

**The Basic Representation of the Current Group
 $O(n,1)^x$ in the L^2 Space
over the Generalized Lebesgue Measure**

**M.I. Graev
A.M. Vershik**

Vienna, Preprint ESI 1659 (2005)

June 1, 2005

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via <http://www.esi.ac.at>

CONTENTS

1.	Introduction	1
2.	Starting definitions and formulas	2
3.	The commutative model of the complementary series of irreducible unitary representations of the group $O(n, 1)$	8
4.	The measures μ and ν on the space of vector distributions	13
5.	Construction of the basic representation of the current group G^X	19
6.	Appendices	21
	References	25

1. INTRODUCTION

The current group G^X , where X is a smooth manifold with a finite continuous measure m and G is a Lie group, is the collection of bounded piecewise continuous or, more generally, Borel G -valued functions on X . The group operation is the pointwise multiplication. The general approach to representations of current groups in the Fock space was suggested in the pioneering paper [1], but the most important case of semisimple groups and the notion of canonical states were first considered in the series of papers [14, 15, 16, 17, 5, 13], in which irreducible unitary representations of the current groups G^X for some simple Lie groups G were constructed that are invariant under any m -preserving transformations of the space X : in the first paper of this series [14], this was done for $G = SL(2, \mathbb{R})$, and in [15, 5], for $G = SO(n, 1)$ and $SU(n, 1)$. We called them the basic representations of the current groups G^X and described several models of these representations. Note that basic representations are uniquely determined by their spherical functions, which in [14] were called canonical states; see [5].

As shown in [15] (see also [18, 12]), the series $G = SO(n, 1)$ and $SU(n, 1)$ exhaust all simple Lie groups G for which such representations of the current groups do exist (in other words, groups with nontrivial first homology with coefficients in some irreducible unitary representation). Canonical states for a close group, the group of automorphisms of an infinite homogeneous tree, were considered in [10]. See also [11], where canonical states on the group $U(\infty)$ are considered.

In this paper, we present a detailed study of the so-called commutative model of the basic representation of the group G^X for the series $G = O(n, 1)$. We will return to the analysis of the series $U(n, 1)$ elsewhere. For convenience, we replaced the groups $SO(n, 1)$ with $O(n, 1)$.

Among different realizations of the same unitary representation π of an arbitrary metrizable topological group G , of special interest are commutative models. Namely, let $G_0 \subset G$

¹⁾key words: current groups, canonical state, special representation

²⁾Supported by RFBR, project 04-01-00363.

³⁾Supported by the grants NSh.-2251.2003.1 and RUMI-2662-ST-04.

be a commutative subgroup such that the restriction of π to G_0 is cyclic. Then, by an isometric operator, one can bring the operators $\pi(a)$, $a \in G_0$, to diagonal form, i.e., realize the Hilbert space of the representation π as the space of square-integrable measurable functions with respect to the spectral measure of the representation $\pi|_{G_0}$, in which the representation operators corresponding to elements of the subgroup G_0 act as multipliers. We call such a realization the commutative model of the representation π with respect to the subgroup G_0 . The commutative models of irreducible unitary representations of the group $SL(2, \mathbb{R})$ with respect to its commutative subgroups — orthogonal, unipotent, and diagonal — are well known; see, for example, [4]. Clearly, it makes sense to choose the subgroup G_0 so that the representation operators corresponding to other elements of the group G will have simple expressions.

In what follows, such subgroups are the unipotent subgroups Z and Z^X of the groups $O(n, 1)$ and $O(n, 1)^X$, respectively. The subgroup Z is commutative (in contrast to the maximal unipotent subgroup in $U(n, 1)$) and isomorphic to the additive group \mathbb{R}^{n-1} . The case $n = 2$, more exactly, that of the group $SL(2, \mathbb{R})$, was earlier considered in [16, 13].

The interest to this realization is also due to the fact that the diagonalization of operators corresponding to the chosen commutative subgroup leads to remarkable infinite σ -finite measures ν in the space of distributions, which have a large group of linear symmetries. In the case of $SL(2, \mathbb{R})$, this measure ν turned out to be related to the well-known gamma process; see [16] and, for more details, [13]. More exactly, this is a σ -finite measure equivalent to the law of the gamma process and invariant under the multiplication of realizations of the process by functions with zero integral of the absolute value of logarithm (an analog of the linear transformation corresponding to a diagonal matrix with determinant 1), which allows one to call it the Lebesgue measure in the infinite-dimensional space.

In the general case $G = O(n, 1)$ considered in this paper, this measure ν is concentrated on vector distributions, and in addition to the above property it is also invariant under the pointwise action of the group $O(n-1)^X$. In all cases, the measure ν is concentrated on distributions that are linear combinations of δ -functions.

From the viewpoint of representation theory, the advantage of the model constructed in this paper is that the representation operators have an explicit description, in contrast to the realization in the Fock space (see [5]). Moreover, this model implicitly suggests certain advantages of the similar realization of unitary representations of the Lie group $O(n, 1)$ itself.

In the appendices we briefly discuss additional properties of the measures arising in the construction of commutative models of representations and describe the limiting case of the group $O(\infty, 1)$.

The short version of the paper was published in *Funct. Anal. et Appl.* v 39(2), 2005. The author express their gratitude to N. Tsilevich for the help in the preparation of this text.

2. STARTING DEFINITIONS AND FORMULAS

2.1. Current groups. In this and the next sections we recall some definitions from papers of the 60s and 70s (see [14, 18, 12, 10, 19, 9] and references therein).

The current group G^X , where X is a smooth manifold with a finite continuous measure m and G is a Lie group, is the collection of bounded Borel G -valued functions on X . The

group operation is the pointwise multiplication. At first we do not introduce any topology on G^X .

The problem is to construct irreducible unitary representations of current groups that are invariant under m -preserving transformations of the space X . Such (invariant) representations are nonlocal, and one usually constructs them in some or other realization of the Fock space, more exactly, in a Hilbert space with factorization structure. It can be not only the ordinary Fock space realized as the L^2 space over the standard Gaussian measure, but also the L^2 space over another measure corresponding to a Lévy process, i.e., a process constructed from an infinitely divisible distribution on \mathbb{R} or \mathbb{R}^n ; for information on factorizations, see [19]. In this paper, such measures will be multidimensional analogs of the classical gamma process and σ -finite measures constructed from them.

2.2. Canonical states. A state on a topological group is a *positive definite continuous complex-valued Hermitian* ($f(x^{-1}) = \bar{f}(x)$) *function normalized to the unity at the group identity*; a state is called *infinitely divisible* if it can be included into a continuous one-parameter semigroup of states. In the representation-theoretic language, the condition of infinite divisibility means that the group representation corresponding to this state (according to the GNS construction) can be included into an arbitrarily high tensor power of some representation of the same group.

It is easy to show that the generator of the semigroup, i.e., the derivative of the one-parameter family at the group identity, is a *conditionally positive definite function on the group* (sometimes it is called a function of negative type); at the same time, it is the squared norm of some 1-cocycle of the group with values in some unitary representation. This cocycle is cohomological to zero if and only if the generator, regarded as a function on the group, is bounded in absolute value. In this case, it coincides, up to sign and constant summand, with a positive definite function. That is why we are interested in unbounded conditionally positive definite functions and the corresponding nontrivial 1-cocycles with values in irreducible unitary representations. Such cocycles do not always exist and must lie in representations “glued” to the identity representation.

A canonical state on a group is an infinitely divisible state such that the generator of the corresponding semigroup is an unbounded conditionally positive definite function.

Given a canonical state on a group G , we construct an irreducible unitary nonlocal representation (integral) of the corresponding current group — the group of G -valued measurable functions. Conversely, each invariant (see above) representation of the current group realized in the Fock space is generated by some canonical state on G .

Thus a canonical state is the exponential of the squared norm of a nontrivial cocycle of the group with values in an irreducible unitary representation of the group. The complete list of groups for which a canonical state does exist is not yet known; however, there is a number of examples, the most important of which are some classical semisimple Lie groups, nontrivial 1-cocycles on which were first found in [15, 14]. Another example — the group of automorphisms of a tree — is considered in [10].

In the case of semisimple groups, one requires additionally that a canonical state should be constant on a chosen maximal compact subgroup. This condition leads to a unique (up to the choice of a positive degree) canonical state; namely, the corresponding one-parameter subgroup is the subgroup of spherical functions, and its generator is the derivative of the

family of spherical functions of complementary series with respect to the parameter at the group identity. Thus, among semisimple groups, canonical states and the corresponding nonlocal representations of current groups exist only on groups of rank 1 (and not all at that), namely, on $SO(n, 1)$ and $SU(n, 1)$ ⁴.

It is clear from above that the restriction of a canonical state to any commutative subgroup is an infinitely divisible characteristic (=positive definite) function, i.e., the Bochner–Fourier transform of an infinitely divisible *measure* on the group of characters of the commutative subgroup; hence this measure determines a Lévy process with values in the group of characters. In our case of the group $SO(n, 1)$, the commutative subgroup and its group of characters is \mathbb{R}^{n-1} , and the representation of the current group is realized in the L^2 space over some vector Lévy process.

Let $\varphi(\cdot)$ be a canonical state on some group G ; considering its restriction to a commutative subgroup, we can write the Fourier transform of the law of the corresponding Lévy process as

$$\Phi(g(\cdot)) = \exp\left(\int_X -\log \varphi(g(x)) dx\right).$$

2.3. The group $G = O(n, 1)$. In this paper the group $G = O(n, 1)$ is realized as the collection of all real matrices of order $n + 1$ preserving the quadratic form

$$2x_1x_{n+1} + x_2^2 + \dots + x_n^2.$$

In another formulation, G is the group of all real matrices satisfying the relation

$$gsg^* = s, \quad s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & e & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (2.1)$$

where e is the identity matrix of order $n - 1$ and $*$ stands for transposition.

We write elements $g \in G$ as block matrices

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix},$$

where the diagonal blocks are quadratic matrices of orders 1, $n - 1$, and 1, respectively.

⁴It is possible to define a canonical state on groups with the Kazhdan property (= the identity representation is an isolated point in the space of all irreducible unitary representations, as in a semisimple group of rank more than one), but in this case we must omit the positivity condition and, consequently, allow considering nonunitary representations. For example, for $SL(n, R)$, $n > 2$, the derivative of the family of spherical functions of complementary series representations makes sense, but it will not be a conditionally positive function on the group. Thus the exponential of this function will not be a state on the group in the true sense, but nevertheless it still determines an invariant bilinear form on the group algebra, which is interesting for further construction of nonunitary representations of current groups. Note that F. A. Berezin [3] considered presumably similar notions, but the relation between his construction and our notions is still not clear.

Condition (2.1) is equivalent to the following relations between the elements of these matrices:

$$\begin{aligned} g_{13}g_{31} + g_{11}g_{33} + g_{12}g_{32}^* &= 1, & g_{23}g_{21}^* + g_{21}g_{23}^* + g_{22}g_{22}^* &= e, \\ 2g_{11}g_{13} + g_{12}g_{12}^* &= 0, & 2g_{31}g_{33} + g_{32}g_{32}^* &= 0, \\ g_{11}g_{23} + g_{13}g_{21} + g_{22}g_{12}^* &= 0, & g_{31}g_{23} + g_{33}g_{21} + g_{22}g_{32}^* &= 0. \end{aligned}$$

The group G contains as a subgroup the group Z of all block matrices of the form

$$z = \begin{pmatrix} 1 & 0 & 0 \\ -\gamma^* & e & 0 \\ -\frac{|\gamma|^2}{2} & \gamma & 1 \end{pmatrix}, \quad \gamma \in \mathbb{R}^{n-1},$$

where $|\gamma| = (\sum \gamma_i^2)^{1/2}$. The group Z is commutative and isomorphic to the additive group \mathbb{R}^{n-1} , and it is the maximal unipotent subgroup of G . We identify elements $z \in Z$ with vectors $\gamma \in \mathbb{R}^{n-1}$ and write $z(\gamma)$ or simply γ instead of z .

By D we denote the subgroup of block diagonal matrices from G , and by B the subgroup of all block lower triangular matrices. Elements of the subgroup D are matrices of the form

$$d = \text{diag}(\epsilon^{-1}, u, \epsilon), \quad \epsilon \in \mathbb{R} \setminus 0, \quad u \in O(n-1).$$

The group B is the semidirect product $B = Z \rtimes D$, and the group D is the direct product of the groups \mathbb{R}^* and $O(n-1)$.

The canonical state $\varphi(g)$ on the group G is given by the following formula:

$$\varphi(g) = \exp\left(\left.\frac{d\psi_\lambda(g)}{d\lambda}\right|_{\lambda=0}\right),$$

where ψ_λ is the spherical function of the complementary series representation of the group G with parameter λ , the value $\lambda = 0$ corresponding to the special representation of G glued to the identity representation; see [14] and [5].

Note that the special representation of the group G corresponding to $\lambda = 0$ has a nontrivial 1-cocycle and is the unique irreducible unitary representation of G with this property.

2.4. The current group $G^X = O(n, 1)^X$. We consider the group $G = O(n, 1)$ in the realization described above. Denote by Z^X , D^X , and B^X the subgroups of functions $g(x) \in G^X$ with values in the subgroups Z , D , and B , respectively. Note that the group Z^X is isomorphic to the additive group $(\mathbb{R}^{n-1})^X$ of bounded Borel functions on X with values in \mathbb{R}^{n-1} , and the group B^X is the semidirect product $B^X = Z^X \rtimes D^X$.

In [15, 5], for each group $O(n, 1)^X$, an irreducible unitary representation was constructed that is invariant under m -preserving transformations of the space X . It was called the basic representation; several models of this representation were described.

In the chosen realization of the group $O(n, 1)$, the spherical function $\Psi(g(\cdot))$ (canonical state) on the current group G^X that determines this representation is given by the following formula:

$$\Psi(g(\cdot)) = \exp\left(-\frac{1}{2} \int_X \log \left| \frac{g_{11}(x) + g_{33}(x) - g_{13}(x) - g_{31}(x)}{2} \right| dm(x)\right), \quad (2.2)$$

where g_{ij} are elements of the block matrix $g \in G^X$. Its restriction to the infinite-dimensional commutative current group Z^X is given by the formula

$$\Psi(\gamma(\cdot)) = \exp\left(-\frac{1}{2} \int_X \log\left(1 + \frac{|\gamma(x)|^2}{4}\right) dm(x)\right), \quad (2.3)$$

where γ is the function determining the block matrix

$$z = \begin{pmatrix} 1 & 0 & 0 \\ -\gamma^* & e & 0 \\ -\frac{|\gamma|^2}{2} & \gamma & 1 \end{pmatrix} \in Z^X.$$

Automatically, this restriction is the characteristic function of an infinitely divisible distribution, because the canonical state on the group G is infinitely divisible.

We construct a commutative model of the basic representation of the group $O(n, 1)^X$. It is realized in the Hilbert space $L^2(\nu)$ of functions on the space of vector distributions square-integrable with respect to the measure ν introduced below. By the properties of this measure, on $L^2(\nu)$ there is a natural unitary representation U_g of the block triangular subgroup B^X , where elements $z \in Z^X$ act as diagonal operators. The desired commutative model of the basic representation of the group G^X is obtained by extending this representation from the subgroup B^X to the whole group $O(n, 1)^X$.

The measure ν is determined by its density with respect to another measure μ introduced in this paper. By definition, μ is the finite measure on the space of vector distributions whose Fourier transform is the functional $\Psi(\gamma(\cdot))$ given by (2.3).

Note that the measure μ is the law of the Lévy process obtained by the canonical construction from the infinitely divisible measure α on the additive group \mathbb{R}^{n-1} whose Fourier transform is $(1 + \frac{|\gamma|^2}{4})^{1-n}$, $\gamma \in \mathbb{R}^{n-1}$. The density of this measure α with respect to the standard Lebesgue measure is given, up to constant factor, by the following formula:

$$\tilde{\psi}(\xi) = |\xi|^{\frac{n-1}{2}} K_{\frac{n-1}{2}}(2|\xi|),$$

where $K_\nu(x)$ is the modified Bessel function of the third kind, see [2]. In particular, $\tilde{\psi}(\xi) = e^{-2|\xi|}$ in the case $n = 2$. The measure α is $O(n-1)$ -invariant, and it is a multidimensional generalization of the symmetrized gamma distribution.

2.5. Several integral relations. In further constructions we will use formulas for the Fourier transform of the functions $(1 + \frac{|\gamma|^2}{4})^{-\lambda/2}$ and $|\gamma|^{-\lambda}$ on \mathbb{R}^{n-1} , where $|\gamma| = (\sum_{i=1}^{n-1} \gamma_i^2)^{1/2}$. Namely,

$$\int_{\mathbb{R}^{n-1}} \left(1 + \frac{|\gamma|^2}{4}\right)^{-\lambda/2} e^{i\langle \xi, \gamma \rangle} d\gamma = c_n \frac{2}{\Gamma(\lambda/2)} |\xi|^{(\lambda-n+1)/2} K_{(n-1-\lambda)/2}(2|\xi|), \quad (2.4)$$

$$\int_{\mathbb{R}^{n-1}} |\gamma|^{-\lambda} e^{i\langle \xi, \gamma \rangle} d\gamma = c_n \frac{2^{-\lambda} \Gamma((n-1-\lambda)/2)}{\Gamma(\lambda/2)} |\xi|^{\lambda-n+1}, \quad (2.5)$$

where the coefficient c_n , the same in (2.4) and (2.5), depends only on n .

The Bessel function K_ρ is given by the following equation (see [2]):

$$K_\rho(2z) = \frac{\pi}{2 \sin(\pi\rho)} (I_{-\rho}(2z) - I_\rho(2z)), \quad (2.6)$$

where

$$I_\rho(2z) = \sum_{m=0}^{\infty} \frac{z^{2m+\rho}}{m! \Gamma(m+\rho+1)}. \quad (2.7)$$

It is important for the sequel that this function is continuous and strictly positive on the half-line $0 < x < \infty$.

Remark. For integer values of ρ , the series for K_ρ contains terms with $\log z$; for half-integer values of ρ , the expression for K_ρ can be simplified:

$$K_{n+\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \sum_{k=0}^n \frac{(n+k)!}{k!(n-k)!(2z)^k}.$$

It is convenient to write equation (2.4) in another form. Namely, setting $\lambda = 2\rho + n - 1$, we obtain

$$\int_{\mathbb{R}^{n-1}} \left(1 + \frac{|\gamma|^2}{4}\right)^{-\frac{n-1}{2}-\rho} e^{i\langle \xi, \gamma \rangle} d\gamma = c_n \frac{2}{\Gamma(\frac{n-1}{2} + \rho)} |\xi|^\rho K_\rho(2|\xi|). \quad (2.8)$$

Let us give a brief derivation of formulas (2.4) and (2.5) for $n > 2$. In spherical coordinates the first integral takes the form

$$J_1 = c \int_0^\infty \int_0^\varphi \left(1 + \frac{r^2}{4}\right)^{-\lambda/2} e^{i|\xi| \cos \varphi} r^{n-2} \sin^{n-3} \varphi d\varphi dr.$$

Integrating with respect to φ yields (see [8, formula 3.915.5])

$$J_1 = c |\xi|^{-\frac{n-3}{2}} \int_0^\infty \left(1 + \frac{r^2}{4}\right)^{-\lambda/2} r^{\frac{n-1}{2}} J_{\frac{n-3}{2}}(|\xi|r) dr,$$

where $J_\rho(\gamma)$ is the Bessel function of the first kind (see [2]). Similarly, for the integral (2.5) we obtain

$$J_2 = c |\xi|^{-\frac{n-3}{2}} \int_0^\infty r^{\frac{n-1}{2}-\lambda} J_{\frac{n-3}{2}}(|\xi|r) dr.$$

Integrating with respect to r yields the expressions (2.4) and (2.5) for J_1 and J_2 , respectively; see [8, formulas 6.565.4 and 6.561.14].

Theorem 2.1 (Multidimensional analog of the Lévy–Khintchin formula). *The function $\log(1 + \frac{|\gamma|^2}{4})$ on \mathbb{R}^{n-1} has the following integral representation:*

$$\log\left(1 + \frac{|\gamma|^2}{4}\right) = \int_{\mathbb{R}^{n-1}} \left(e^{i\langle \xi, \gamma \rangle} - 1\right) g(\xi) d\xi, \quad \text{where } g(\xi) = |\xi|^{-\frac{n-1}{2}} K_{-\frac{n-1}{2}}(2|\xi|). \quad (2.9)$$

Indeed, it follows from (2.4) that

$$\int_{\mathbb{R}^{n-1}} \left(e^{i\langle \xi, \gamma \rangle} - 1\right) |\xi|^{\frac{\lambda-n+1}{2}} K_{\frac{\lambda-n+1}{2}}(2|\xi|)^{1/2} d\xi = c \Gamma(\lambda/2) \left(\left(1 + \frac{|\gamma|^2}{4}\right)^{-\lambda/2} - 1\right).$$

As $\lambda \rightarrow 0$, we obtain (2.9).

3.1. The action of the group G on \mathbb{R}^{n-1} and the 1-cocycle. Let Y be the manifold of one-dimensional subspaces in \mathbb{R}^{n+1} lying inside the light cone

$$2x_1x_{n+1} + x_2^2 + \dots + x_n^2 = 0.$$

The group $G = O(n, 1)$, regarded as a group of linear transformations in \mathbb{R}^{n+1} , acts transitively on Y . We use the right notation for this action: $y \rightarrow y\bar{g}$. Note that in another interpretation Y is the absolute of the n -dimensional Lobachevsky space realized as the collection of one-dimensional subspaces in \mathbb{R}^{n+1} lying inside the light cone.

Let us realize $Y \setminus y_0$, where $y_0 = (\lambda, 0, \dots, 0)$, as the intersection of the cone with the hyperplane $x_{n+1} = 1$, i.e., as the set of points in \mathbb{R}^{n+1} of the form

$$\left(-\frac{|\gamma|^2}{2}, \gamma_i, \dots, \gamma_{n-1}, 1\right),$$

where $\gamma = (\gamma_i, \dots, \gamma_{n-1}) \in \mathbb{R}^{n-1}$ and $|\gamma| = (\sum \gamma_i^2)^{1/2}$. According to this realization, there is a natural bijection $Y \setminus y_0 \rightarrow \mathbb{R}^{n-1}$; hence the action of the group G on Y induces an action $\gamma \rightarrow \gamma\bar{g}$ of G on the space \mathbb{R}^{n-1} . We emphasize that this action is not linear.

It follows from the definition that the vector $\gamma\bar{g}$ is given by the following formula:

$$\gamma\bar{g} = \left(-\frac{|\gamma|^2}{2}g_{13} + \gamma g_{23} + g_{33}\right)^{-1} \left(-\frac{|\gamma|^2}{2}g_{12} + \gamma g_{22} + g_{32}\right), \quad (3.1)$$

where g_{ij} are elements of the block matrix g . In particular,

$$\begin{aligned} \gamma\bar{g} &= \gamma + \gamma_0 & \text{for } g &= z(\gamma_0) \in Z; \\ \gamma\bar{g} &= \epsilon^{-1}\gamma u & \text{for } g &= \text{diag}(\epsilon^{-1}, u, \epsilon); \\ \gamma\bar{s} &= -\frac{2\gamma}{|\gamma|^2}. \end{aligned}$$

Now let us define a function $\beta(\gamma, g)$ by the formula

$$\beta(\gamma, g) = \left| -\frac{|\gamma|^2}{2}g_{13} + \gamma g_{23} + g_{33} \right|, \quad \gamma \in \mathbb{R}^{n-1}, \quad g \in G. \quad (3.2)$$

In particular,

$$\begin{aligned} \beta(\gamma, g) &= 1 & \text{for } g &\in Z; \\ \beta(\gamma, g) &= |\epsilon| & \text{for } g &= \text{diag}(\epsilon^{-1}, u, \epsilon); \\ \beta(\gamma, s) &= \frac{|\gamma|^2}{2}. \end{aligned}$$

It follows from the definition that $\beta(\gamma, g)$ is a 1-cocycle of the group G with values in \mathbb{R}^* , i.e.,

$$\beta(\gamma, g_1g_2) = \beta(\gamma, g_1)\beta(\gamma\bar{g}_1, g_2) \quad \text{for any } \gamma \in \mathbb{R}^{n-1} \quad \text{and } g_1, g_2 \in G. \quad (3.3)$$

3.2. The standard model of complementary series representations. Each irreducible unitary complementary series representation of the group $G = O(n, 1)$ is determined by a number λ from the interval $0 < \lambda < n - 1$. In the standard realization, the representation T^λ with parameter λ acts in the Hilbert space L_λ of real-valued functions $f(\gamma)$ on $\mathbb{R}^{n-1} \simeq Z$ with scalar product

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} |\gamma' - \gamma''|^{-\lambda} f_1(\gamma') f_2(\gamma'') d\gamma' d\gamma'', \quad (3.4)$$

where $d\gamma = d\gamma_1 \dots d\gamma_{n-1}$ is the Lebesgue measure on \mathbb{R}^{n-1} . The operators of this representation have the form

$$T_g^\lambda f(\gamma) = f(\gamma \bar{g}) \beta^{1-n+\frac{\lambda}{2}}(\gamma, g), \quad (3.5)$$

where $\gamma \bar{g}$ and $\beta(\gamma, g)$ are given by equations (3.1) and (3.2), respectively. In particular,

$$T_z^\lambda f(\gamma) = f(\gamma + \gamma_0) \quad \text{for } z = z(\gamma_0) \in Z; \quad (3.6)$$

$$T_d^\lambda f(\gamma) = |\epsilon|^{1-n+\frac{\lambda}{2}} f(\epsilon^{-1} \gamma u) \quad \text{for } d = \text{diag}(\epsilon^{-1}, u, \epsilon); \quad (3.7)$$

$$T_s^\lambda f(\gamma) = f\left(-\frac{2\gamma}{|\gamma|^2}\right) \left(\frac{|\gamma|^2}{2}\right)^{1-n+\frac{\lambda}{2}} \quad \text{for } s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & e & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (3.8)$$

The group property of these operators follows immediately from the property (3.3) of the function $\beta(\gamma, g)$, and their unitarity follows from the relations

$$d(\gamma \bar{g}) = \beta^{1-n}(\gamma, g) d\gamma \quad \text{for any } g \in G, \quad (3.9)$$

where $d\gamma = d\gamma_1 \dots d\gamma_{n-1}$, and

$$|x - y|^2 = |x \bar{g} - y \bar{g}|^2 \beta(x, g) \beta(y, g) \quad (3.10)$$

for any $x, y \in \mathbb{R}^{n-1}$ and $g \in G$.

Relations (3.9) and (3.10) are easily verified for elements from Z and D and the element s . It follows from the properties of the 1-cocycle $\beta(\gamma, g)$ that they hold for any element $g \in G$.

3.3. Construction of the commutative model of complementary series representations. Let us describe the commutative model of a complementary series representation T^λ of the group $G = O(n, 1)$ with respect to the subgroup Z , i.e., the model in which the operators T_z^λ , $z \in Z$, act as multipliers.

This model is obtained by passing from functions $f(\gamma)$ in the standard model to their Fourier transforms

$$\varphi(\xi) = \int_{\mathbb{R}^{n-1}} e^{i\langle \xi, \gamma \rangle} f(\gamma) d\gamma.$$

Theorem 3.1. *In the commutative model, the complementary series representation T^λ is realized in the Hilbert space L_λ of complex-valued functions on \mathbb{R}^{n-1} with the norm*

$$\|\varphi\|^2 = \frac{2^{-\lambda} \Gamma((n-1-\lambda)/2)}{\Gamma(\lambda/2)} \int_{\mathbb{R}^{n-1}} |\xi|^{1-n+\lambda} |\varphi(\xi)|^2 d\xi, \quad |\xi| = \langle \xi, \xi \rangle^{1/2}, \quad (3.11)$$

satisfying the condition

$$\overline{\varphi(\xi)} = \varphi(-\xi). \quad (3.12)$$

The operators of the representation are given by the formula

$$T_g^\lambda \varphi(\xi) = \int_{\mathbb{R}^{n-1}} A^\lambda(\xi, \xi', g) \varphi(\xi') d\xi', \quad (3.13)$$

where

$$A^\lambda(\xi, \xi', g) = \int_{\mathbb{R}^{n-1}} e^{i(\langle \xi, \gamma \rangle - \langle \xi', \gamma \bar{g} \rangle)} \beta^{1-n+\lambda/2}(\gamma, g) d\gamma. \quad (3.14)$$

In particular,

$$\begin{aligned} T_z^\lambda \varphi(\xi) &= e^{-i\langle \xi, \gamma_0 \rangle} \varphi(\xi) & \text{for } z = z(\gamma_0) \in Z; \\ T_d^\lambda \varphi(\xi) &= |\epsilon|^{\lambda/2} \varphi(\epsilon \xi u) & \text{for } d = \text{diag}(\epsilon^{-1}, u, \epsilon). \end{aligned}$$

Proof. In the new model, the squared norm is given by the formula

$$\|\varphi\|^2 = \int_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} R(\xi, \xi') \varphi(\xi) \overline{\varphi(\xi')} d\xi d\xi',$$

where

$$R(\xi, \xi') = \int_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} |\gamma - \gamma'|^{-\lambda} e^{i(\langle \xi, \gamma \rangle - \langle \xi', \gamma' \rangle)} d\gamma d\gamma' = \delta(\xi - \xi') \int_{\mathbb{R}^{n-1}} |\gamma|^{-\lambda} e^{i\langle \xi, \gamma \rangle} d\xi.$$

This implies (3.11) in view of (2.5).

Relation (3.12) is equivalent to the condition that the original space is real.

The formulas for representation operators in the new model can be obtained directly from the formulas for these operators in the original model by passing from functions $f(\gamma)$ to their Fourier transforms. \square

Proposition 3.1. *In the commutative model of the representation T^λ , the kernel $A(\xi, \xi') =$*

$A^\lambda(\xi, \xi', s)$ of the operator T_s^λ corresponding to the element $s = \begin{pmatrix} 0 & 0 & 1 \\ 0 & e & 0 \\ 1 & 0 & 0 \end{pmatrix}$ has the fol-

lowing form:

$$A(\xi, \xi') = 2^{1-\frac{\lambda}{2}} \int_0^\infty \cos\left(\xi x + \frac{2\xi'}{x}\right) x^{\lambda-2} dx \quad \text{for } n = 2, \quad (3.15)$$

$$A(\xi, \xi') = c_n 2^{-\lambda/2} \int_0^\infty r^{\lambda-n} |r\xi + 2r^{-1}\xi'|^{-\frac{n-3}{2}} J_{\frac{n-3}{2}}(|r\xi + 2r^{-1}\xi'|) dr \quad (3.16)$$

for $n > 2$, where $J_{\frac{n-3}{2}}$ is the Bessel function of the first kind.

Indeed, since $\gamma \bar{s} = \frac{-2\gamma}{|\gamma|^2}$ and $\beta(\gamma, s) = \frac{|\gamma|^2}{2}$, it follows from (3.14) that

$$A(\xi, \xi') = 2^{n-1-\lambda/2} \int_{\mathbb{R}^{n-1}} e^{i(\langle \xi, \gamma \rangle + \langle \xi', \frac{2\gamma}{|\gamma|^2} \rangle)} |\gamma|^{2-2n+\lambda} d\gamma.$$

For $n = 2$, (3.15) follows immediately. For $n > 2$, in spherical coordinates we obtain

$$A(\xi, \xi) = c_n 2^{-\lambda/2} \int_0^\infty \int_0^\pi e^{i|r\xi + 2r^{-1}\xi'|\cos\varphi} r^{\lambda-n} \sin^{n-3}\varphi d\varphi dr.$$

Integrating with respect to φ yields (3.16).

Remark. For $n = 2$, the kernel $A(\xi, \xi')$ can be expressed in terms of Bessel functions:

$$A(\xi, \xi') = c \left(\cos \frac{\pi\lambda}{2} \right)^{-1} |\xi'\xi^{-1}|^{1/2} [J_{\lambda-1}(2^{3/2}|\xi\xi'|) - J_{1-\lambda}(2^{3/2}|\xi\xi'|)] \quad \text{for } \xi\xi' < 0,$$

$$A(\xi, \xi') = c \left(\cos \frac{\pi\lambda}{2} \right)^{-1} |\xi'\xi^{-1}|^{1/2} [I_{\lambda-1}(2^{3/2}|\xi\xi'|) - I_{1-\lambda}(2^{3/2}|\xi\xi'|)] \quad \text{for } \xi\xi' > 0.$$

The representation T^λ is uniquely determined by its spherical function

$$\psi_\lambda(g) = \langle T_g^\lambda \mathbb{1}_\lambda, \mathbb{1}_\lambda \rangle,$$

where $\mathbb{1}_\lambda \in L_\lambda$ is a vector of norm 1 that is invariant under the maximal compact subgroup of G (vacuum vector). In the chosen realization of the group G , this spherical function takes the form

$$\psi_\lambda(g) = \left| \frac{g_{11}(x) + g_{33}(x) - g_{13}(x) - g_{31}(x)}{2} \right|^{-\lambda/2}, \quad (3.17)$$

where g_{ij} are elements of the block matrix g . In particular,

$$\psi_\lambda(z(\gamma)) = \left(1 + \frac{|\gamma|^2}{4} \right)^{-\lambda/2}.$$

Let us introduce the following vector in the space L_λ :

$$f_\lambda(\xi) = (|\xi|^{-\frac{\lambda-n+1}{2}} K_{\frac{\lambda-n+1}{2}}(2|\xi|))^{1/2}, \quad (3.18)$$

where $K_\rho(x)$ is the Bessel function defined above.

Proposition 3.2. *The vector f_λ is invariant under the maximal compact subgroup of the group G , hence it is proportional to the vacuum vector $\mathbb{1}_\lambda$.*

Corollary. *The following equation holds:*

$$\langle T_{z\gamma}^\lambda f_\lambda, f_\lambda \rangle = \|f_\lambda\|^2 \psi_\lambda(z(\gamma)),$$

i.e.,

$$\int_{\mathbb{R}^{n-1}} e^{i\langle \xi, \gamma \rangle} |\xi|^{\frac{\lambda-n+1}{2}} K_{\frac{\lambda-n+1}{2}}(2|\xi|)^{1/2} d\xi = \|f_\lambda\|^2 \left(1 + \frac{|\gamma|^2}{4} \right)^{-\lambda/2}. \quad (3.19)$$

Proposition 3.3. *We have $\|f_\lambda\|^2 = c\Gamma(\lambda/2) \left(1 + \frac{|\gamma|^2}{4} \right)^{-\lambda/2}$.*

Indeed, it follows from (2.4) that the left-hand side of (3.19) is equal to $(2c_n)^{-1} \Gamma(\lambda/2)$.

Proposition 3.4. *For any positive real numbers $\lambda_1, \dots, \lambda_l$, $\sum \lambda_i < n - 1$, there exists an isometric embedding*

$$\tau : L_\lambda \rightarrow \bigotimes_{i=1}^l L_{\lambda_i}, \quad \lambda = \sum \lambda_i,$$

that commutes with the action of the group G . In the standard realization of representations it is given by the formula

$$\tau f(\gamma^1, \dots, \gamma^l) = f(\gamma^1) \prod_{i=2}^l \delta(\gamma^1 - \gamma^i), \quad (3.20)$$

and in the commutative model, by the formula

$$\tau \varphi(\xi_1, \dots, \xi_l) = \varphi(\xi_1 + \dots + \xi_l). \quad (3.21)$$

Proof. First let us consider the standard model of representations. Let \langle, \rangle and \langle, \rangle_l be the scalar products in the spaces L_λ and $\bigotimes_{i=1}^l L_{\lambda_i}$, respectively. Then it is obvious that

$$\langle \tau f, \tau f \rangle_l = \int_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} \prod_{i=1}^l |\gamma' - \gamma''|^{-\lambda_i} f(\gamma') f(\gamma'') d\gamma' d\gamma'' = \langle f, f \rangle.$$

Thus the mapping τ is isometric. Obviously, it commutes with the action of G .

Now let us consider the commutative model of representations. Let $\varphi(\xi)$ be the Fourier transform of a function $f(\gamma)$. Then, according to (3.20), the image of φ under this embedding is equal to

$$\int f(\gamma_1) \prod_{i=2}^l \delta(\gamma_1 - \gamma_i) \prod_{i=1}^l e^{i \langle \xi_i, \gamma_i \rangle} d\gamma_i = \varphi(\xi_1 + \dots + \xi_l).$$

□

3.5. The commutative model of the special representation of the group $O(n, 1)$.

The special representation of the group $O(n, 1)$ is the irreducible unitary representation of this group glued to the identity representation. It is obtained from the complementary series representations in the $\lambda \rightarrow 0$ limit. Thus Theorem 3.1 implies the following result.

Theorem 3.2. *In the commutative model, the special representation T^0 of the group $O(n, 1)$ is realized in the Hilbert space of complex-valued functions on \mathbb{R}^{n-1} with the norm*

$$\|\varphi\|^2 = \int_{\mathbb{R}^{n-1}} |\xi|^{1-n} |\varphi(\xi)|^2 d\xi \quad (3.22)$$

satisfying the condition $\overline{\varphi(\xi)} = \varphi(-\xi)$. Operators of the representation are given by the formula

$$T_g \varphi(\xi) = \int_{\mathbb{R}^{n-1}} A^0(\xi, \xi', g) \varphi(\xi') d\xi', \quad (3.23)$$

where

$$A^0(\xi, \xi', g) = \int_{\mathbb{R}^{n-1}} e^{i(\langle \xi, \gamma \rangle - \langle \xi', \gamma \bar{g} \rangle)} \beta^{1-n}(\gamma, g) d\gamma. \quad (3.24)$$

In particular,

$$\begin{aligned} T_z \varphi(\xi) &= e^{-i\langle \xi, \gamma_0 \rangle} \varphi(\xi) \quad \text{for } z = z(\gamma_0) \in Z; \\ T_d \varphi(\xi) &= \varphi(\epsilon \xi u) \quad \text{for } d = \text{diag}(\epsilon^{-1}, u, \epsilon). \end{aligned}$$

The special representation T^0 has a nontrivial 1-cocycle $\beta : G \rightarrow L_0$, where L_0 is the space of T^0 . Namely, set $f_0(\xi) = \lim_{\lambda \rightarrow 0} f_\lambda(\xi)$, where $f_\lambda(\xi) \in L_\lambda$ is given by (3.18), i.e.,

$$f_0(\xi) = (|\xi|^{\frac{n-1}{2}} K_{\frac{n-1}{2}}(2|\xi|))^{1/2}.$$

This vector is invariant under the maximal compact subgroup of G , but it does not belong to the space L_0 of the special representation. The desired nontrivial 1-cocycle $G \rightarrow L_0$ is given by the equation

$$\beta(\gamma, g) = T_g^0 f_0(\xi) - f_0(\xi).$$

It is known that T^0 is the unique irreducible unitary representation of the group G possessing a nontrivial 1-cocycle.

4. THE MEASURES μ AND ν ON THE SPACE OF VECTOR DISTRIBUTIONS

4.1. The spaces F and Φ . The construction of the commutative model of the basic representation of the current groups G^X will be based on two remarkable measures in the space of vector distributions.

Denote by F the space of smooth bounded real-valued functions on X and by Φ the dual space with the ordinary topology (the space of distributions on X). We will denote by \langle, \rangle the pairing of elements from F and Φ .

Let us introduce the space F^{n-1} of vector functions $\gamma(x) = (\gamma_1(x), \dots, \gamma_{n-1}(x))$, $\gamma_i \in F$, and denote by Φ^{n-1} the dual space. Elements from Φ^{n-1} are realized as vector distributions $\xi(x) = (\xi_1(x), \dots, \xi_{n-1}(x))$, $\xi_i \in \Phi$, with the pairing

$$\langle \xi, \gamma \rangle = \sum_{k=1}^{n-1} \langle \xi_k, \gamma_k \rangle.$$

For example, if $\xi = c \delta_{x_0}$, where $c = (c_1, \dots, c_{n-1}) \in \mathbb{R}^{n-1}$ and δ_{x_0} is a δ -function on X , then $\langle \xi, \gamma \rangle = \sum_{k=1}^{n-1} c_k \gamma_k(x_0)$.

Since in what follows we will consider only the spaces F^{n-1} and Φ^{n-1} , we will omit the index $n-1$ in their notation.

4.2. The measure μ on Φ . Let us introduce the following function on \mathbb{R}^{n-1} :

$$l(\gamma) = \left(1 + \frac{|\gamma|^2}{4}\right)^{-1/2}, \quad \gamma \in \mathbb{R}^{n-1}. \quad (4.1)$$

It is known that this function is positive definite. According to (2.4), its Fourier transform equals

$$\int_{\mathbb{R}^{n-1}} \left(1 + \frac{|\gamma|^2}{4}\right)^{-1/2} e^{i\langle \xi, \gamma \rangle} d\gamma = c_n \frac{2}{\Gamma(1/2)} |\xi|^{(-n+2)/2} K_{(n-2)/2}(2|\xi|).$$

Let us introduce a functional $L(\gamma(\cdot))$ on F by the formula

$$L(\gamma(\cdot)) = \exp\left(\int_X \log l(\gamma(x)) d m(x)\right),$$

i.e.,

$$L(\gamma(\cdot)) = \exp\left(-\frac{1}{2} \int_X \log\left(1 + \frac{1}{4} |\gamma(x)|^2\right) dm(x)\right). \quad (4.2)$$

Note that $L(\gamma(\cdot)) = \Psi(\gamma(\cdot))$, where $\Psi(\gamma(\cdot))$ is the restriction of the spherical function of the basic representation of the group G^X to the subgroup Z^X , see (2.3). This functional is positive definite and continuous (see [6]); hence, by the Minlos theorem on measures on the space of distributions [6], it is the Fourier transform of a finite normalized measure μ on Φ , i.e.,

$$L(\gamma(\cdot)) = \int_{\Phi} e^{i\langle \xi, \gamma \rangle} d\mu(\xi). \quad (4.3)$$

Theorem 4.1. *The measure μ is concentrated on the set $\Phi_0 \subset \Phi$ of distributions of the form*

$$\xi = \sum c^i \delta_{x_i}, \quad c^i = (c_1^i, \dots, c_{n-1}^i) \in \mathbb{R}^{n-1}, \quad \text{where} \quad \sum |c^i| < \infty. \quad (4.4)$$

Proof. The series in (4.4) converges if and only if the series for each coordinate converges; hence it suffice to verify the condition for the one-dimensional processes obtained by projecting to coordinates. The characteristic functions of these processes are the restrictions of the function $l(\cdot)$ (given by (4.1)) to the one-dimensional subspaces; but all of them determine the classical gamma process, which satisfies the convergence condition (see [13]; for a general convergence condition for one-dimensional Lévy processes, see [7]). \square

Let us introduce the space $\mathcal{F} \supset F$ of real bounded Borel vector functions on X . Since μ -almost every distribution $\xi \in \Phi$ is of the form (4.4), each function $\gamma \in \mathcal{F}$ corresponds to a measurable linear functional $\langle \xi, \gamma \rangle$ on Φ defined μ -almost everywhere on Φ .

Note that on \mathcal{F} and Φ there are two natural operations :

1) the multiplication by a bounded Borel \mathbb{R}^* -valued function $\epsilon(x)$ on X :

$$\gamma \rightarrow \epsilon\gamma, \quad \xi \rightarrow \epsilon\xi;$$

2) the action of the group $O(n-1)^X$ of Borel functions on X with values in the compact group $O(n-1)$:

$$\gamma \rightarrow \gamma u, \quad \xi \rightarrow \xi u, \quad u \in O(n-1)^X.$$

According to this definition,

$$\langle \epsilon\xi, \gamma \rangle = \langle \xi, \epsilon\gamma \rangle \quad \text{and} \quad \langle \xi u, \gamma \rangle = \langle \xi, \gamma u \rangle \quad \text{for any} \quad \gamma \in \mathcal{F}, \quad \xi \in \Phi.$$

The following proposition follows from the definition of the measure μ .

Proposition 4.1. *The measure μ is invariant under the transformations $\xi \rightarrow \xi u$, $u \in O(n-1)^X$.*

Remark. In fact, we construct a vector gamma process such that the measure in the space of trajectories of this process enjoys the additional property as compared with the symmetrized gamma process: it is invariant under pointwise orthogonal transformations.

4.3. Projections of the measure μ to finite-dimensional function spaces. Let us consider all finite partitions $X = \bigcup_{i=1}^l X_i$ of the space X such that $m(X_i) < n - 1$ for all elements of the partition. Let us associate with each partition $\alpha : X = \bigcup_{i=1}^l X_i$, where $m(X_i) < n - 1$, the subspace $\mathcal{F}_\alpha \cong (\mathbb{R}^{n-1})^l$ of vector functions $\gamma(x) \in \mathcal{F}$ that are constant on elements of α , and the dual space $\Phi_\alpha \cong (\mathbb{R}^{n-1})^l$, realized as a quotient space of Φ .

Denote by μ_α the projection of the measure μ to the quotient space Φ_α .

Proposition 4.2. *The measure μ_α on $\Phi_\alpha \cong (\mathbb{R}^{n-1})^l$ has the form*

$$d\mu_\alpha(\xi_1, \dots, \xi_l) = \prod_{k=1}^l \left(\frac{2}{\Gamma(\lambda_k/2)} |\xi_k|^{(\lambda_k - n + 1)/2} K_{\frac{n-1-\lambda_k}{2}}(2|\xi_k|) d\xi_k \right), \quad (4.5)$$

where $\xi_i \in \mathbb{R}^{n-1}$, $d\xi_i$ is the Lebesgue measure on \mathbb{R}^{n-1} , $\lambda_k = m(X_k)$, and K_ν is the modified Bessel function of the third kind defined above.

Proof. For every $\gamma(x) \in \mathcal{F}_\alpha$ we have

$$\exp\left(-\frac{1}{2} \int_X \log\left(1 + \frac{|\gamma(x)|^2}{4}\right) dm(x)\right) = \prod_{i=1}^l \left(1 + \frac{|\gamma^i|^2}{4}\right)^{-\lambda_i/2},$$

where $\gamma^i = \gamma(x)|_{X_i}$ and $\lambda_i = m(X_i)$.

Therefore, in view of (4.3), for every $\gamma(x) \in \mathcal{F}_\alpha$ we have

$$\int_{\Phi_\alpha} e^{i\langle \xi, \gamma \rangle} d\mu_\alpha(\xi) = \int_{\Phi} e^{i\langle \xi, \gamma \rangle} d\mu(\xi) = \prod_{i=1}^l \left(1 + \frac{|\gamma^i|^2}{4}\right)^{-\lambda_i/2}.$$

Hence $d\mu_\alpha(\xi^1, \dots, \xi^l) = \prod_{i=1}^l \psi(\xi^i) d\xi^i$, where $\psi(\xi^i)$ is the Fourier transform of the function $(1 + \frac{|\gamma^i|^2}{4})^{-\lambda_i/2}$ on \mathbb{R}^{n-1} . Now (4.5) follows from (2.4). \square

4.4. The function $V_\rho(x)$ and the measure ν on Φ . For $\rho > 0$, introduce a function $V_\rho(x)$ on the half-line $0 \leq x < \infty$ by the formula

$$V_\rho(x) = \left(\frac{2}{\Gamma(\rho)} x^\rho K_\rho(2x) \right)^{-1} = (\Gamma(1 - \rho) x^\rho [I_{-\rho}(2x) - I_\rho(2x)])^{-1}. \quad (4.6)$$

In particular, $V_{1/2}(x) = e^{2x}$.

Theorem 4.2. *The Fourier transform of the function $V_\rho^{-1}(|\xi|)$ on \mathbb{R}^{n-1} equals*

$$\int_{\mathbb{R}^{n-1}} V_\rho^{-1}(|\xi|) e^{i\langle \xi, x \rangle} d\xi = c_n \frac{\Gamma(\rho)}{\Gamma(\frac{n-1}{2} + \rho)} \left(1 + \frac{|x|^2}{4}\right)^{-\frac{n-1}{2} - \rho}. \quad (4.7)$$

In particular,

$$\int_{\mathbb{R}^{n-1}} V_{\frac{n-1}{2}}^{-1}(|\xi|) e^{i\langle \xi, x \rangle} d\xi = c_n \frac{\Gamma(\frac{n-1}{2})}{\Gamma(n-1)} \left(1 + \frac{|x|^2}{4}\right)^{-n+1}.$$

Indeed, according to (2.8),

$$V_\rho^{-1}(|\xi|) = c_n^{-1} \frac{\Gamma(\frac{n-1}{2} + \rho)}{\Gamma(\rho)} \int_{\mathbb{R}^{n-1}} \left(1 + \frac{|x|^2}{4}\right)^{-\frac{n-1}{2} - \rho} e^{i\langle \xi, x \rangle} dx. \quad (4.8)$$

Applying the inverse Fourier transform yields (4.7).

Corollary. The measure $V_{\frac{n-1}{2}}(|\xi|) d\xi$ is an infinitely divisible measure on \mathbb{R}^{n-1} .

(Since its Fourier transform has a Lévy–Khintchin representation; see (2.9).)

Proposition 4.3. The function $V_\rho(x)$ is continuous and strictly positive on the half-line $0 \leq x < \infty$, satisfies $V_\rho(0) = 1$ for every $\rho > 0$, and has the following asymptotic estimates as $x \rightarrow 0$:

$$V_\rho(x) \sim \begin{cases} 1 + x^{2\rho} \Gamma(1-\rho)/\Gamma(1+\rho) & \text{for } \rho < 1, \\ 1 - 2x^2 \log(x) & \text{for } \rho = 1, \\ 1 + 2x^2/(\rho-1) & \text{for } \rho > 1. \end{cases} \quad (4.9)$$

Proof. Let us prove (4.9). If $\rho \notin \mathbb{Z}$, then we use the estimate for the functions $I_{-\rho}$ and I_ρ that follows from their power series representation:

$$I_{-\rho}(2x) \sim \frac{x^{-\rho}}{\Gamma(1-\rho)} + \frac{x^{2-\rho}}{\Gamma(2-\rho)} x^2, \quad I_\rho(2x) \sim \frac{x^\rho}{\Gamma(1+\rho)}.$$

Hence we have

$$V_\rho^{-1}(x) \sim 1 + \frac{\Gamma(1-\rho)}{\Gamma(2-\rho)} x^2 - \frac{\Gamma(1-\rho)}{\Gamma(1+\rho)} x^{2\rho}.$$

This estimate implies (4.9) for $\rho \notin \mathbb{Z}$.

If $\rho \in \mathbb{Z}$, then we use the series representation of $K_n(2x)$, see [2, §7.2.5, formula (37)]. This representation implies the estimate

$$2K_n(2x) \sim (n-1)!x^{-n} + (n-2)!x^{2-n} + \frac{2(-1)^{n+1}}{n!}x^n \log x.$$

Therefore

$$V_n^{-1}(x) \sim 1 + \frac{x^2}{n-1} + \frac{2(-1)^{n+1}}{n!(n-1)!}x^{2n} \log x,$$

whence $V_1^{-1} \sim 1 + 2x^2 \log x$ and $V_n^{-1} \sim 1 + \frac{x^2}{n-1}$ for $n > 1$. \square

Corollary. For every $\rho \geq 1/2$, the infinite product $\prod_{i=1}^{\infty} V_\rho(x_i)$ converges provided that the series $\sum_{i=1}^{\infty} x_i$ converges.

Definition. Consider the infinite σ -finite measure ν on the space of vector distributions Φ whose density $v = \frac{d\nu}{d\mu}$ with respect to the measure μ is defined on the support $\Phi_0 \subset \Phi$ of μ by the following formula:

$$v\left(\sum_{i=1}^{\infty} c^i \delta_{x_i}\right) = 2^{-m(X)} \prod_{i=1}^{\infty} V_{\frac{n-1}{2}}(|c^i|), \quad c^i \in \mathbb{R}^{n-1}, \quad (4.10)$$

where $V_\rho(x)$ is given by (4.6). In particular, for $n = 2$

$$v\left(\sum c^i \delta_{x_i}\right) = \exp\left(2 \sum |c^i|\right), \quad c^i \in \mathbb{R}.$$

Since $\sum_{i=1}^{\infty} |c^i| < \infty$ on the support of μ , it follows from Proposition 4.3 that the infinite product $\prod_{i=1}^{\infty} V_{\frac{n-1}{2}}(|c^i|)$ converges.

By definition, the measure ν is absolutely continuous with respect to μ , and its density is positive μ -almost everywhere.

4.5. **Approximative construction of the measure ν .** Let $\alpha : X = \bigcup_{i=1}^l X_i$ be an arbitrary finite partition of the space X , Φ_α be the quotient space of Φ associated with α , and μ_α be the projection of the measure μ to Φ_α . Let us introduce a new measure ν_α on Φ_α with density

$$\frac{d\nu_\alpha(\xi^1, \dots, \xi^l)}{d\mu_\alpha(\xi^1, \dots, \xi^l)} = 2^{-m(X)} \prod_{k=1}^l V_{(n-1-\lambda_k)/2}(|\xi^k|).$$

(By the condition imposed on α , we have $n - 1 - \lambda_k > 0$ for all k .)

It follows from the explicit formula for $d\mu_\alpha(\xi^1, \dots, \xi^l)$ that this measure is of the form

$$d\nu_\alpha(\xi^1, \dots, \xi^l) = \prod_{i=1}^l \frac{2^{-\lambda_i} \Gamma((n-1-\lambda_i)/2)}{\Gamma(\lambda_i/2)} |\xi^i|^{\lambda_i-n+1} d\xi^i, \quad (4.11)$$

where $\xi^i \in \mathbb{R}^{n-1}$ and $\lambda_i = m(X_i)$.

Let us write $\beta \geq \alpha$ if β is a refinement of a partition α . For $\beta \geq \alpha$, there is a natural embedding $\mathcal{F}_\alpha \rightarrow \mathcal{F}_\beta$ and a natural epimorphism $\Phi_\beta \rightarrow \Phi_\alpha$. Obviously, the measure μ_β on Φ_β and the measure μ_α on Φ_α are coherent with respect to this epimorphism.

Proposition 4.4. *The measures ν_α on the quotient spaces Φ_α are coherent, i.e., the epimorphism $\Phi_\beta \rightarrow \Phi_\alpha$, $\beta \geq \alpha$, sends ν_β to ν_α .*

Proof. It suffices to prove that

$$\int_{\Phi_\alpha} e^{i\langle \xi, \gamma \rangle} d\nu_\alpha(\xi) = \int_{\Phi_\beta} e^{i\langle \xi, \gamma \rangle} d\nu_\beta(\xi) \quad \text{for every } \gamma \in \mathcal{F}_\alpha.$$

Let $\alpha : X = \bigcup_{i=1}^l X_i$ and $\beta : X = \bigcup_{i,j} X_{ij}$, where $\bigcup_j X_{ij} = X_i$, $i = 1, \dots, l$. By (2.5), it follows from the expression (4.11) for $d\nu_\alpha$ and $d\mu_\alpha$ that

$$\begin{aligned} \int_{\Phi_\alpha} e^{i\langle \xi, \gamma \rangle} d\nu_\alpha(\xi) &= \prod_{i=1}^l |\gamma^i|^{-\lambda_i}, \quad \text{where } \gamma^i = \gamma|_{X_i}, \quad \lambda_i = m(X_i); \\ \int_{\Phi_\beta} e^{i\langle \xi, \gamma \rangle} d\nu_\beta(\xi) &= \prod_{i,j} |\gamma^{ij}|^{-\lambda_{ij}}, \quad \text{where } \gamma^{ij} = \gamma|_{X_{ij}}, \quad \lambda_{ij} = m(X_{ij}). \end{aligned}$$

Since $\gamma^{ij} = \gamma^i$ for all i and j and $\sum_j \lambda_{ij} = \lambda_i$, $i = 1, \dots, l$, the right-hand sides of these equations coincide. \square

Definition. Let us define a σ -finite measure $\tilde{\nu}$ on Φ as the weak limit of the coherent family of measures ν_α .

Theorem 4.3. *The measure $\tilde{\nu}$ coincides with the measure ν , i.e., on the support of μ ,*

$$\frac{d\tilde{\nu}}{d\mu}(\sum c^i \delta_{x_i}) = 2^{-m(X)} \prod V_{\frac{n-1}{2}}(|c^i|), \quad c^i \in \mathbb{R}^{n-1}, \quad (4.12)$$

where $V_\rho(x)$ is given by (4.6).

Proof. It suffices to prove (4.12) only for finite sums $\xi = \sum_{i=1}^k c^i \delta_{x_i}$.

Given such a sum, consider partitions $\alpha : X = \bigcup_{i=1}^k X_i$ such that each element of α contains at most one point x_i . For definiteness, let $x_i \in X_i$, $i = 1, \dots, k$. Then, since $V_\rho(0) = 1$, we have

$$\frac{d\nu_\alpha}{d\mu_\alpha} \left(\sum_{i=1}^k c^i \delta_{x_i} \right) = 2^{-m(X)} \prod_{i=1}^k V_{\frac{n-1-\lambda_i}{2}}(|c^i|).$$

Taking the inductive limit with respect to α , we obtain

$$\frac{d\tilde{\nu}}{d\mu} \left(\sum_{i=1}^k c^i \delta_{x_i} \right) = 2^{-m(X)} \prod_{i=1}^k V_{\frac{n-1}{2}}(|c^i|).$$

□

4.6. The Fourier transform of the measure ν . If $\gamma \in \mathcal{F}_\alpha$, where α is an arbitrary finite partition of the space X , then by (2.5) we have

$$\int_{\Phi} e^{i \langle \xi, \gamma \rangle} d\nu(\xi) = \int_{\Phi_\alpha} e^{i \langle \xi, \gamma \rangle} d\nu_\alpha(\xi) = \prod_{i=1}^l |\gamma^i|^{-\lambda_i},$$

where $\gamma^i = \gamma|_{X_i}$ and $\lambda_i = m(X_i)$. The right-hand side of this equation can be represented in the form

$$\prod_{i=1}^l |\gamma^i|^{-\lambda_i} = \exp \left(- \int_X \log |\gamma(x)| dm(x) \right).$$

Thus

$$\int_{\Phi} e^{i \langle \xi, \gamma \rangle} d\nu(\xi) = \exp \left(- \int_X \log |\gamma(x)| dm(x) \right). \quad (4.13)$$

Equation (4.13) determines the Fourier transform of the measure ν whenever the integral in the right-hand side is finite.

Remark. One can take this equation as the definition of the measure ν .

4.7. The invariance properties of the measure ν .

Theorem 4.4. *The measure ν is invariant under the action of the group $O(n-1)^X$ and projective invariant under the multiplication by bounded Borel functions $\epsilon(x) \in (\mathbb{R}^*)^X$ such that the integral $\int_X \log |\epsilon(x)| dm(x)$ converges. Namely,*

$$d\nu(\xi u) = d\nu(\xi) \quad \text{for every } u(x) \in O(n-1)^X; \quad (4.14)$$

$$d\nu(\epsilon \xi) = e^{\int_X \log |\epsilon(x)| dm(x)} d\nu(\xi). \quad (4.15)$$

In particular, ν is invariant under the subgroup of multiplications by functions $\epsilon(x)$ satisfying $\int_X \log |\epsilon(x)| dm(x) = 0$.

Proof. It suffices to establish this property for the projections $d\nu_\alpha(\xi) = d\nu_\alpha(\xi_1, \dots, \xi_l)$ of ν to the quotient spaces Φ_α of the space Φ . Let $u(x) \in O(n-1)^X$ and $\epsilon(x) \in (\mathbb{R}^*)^X$ be constant on the elements of a partition α . By definition,

$$d\nu_\alpha(\xi u) = d\nu_\alpha(\xi_1 u_1, \dots, \xi_l u_l),$$

$$d\nu_\alpha(\epsilon \xi) = d\nu_\alpha(\epsilon_1 \xi_1, \dots, \epsilon_l \xi_l),$$

where $u_i = u|_{X_i}$, $\epsilon_i = \epsilon|_{X_i}$, X_i are elements of α .

It follows immediately from the explicit expression (4.11) for $d\nu_\alpha$ that $d\nu_\alpha(\xi u) = d\nu_\alpha(\xi)$ and $d\nu_\alpha(\epsilon\xi) = \prod_{i=1}^l |\xi_i|^{\lambda_i} d\nu_\alpha(\epsilon\xi)$, where $\lambda_i = m(X_i)$. It suffices to observe that $\prod_{i=1}^l |\xi_i|^{\lambda_i} = \int_X \log |\epsilon(x)| dm(x) d\nu(\xi)$. \square

5. CONSTRUCTION OF THE BASIC REPRESENTATION OF THE CURRENT GROUP G^X

5.1. The basic representation of the block triangular group B^X . First let us describe the representation of the subgroup $B^X = Z^X \ltimes D^X$ of block triangular matrices. We will write elements of this subgroup as triples

$$g(x) = (\epsilon, u, \gamma), \quad \epsilon \in (\mathbb{R}^*)^X, \quad u(x) \in O(n-1)^X, \quad \gamma \in (\mathbb{R}^{n-1})^X.$$

In this notation, the product of group elements takes the form

$$(\epsilon_1, u_1, \gamma_1) (\epsilon_2, u_2, \gamma_2) = (\epsilon_1 \epsilon_2, u_1 u_2, \gamma_1 + \epsilon_1 \gamma_2 u_1^{-1}).$$

The representation of the group B^X is realized in the Hilbert space $L^2(\nu)$ of all functions on Φ square-integrable with respect to the measure ν introduced above.

Let us associate with elements of the subgroup B^X the following operators U_g in the space of functions $f(\xi)$ on Φ :

$$U_{\epsilon, u, \gamma} f(\xi) = e^{1/2 \int_X \log |\epsilon(x)| dm(x) + i \langle \xi, \gamma \rangle} f(\epsilon \xi u). \quad (5.1)$$

In particular, elements $z = z(\gamma) \in Z^X$ give rise to the operators

$$U_z f(\xi) = e^{i \langle \xi, \gamma \rangle} f(\xi),$$

where $\gamma \in (\mathbb{R}^{n-1})^X$ is the parameter of the block matrix z , and elements $d = \text{diag}(\epsilon^{-1}, u, \epsilon) \in D^X$, to the operators

$$U_d f(\xi) = e^{1/2 \int_X \log |\epsilon(x)| dm(x)} f(\epsilon \xi u).$$

Note that the integral $\int_X \log |\epsilon(x)| dm(x)$ converges, because the elements $\epsilon(x)$ and $\epsilon^{-1}(x)$ of the matrix d are bounded functions on X .

Theorem 5.1. *The operators U_g , $g \in B^X$, given by (5.1) are unitary with respect to the norm in $L^2(\nu)$ and form an irreducible representation of the group B^X in $L^2(\nu)$.*

Indeed, the group property of the operators U_g follows immediately from their definition, and the unitarity follows from the invariance properties of the measure ν established in Theorem 4.4. The representation U_g is irreducible, because the action of the subgroup Z^X in $L^2(\nu)$ is ergodic and the algebra of multipliers is maximal.

5.2. Extension of the representation U_g to the whole group G^X . In order to extend the representation U_g from the subgroup B^X to the whole group G^X , we use the following obvious lemma.

Lemma. *The group $O(n, 1)^X$ is algebraically generated by the subgroup B^X and the unique element $g(x) \equiv s$.*

For example, every element g of the block upper triangular subgroup can be represented in the form $g = sg^*s$, where $g^* \in B^X$.

By this lemma, every operator U_g , $g \in G^X$, can be represented as the product of an operator from the subgroup B^X described above and the operator U_s ; hence, in order to define the representation of the whole group G^X , it suffices to describe only the operator U_s .

We call U_s the involution operator in the space $L^2(\nu)$ and denote it by I .

Proposition 5.1. *The operator $I = U_s$ and the operators U_g , $g \in B^X$, satisfy the following relations:*

$$IU_d = U_{sd}I \quad \text{for every } d \in D^X, \quad (5.2)$$

$$U_{z(\gamma)}I = U_{d(\gamma)}IU_{z(-\gamma)}IU_{z(j\gamma)} \quad \text{for every } z(\gamma) \in Z^X, \quad (5.3)$$

where $j\gamma = -\frac{2\gamma}{|\gamma|^2}$ (involution) and $d(\gamma) \in D^X$ is given by

$$d(\gamma) = \text{diag}\left(-\frac{2}{|\gamma|^2}, u_\gamma, -\frac{|\gamma|^2}{2}\right), \quad u_\gamma = e - \frac{2\gamma^*\gamma}{|\gamma|^2}. \quad (5.4)$$

Indeed, (5.2) is obvious and (5.3) follows from the corresponding relation in the group $O(n, 1)^X$:

$$z(\gamma)s = d(\gamma)sz(-\gamma)sz(j\gamma).$$

Note that for every $x \in X$ the matrix $u_\gamma(x)$ determines the reflection in \mathbb{R}^{n-1} with respect to the hyperplane orthogonal to the vector $\gamma(x)$.

Relations (5.2) and (5.3) uniquely determine the operator I ; however, they do not give an explicit expression for this operator.

An explicit description for I can be obtained from the description of the operator T_s in the commutative model of complementary series representations of the group $O(n, 1)$. Namely, let us associate with each partition $\alpha : X = \bigcup_{i=1}^l X_i$ the Hilbert space $L^2(\nu_\alpha) \subset L^2(\nu)$ of functions on Φ_α square-integrable with respect to the measure ν_α . The spaces $L^2(\nu_\alpha)$ are invariant under I , and in order to describe the operator I on the whole space $L^2(\nu)$, it suffices to describe its action on each of these subspaces. By construction, each space $L^2(\nu_\alpha)$ is isomorphic to the tensor product of finitely many spaces on which the commutative model of a complementary series representation of the group $O(n, 1)$ acts.

Proposition 5.2. *On each subspace $L^2(\nu_\alpha)$, the involution operator I coincides with the operator T_s of the representation of the group $O(n, 1)$ in this subspace.*

Starting from formulas (3.15) and (3.16) for the operator T_s in the commutative model of complementary series representations of the group $O(n, 1)$, we obtain the following theorem.

Theorem 5.2. *For every partition $\alpha : X = \bigcup_{i=1}^l X_i$, the action of the operator I on the subspace $L^2(\nu_\alpha)$ is given by the following formula:*

$$If(\xi_1, \dots, \xi_l) = \int_{(\mathbb{R}^{n-1})^l} \left(f(\xi'_1, \dots, \xi'_l) \prod_{i=1}^l A^{\lambda_i}(\xi_i, \xi'_i) d\xi_i \right), \quad \xi_i \in \mathbb{R}^{n-1}, \quad \lambda_i = m(X_i),$$

where

$$A^\lambda(\xi, \xi') = c_n 2^{-\lambda/2} \int_0^\infty r^{\lambda-n} |r\xi + 2r^{-1}\xi'|^{-\frac{n-3}{2}} J_{\frac{n-3}{2}}(|r\xi + 2r^{-1}\xi'|) dr.$$

We see that the operator I is well defined on the whole space $L^2(\nu)$, because on all subspaces $L^2(\nu_\alpha)$ it is unitary and satisfies the required relations. Thus we have constructed an irreducible unitary representation of the group G^X in the space $L^2(\nu)$.

Theorem 5.3. *The constructed representation U_g of the group G^X is equivalent to the basic representation of this group introduced in [5].*

Proof. The representations are equivalent, because their spherical functions coincide on the subgroup Z^X . Namely, set $\varphi(\xi) = v^{-1/2}(\xi)$, where $v(\xi)$ is the density of the measure ν with respect to μ . The function φ belongs to the space $L^2(\nu)$, is of norm 1, and satisfies the equation

$$\langle U_{z(\gamma)} v^{-1/2}, v^{-1/2} \rangle = \Psi(\gamma(\cdot)), \quad z(\gamma) \in Z^X,$$

where $\Psi(\gamma(\cdot))$ is the restriction of the spherical function of the basic representation of G^X to Z^X . Indeed,

$$\langle U_{z(\gamma)} v^{-1/2}, v^{-1/2} \rangle = \int_{\Phi} e^{i\langle \xi, \gamma \rangle} v^{-1}(\xi) d\nu(\xi) = \int_{\Phi} e^{i\langle \xi, \gamma \rangle} d\mu(\xi) = \Psi(\gamma(\cdot)).$$

□

Remark. One can also construct a commutative model of the basic representation of the group G^X in the L^2 space over the probability measure μ . However, μ is only quasi-invariant under the transformations $\xi \rightarrow \epsilon\xi$, $\epsilon \in (\mathbb{R}^*)^X$. Thus, in order to obtain a unitary representation, one must introduce an additional factor. In the case of the σ -finite measure ν , there is no need to do this.

6. APPENDICES

6.1. Approximative construction of the commutative model. Let us give another, independent construction of the commutative model of the basic representation of the group G^X . With each finite partition $\alpha : X = \bigcup_{i=1}^l X_i$ of the space X associate the Hilbert space $L^2(\nu_\alpha)$ of functions on Φ_α introduced above and the subgroup $G_\alpha^X \subset G^X$ of functions constant on the elements of α .

For $\beta \geq \alpha$, there is a natural embedding of groups $G_\alpha^X \rightarrow G_\beta^X$ and a natural isometric embedding of spaces $L^2(\nu_\alpha) \rightarrow L^2(\nu_\beta)$. Denote by L_0 the inductive limit of the subspaces $L^2(\nu_\alpha)$, and by G_0^X the inductive limit of the subgroups G_α^X . Note that the group G_0^X is everywhere dense in G^X .

We will define a unitary representation U_g of the group G^X on the completion L of the space L_0 with respect to the norm of L_0 . In order to define it, it suffices to describe the action of the operators U_g , $g \in G_0^X$, on the subspace L_0 .

Proposition 6.1. *The space $L^2(\nu_\alpha)$ is the tensor product*

$$L^2(\nu_\alpha) = \bigotimes_{i=1}^l L_{\lambda_i}, \quad \lambda_i = m(X_i), \quad (6.1)$$

where L_{λ_i} are the Hilbert spaces defined in Theorem 3.1.

Starting from the decomposition (6.1), define a unitary representation U_g of the group G_α^X in $L^2(\nu_\alpha)$ by the formula

$$U_{g(x)} = T_{g_1}^{\lambda_1} \otimes \dots \otimes T_{g_l}^{\lambda_l}, \quad g_i = g(x)|_{X_i}, \quad (6.2)$$

where T^{λ_i} are operators of the complementary series representations of the group $G = O(n, 1)$ in the spaces L_{λ_i} defined in Theorem 3.1.

Proposition 6.2. *For $\beta \geq \alpha$, the embedding $L^2(\nu_\alpha) \rightarrow L^2(\nu_\beta)$ commutes with the representations \tilde{U}_g of the groups G_α^X and G_β^X in these spaces.*

Corollary. *The unitary representations U_g of the groups G_α^X in the spaces $L^2(\nu_\alpha)$ are coherent and hence generate a unitary representation of the group G_0^X in L_0 .*

Theorem 6.1. *The constructed representation U_g of the group G^X is irreducible and equivalent to the basic representation of this group defined in [15, 5].*

6.2. The dual description of the representation U_g . Let us give a description of the representation U_g in terms of the Fourier transform sending functions $f(\xi)$ on Φ to functions $\varphi(\gamma)$ on \mathcal{F} :

$$\mathcal{R}f(\gamma) = \int_{\Phi} f(\xi) e^{i\langle \xi, \gamma \rangle} d\nu(\xi), \quad \gamma \in \mathcal{F}.$$

The operator \mathcal{R} is defined on an everywhere dense subset of functions $f \in L^2(\nu)$; in particular, if the integral $\int_{\Phi} |f(\xi)| d\nu(\xi)$ converges, then the function $\mathcal{R}f(\gamma)$ is defined on the whole space \mathcal{F} .

Theorem 6.2. *Let $f_1 \in L^2(\nu)$, $g \in G^X$, and $f_2 = U_g f_1$. Then on the subset of $\gamma \in \mathcal{F}$ for which the functions $\mathcal{R}f_1$ and $\mathcal{R}f_2$ are defined, they satisfy the relation*

$$\mathcal{R}f_2(\gamma) = \mathcal{R}f_1(\gamma \bar{g}) e^{-1/2 \int_X \log \beta(\gamma(x), g(x)) dm(x)}, \quad (6.3)$$

where, according to formulas (3.1) and (3.2) for the group $G = O(n, 1)$,

$$\begin{aligned} & \gamma(x) \overline{g(x)} \\ &= \left(-\frac{|\gamma(x)|^2}{2} g_{13}(x) + \gamma(x) g_{23}(x) + g_{33}(x) \right)^{-1} \left(-\frac{|\gamma(x)|^2}{2} g_{12}(x) + \gamma(x) g_{22}(x) + g_{32}(x) \right), \\ & \beta(\gamma(x), g(x)) = \left| -\frac{|\gamma(x)|^2}{2} g_{12}(x) + \gamma(x) g_{22}(x) + g_{32}(x) \right|. \end{aligned}$$

In particular,

$$\mathcal{R}f_2(\gamma) = \mathcal{R}f_1(\gamma + \gamma_0) \quad \text{for } g = z(\gamma_0) \in Z^X; \quad (6.4)$$

$$\mathcal{R}f_2(\gamma) = \mathcal{R}f_1(\epsilon^{-1} \gamma u) e^{-1/2 \int_X \log |\epsilon(x)| dm(x)} \quad \text{for } g = \text{diag}(\epsilon^{-1}, u, \epsilon) \in D^X;$$

$$\mathcal{R}f_2(\gamma) = 2^{1/2m(X)} \mathcal{R}f_1\left(-\frac{2\gamma}{|\gamma|^2}\right) e^{-\int_X \log |\gamma(x)| dm(x)} \quad \text{for } g(x) \equiv s.$$

Proof. It suffices to prove (6.3) for functions $f_1(\xi) = f_1(\xi^1, \dots, \xi^l)$ from $L^2(\nu_\alpha)$, where $\alpha : \bigcup_{i=1}^l X_i$ is an arbitrary finite partition of the space X , and matrices $g(x)$ that are constant on the elements of α .

We have

$$\mathcal{R}f_i(\gamma) = \mathcal{R}f_i(\gamma^1, \dots, \gamma^l) = \int_{\Phi_\alpha} f_i(\xi^1, \dots, \xi^l) e^{i\langle \xi, \gamma \rangle} d\nu_\alpha(\xi^1, \dots, \xi^l), \quad i = 1, 2,$$

where the measure $d\nu_\alpha(\xi^1, \dots, \xi^l)$ is given by (4.11). Since

$$\int_{\Phi_\alpha} e^{i\langle \xi, \gamma \rangle} d\nu(\xi^1, \dots, \xi^l) = \prod_{i=1}^l |\gamma^i|^{-\lambda_i}, \quad \lambda_i = m(X_i),$$

it follows that

$$\mathcal{R}f_i(\gamma^1, \dots, \gamma^l) = \int_{\mathcal{F}_\alpha} \varphi_i(\zeta^1, \dots, \zeta^l) \prod_{i=1}^l |\gamma^i - \zeta^i|^{-\lambda_i} d\zeta^i, \quad i = 1, 2, \quad (6.5)$$

where

$$\varphi_i(\zeta^1, \dots, \zeta^l) = \int_{\mathcal{F}_\alpha} \prod_{i=1}^l e^{i\langle \xi^i, \zeta^i \rangle} f_i(\gamma^1, \dots, \gamma^l) d\zeta^i.$$

The equation $f_2 = U_g f_1$ and formula (3.5) for operators of the complementary series representations of $O(n, 1)$ imply that the functions φ_i satisfy the relations

$$\varphi_2(\zeta^1, \dots, \zeta^l) = \varphi_1(\zeta^1 \bar{g}_1, \dots, \zeta^l \bar{g}_l) \prod_{i=1}^l \beta^{1-n+\frac{\lambda_i}{2}}(\gamma^i, g_i),$$

where $g_i = g|_{X_i}$. Thus

$$\mathcal{R}f_2(\gamma^1, \dots, \gamma^l) = \int_{\mathcal{F}_\alpha} \varphi_1(\zeta^1 \bar{g}_1, \dots, \zeta^l \bar{g}_l) \prod_{i=1}^l \beta^{1-n+\frac{\lambda_i}{2}}(\gamma^i, g_i) |\gamma^i - \zeta^i|^{-\lambda_i} d\zeta^i.$$

Applying the transformation $\zeta_i \bar{g}_i \rightarrow \zeta_i$, we obtain

$$\mathcal{R}f_2(\gamma^1, \dots, \gamma^l) = \prod_{i=1}^l \beta^{-\lambda_i/2}(\gamma^i, g_i) = \int_{\mathcal{F}_\alpha} \varphi_1(\zeta^1, \dots, \zeta^l) \prod_{i=1}^l |\gamma^i \bar{g}_i - \zeta^i|^{-\lambda_i} d\zeta^i,$$

i.e.,

$$\mathcal{R}f_2(\gamma^1, \dots, \gamma^l) = \left(\prod_{i=1}^l \beta^{-\lambda_i/2}(\gamma^i, g_i) \right) \mathcal{R}f_1(\gamma^1 \bar{g}_1, \dots, \gamma^l \bar{g}_l).$$

It remains to observe that

$$\prod_{i=1}^l \beta^{-\lambda_i/2}(\gamma^i, g_i) = e^{-1/2 \int_X \log \beta(\gamma(x), g(x)) dm(x)}.$$

□

Corollary. *On the set of all $\gamma \in \mathcal{F}$ for which the functions $\mathcal{R}f$, $f \in L^2(\nu)$, and $\mathcal{R}If$ are defined, they satisfy the relation*

$$\mathcal{R}If(\gamma) = 2^{1/2m(X)} \mathcal{R}f\left(-\frac{2\gamma}{|\gamma|^2}\right) e^{-\int_X \log|\gamma(x)| dm(x)} \quad \text{for } g(x) \equiv s. \quad (6.6)$$

6.3. On the properties of the measures μ and ν .

6.3.1. Consider the subgroup $O(n-1)^X$ of the current group $O(n, 1)^X$. For $n = 2$, this is the group of functions on X taking values $+1$ and -1 .

The subgroup $O(n-1)^X$ acts pointwise in the space of vector distributions of dimension $n-1$ on the manifold X , and the measures μ and ν are invariant under this action.

Note that this action is not free. Indeed, since almost every, with respect to μ and ν , realization is a linear combination of a countable family of δ -measures, it follows that for every distribution ξ , only the compact quotient group of $O(n-1)^X$ that consists of the restrictions of currents to the countable support of ξ acts freely on ξ . Therefore the orbit is compact as the product of a countable family of $(n-1)$ -dimensional spheres and hence has an invariant (product) measure.

Thus almost every ergodic component of the action of $O(n-1)^X$ consists of distributions with equal values of the norm at all points of their common support, i.e., the function

$$x \rightarrow \|c(x)\|, \quad c(x) = (c_1(x), \dots, c_{n-1}(x)),$$

is an invariant of the orbit.

At the same time, as noted above, the action of the group of homotheties together with rotations, i.e., the group $(\mathbb{R}_+ \times O(n-1))^X$, on the space Φ is already ergodic. Recall that, as was proved in [13] for the case $n = 2$, ν is the unique, up to normalization, measure that is invariant and ergodic (Theorem 5). Apparently, a similar theorem holds for an arbitrary n .

6.3.2. The results of this paper can be translated to the infinite-dimensional group $O(\infty, 1)^X$, because (as observed by G. Olshanski) all our products involve the dimension n in a controllable way. In other words, for different n only the natural dimension $n - 1$ of the space of vector distribution changes, but the form of the restriction of the spherical function to the subgroup Z^X , and hence the characteristic functionals of the measures μ and ν remain the same. Therefore our theory can be considered for infinite n .

In this case, μ -almost every vector distribution is still a countable linear combination of δ -measures on X , but taking values in an infinite-dimensional Hilbert space equipped with a mixture of Gaussian measures with some weight.

The role of the group $O(n-1)^X$ is played by the group $O(\infty)^X$ of orthogonal matrices of the form $I + K$, where K is a finite-dimensional operator. Remarkably, the decomposition into ergodic components under the action of the group $O(\infty)^X$ is the decomposition into the Gaussian measures with characteristic functionals of the form $e^{-c\|x\|^2}$, where c is distributed on $(0, \infty)$ according to the measure with density e^{-c} .

6.4. Remark on the group $U(n, 1)^X$. The above construction of an irreducible unitary representation of the subgroup B^X of $O(n, 1)^X$ in the space $L^2(\mu)$ can also be used for the group $U(n, 1)^X$. However, in this case there appears a new phenomenon.

Like $O(n, 1)$, we realize $U(n, 1)$ as the group of linear transformations in \mathbb{C}^{n+1} preserving the Hermitian form

$$x_1 \overline{x_{n+1}} + x_2 \overline{x_2} + \dots + x_n \overline{x_n}$$

and represent its elements as block matrices. In this realization, $B^X \subset U(n, 1)^X$ is the semidirect product $B^X = Z^X \ltimes D^X$, where Z^X is the group of matrices of the form

$$z = \begin{pmatrix} 1 & 0 & 0 \\ -\gamma^* & e & 0 \\ it - \frac{|\gamma|^2}{2} & \gamma & 1 \end{pmatrix}, \quad t \in \mathbb{R}^X, \quad \gamma \in (\mathbb{C}^{n-1})^X$$

(the Heisenberg group) and D^X is the subgroup of block diagonal matrices

$$d = \text{diag}(\bar{\epsilon}^{-1}, u, \epsilon), \quad \epsilon \in (\mathbb{C}^*)^X, \quad u \in U(n-1)^X.$$

Accordingly, elements of the group B^X are 4-tuples $b = (t, \gamma, \epsilon, u)$. If we now try to use a formula similar to (5.1),

$$U_b f(\xi) = e^{1/2 \int_X \log |\epsilon(x)| dm(x) + i \text{Re} \langle \xi, \gamma \rangle} f(\epsilon \xi u), \quad b = (t, \gamma, \epsilon, u),$$

for representation operators in the space $L^2(\nu)$, where the measure ν is constructed as above, then we will find out that this representation is not a *faithful representation* of the group B^X , and it is faithful only on the quotient group with respect to the center of the Heisenberg subgroup, since the operators of the representation do not involve the parameter t . Thus the direct translation of the construction does not use the symplectic structure on $\mathbb{C}^{n-1} \times \mathbb{C}^{n-1}$, which is used in the definition of the group $U(n, 1)$, and cannot be extended to the whole group $U(n, 1)^X$. Nevertheless, the construction can be modified appropriately; we will return to this question elsewhere.

REFERENCES

- [1] H. Araki, "Factorizable representations of current algebra," *Publ. Res. Inst. Math. Sci. Ser. A* **5** (1969/70), 361–422.
- [2] H. Bateman and A. Erdelyi, *Higher Transcendental Functions* [in Russian], Vol. 2, Moscow (1974).
- [3] F. A. Berezin, "Representations of the continuous direct product of universal coverings of the group of motions of a complex ball," *Tr. Mosk. Mat. Ob-va* **36**, (1978), 275–293. English transl. in: *Trans. Mosc. Math. Soc.* **36** (1979), 281–298.
- [4] I. M. Gelfand, M. I. Graev, and I. I. Pyatetskij-Shapiro, *Representation Theory and Automorphic Functions*, W. B. Saunders Company, Philadelphia (1969).
- [5] I. M. Gelfand, M. I. Graev, and A. M. Vershik, "Models of representations of current groups," in: *Representations of Lie Groups and Lie Algebras* (A. A. Kirillov, ed.), Akadémiai Kiadó, Budapest (1985), pp. 121–179.
- [6] I. M. Gelfand and N. Ya. Vilenkin, *Some Applications of Harmonic Analysis. Generalized Hilbert Spaces* [in Russian], Fizmatgiz, Moscow (1951).
- [7] I. I. Gikhman and A. V. Skorokhod, *Theory of Random Processes* [in Russian], Vol. 2, Nauka, Moscow (1973).
- [8] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Sums, Series, and Products* [in Russian], Fizmatgiz, Moscow (1962).
- [9] A. Guichardet, *Symmetric Hilbert Spaces and Related Topics*, Lecture Notes in Math. **261**, Springer-Verlag, Berlin–Heidelberg–New York (1972).
- [10] G. Kuhn and A. Vershik, "Canonical semigroups of states and cocycles for the group of automorphisms of a homogeneous tree," *Algebras and Representation Theory* **6**, No. 3 (2003), 333–352.

- [11] G. Olshanski, "Spherical functions and characters on the group $U(\infty)^*$," *Uspekhi Mat. Nauk* **37**, No. 2 (1982), 217–218. English transl. in: *Russian Math. Surveys* **37**, No. 2 (1982), 233–234.
- [12] Y. Shalom, "Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group," *Ann. of Math (2)*, **152**, No. 1 (2000), 113–182.
- [13] N. Tsilevich, A. Vershik, and M. Yor, "An infinite-dimensional analogue of the Lebesgue measure and distinguished properties of the gamma process," *J. Funct. Anal.* **185**, No. 1 (2001), 274–296.
- [14] A. M. Vershik, I. M. Gelfand, and M. I. Graev, "Representations of the group $SL(2, R)$, where R is a ring of functions," *Uspekhi Mat. Nauk* **28**, No. 5 (1973), 83–128. English transl. in: *Representation theory*, London Math. Soc. Lect. Note Ser. **69**, Cambridge Univ. Press (1982), pp. 15–60.
- [15] A. M. Vershik, I. M. Gelfand, and M. I. Graev, "Irreducible representations of the group G^X and cohomology," *Funkts. Anal. i Prilozh.* **8**, No. 2 (1974), 67–69. English transl. in: *Funct. Anal. Appl.* **8** (1974), 151–153.
- [16] A. M. Vershik, I. M. Gelfand, and M. I. Graev, "A commutative model of the basic representation of the group $SL(2, \mathbb{R})^X$ with respect to a unipotent subgroup," in: *Group Theoretical Methods in Physics*, Vol. II, Proc. Int. Semin., Zvenigorod (1983), pp. 472–487.
- [17] A. M. Vershik, I. M. Gelfand, and M. I. Graev, "Commutative model of the representation of the group of flows $SL(2, R)^X$ connected with a unipotent subgroup," *Funkts. Anal. Prilozh.* **17**, No. 2 (1983), 70–72. English transl. in: *Funct. Anal. Appl.* **17** (1983), 137–139.
- [18] A. M. Vershik and S. I. Karpushev, "Cohomology of groups in unitary representations, the neighborhood of the identity, and conditionally positive definite functions," *Mat. Sb.* **119**, No. 4 (1982), 521–533. English transl. in: *Math. in USSR* **47** (1984), 513–526.
- [19] A. M. Vershik and N. V. Tsilevich, "Fock factorizations, and decompositions of the L^2 spaces over general Lévy processes," *Uspekhi Mat. Nauk* **58**, No. 3 (2003), 3–50. English transl. in: *Russian Math. Surveys* **58**, No. 3 (2003), 427–472.

A. M. VERSHIK, ST. PETERSBURG DEPARTMENT OF STEKLOV INSTITUTE OF MATHEMATICS, 27 FONTANKA, 191023 ST. PETERSBURG, RUSSIA, ERWIN SCHRÖDINGER INST. 9, BOLTZMANGASSE, 1090 WIEN, AUSTRIA

E-mail address: `vershik@pdmi.ras.ru`

M. I. GRAEV, INSTITUTE FOR SYSTEM STUDIES, 36-1 NAKHIMOVSKY PR., 117218 MOSCOW, RUSSIA

E-mail address: `graev_36@mtu-net.ru`