \tilde{\mathfrak{p}})$, $n > 1$

homog.	contained in
3	$\tilde{\mathfrak{g}}_1^V \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_0$
2	$\tilde{\mathfrak{g}}_1^E \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_{-1}^V$
0	$\Lambda^2\tilde{\mathfrak{g}}_1^V \otimes \tilde{\mathfrak{g}}_{-2}$

3.8. We are now ready to prove the main result of this article:

Theorem. *Let $(p : \mathcal{G} \to M, \omega)$ be a regular normal parabolic geometry of type (G, P) and let $(\tilde{\mathcal{G}} := \mathcal{G} \times_Q \tilde{P} \to \mathcal{P}_0(TM), \tilde{\omega}_\alpha)$ be the parabolic geometry obtained using the extension functor associated to the pair (i, α) defined in 3.5. Then this geometry is regular and normal if and only if $(p : \mathcal{G} \to M, \omega)$ is torsion free.*

Proof. We first prove necessity of torsion freeness. From the tables in 3.7 we see that for $n = 1$ a regular normal parabolic geometry of type (G, P) is automatically torsion free, so we only have to consider the case $n > 1$. If $\tilde{\omega}_\alpha$ is regular and normal, then all nonzero homogeneous components of $\tilde{\kappa}$ are homogeneous of positive degrees. The table in 3.7 shows that then the homogeneity is at least two, and by part (1) of Lemma 3.7 the homogeneous component of degree two sits in the subspace $\tilde{\mathfrak{g}}_1^E \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_{-1}^V$. In particular, for any $\tilde{u} \in \tilde{\mathcal{G}}$, the restriction of $\tilde{\kappa}(\tilde{u})$ to $\Lambda^2\tilde{\mathfrak{g}}_{-2}$ is homogeneous of degree at least three, which implies that $\tilde{\kappa}(\tilde{u})$ has values in $\tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{p}}$, i.e. for the natural projection $\pi : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}/(\tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{p}})$ we get $\pi \circ \tilde{\kappa}(\tilde{u}) = 0$.

Using the notation of the proof of Lemma 3.6, consider two elements $X, Y \in \tilde{\mathfrak{g}}_{-2}$. From that proof, we see that

$$(\pi \circ \tilde{\kappa}(j(u)))(X,Y) = (\pi \circ \alpha \circ \kappa(u)) \left(\begin{pmatrix} 0 & 0 & 0 \\ X_1 & 0 & 0 \\ 0 & X_2^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ Y_1 & 0 & 0 \\ 0 & Y_2^t & 0 \end{pmatrix} \right).$$

By regularity, $\kappa(u)(\Lambda^2\mathfrak{g}_{-1}) \subset \mathfrak{g}_{-1} \oplus \mathfrak{p}$. From the definition in 3.5 it is evident that α induces a linear isomorphism $\mathfrak{g}/(\mathfrak{g}_{-2} \oplus \mathfrak{p}) \to \tilde{\mathfrak{g}}/(\tilde{\mathfrak{g}}_{-1} \oplus \tilde{\mathfrak{p}})$. Hence we conclude that if $\tilde{\omega}_\alpha$ is regular and normal, then $\kappa(u)(\Lambda^2\mathfrak{g}_{-1}) \subset \mathfrak{p}$. From the table in 3.7 we see that this implies that the homogeneous component of degree one of κ has to vanish identically, and then further that the homogeneous component of degree two has values in $\mathfrak{p}$. Since $\Lambda^2\mathfrak{g}_{-2} = 0$, components of homogeneity at least three automatically have values in $\mathfrak{p}$, so we see that ω is torsion free.

To prove sufficiency, we first need two facts on the curvature function κ of a torsion free normal parabolic geometry of type (G, P). On the one hand, the map ∂^* as defined in 3.6 can be written as the sum $\partial_1^* + \partial_2^*$ of two P–equivariant maps, with ∂_1^* corresponding to the first two summands and ∂_2^* corresponding to the last summand in the definition. We claim that κ has values in the kernels of both operators ∂_i^*. On the other hand, one easily verifies that the subspace $\widehat{\mathfrak{p}} \subset \mathfrak{p}$ formed by all matrices of the form $\begin{pmatrix} 0 & u & d \\ 0 & B & v \\ 0 & 0 & 0 \end{pmatrix}$ is a P–submodule. (Indeed, this is the preimage in $\mathfrak{p}$ of the semisimple part of the reductive algebra $\mathfrak{g}_0 = \mathfrak{p}/\mathfrak{p}_+$.) Our second claim is that $\kappa(u)(X, Y) \in \widehat{\mathfrak{p}}$ for all $u \in \mathcal{G}$ and all X, Y.

To prove both claims, it suffices to show that κ has values in the P–submodule $\Lambda_0^2\mathfrak{p}_+ \otimes \widehat{\mathfrak{p}} \subset \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{p}$. Here $\Lambda_0^2\mathfrak{p}_+$ is the kernel of the P–homomorphism $\Lambda^2\mathfrak{p}_+ \to \mathfrak{p}_+$ defined by the Lie bracket on $\mathfrak{p}_+$, so $\Lambda_0^2\mathfrak{p}_+ \otimes \mathfrak{g} = \ker(\partial_2^*)$.

In the case $n = 1$, this is evident, since from the table in 3.7 we see that the lowest nonzero homogeneous component of $\kappa(u)$ is of degree 4, vanishes on $\Lambda^2\mathfrak{g}_{-1}$ and has values in $\mathfrak{p}_+$. For homogeneous components of higher degree, these two properties are automatically satisfied, and we conclude that $\kappa(u) \in \mathfrak{g}_1 \wedge \mathfrak{g}_2 \otimes \mathfrak{p}_+ \subset \Lambda_0^2\mathfrak{p}_+ \otimes \widehat{\mathfrak{p}}$.

In the case $n > 1$, we see from the table in 3.7 that by torsion freeness the lowest homogeneous component of $\kappa(u)$ must be of homogeneity 2. By part (1) of Lemma 3.7 it has values in $\ker(\Box) \subset \Lambda^2\mathfrak{g}_1 \otimes \mathfrak{g}_0$. Since this component of $\ker(\Box)$ is a highest weight part, it lies in the kernel of all possible traces, and hence it must be contained in the tensor product of $\Lambda^2\mathfrak{g}_1 \cap \Lambda_0^2\mathfrak{p}_+$ with the semisimple part of $\mathfrak{g}_0$. Hence $\ker(\Box)$ is contained in the P–submodule $\Lambda_0^2\mathfrak{p}_+ \otimes \widehat{\mathfrak{p}}$ so, by part (2) of Lemma 3.7, the curvature function κ has values in that submodule.

In view of Proposition 3.3 and the proof of Theorem 3.6, to prove that $\tilde{\omega}_\alpha$ is regular and normal, it suffices to verify that the map $F(u) : \Lambda^2\tilde{\mathfrak{g}}_- \to \tilde{\mathfrak{g}}$ defined by $F(u)(X,Y) := \alpha(\kappa(u)(\underline{\alpha}^{-1}(X), \underline{\alpha}^{-1}(Y)))$ lies in the kernel of ∂^* for all $u \in \mathcal{G}$. To compute $\partial^* F(u)$, it is better to view $F(u)$ as an element of $\Lambda^2\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{g}}$, and we want to relate this to $\kappa(u)$, viewed as an element of $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$. Therefore, we have to compute the map $\varphi : \mathfrak{p}_+ \to \tilde{\mathfrak{p}}_+$, which is dual to the composition of the canonical projection $\mathfrak{g}/\mathfrak{q} \to \mathfrak{g}/\mathfrak{p}$ with $\underline{\alpha}^{-1} : \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \to \mathfrak{g}/\mathfrak{q}$, since by construction $F(u) = (\Lambda^2\varphi \otimes \alpha)(\kappa(u))$. Recall that the duality between $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{p}_+$ (and likewise for the other algebra) is induced by the Killing form. Since the Killing form of a simple Lie algebra is uniquely determined up to a nonzero multiple by invariance, we may as well use the trace form on both sides, which leads to a nonzero multiple of φ. But then the computation is very easy, showing that

$$\varphi\begin{pmatrix}0 & Z & \psi\\ 0 & 0 & W\\ 0 & 0 & 0\end{pmatrix}=\begin{pmatrix}0 & \psi & Z & W^t\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

In particular, $\varphi(\mathfrak{p}_+)\subset\tilde{\mathfrak{g}}_1^E\oplus\tilde{\mathfrak{g}}_2$, which implies that $\partial_2^*(F(u))=0$ for all u.

On the other hand, the formula for α from 3.5 shows that $\alpha(\widehat{\mathfrak{p}})\subset\tilde{\mathfrak{g}}_{-1}^V\oplus\tilde{\mathfrak{g}}_0\oplus\tilde{\mathfrak{g}}_1^E\oplus\tilde{\mathfrak{g}}_2$, and the $\tilde{\mathfrak{g}}_0$–component is contained in the bottom right $2n\times 2n$ block. This shows that for $Z\in\mathfrak{p}_+$ and $A\in\widehat{\mathfrak{p}}$ we have $[\varphi(Z),\alpha(A)]\in\tilde{\mathfrak{g}}_1^E\oplus\tilde{\mathfrak{g}}_2$. One immediately verifies directly that the $\tilde{\mathfrak{g}}_1^E$–component of $[\varphi(Z),\alpha(A)]$ equals the $\tilde{\mathfrak{g}}_1^E$–component of $\alpha([Z,A])$, while the $\tilde{\mathfrak{g}}_2$–component of $[\varphi(Z),\alpha(A)]$ equals twice the $\tilde{\mathfrak{g}}_2$–component of $\alpha([Z,A])$. From the definition of ∂_1^* we now conclude that $\Lambda^2\varphi\otimes\alpha$ maps $\ker(\partial_1^*)$ to $\ker(\partial_1^*)$, so we also get $\partial_1^*(F(u))=0$ for all u. □

4. Applications

For torsion free Lagrangean contact structures, Theorem 3.8 provides us with an explicit description of the parabolic geometry determined by the path geometry of chains. In particular, we obtain an explicit formula for the Cartan curvature which is the basis for the applications discussed in this section. The main result is that one can essentially reconstruct the torsion free Lagrangean contact structure from the harmonic curvature of this parabolic geometry. In particular, this implies that a contact diffeomorphism which maps chains to chains has to either preserve or swap the subbundles defining the Lagrangean contact structure. On the way, we can prove that chains can never be described by linear connections and that only locally flat Lagrangean contact structures give rise to torsion free path geometries of chains.

4.1. Decomposing the Cartan curvature. For a torsion free Lagrangean contact structure with curvature κ, the curvature $\tilde{\kappa}$ of the normal Cartan connection associated to the path geometry of chains is determined by the formula from Proposition 3.3, which holds on $j(\mathcal{G})\subset\mathcal{G}\times_i\tilde{P}$. In this formula, there are two terms, one of which depends on κ while the other one only comes from the map α. Our main task is to extract parts of $\tilde{\kappa}$ which only depend on one of the two terms. The difficulty is that this has to be done in a geometric way without knowing the subset $j(\mathcal{G})$ in advance.

The curvature function $\tilde{\kappa}$ has values in the P–module $\Lambda^2\tilde{\mathfrak{p}}_+\otimes\tilde{\mathfrak{g}}$, and using the map φ from the proof of Theorem 3.8, the formula from Proposition 3.3 reads as $\tilde{\kappa}(j(u))=(\Lambda^2\varphi\otimes\alpha)(\kappa(u))+\Psi_\alpha$. Now $\tilde{\mathfrak{p}}_+$ contains the P–invariant subspace $\tilde{\mathfrak{g}}_2$. Correspondingly, we obtain P–invariant subspaces $\Lambda^2\tilde{\mathfrak{g}}_2\subset\tilde{\mathfrak{p}}_+\wedge\tilde{\mathfrak{g}}_2\subset\Lambda^2\tilde{\mathfrak{p}}_+$. In the proof of Theorem 3.8, we have seen that φ has values in $\tilde{\mathfrak{g}}_1^E\oplus\tilde{\mathfrak{g}}_2$, whence $\Lambda^2\varphi$ has values in $\tilde{\mathfrak{p}}_+\wedge\tilde{\mathfrak{g}}_2$. From Lemma 3.6 we know that $\Psi_\alpha\in\tilde{\mathfrak{p}}_+\wedge\tilde{\mathfrak{g}}_2\otimes\tilde{\mathfrak{g}}$, so we conclude that $\tilde{\kappa}(j(u))$ lies in this $\tilde{P}$–submodule. By equivariancy, all values of the curvature function lie in $\tilde{\mathfrak{p}}_+\wedge\tilde{\mathfrak{g}}_2\otimes\tilde{\mathfrak{g}}\subset\Lambda^2\tilde{\mathfrak{p}}_+\otimes\tilde{\mathfrak{g}}$.

On the quotient $\tilde{\mathfrak{p}}_+/\tilde{\mathfrak{g}}_2$, the subgroup $\tilde{P}_+\subset\tilde{P}$ acts trivially, so we can identify this quotient with the $\tilde{G}_0$–module $\tilde{\mathfrak{g}}_1=\tilde{\mathfrak{g}}_1^E\oplus\tilde{\mathfrak{g}}_1^V$. Correspondingly, we get $\tilde{P}$–equivariant projections

$$\pi^E:\tilde{\mathfrak{p}}_+\wedge\tilde{\mathfrak{g}}_2\otimes\tilde{\mathfrak{g}}\to\tilde{\mathfrak{g}}_1^E\wedge\tilde{\mathfrak{g}}_2\otimes\tilde{\mathfrak{g}}$$
$$\pi^V:\tilde{\mathfrak{p}}_+\wedge\tilde{\mathfrak{g}}_2\otimes\tilde{\mathfrak{g}}\to\tilde{\mathfrak{g}}_1^V\wedge\tilde{\mathfrak{g}}_2\otimes\tilde{\mathfrak{g}}.$$

From the description of the image of φ in the proof of Theorem 3.8 we conclude that $(\Lambda^2\varphi\otimes\alpha)(\kappa(u))\in\ker(\pi^V)$. On the other hand, Lemma 3.6 in particular shows that $\pi^V(\Psi_\alpha)\neq 0$ and $\Psi_\alpha\in\ker(\pi^E)$.

Theorem. *Let (M,L,R) be a torsion free Lagrangean contact structure.*
(1) There is no linear connection on the tangent bundle TM which has the chains among its geodesics.
(2) The parabolic geometry associated to the path geometry of chains on $\tilde{M} = \mathcal{P}_0(TM)$ is torsion free if and only if (M,L,R) is locally flat, i.e. locally isomorphic to the homogeneous model G/P.

Proof. (1) Suppose that ∇ is a linear connection on TM whose geodesics in directions transverse to $L \oplus R$ are parametrizations of the chains. Since symmetrizing a connection does not change the geodesics, we may without loss of generality assume that ∇ is torsion free. Then we can look at the associated projective structure $[\nabla]$ on M and use the machinery of correspondence space from [3]. The fact that the geodesics of ∇ are the chains exactly means that the path geometry of chains on $\tilde{M}$ is isomorphic to an open subgeometry of the correspondence space $\mathcal{C}(M,[\nabla])$, see 4.7 of [3]. In particular, the Cartan curvature $\tilde{\kappa}$ is the restriction of the curvature of this correspondence space. By [3, Proposition 2.4] this curvature has the property that it vanishes upon insertion of one tangent vector contained in the vertical bundle of $\tilde{M} \to M$. But this contradicts the fact that $\pi^V \circ \tilde{\kappa} \neq 0$ we have observed above.
(2) By Theorem 3.6, the path geometry of chains associated to a locally flat Lagrangean contact structure is torsion free. Conversely, if the Cartan connection $\tilde{\omega}$ is torsion free, then according to part (1) of Lemma 3.7 and the tables in 3.7, the lowest nonzero homogeneous component of $\tilde{\kappa}$ must be of degree at least three, and the harmonic curvature must have values in $\tilde{\mathfrak{g}}_1^V \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_0 \subset \ker(\pi^E)$. By part (2) of Lemma 3.7 the whole curvature $\tilde{\kappa}$ has values in $\ker(\pi^E)$. Above, we have observed that $\Psi_\alpha \in \ker(\pi^E)$ so we conclude that for each $u \in \mathcal{G}$ we get $\pi^E \circ (\Lambda^2\varphi \otimes \alpha)(\kappa(u)) = 0$.

In the proof of Theorem 3.8 we see that φ is a linear isomorphism $\mathfrak{p}_+ \to \tilde{\mathfrak{g}}_1^E \oplus \tilde{\mathfrak{g}}_2$, and hence $\tilde{\mathfrak{g}}_1^E \wedge \tilde{\mathfrak{g}}_2$ is contained in the image of $\Lambda^2\varphi$. Hence we conclude that $\alpha \circ \kappa(u) = 0$ and since α is injective, the result follows. □

4.2. Harmonic curvature. We have discussed the definition of harmonic curvature already in 3.7. Let π_H be the natural projection from $\ker(\partial^*) \subset \Lambda^2\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{g}}$ to the quotient $\ker(\partial^*)/\operatorname{im}(\partial^*)$. Since this is a $\tilde{P}$–equivariant map, the composition $\tilde{\kappa}_H = \pi_H \circ \tilde{\kappa} : \tilde{\mathcal{G}} \to \ker(\partial^*)/\operatorname{im}(\partial^*)$ defines a smooth section of the associated bundle $\tilde{\mathcal{G}} \times_{\tilde{P}} \ker(\partial^*)/\operatorname{im}(\partial^*)$, which is the main geometric invariant of the parabolic geometry associated to the path geometry of chains.

From 3.7 we also know that $\tilde{P}_+$ acts trivially on the quotient $\ker(\partial^*)/\operatorname{im}(\partial^*)$ and we may identify it with the $\tilde{G}_0$–module $\ker(\Box) \subset \Lambda^2\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{g}}$. From the table in 3.7, we see that this module contains two irreducible components in positive homogeneity, which are the highest weight components of the subrepresentations $\tilde{\mathfrak{g}}_1^E \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_{-1}^V$ respectively $\tilde{\mathfrak{g}}_1^V \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_0$. Correspondingly, we obtain decompositions $\pi_H = \pi_H^E + \pi_H^V$ and $\tilde{\kappa}_H = \tilde{\kappa}_H^E + \tilde{\kappa}_H^V$.

Lemma. *Let π^E and π^V be the projections on $\tilde{\mathfrak{p}}_+ \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}$ defined in 4.1. Then the restriction of π_H^E (respectively π_H^V) to $\ker(\partial^*) \cap (\tilde{\mathfrak{p}}_+ \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}})$ factorizes through π^E (respectively π^V).*

Proof. By Kostant's version of the Bott–Borel–Weil theorem, see [13], the $\tilde{G}_0$–irreducible components contained in $\ker(\Box)$ occur with multiplicity one, even within $\Lambda^*\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{g}}$. To obtain π^E and π^V, we used the projection $\tilde{\mathfrak{p}}_+ \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}_1 \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}$ with kernel $\Lambda^2\tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}$. By the multiplicity one result and the fact that both components of $\ker(\Box)$ are contained in $\tilde{\mathfrak{g}}_1 \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}$, there is no nonzero $\tilde{G}_0$–equivariant map $\Lambda^2\tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}} \to \ker(\partial^*)/\operatorname{im}(\partial^*)$. Hence each of the projections π_H, π_H^E and π_H^V factorizes through $\tilde{\mathfrak{g}}_1 \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}$. Looking at the resulting map for π^E_H, we see that again by multiplicity one, the subspace $\tilde{\mathfrak{g}}^V_1 \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}$ must be contained in the kernel, so we conclude that π^E_H factorizes through π^E. In the same way one shows that π^V_H factorizes through π^V. □

Proposition. *Let (M, L, R) be a torsion free Lagrangean contact structure, and let $\tilde{\kappa}_H = \tilde{\kappa}^E_H + \tilde{\kappa}^V_H$ be the harmonic curvature of the regular normal parabolic geometry determined by the path geometry of chains.*

Then the function $\tilde{\mathcal{G}} \to \tilde{\mathfrak{g}}^V_1 \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_0$ corresponding to $\tilde{\kappa}^V_H$ is a nonzero multiple of the unique equivariant extension of the constant function Ψ_α (compare with Lemma 3.6) on $j(\mathcal{G})$.

Proof. We have to compute the function $\pi^V_H \circ \tilde{\kappa}$. By the lemma, π^V_H factorizes through the projection π^V introduced in 4.1, and from there we know that $\pi_V(\tilde{\kappa}(j(u))) = \pi_V(\Psi_\alpha)$. Hence we see that $(\pi^V_H \circ \tilde{\kappa})|_{j(\mathcal{G})} = \pi^V_H(\Psi_\alpha)$. Now $\Psi_\alpha \in \tilde{\mathfrak{g}}^V_1 \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_0$ by Lemma 3.6, and the values even lie in the semisimple part of $\tilde{\mathfrak{g}}_0$, which may be identified with $\mathfrak{sl}(\tilde{\mathfrak{g}}_{-2})$. Evidently, $\tilde{\mathfrak{g}}^V_1 \cong \tilde{\mathfrak{g}}^E_{-1} \otimes \tilde{\mathfrak{g}}_2$ as a $\tilde{G}_0$–module, so we may interpret Ψ_α as an element of $\tilde{\mathfrak{g}}^E_{-1} \otimes (\otimes^3 \tilde{\mathfrak{g}}_2) \otimes \tilde{\mathfrak{g}}_{-2}$. In Lemma 3.6 and the proof of Theorem 3.6 we have seen that in this picture Ψ_α lies in the irreducible component $\tilde{\mathfrak{g}}^E_{-1} \otimes (S^3 \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_{-2})_0$, where the subscript denotes the trace free part. Passing back to $\tilde{\mathfrak{g}}^V_1 \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_0$ this exactly means that Ψ_α lies in the highest weight subspace, which is the intersection with $\ker(\Box)$. Now π^V_H restricts to $\tilde{G}_0$–equivariant linear isomorphism on this intersection, which implies the result. □

Remark. Similarly to the proof above, one shows that the harmonic curvature component $\tilde{\kappa}^E_H$ is the extension of a component of $j(u) \mapsto (\Lambda^2 \varphi \otimes \alpha)(\kappa(u))$. Since we explicitly know $\Lambda^2 \varphi \otimes \alpha$, this can be used to obtain a more explicit description of the second harmonic curvature component. From part (2) of Theorem 4.1 and [3, 4.7] we see that vanishing of $\tilde{\kappa}^E_H$ is equivalent to local flatness of the original Lagrangean contact structure, so κ is completely encoded in $\tilde{\kappa}^E_H$.

4.3. Passing to the underlying manifold. The harmonic curvature component determined by the function $\tilde{\kappa}^V_H$ is a section of the bundle associated to $\tilde{\mathfrak{g}}^V_1 \wedge \tilde{\mathfrak{g}}_2 \otimes \tilde{\mathfrak{g}}_0$. In the proof of Proposition 4.2 we have seen that we can replace that space by $\tilde{\mathfrak{g}}^E_{-1} \otimes (\otimes^3 \tilde{\mathfrak{g}}_2) \otimes \tilde{\mathfrak{g}}_2$. The corresponding bundle is $E \otimes \otimes^3 F^* \otimes F \to \tilde{M}$, where $F := T\tilde{M}/(E \oplus V)$. Since $E \subset TM$ is a line bundle, we can view $\tilde{\kappa}^V_H$ as a section of $\otimes^3 F^* \otimes F$ which is determined up to a nonzero multiple.

To relate this to the underlying manifold M, recall that $\tilde{M}$ is an open subset in the projectivized tangent bundle of M. A point in $\tilde{M}$ is a line in some tangent space $T_x M$ that is transversal to $L_x \oplus R_x$. We have noted in 2.4 that $TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$ and $T\tilde{M} \cong \mathcal{G} \times_Q \mathfrak{g}/\mathfrak{q}$, and the tangent map of the projection $\pi : \tilde{M} \to M$ corresponds to the natural projection $\mathfrak{g}/\mathfrak{q} \to \mathfrak{g}/\mathfrak{p}$. Fix a point $\ell \in \pi^{-1}(x)$. Then for each $\xi \in T_x M$ there is a lift $\tilde{\xi} \in T_\ell \tilde{M}$ and we can consider the class of $\tilde{\xi}$ in $F_\ell = T_\ell \tilde{M}/(E_\ell \oplus V_\ell)$. Since V_ℓ is the vertical subbundle, this class is independent of the choice of the lift and from the explicit description of $T\pi$ we see that restricting to $L_x \oplus R_x$, we obtain a linear isomorphism $L_x \oplus R_x \cong F_\ell$.

Fixing x and ℓ we therefore see that the harmonic curvature component corresponding to $\tilde{\kappa}^V_H$ gives rise to an element of $\otimes^3(L_x \oplus R_x)^* \otimes (L_x \oplus R_x)$, which is determined up to a nonzero multiple. To write down this map explicitly, we first need the Levi bracket

$$\mathcal{L} : (L_x \oplus R_x) \times (L_x \oplus R_x) \to T_x M/(L_x \oplus R_x).$$

Since this has values in a one–dimensional space, we may view it as a real valued bilinear map determined up to a nonzero multiple. Further, we denote by $\mathbb{J}$ the almost product structure corresponding to the decomposition $L \oplus R$. This means that $\mathbb{J}$ is the endomorphism of $L \oplus R$ which is the identity on L and minus the identity on R. Using this we can now formulate:

Lemma. *The element of $\otimes^3(L_x \otimes R_x)^* \otimes (L_x \oplus R_x)$ obtained from $\tilde{\kappa}_H^V$ above is (a nonzero multiple of) the complete symmetrization of the map*

$$(\xi, \eta, \zeta) \mapsto \mathcal{L}(\xi, \mathbb{J}(\eta))\mathbb{J}(\zeta).$$

Proof. This is a reinterpretation of the proof of Lemma 3.6. Observe that $\mathbb{J}$ corresponds to the map $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \mapsto \begin{pmatrix} X_1 \\ -X_2 \end{pmatrix}$ in the notation there. Since $\mathcal{L}$ corresponds to $[\ ,\ ] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$, computing the bracket

$$\left[\begin{pmatrix} 0 & 0 & 0 \\ X_1 & 0 & 0 \\ 0 & X_2^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ Y_1 & 0 & 0 \\ 0 & -Y_2^t & 0 \end{pmatrix} \right],$$

we see that the expression $\langle X_1, Y_2 \rangle + \langle Y_1, X_2 \rangle$ in the proof of Lemma 3.6 corresponds to $\mathcal{L}(\xi, \mathbb{J}(\eta))$. □

4.4. Reconstructing the Lagrangean contact structure. Now we can finally show that the Cartan curvature of the path geometry of chains can be used to (almost) reconstruct the Lagrangean contact structure on M that we have started from:

Theorem. *Let (M, L, R) be a torsion free Lagrangean contact structure. Then for each $x \in M$, the subset $L_x \cup R_x \subset T_xM$ can be reconstructed from the harmonic curvature of the normal parabolic geometry associated to the path geometry of chains.*

Proof. In view of the results in 4.2 and 4.3 it suffices to show that $L_x \cup R_x$ can be recovered from the complete symmetrization S of the map

$$(\xi, \eta, \zeta) \mapsto \mathcal{L}(\xi, \mathbb{J}(\eta))\mathbb{J}(\zeta).$$

First we see that $S(\xi, \xi, \xi) = 0$ if and only if $\mathcal{L}(\xi, \mathbb{J}(\xi)) = 0$. Note that this is always satisfied for $\xi \in L_x \cup R_x$. Fixing an element ξ with this property, we see that

$$S(\xi, \xi, \eta) = 2\mathcal{L}(\xi, \mathbb{J}(\eta))\mathbb{J}(\xi).$$

By non–degeneracy of $\mathcal{L}$, given a nonzero element ξ we can always find η such that $\mathcal{L}(\xi, \mathbb{J}(\eta)) \neq 0$. Hence we see that ξ is an eigenvector for $\mathbb{J}$ (which by definition is equivalent to $\xi \in L_x \cup R_x$) if and only if $S(\xi, \xi, \xi) = 0$ and there is an element η such that $S(\xi, \xi, \eta)$ is a nonzero multiple of ξ. □

Corollary. *Let (M, L, R) be a torsion free Lagrangean contact structure and let $f : M \to M$ be a contact diffeomorphism which maps chains to chains. Then either f is an automorphism or an anti–automorphism of the Lagrangean contact structure. Here anti–automorphism means that $T_xf(L_x) = R_{f(x)}$ and $T_xf(R_x) = L_{f(x)}$ for all $x \in M$.*

Proof. By assumption, f induces an automorphism $\tilde{f}$ of the path geometry of chains associated to (M, L, R). This automorphism has to pull back the Cartan curvature $\tilde{\kappa}$ and also the harmonic curvature κ_H to itself. From the theorem we conclude that this implies $T_xf(L_x \cup R_x) = L_{f(x)} \cup R_{f(x)}$, and this is only possible if f is an automorphism or an anti–automorphism. □

5. Partially integrable almost CR structures

What we have done for Lagrangean contact structures so far can be easily adapted to deal with partially integrable almost CR structure. We will only briefly sketch the necessary changes in this section.

5.1. A non–degenerate partially integrable almost CR structure on a smooth manifold M is given by a contact structure $H \subset TM$ together with an almost complex structure J on H such that the Levi bracket $\mathcal{L}$ has the property that $\mathcal{L}(J\xi, J\eta) = \mathcal{L}(\xi, \eta)$ for all ξ, η. Then $\mathcal{L}$ is the imaginary part of a non–degenerate Hermitian form and we denote the signature of this form by (p, q). Such a structure of signature (p, q) is equivalent to a regular normal parabolic geometry of type (G, P), where $G = PSU(p + 1, q + 1)$ and $P \subset G$ is the stabilizer of a point in $\mathbb{C}P^{n+1}$, $n = p + q$, corresponding to a null line, see [4, 4.15]. The group G is the quotient of $SU(p+1, q+1)$ by its center (which is isomorphic to $\mathbb{Z}_{n+2}$) and we will work with representative matrices as before.

We will use the Hermitian form of signature (p, q) on $\mathbb{C}^{n+1}$ corresponding to

$$(z_0, \dots, z_{n+1}) \mapsto z_0\bar{z}_{n+1} + z_{n+1}\bar{z}_0 + \textstyle\sum_{j=1}^{p} |z_j|^2 - \sum_{j=p+1}^{n} |z_j|^2.$$

Then the decomposition on $\mathfrak{sl}(n + 2, \mathbb{C})$ with block sizes 1, n, and 1 restricts to a contact grading on the Lie algebra $\mathfrak{g}$ of G. The explicit form for signature $(n, 0)$ can be found in [4, 4.15]. In general, $\mathfrak{g}$ consists of all matrices of the form

$$\begin{pmatrix} w & Z & iz \\ X & A & -\mathbb{I}Z^* \\ ix & -X^*\mathbb{I} & -\bar{w} \end{pmatrix}$$

with blocks of sizes 1, n, and 1, $w \in \mathbb{C}$, $x, z \in \mathbb{R}$, $X \in \mathbb{C}^n$, $Z \in \mathbb{C}^{n*}$, and $A \in \mathfrak{u}(p, q)$ such that $w - \bar{w} + \operatorname{tr}(A) = 0$. Here $\mathbb{I}$ is the diagonal matrix with the first p entries equal to 1 and the remaining q entries equal to -1.

It is easy to show that the subgroup $Q \subset G$ corresponds to matrices of the form

$$\begin{pmatrix} \varphi & 0 & ia\varphi \\ 0 & \Phi & 0 \\ 0 & 0 & \bar{\varphi}^{-1} \end{pmatrix},$$

with $\varphi \in \mathbb{C} \setminus \{0\}$, $a \in \mathbb{R}$ and $\Phi \in U(p, q)$ such that $\frac{\varphi^2}{|\varphi|^2} \det(\Phi) = 1$.

5.2. Next we need an analog of the pair (i, α) introduced in 3.5. As before we start with a manifold M of dimension $2n + 1$, so again $\tilde{G} = PGL(2n + 2, \mathbb{R})$. We will use a block decomposition into blocks of sizes 1, 1, n, and n as before. The right choice turns out to be

$$i\begin{pmatrix} \varphi & 0 & ia\varphi \\ 0 & \Phi & 0 \\ 0 & 0 & \bar{\varphi}^{-1} \end{pmatrix} := \begin{pmatrix} |\varphi| & -a|\varphi| & 0 & 0 \\ 0 & |\varphi|^{-1} & 0 & 0 \\ 0 & 0 & \Re(\frac{|\varphi|}{\varphi}\Phi) & -\Im(\frac{|\varphi|}{\varphi}\Phi) \\ 0 & 0 & \Im(\frac{|\varphi|}{\varphi}\Phi) & \Re(\frac{|\varphi|}{\varphi}\Phi) \end{pmatrix},$$

$$\alpha\begin{pmatrix} w & Z & iz \\ X & A & -\mathbb{I}Z^* \\ ix & -X^*\mathbb{I} & -\bar{w} \end{pmatrix} := \begin{pmatrix} \Re(w) & -z & \Re(Z) & -\Im(Z) \\ x & -\Re(w) & -\Im(X^*\mathbb{I}) & -\Re(X^*\mathbb{I}) \\ \Re(X) & \Im(\mathbb{I}Z^*) & \Re(A) & -\Im(A) + \Im(w) \\ \Im(X) & -\Re(\mathbb{I}Z^*) & \Im(A) - \Im(w) & \Re(A) \end{pmatrix},$$

where $\Re$ and $\Im$ denote real and imaginary part, respectively, and we write $\Im(w)$ for the appropriate multiple of the identity matrix.

There is an analog of Lemma 3.6 (with similar proof), the only change one has to make is that the map whose alternation has to be used is given by

$$(X, Y, Z) \mapsto (\langle X_1, \mathbb{I}Y_1\rangle + \langle X_2, \mathbb{I}Y_2\rangle) \begin{pmatrix} -Z_2 \\ Z_1 \end{pmatrix}.$$

This map has similar properties as the one from 3.6 so the analogs of Theorem 3.6 and Corollary 3.6 hold.

Concerning the structure of $\ker(\Box)$ the situation is also similar to the case of Lagrangean contact structures, since the decomposition of $\ker(\Box)$ can be determined from the complexifications of $\mathfrak{g}$ and $\mathfrak{p}$ which are the same in both cases. The only difference is that the two irreducible components for $n = 1$ respectively the two irreducible components contained in homogeneity 1 in the case $n > 1$ in the Lagrangean case correspond to only one component here. This component however has a complex structure and it consists of maps $\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-2} \to \mathfrak{g}_1$ which are complex linear in the first variable respectively maps $\Lambda^2\mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$, which are conjugate linear in both variables. For $n > 1$ this component is a torsion which is up to a nonzero multiple given by the Nijenhuis tensor. Vanishing of this component is equivalent to torsion freeness and to integrability of the almost CR structure, see [4, 4.16].

Theorem. *Let (M, H, J) be a partially integrable almost CR structure and let $(p : \mathcal{G} \to M, \omega)$ be the corresponding regular normal parabolic geometry of type (G, P). Then the parabolic geometry $(\mathcal{G} \times_Q \tilde{P} \to \mathcal{P}_0(TM), \tilde{\omega}_\alpha)$ constructed using the extension functor associated to the pair (i, α) from 5.1 is regular and normal if and only if ω is torsion free, i.e. the almost CR structure is integrable.*

Proof. Apart from some numerical factors which cause no problems, this is completely parallel to the proof of Theorem 3.8. □

Hence the direct relation between the regular normal parabolic geometries associated to a partially integrable almost CR structure respectively to the associated path geometry of chains works exactly on the the subclass of CR structures.

5.3. Applications. The developments of section 4 can be applied to the CR case with only minimal changes. In analog of Lemma 4.3, one obtains $S \in \otimes^3 H_x^* \otimes H_x$, which is the complete symmetrization of

$$(\xi, \eta, \zeta) \mapsto \mathcal{L}(\xi, J(\eta))J(\zeta),$$

where J is the almost complex structure on H.

Theorem. *Let (M, H, J) be a CR structure.*
(1) There is no linear connection on TM which has the chains among its geodesics.
(2) The path geometry of chains is torsion free if and only if the CR structure is locally flat.
(3) The almost complex structure J can be reconstructed up to sign from the harmonic curvature of the associated path geometry of chains.

Proof. The only change compared to section 4 is that one has to extend S to the complexified bundle $H \otimes \mathbb{C}$. As in the proof of Theorem 4.4 one then reconstructs the subset $H_x^{1,0} \cup H_x^{0,1} \subset H_x \otimes \mathbb{C}$ for each $x \in M$, i.e. the union of the holomorphic and the anti–holomorphic part. This union determines J up to sign. □

This theorem now also implies that the signature of the CR structure, which is encoded in $\mathcal{L}(-, J(-))$, can be reconstructed from the path geometry of chains. As a corollary, we obtain a completely independent proof of the analog of Corollary 4.4, which is due to [7] for CR structures:

Corollary. *A contact diffeomorphism between two CR manifolds which maps chains to chains is either a CR isomorphism or a CR anti–isomorphism.*

References

[1] R. J. Baston, M. G. Eastwood: "The Penrose Transform" Its Interaction with Representation Theory. Oxford Science Publications, Clarendon Press, 1989.

[2] A. Čap, Parabolic geometries, CR–tractors, and the Fefferman construction, Differential Geom. Appl. **17** (2002) 123–138.

[3] A. Čap, Correspondence spaces and twistor spaces for parabolic geometries. to appear in J. Reine Angew. Math., electronically available as preprint ESI 989 at http://www.esi.ac.at

[4] A. Čap, H. Schichl, Parabolic Geometries and Canonical Cartan Connections. Hokkaido Math. J. **29** no.3 (2000), 453–505.

[5] A. Čap, J. Slovák, V. Souček, Bernstein–Gelfand–Gelfand sequences. Ann. of Math. **154** no. 1 (2001), 97–113.

[6] A. Čap, J. Slovák, V. Žádník, On Distinguished Curves in Parabolic Geometries, Transform. Groups **9** no. 2 (2004) 143–166

[7] J. Cheng, Chain-preserving diffeomorphisms and CR equivalence, Proc. Amer. Math. Soc. **103**, no. 1 (1988) 75–80.

[8] S. S. Chern, J. Moser, Real hypersurfaces in complex manifolds, Acta Math. **133** (1974), 219–271

[9] J. Douglas, The general geometry of paths. Ann. of Math. **29** no. 1-4 (1927/28) 143–168

[10] C. Fefferman, Monge–Ampère equations, the Bergman kernel and geometry of pseudoconvex domains, Ann. of Math. **103** (1976) 395–416; Erratum **104** (1976) 393–394.

[11] D. Grossman, Torsion–free path geometries and integrable second order ODE systems. Selecta Math. **6** no. 4 (2000), 399–442.

[12] S. Kobayashi, On connections of Cartan. Canad. J. Math. **8** (1956), 145–156.

[13] B. Kostant, Lie algebra cohomology and the generalized Borel–Weil theorem. Ann. of Math. **74** no. 2 (1961), 329–387.

[14] T. Morimoto, Geometric structures on filtered manifolds, Hokkaido Math. J. 22, (1993), 263–347

[15] M. Takeuchi, Lagrangean contact structures on projective cotangent bundles. Osaka J. Math. **31** (1994), 837–860.

[16] N. Tanaka, On non–degenerate real hypersurfaces, graded Lie algebras and Cartan connections, Japan J. Math. **2** (1976), 131–190

[17] N. Tanaka, On the equivalence problem associated with simple graded Lie algebras, Hokkaido Math. J., **8** (1979), 23–84.

[18] H. Wang, On invariant connections over a principal fibre bundle. Nagoya Math. J. **13** (1958) 1–19.

[19] K. Yamaguchi, Differential systems associated with simple graded Lie algebras, Advanced Studies in Pure Mathematics 22 (1993), 413–494

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